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POSITIVE SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

Yisheng Huang

1. Introduction

In this paper we are concerned with the existence of positive solutions for a class of quasilinear elliptic equations of the form

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u + f(x,u,\lambda), \\ \qquad u \in \mathcal{D}_0^{1,p}(\Omega), \end{cases} \tag{1.1}$$

where Ω is an unbounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ $(p > 1)$ is the p-Laplacian, $\mathcal{D}_0^{1,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm $\|u\| = \{\int_\Omega |\nabla u|^p\}^{1/p}$, $0 < a(x) \in L^\infty(\Omega) \cap L^1(\Omega)$, $\lambda \geq 0$ is a real parameter and f satisfies some conditions to be given later.

It is not difficult to show that the eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u, \\ \qquad u \in \mathcal{D}_0^{1,p}(\Omega), \end{cases} \tag{1.2}$$

has the least eigenvalue $\lambda_1 > 0$ with a positive eigenfunction e_1 and λ_1 is the only eigenvalue having this property (cf. Proposition 3.1). This gives us a possibility to study the existence of an unbounded branch of positive solutions bifurcating from $(\lambda_1, 0)$. When Ω is bounded, the result is well-known, we refer to the survey article of Amann [2] and the paper of Ambrosetti and Hess [4] for the case $p = 2$,

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and to the recent paper of Ambrosetti, Azorero and Peral [3] for the general case $p > 1$. When $\Omega = \mathbb{R}^N$, the problem was studied by Drábek and Huang [10] in a situation where a and f may change sign. In [10] an extra assumption was needed that, roughly speaking, (1.1) has no nonzero solution for $\lambda = \lambda_1$ when u is small (see [10, (4.12) of Theorem 4.5]). It seems that this condition is essential in the proof in [10] even if a and f are positive. On the other hand, if Ω is bounded, we know (cf. [11, Theorem 1]) that when $h > 0$ satisfies appropriate conditions, the equation $-\Delta_p u = \lambda |u|^{p-2}u + h(x)$ has no solution for $\lambda = \lambda_1$, where λ_1 is the first eigenvalue of the equation $-\Delta_p u = \lambda |u|^{p-2}u$. A similar result is given in this paper when Ω is unbounded (see Lemma 3.5). Using this we will be able to obtain the existence of a branch of positive solutions without the assumption of Drábek and Huang mentioned above (see Theorem 3.2 for the details). Our approach in this paper is via a fixed point index that is based on the one of Amann [2], which we give in Section 2. In Section 4, using the fixed point index we established, we obtain several existence results for positive solutions of equations involving the p-Laplacian.

2. Preliminaries

Throughout this paper we denote by Ω an unbounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$. $X = \mathcal{D}_0^{1,p}(\Omega)$, where $p > 1$, is the completion of $C_0^\infty(\Omega)$ in the norm $\|u\| = \{\int_\Omega |\nabla u|^p\}^{1/p}$. We denote by $\langle \cdot\,, \cdot \rangle$ the duality pairing between X^* and X, and let $P = \{u \in X \mid u(x) \geq 0 \quad \text{a.e. in } \Omega\}$, $P^* = \{f \in X^* \mid \langle f, u\rangle \geq 0\ \forall u \in P\}$, $P_\varepsilon = \{u \in P \mid \|u\| < \varepsilon\}$. A mapping $F : X \to X^*$ is said to be completely continuous if it maps weakly convergent subsequences to strongly convergent ones.

Similarly as in Lemma 3.3 of [9], we have

Proposition 2.1. *Let $J : X \to X^*$ be a mapping defined by*

$$\langle J(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx, \quad \forall u, v \in X. \tag{2.1}$$

Then J is bounded (i.e., J maps bounded sets to bounded ones), strictly monotone and continuous. Furthermore, $J^{-1} : X^* \to X$ is bounded and continuous.

Proposition 2.2. *$J^{-1}(P^*) \subset P$.*

Proof. For all $h \in P^*$, we want to show the solution u of the equation $J(u) = h$ is nonnegative. We have

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \int_\Omega h \cdot v, \quad \forall v \in X.$$

Let $v = u^-$, where $u^- = max\{-u, 0\}$. Then $\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u^- = \int_\Omega hu^-$. Notice that

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla u^- = \int_{u\le 0} |\nabla u|^{p-2}\nabla u \cdot \nabla u^- = -\int_\Omega |\nabla u^-|^p = -\|u^-\|^p$$

which yields $-\|u^-\|^p = \int_\Omega h \cdot u^- \ge 0$, hence we obtain that $u^- = 0$ and $u \ge 0$.□

Now consider the operator equation

$$J(u) = F(u), \quad u \in P. \tag{2.2}$$

Since P is a closed convex subset of X, it is a retract of X. Let U be a bounded open subset of P. If $F : \overline{U} \to P^*$ is completely continuous and (2.2) has no solution on ∂U, then $J^{-1} \circ F : \overline{U} \to P$ is completely continuous and has no fixed point on ∂U. Therefore, according to Amann [2, Section 11], the fixed point index $\mathrm{i}(J^{-1} \circ F, U)$, where $\mathrm{i}(J^{-1} \circ F, U) = \deg(\mathrm{id} - J^{-1} \circ F \circ \rho, \rho^{-1}(U), 0)$ and $\rho : X \to P$ is an arbitrary retraction, is well defined.

We define the solution index of (2.2) relative to F, $\mathrm{ind}(F, U)$, by

$$\mathrm{ind}(F, U) = \mathrm{i}(J^{-1} \circ F, U).$$

The index $\mathrm{ind}(F, U)$ has the following properties which are an immediate consequence of the definition of $\mathrm{ind}(F, U)$ and the corresponding properties of the fixed point index (cf. [2, Section 11]).

PROPOSITION 2.3.

(i) *If $q \in J(U)$, then the constant mapping $F(u) \equiv q$ has index* $\mathrm{ind}(F, U) = 1$.
(ii) *If* $\mathrm{ind}(F, U) \neq 0$, *then* (2.2) *has a solution $u \in U$.*
(iii) *For every open subset $V \subset U$ such that* (2.2) *has no solution in $\overline{U} \setminus V$,* $\mathrm{ind}(F, U) = \mathrm{ind}(F, V)$.
(iv) *For every pair of disjoint open subsets U_1, U_2 of U such that the equation* (2.2) *has no solution on $\overline{U} \setminus (U_1 \cup U_2)$,* $\mathrm{ind}(F, U) = \mathrm{ind}(F, U_1) + \mathrm{ind}(F, U_2)$.
(v) *For every compact interval I and every completely continuous homotopy $H : I \times \overline{U} \to P^*$ such that the equation $J(u) = H(t, u)$ has no solution for $(t, u) \in I \times \partial U$, the index* $\mathrm{ind}(H(\,\cdot\,, u))$ *is independent of $t \in I$.*
(vi) *Let Λ be a nonempty compact interval and U a bounded open subset of $\Lambda \times P$. For a fixed $\lambda \in \Lambda$, we denote $U_\lambda = \{u \in P \mid (\lambda, u) \in U\}$ (the slice of U at λ). If $h : \overline{U} \to P^*$ is completely continuous and the equation $J(u) = h(\lambda, u)$ has no solution for $(\lambda, u) \in \partial U$, then* $\mathrm{ind}(h(\lambda, \cdot\,), U_\lambda)$ *is well-defined and independent of $\lambda \in \Lambda$.*

As a consequence of Proposition 2.3, we give a result which will be used later.

PROPOSITION 2.4. *Let P, J be as above, U a bounded open subset of P, $0 \in U$, and $Q : \overline{U} \to P^*$ a completely continuous mapping. Suppose that*

$$\langle J(u), u\rangle > \langle Q(u), u\rangle \quad \forall u \in \partial U.$$

Then $\operatorname{ind}(Q, U) = 1$.

PROOF. Since $0 \in U$, $0 = J(0) \in J(U)$ and we see by (i) of Proposition 2.3 that $\operatorname{ind}(0, U) = 1$. Set $H(t, u) = tQ(u)$. Then

$$\langle J(u) - tQ(u), u\rangle = (1 - t)\langle J(u), u\rangle + t\langle J(u) - Q(u), u\rangle > 0 \quad \forall u \in \partial U$$

since $\langle J(u), u\rangle > 0$ unless $u = 0$. Thus we obtain that the equation $J(u) = H(t, u)$ has no solutions on $[0, 1] \times \partial U$, and this implies by (v) of Proposition 2.3 that $\operatorname{ind}(Q, U) = \operatorname{ind}(0, U) = 1$. □

Let $F : \mathbb{R}_+ \times P \to P^*$ and consider the equation

$$J(u) = F(\lambda, u), \quad (\lambda, u) \in \mathbb{R}_+ \times P. \tag{2.3}$$

Suppose that $F(\lambda, 0) = 0$ for all $\lambda \in \mathbb{R}_+$. Then the pairs $(\lambda, 0) \in \mathbb{R}_+ \times P$ are solutions of (2.3); they will be called the trivial solutions. $(\lambda, 0) \in \mathbb{R}_+ \times P$ is said to be a bifurcation point of (2.3) if there exists a sequence $\{(\lambda_n, u_n)\}$ of solutions of (2.3) such that $u_n \neq 0$ and $(\lambda_n, u_n) \to (\lambda, 0)$.

PROPOSITION 2.5. *Let F be a completely continuous mapping with $F(0, u) = F(\lambda, 0) = 0$. Suppose that there is a positive number λ_0 such that if $\lambda > \lambda_0$, then $(\lambda, 0)$ is not a bifurcation point for equation* (2.3) *and* $\operatorname{ind}(F(\lambda, \cdot\,), P_\varepsilon) = 0$ *for all ε small enough. Then there exists $\lambda_1 \in [0, \lambda_0]$ such that the set of nontrivial solutions of* (2.3) *contains an unbounded subcontinuum bifurcating from* $(\lambda_1, 0)$.

PROOF. Let Σ^+ be the closure of the set of nontrivial solutions of (2.3) in $\mathbb{R}_+ \times P$ and $\mathcal{C}$ the component of $\Sigma^+ \cup ([0, \lambda_0] \times \{0\})$ containing $[0, \lambda_0] \times \{0\}$. Suppose that $\mathcal{C}$ is bounded, then there exist $r > 0$ and $\mu > \lambda_0$ such that the boundary of $[0, \mu] \times \overline{P_r}$ (in $\mathbb{R}_+ \times P$) does not meet $\mathcal{C}$. Let $\mathcal{C}_1 = \mathcal{C} \cup ([0, \mu] \times \{0\})$, then there exists a bounded open subset U of $[0, \mu] \times P$ such that $\mathcal{C}_1 \subset U$ and (2.3) has no solution for $(\lambda, u) \in \partial U \cup (\{\mu\} \times (U_\mu \backslash \{0\}))$ (this follows from a well-known argument in bifurcation theory, see e.g. [2, proof of Theorem 18.3]). If ε is small enough, $\overline{P_\varepsilon} \subset U_\mu$ and hence, by (i), (iii) and (vi) of Proposition 2.3

$$1 = \operatorname{ind}(F(0, \cdot\,), U_0) = \operatorname{ind}(F(\mu, \cdot\,), U_\mu) = \operatorname{ind}(F(\mu, \cdot\,), P_\varepsilon).$$

This contradicts the assumption that $\operatorname{ind}(F(\lambda, \cdot\,), P_\varepsilon) = 0$ for $\lambda > \lambda_0$ and sufficiently small ε. □

3. Bifurcation of Positive Solutions

In this section we consider the equation

$$\begin{cases} -\Delta_p u = \lambda a(x) u^{p-1} + f(x,u,\lambda), \\ \quad u \geq 0 \text{ in } \Omega, \\ \quad u \in \mathcal{D}_0^{1,p}(\Omega). \end{cases} \tag{3.1}$$

Let $1 < p < N$, denote $p^* = Np/(N-p)$ and $p' = p/(p-1)$. We assume $0 < a(x) \in L^\infty(\Omega) \cap L^1(\Omega)$ and f satisfies

(f1) $f : \Omega \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a Carathéodory function, i.e., $f(x,\cdot\,,\cdot\,)$ is continuous for a.e. $x \in \Omega$ and $f(\,\cdot\,, s, \lambda)$ is measurable for all $(s,\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+$;

(f2) $f(x,s,\lambda) \leq c(\lambda)(\sigma(x) + \rho(x)\, s^{q-1})$ for a.e. $x \in \Omega$ and $s \in \mathbb{R}_+$, where $c(\lambda) \geq 0$ is continuous on $\mathbb{R}_+$, $p < q < p^*$, $0 \leq \rho(x) \in L^r(\Omega) \cap L^\infty_{\text{loc}}(\Omega)$, where $r = p^*/(p^*-q)$, $0 \leq \sigma(x) \in L^{(p^*)'}(\Omega) \cap L^{N/p}(\Omega)$;

(f3) the following limit exists:
$$\lim_{s \to 0^+} \frac{f(x,s,\lambda)}{a(x)\, s^{p-1}} = 0$$
uniformly with respect to a.e. $x \in \Omega$ and λ on bounded intervals.

By a solution of (3.1) we understand a pair $(\lambda, u) \in \mathbb{R}_+ \times P$ satisfying (3.1) in the weak sense, i.e.,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v = \int_\Omega (\lambda a(x) u^{p-1} + f(x,u,\lambda)) v, \quad \forall v \in X.$$

We define the operator $F : \mathbb{R}_+ \times P \to P^*$ as

$$F = \lambda G_1 + G_2, \tag{3.2}$$

where the operators $G_1 : P \to P^*$, $G_2 : \mathbb{R}_+ \times P \to P^*$ are given by

$$\langle G_1(u), v \rangle = \int_\Omega a(x) u^{p-1} v \quad \forall v \in X, \tag{3.3}$$

$$\langle G_2(\lambda, u), v \rangle = \int_\Omega f(x,u,\lambda) v \quad \forall v \in X. \tag{3.4}$$

Under conditions (f1) and (f2), we shall show that G_1 and G_2 are well defined and completely continuous, hence so is F. Using Hölder's and Sobolev's inequalities, we have

$$\begin{aligned} |\langle G_1(u), v \rangle| &\leq \int_\Omega a u^{p-1}\, |v| \leq \left(\int_\Omega a\, u^p \right)^{1/p'} \left(\int_\Omega a\, |v|^p \right)^{1/p} \\ &\leq \left(\int_\Omega a^{N/p} \right)^{p/N} \left(\int_\Omega u^{p^*} \right)^{(p-1)/p^*} \left(\int_\Omega |v|^{p^*} \right)^{1/p^*} \\ &\leq c_1\, \|u\|^{p-1} \|v\|, \end{aligned} \tag{3.5}$$

which yields that G_1 is well defined. For G_2 we have

$$|\langle G_2(\lambda, u), v\rangle| \le c(\lambda)\bigg(\int_\Omega \sigma\,|v| + \int_\Omega \rho\, u^{q-1}\,|v|\bigg).$$

By (f_2),

$$\int_\Omega \sigma|v| \le \bigg(\int_\Omega \sigma^{(p^*)'}\bigg)^{1/(p^*)'}\bigg(\int_\Omega |v|^{p^*}\bigg)^{1/p^*} \le c_2\|v\| \tag{3.6}$$

and

$$\begin{aligned}\int_\Omega \rho u^{q-1}|v| \le&\bigg(\int_\Omega \rho^{(p^*)'}u^{(q-1)(p^*)'}\bigg)^{1/(p^*)'}\bigg(\int_\Omega |v|^{p^*}\bigg)^{1/p^*}\\ \le&\bigg(\int_\Omega \rho^r\bigg)^{1/r}\bigg(\int_\Omega u^{p^*}\bigg)^{(q-1)/p^*}\bigg(\int_\Omega |v|^{p^*}\bigg)^{1/p^*}\\ \le& c_3\|u\|^{q-1}\|v\|.\end{aligned} \tag{3.7}$$

Hence G_2 is well defined. We will show the complete continuity of G_2.

Let $u_n \rightharpoonup u_0$ in X. Denote $\Sigma_k = \Omega \cap B(0,K)$, where $B(0,K)$ is the ball centered at 0 and having radius $K > 0$. We get

$$\begin{aligned}\|G_2(\lambda, u_n) - G_2(\lambda, u_0)\|_* &= \sup_{\|v\|\le 1} |\langle G_2(\lambda, u_n) - G_2(\lambda, u_0), v\rangle|\\ &\le \sup_{\|v\|\le 1}\int_{\Sigma_K} |f(x,u_n,\lambda) - f(x,u_0,\lambda)||v|\\ &\quad + \sup_{\|v\|\le 1}\int_{\Omega\setminus\Sigma_K} |f(x,u_n,\lambda) - f(x,u_0,\lambda)||v|.\end{aligned} \tag{3.8}$$

Noting that $\{u_n\}$ is bounded, we obtain as in (3.6) and (3.7) that

$$\begin{aligned}&\sup_{\|v\|\le 1}\int_{\Omega\setminus\Sigma_K} |f(x,u_n,\lambda) - f(x,u_0,\lambda)||v|\\ &\qquad \le c_4\bigg(\int_{\Omega\setminus\Sigma_K} \sigma^{(p^*)'}\bigg)^{1/(p^*)'} + c_5\bigg(\int_{\Omega\setminus\Sigma_K} \rho^r\bigg)^{1/r},\end{aligned} \tag{3.9}$$

where c_4 and c_5 are constants independent of K and n. For all $\varepsilon > 0$ we can choose K such that the right-hand side of (3.9) is $< \varepsilon/2$. By the compact embedding theorem, going if necessary to a subsequence, we can assume that $u_n \to u_0$ in $L^s(\Sigma_K)$, where $s = (q-1)(p^*)'$ (note that $s < p^*$). Using the continuity of the Nemytskiĭ operator $u \mapsto f(x,u,\lambda)$ from $L^s(\Sigma_K)$ to $L^{(p^*)'}(\Sigma_K)$ (cf. [12, Theorem 2.1]), we can choose N_0 so that

$$\begin{aligned}&\sup_{\|v\|\le 1}\int_{\Sigma_K} |f(x,u_n,\lambda) - f(x,u_0,\lambda)|\,|v|\\ &\qquad \le c_6\bigg(\int_{\Sigma_K} |f(x,u_n,\lambda) - f(x,u_0,\lambda)|^{(p^*)'}\bigg)^{1/(p^*)'} < \varepsilon/2\end{aligned}$$

if $n > N_0$. Thus G_2 is completely continuous. Since $a(x) \in L^\infty(\Omega) \cap L^{N/p}(\Omega)$, using the same argument we get that G_1 is completely continuous.

We notice that the existence of the first eigenvalue λ_1 of the equation (1.2) can be established by solving the constrained variational problem

$$\lambda_1 = \inf\bigg\{ \int_\Omega |\nabla u|^p \mid \int_\Omega a|u|^p = 1,\ u \in X \bigg\}. \tag{3.10}$$

Indeed, $\lambda_1 > 0$ is obvious by (3.10) and Sobolev's inequality. The boundedness of a minimizing sequence $\{u_n\}$ for (3.10) and the weak continuity of the functional $u \mapsto \int_\Omega a|u|^p$ (cf. [6, Proposition 2.1]) imply that there exists some $u_0 \in X$ for which the infimum in (3.10) is attained, and then u_0 is a (weak) solution of (1.2) by the Euler–Lagrange principle. If u_0 minimizes (3.10), so does $|u_0|$. Hence it can be assumed that $u_0 \geq 0$, and then $u_0 > 0$ in Ω by Harnack's inequality [15, Theorem 1.1]. Thus there exists a positive eigenfunction corresponding to λ_1. Using the same argument as in [13] (where Ω was assumed to be bounded and $a \equiv 1$) we can show that λ_1 is simple and there are no other eigenvalues having nonnegative eigenfunctions, here we have used that $a(x) \in L^\infty(\Omega) \cap L^1(\Omega)$. Therefore, we get

Proposition 3.1.

(i) *The first eigenvalue* λ_1 *of* (1.2) *is positive and simple.*
(ii) *The corresponding eigenfunction* e_1 *can be chosen so that* $e_1 > 0$ *in* Ω*; moreover,* λ_1 *is the only eigenvalue having an eigenfunction not changing sign in* Ω.

The main result of this section is the following theorem.

Theorem 3.2. *We suppose that* f *satisfies the conditions* (f1)–(f3) *and* $f(x,s,0) = 0$. *Then the set of nontrivial solutions of* (3.1) *contains an unbounded subcontinuum bifurcating from* $(\lambda_1, 0)$.

Before proving Theorem 3.2, we show the following results.

Lemma 3.3. *There exists a sequence* $\{\Omega_n\}$ *of open bounded subsets of* Ω *such that* $\Omega = \bigcup_{n\geq 1} \Omega_n$, $\Omega_n \subset \Omega_{n+1}$ *and* $\partial\Omega_n$ *is smooth.*

This result is well-known but since we could not find any convenient reference, we give a brief proof below.

Proof. For each $n \in \mathbb{N}$, let $\Sigma_n = \Omega \cap B(0,n)$, and let $d_n(x)$ be the distance from x to $\mathbb{R}^N \backslash \Sigma_n$. It follows from [14, Theorem 2 of Chapter 6] that there exist functions $\delta_n(x)$ and constants c_7, c_8 ($c_7 < c_8$) independent of n such that

$$c_7 d_n(x) \leq \delta_n(x) \leq c_8 d_n(x)$$

and $\delta_n(x) \in C^\infty(\Sigma_n)$. It follows from Sard's theorem that for each $n \in \mathbb{N}$ there exist $\varepsilon_n > 0$ such that $\delta_n^{-1}(\varepsilon_n)$ is smooth, and we can choose ε_n so that $\varepsilon_n \le c_7\varepsilon_{n-1}/c_8$. We complete the proof by taking $\Omega_n = \{x \in \mathbb{R}^N \mid \delta_n(x) > \varepsilon_n\}$. $\square$

LEMMA 3.4. *Let Ω_n be as in Lemma 3.3. Define*

$$\lambda_1(n) = \inf_{\substack{u \in W_0^{1,p}(\Omega_n) \\ u \neq 0}} \frac{\int_{\Omega_n} |\nabla u|^p}{\int_{\Omega_n} a|u|^p}.$$

Then $\lim\limits_{n\to\infty} \lambda_1(n) = \lambda_1$.

PROOF. For each $n \in \mathbb{N}$, $\lambda_1(n)$ is the first eigenvalue of the problem

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u & \text{in } \Omega_n, \\ \quad u \in W_0^{1,p}(\Omega_n). \end{cases}$$

Since $W_0^{1,p}(\Omega_n) \subset W_0^{1,p}(\Omega_{n+1}) \subset X$, it is clear that $\lambda_1(n) \ge \lambda_1$ for all n and $\lambda_1(n)$ is decreasing. Hence $\lim \lambda_1(n) = \overline{\lambda} \ge \lambda_1$. Let $e_1 \in X$ be the positive eigenfunction corresponding to λ_1 (cf. Proposition 3.1). There exists a sequence $\{\varphi_n\} \subset C_0^\infty(\Omega)$ such that $\varphi_n \to e_1$ in X. So $\int_\Omega |\nabla\varphi_n|^p \to \int_\Omega |\nabla e_1|^p$ and $\int_\Omega a|\varphi_n|^p \to \int_\Omega ae_1^p$. It follows that

$$\lambda_1 = \frac{\int_\Omega |\nabla e_1|^p}{\int_\Omega ae_1^p} = \lim_{n\to\infty} \frac{\int_\Omega |\nabla\varphi_n|^p}{\int_\Omega a|\varphi_n|^p}.$$

If $\overline{\lambda} > \lambda_1$, we may pick φ_{n_0} such that $\dfrac{\int_\Omega |\nabla\varphi_{n_0}|^p}{\int_\Omega a|\varphi_{n_0}|^p} < \overline{\lambda}$. On the other hand, we can take n so large that $\varphi_{n_0} \in C_0^\infty(\Omega_n) \subset W_0^{1,p}(\Omega_n)$. Then $\lambda_1(n) \le \dfrac{\int_{\Omega_n} |\nabla\varphi_{n_0}|^p}{\int_{\Omega_n} a|\varphi_{n_0}|^p} < \overline{\lambda}$ which is impossible. Thus we get $\lim\limits_{n\to\infty} \lambda_1(n) = \lambda_1$. $\square$

LEMMA 3.5. *Let $\Phi \in C_0^\infty(\Omega)$, $\Phi \ge 0$, $\Phi \not\equiv 0$. Then the equation*

$$-\Delta_p u = \lambda a(x)|u|^{p-2}u + \Phi(x) \tag{3.11}$$

has no solution $u \in P$ if $\lambda > \lambda_1$.

PROOF. Suppose that $u \in P$ is a solution of (3.11), then $u \not\equiv 0$. Since $\lambda > \lambda_1$, it follows from Lemma 3.4 that we can choose n_0 such that $\lambda_1(n_0) < \lambda$. Denote $u_0 = u|_{\Omega_{n_0}}$, then u_0 is a supersolution of the equation

$$\begin{cases} -\Delta_p u = \lambda a(x)|u|^{p-2}u + \Phi(x) & \text{in } \Omega_{n_0}, \\ \quad u = 0 & \text{on } \partial\Omega_{n_0}, \end{cases} \tag{3.12}$$

and obviously 0 is a subsolution of (3.12). It follows from [7] that there exists a solution u of (3.12) such that $0 \le u \le u_0$. Furthermore, we know that $u \in C^{1,\alpha}(\overline{\Omega}_{n_0})$ for some $\alpha \in (0,1)$ (cf. [5, Corollary (A.1)]). It follows from the strong maximum principle (cf. [16, Theorem 5]) that $u > 0$ in Ω_{n_0} and $\partial u(x)/\partial\nu > 0$

for all $x \in \partial\Omega_{n_0}$; here ν is the unit interior normal at x. Let $h(x) = (\lambda - \lambda_1(n_0))a(x)u^{p-1} + \Phi(x)$, then $h(x) \geq 0$, $h \not\equiv 0$ and u is a solution of the equation

$$(3.13)\qquad \begin{cases} -\Delta_p u = \lambda_1(n_0)a(x)u^{p-1} + h(x) & \text{in } \Omega_{n_0}, \\ u = 0 & \text{on } \partial\Omega_{n_0}. \end{cases}$$

On the other hand, let e_1^0 be the first eigenfunction corresponding to $\lambda(n_0)$. Applying an inequality due to Díaz and Saa [8, Lemma 2] to u and te_1^0, $t > 0$, we have

$$\int_{\Omega_{n_0}} \bigg(\frac{-\Delta_p u}{u^{p-1}} - \frac{-\Delta_p(te_1^0)}{(te_1^0)^{p-1}}\bigg)(u^p - (te_1^0)^p) \geq 0$$

which leads (letting $t \to \infty$) to $\int_{\Omega_{n_0}} h(x)(e_1^0)^p/u^{p-1} = 0$, but this is impossible because $h \geq 0$ and $h \not\equiv 0$. This contradiction completes the proof. □

Lemma 3.6. *If* $(\overline{\lambda}, 0)$ *is a bifurcation point for* (3.1), *then* $\overline{\lambda}$ *is an eigenvalue of* (1.2) *with some eigenfunction* $\overline{v} \in P\backslash\{0\}$; *hence* $\overline{\lambda} = \lambda_1$.

Proof. By the assumption there exists a sequence $\{(\lambda_n, u_n)\}$ of nontrivial solutions of the equation (3.1) such that $\lambda_n \to \overline{\lambda}$, $u_n \neq 0$ and $u_n \to 0$ in X, and then

$$(3.14)\qquad \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \cdot \nabla\varphi = \lambda_n \int_\Omega a u_n^{p-1}\varphi + \int_\Omega f(x, u_n, \lambda_n)\varphi, \quad \forall \varphi \in X.$$

Let $v_n = u_n/\|u_n\|$. (3.14) yields that

$$(3.15)\qquad \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla\varphi = \lambda_n \int_\Omega a v_n^{p-1}\varphi + \int_\Omega \frac{f(x, u_n, \lambda_n)}{\|u_n\|^{p-1}}\varphi, \quad \forall \varphi \in X.$$

We claim that for all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|u\| < \delta$ yields $\sup_{\|\varphi\|\leq 1} \int_\Omega \frac{f(x, u, \lambda)}{\|u\|^{p-1}}|\varphi| < \varepsilon$, i.e.,

$$(3.16)\qquad \lim_{\|u\|\to 0} \sup_{\|\varphi\|\leq 1} \int_\Omega \frac{f(x, u, \lambda)}{\|u\|^{p-1}}|\varphi| = 0.$$

Indeed, by (f3), given any $\widehat{\varepsilon} > 0$, there exists a $\widehat{\delta} > 0$ such that

$$\frac{f(x, s, \lambda)}{a(x)s^{p-1}} < \widehat{\varepsilon} \quad \text{if } s < \widehat{\delta} \text{ and } x \in \Omega.$$

Let $\|u\| < \delta$, δ being free for now. Set $\Omega_{\widehat{\delta}} = \{x \in \Omega \mid u(x) \geq \widehat{\delta}\}$ and $v = u/\|u\|$. Then we have as in (3.5)

$$\begin{aligned} \int_{\Omega\backslash\Omega_{\widehat{\delta}}} \frac{f(x, u, \lambda)}{\|u\|^{p-1}}|\varphi| &= \int_{\Omega\backslash\Omega_{\widehat{\delta}}} \frac{f(x, u, \lambda)}{a u^{p-1}} a v^{p-1}|\varphi| \\ &\leq \widehat{\varepsilon}\int_\Omega a v^{p-1}|\varphi| \leq c_9\widehat{\varepsilon}\|v\|^{p-1}\|\varphi\|. \end{aligned}$$

Hence

$$\sup_{\|\varphi\|\le 1}\int_{\Omega\setminus\Omega_{\widehat{\delta}}}\frac{f(x,u,\lambda)}{\|u\|^{p-1}}|\varphi|\le c_9\widehat{\varepsilon}.$$

We now choose $\widehat{\varepsilon}$ so that $c_9\widehat{\varepsilon}<\varepsilon/2$ and determine a corresponding $\widehat{\delta}$. Using Hölder's and Sobolev's inequalities as in (3.5) and (3.7) again, we obtain

$$\begin{aligned}\int_{\Omega_{\widehat{\delta}}}\frac{f(x,u,\lambda)}{\|u\|^{p-1}}|\varphi| &\le \frac{c(\lambda)}{\widehat{\delta}^{p-1}}\int_{\Omega_{\widehat{\delta}}}\sigma v^{p-1}|\varphi|+\frac{c(\lambda)}{\|u\|^{p-1}}\int_{\Omega}\rho u^{q-1}|\varphi|\\ &\le c_{10}\bigg(\int_{\Omega_{\widehat{\delta}}}\sigma^{N/p}\bigg)^{p/N}\|v\|^{p-1}\|\varphi\|+c_{11}\|u\|^{q-p}\|\varphi\|.\end{aligned}$$

Therefore

$$\sup_{\|\varphi\|\le 1}\int_{\Omega_{\widehat{\delta}}}\frac{f(x,u,\lambda)}{\|u\|^{p-1}}|\varphi|\le c_{10}\bigg(\int_{\Omega_{\widehat{\delta}}}\sigma^{N/p}\bigg)^{p/N}+c_{11}\delta^{q-p}. \tag{3.17}$$

On the other hand, if we set $\Omega_{\widehat{\delta}}(n)=\Omega_{\widehat{\delta}}\cap B(0,n)$, $n\in\mathbb{N}$, then we have

$$\widehat{\delta}^{p^*}\operatorname{meas}\Omega_{\widehat{\delta}}(n)\le\int_{\Omega_{\widehat{\delta}}(n)}u^{p^*}\le\int_{\Omega}u^{p^*}\le c_{12}\|u\|^{p^*}<c_{12}\delta^{p^*}, \tag{3.18}$$

where c_{12} is a constant independent of n. It follows from (3.18) that $\operatorname{meas}\Omega_{\widehat{\delta}}=\lim\operatorname{meas}\Omega_{\widehat{\delta}}(n)\le c_{12}(\delta\widehat{\delta}^{-1})^{p*}$. Thus we can choose δ so that the right-hand side of (3.17) is $<\varepsilon/2$ and (3.16) is proved.

It follows from (3.16) that $G_2(\lambda_n,u_n)/\|u_n\|^{p-1}\to 0$ in X^* as $n\to\infty$.

Equation (3.15) can be written as

$$J(v_n)=\lambda_n G_1(v_n)+G_2(\lambda_n,u_n)/\|u_n\|^{p-1},$$

or

$$v_n=J^{-1}(\lambda_n G_1(v_n)+G_2(\lambda_n,u_n)/\|u_n\|^{p-1}), \tag{3.19}$$

where the mappings J and G_1 are defined as in (2.1) and (3.3), respectively. Since $\{v_n\}$ is bounded, without any loss of generality we may assume $v_n\rightharpoonup\overline{v}$ in X. Taking the limit in (3.19), using the complete continuity of G_1 and the continuity of J^{-1}, we have $\overline{v}=J^{-1}(\overline{\lambda}G_1(\overline{v}))$ that is, $\overline{v}$ satisfies $-\Delta_p\overline{v}=\overline{\lambda}\overline{v}^{p-1}$.

Taking $\varphi=v_n$ in (3.15), we get

$$1=\int_{\Omega}|\nabla v_n|^p=\lambda_n\int_{\Omega}av_n^p+\int_{\Omega}\frac{f(x,u_n,\lambda_n)}{\|u_n\|^{p-1}}\,v_n.$$

It follows from (3.16) and the weak continuity of the functional $u\mapsto\int_\Omega au^p$ that

$$1=\overline{\lambda}\int_{\Omega}a\overline{v}^p$$

which yields $\overline{v}\ne 0$. Hence $\overline{\lambda}$ is an eigenvalue of (1.2) with some eigenfunction $\overline{v}\in P\setminus\{0\}$. By Proposition 3.1, $\overline{\lambda}=\lambda_1$. □

Lemma 3.7. *Let F be as in* (3.2) *and let $\lambda > \lambda_1$. Then for all $r > 0$ small,* $\operatorname{ind}(F(\lambda, \cdot\,), P_r) = 0$.

Proof. Define $H : [0,1] \times P \to P^*$ as

$$\langle H(t,u), v\rangle = \int_\Omega (\lambda a(x) u^{p-1} + t f(x,u,\lambda))v \quad \forall v \in X.$$

A similar argument as for F gives that H is completely continuous.

We claim that the operator equation $J(u) = H(t,u)$ has no solution on ∂P_r for $r > 0$ small, $0 \le t \le 1$. Indeed, otherwise there exist $\{u_n\}$ and $\{t_n\}$ such that $u_n \ne 0$, $u_n \to 0$ in X, $t_n \to t_0 \in [0,1]$ and $J(u_n) = H(t_n, u_n)$. By the argument of Lemma 3.6 we get that $\lambda (> \lambda_1)$ is an eigenvalue of (1.2) with some eigenfunction $\overline{v} \in P\backslash\{0\}$, but by Proposition 3.1, this is impossible. Thus we obtain from (v) of Proposition 2.3 that for $r > 0$ small,

$$\begin{aligned}\operatorname{ind}(\lambda G_1(u), P_r) &= \operatorname{ind}(H(0,u), P_r)\\ &= \operatorname{ind}(H(1,u), P_r) = \operatorname{ind}(F(\lambda,u), P_r).\end{aligned} \tag{3.20}$$

Now define $K : [0,1] \times P \to P^*$ as

$$\langle K(t,u), v\rangle = \int_\Omega (\lambda\, a(x)\, u^{p-1} + t\Phi(x))v, \quad \forall v \in X,$$

where $\Phi(x)$ is as in Lemma 3.5. Obviously K is completely continuous. It follows from Lemma 3.5 that for all $r > 0$, for all $\lambda > \lambda_1$,

$$\operatorname{ind}(\lambda G_1(u), P_r) = \operatorname{ind}(K(0,u), P_r) = \operatorname{ind}(K(1,u), P_r) = 0. \tag{3.21}$$

Here we use the fact that $u = 0$ is not a solution of equation (3.11). The equalities (3.20) and (3.21) yield $\operatorname{ind}(F(\lambda,u), P_r) = 0$ for all $\lambda > \lambda_1$ and $r > 0$ small. □

Proof of Theorem 3.2. Taking $\lambda_0 = \lambda_1$ in Proposition 2.5, we see by Lemma 3.6 and Lemma 3.7 that all conditions of Proposition 2.5 are satisfied. Hence it follows from Proposition 2.5 that the set of nontrivial solutions of (3.1) contains an unbounded subcontinuum bifurcating from $(\lambda_1, 0)$. □

Remark 3.8.

(a) In order to obtain the compactness of G_2 the condition $\rho \in L^\infty_{\mathrm{loc}}$ can be relaxed to $\rho \in L^{r_1}_{\mathrm{loc}}$, where $r_1 = p^*/\big(p^* - 1 - k(q-1)\big)$, $1 < k < (p^*-1)/(q-1)$.

(b) If $u \in P\backslash\{0\}$ is a solution of (3.1), it follows from the strong maximum principle (cf. [16, Theorem 5]) that $u(x) > 0$ in Ω.

Remark 3.9.

(i) A result similar to Theorem 3.2 but for bounded Ω was obtained by Ambrosetti, Azorero and Peral in a recent paper [3]. They considered the problem (1.1) in a closed subset of $C(\overline{\Omega})$; therefore they did not need the growth restrictions for the nonlinearity f.

(ii) In a very recent paper [10], Drábek and Huang gave a similar result to Theorem 3.3 in the case $\Omega = \mathbb{R}^N$. However, we do not need the assumption in [10] that (3.1) with $\lambda = \lambda_1$ has no solution u satisfying $0 < \|u\| < \delta$.

REMARK 3.10. For the case $p \geq N$, when $\Omega = \mathbb{R}^N$, (1.2) has no eigenvalue $\lambda > 0$ with positive eigenfunction (cf. [1]); hence there is no bifurcation from the set of trivial solutions for (3.1). So our assumption that $p < N$ is essential.

4. Existence results

In this section we let J, λ_1 and $a(x)$ be as previously, i.e., $J : X \to X^*$ is defined by (2.1), λ_1 is the first eigenvalue of equation (1.2) and $0 < a(x) \in L^\infty(\Omega) \cap L^1(\Omega)$. First, we have

THEOREM 4.1. *Suppose that f satisfies* (f1) *and the following conditions:*

(f2)′ $f(x,s,\lambda) \leq c(\lambda)(\alpha(x) + \beta(x)s^{p-1})$ *for a.e.* $x \in \Omega$ *and* $s \in \mathbb{R}_+$, *where* $0 \leq \alpha(x) \in L^{(p^*)'}(\Omega)$ *and* $0 \leq \beta(x) \in L^{N/p}(\Omega) \cap L^\infty_{\mathrm{loc}}(\Omega)$*;*

(f4) $\displaystyle\lim_{s\to+\infty} \frac{f(x,s,\lambda)}{a(x)s^{p-1}} = 0$ *uniformly with respect to a.e.* $x \in \Omega$.

Then the equation (3.1) *has a solution if* $0 \leq \lambda < \lambda_1$.

Note that if $f(x,0,\lambda) = 0$ for almost all x, then the above conclusion is trivially true (since $u = 0$ is a solution).

To prove this theorem we will need the following result.

LEMMA 4.2. *Under the assumptions of Theorem* 4.1*,*

$$\lim_{\|u\|\to\infty} \sup_{\|\varphi\|\leq 1} \int_\Omega \frac{f(x,u,\lambda)}{\|u\|^{p-1}}|\varphi| = 0. \tag{4.1}$$

PROOF. By (f4), for all $\varepsilon > 0$ there exists $A > 0$ such that

$$\frac{f(x,s,\lambda)}{a(x)s^{p-1}} < \varepsilon \quad \forall s > A.$$

Define $\Omega_A = \{x \in \Omega \mid u(x) \leq A\}$ and $v = u/\|u\|$. We split the integral in (4.1) into integrals over $\Omega\backslash\Omega_A$ and Ω_A. Then we have as in (3.5), for each $\varphi \in X$,

$$\begin{aligned}\int_{\Omega\backslash\Omega_A} \frac{f(x,u,\lambda)}{\|u\|^{p-1}}|\varphi| &= \int_{\Omega\backslash\Omega_A} \frac{f(x,u,\lambda)}{au^{p-1}} av^{p-1}|\varphi| \\ &\leq \varepsilon\biggl(\int_\Omega av^p\biggr)^{1/p'}\biggl(\int_\Omega a|\varphi|^p\biggr)^{1/p} \leq c_{13}\varepsilon\|\varphi\|,\end{aligned}$$

where c_{13} is independent of A. Denote $\Omega_A(K) = \Omega_A \cap B(0,K)$. By (f2)$'$, for the second integral we have

$$\int_{\Omega_A} \frac{f(x,u,\lambda)}{\|u\|^{p-1}}|\varphi| \le c(\lambda)\bigg(\int_{\Omega_A} \frac{\alpha}{\|u\|^{p-1}}|\varphi| + \int_{\Omega_A} \frac{\beta u^{p-1}}{\|u\|^{p-1}}|\varphi|\bigg).$$

By using Hölder's and Sobolev's inequalities, we see that

$$\int_{\Omega_A} \frac{\alpha(x)}{\|u\|^{p-1}}|\varphi| \le \frac{c_{14}\|\alpha\|_{L^{(p^*)'}(\Omega)}\|\varphi\|}{\|u\|^{p-1}} \tag{4.2}$$

and

$$\begin{aligned}\int_{\Omega_A} \frac{\beta u^{p-1}}{\|u\|^{p-1}}|\varphi| &= \int_{\Omega_A(K)} \frac{\beta u^{p-1}}{\|u\|^{p-1}}|\varphi| + \int_{\Omega_A\setminus\Omega_A(K)} \frac{\beta u^{p-1}}{\|u\|^{p-1}}|\varphi| \\ &\le \frac{c_{15}(K)\|\varphi\|}{\|u\|^{p-1}} + c_{16}\|\beta\|_{L^{N/p}(\Omega_A\setminus\Omega_A(K))}\|\varphi\|.\end{aligned} \tag{4.3}$$

Now we can choose K so that the second term on the right-hand side of (4.3) is $\le \varepsilon\|\varphi\|$, and then R such that the right-hand side of (4.2) and the first term on the right-hand side of (4.3) are $\le \varepsilon\|\varphi\|$ if $\|u\| \ge R$. Thus we get (4.1). □

Proof of Theorem 4.1. Let $F(\lambda,u) = \lambda G_1(u) + G_2(\lambda,u)$, where G_1, G_2 are defined as in (3.3), (3.4). Then by (f1) and (f2)$'$, $F : \mathbb{R}_+ \times P \to P^*$ is completely continuous. We claim that there exists $R > 0$ such that

$$\langle J(u),u\rangle > \langle F(\lambda,u),u\rangle, \quad \forall u \in \partial P_R. \tag{4.4}$$

Indeed, if not, then there exists $\{u_n\}$, $\|u_n\| \to \infty$, such that

$$\langle J(u_n),u_n\rangle \le \langle F(\lambda,u_n),u_n\rangle.$$

Let $z_n = u_n/\|u_n\|$, then the above inequality yields

$$\langle J(z_n),z_n\rangle \le \lambda\langle G_1(z_n),z_n\rangle + \langle G_2(\lambda,u_n)/\|u_n\|^{p-1},z_n\rangle. \tag{4.5}$$

We may assume that $z_n \rightharpoonup \overline{z}$. Passing to the limit in (4.5), using Lemma 4.2, weak continuity of the functional $z \mapsto \langle G_1(z),z\rangle$ and the characterization (3.10) of λ_1, we obtain

$$\lambda_1\langle G_1(\overline{z}),\overline{z}\rangle \le \|\overline{z}\|^p \le 1 \le \lambda\langle G_1(\overline{z}),\overline{z}\rangle.$$

Hence $\lambda \ge \lambda_1$, a contradiction. We thus conclude that (4.4) holds. By Proposition 2.4, $\operatorname{ind}(F(\lambda,u),P_R) = 1$ which implies the equation (3.1) has a solution. □

Remark 4.3. Suppose that f satisfies all conditions of Theorem 4.1 and $0 < b(x) \in L^\infty_{\mathrm{loc}}(\Omega) \cap L^{p^*/(p^*-\gamma)}(\Omega)$, where $1 < \gamma < p$. Then the equation

$$\begin{cases} -\Delta_p u = \lambda b(x)u^{\gamma-1} + f(x,u,\lambda), \\ \qquad u \ge 0 \text{ in } \Omega, \\ \qquad u \in \mathcal{D}^{1,p}_0(\Omega), \end{cases} \tag{4.6}$$

has a solution for all $\lambda \geq 0$.

This is a consequence of Proposition 2.4 and Lemma 4.2. Indeed, if there exists $\{u_n\}$, $\|u_n\| \to \infty$, such that

$$\|u_n\|^p = \int_\Omega |\nabla u_n|^p \leq \lambda \int_\Omega b(x) u_n^\gamma + \int_\Omega f(x, u_n, \lambda) u_n,$$

then by Hölder's and Sobolev's inequalities and Lemma 4.2, we have

$$1 \leq \lambda c_{16} \|b\|_{L^{p^*/(p^*-\gamma)}(\Omega)} \|u_n\|^{\gamma - p} + \int_\Omega \frac{f(x, u_n, \lambda)}{\|u_n\|^{p-1}} \frac{u_n}{\|u_n\|} \to 0 \quad \text{as } \|u_n\| \to \infty.$$

This contradiction and Proposition 2.4 imply that $\operatorname{ind}(\widetilde{F}(\lambda, u), P_R) = 1$ for large $R > 0$, where $\widetilde{F} : \mathbb{R}_+ \times P \to P^*$ is defined by $\langle \widetilde{F}(\lambda, u), v\rangle = \int_\Omega (\lambda b(x) u^{\gamma-1} + f(x, u_n, \lambda))v$. Hence it follows from (ii) of Proposition 2.3 that (4.6) has a solution.

In the remainder of this section we study the existence positive nontrivial (i.e., $\neq 0$) solutions of the problem

$$\begin{cases} -\Delta_p u = g(x, u), \\ \qquad u \geq 0 \text{ in } \Omega, \\ \qquad u \in \mathcal{D}_0^{1,p}(\Omega), \end{cases} \tag{4.7}$$

where g satisfies

- (g1) $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is a Carathéodory function;
- (g2) $g(x, s) \leq \alpha(x) + \beta(x) s^{p-1}$ for a.e. $x \in \Omega$ and $s \in \mathbb{R}_+$, where $0 \leq \beta(x) \in L^{N/p}(\Omega) \cap L^\infty_{\rm loc}(\Omega)$ and $0 \leq \alpha(x) \in L^{(p^*)'}(\Omega) \cap L^{N/p}(\Omega)$.

Then we have the folowing results.

THEOREM 4.4. *Suppose that g satisfies* (g1), (g2), $0 < a(x) \in L^\infty(\Omega) \cap L^1(\Omega)$ *and the following limits exist uniformly with respect to $x \in \Omega$:*

- (g3) $\displaystyle\lim_{s \to 0} \frac{g(x, s)}{a(x) s^{p-1}} = \underline{\lambda} < \lambda_1$,
- (g4) $\displaystyle\lim_{s \to \infty} \frac{g(x, s)}{a(x) s^{p-1}} = \overline{\lambda} > \lambda_1$.

Then (4.7) *has a nontrival solution.*

PROOF. Define $G : P \to P^*$ as

$$\langle G(u), v\rangle = \int_\Omega g(x, u) v, \quad \forall v \in X. \tag{4.8}$$

It follows from conditions (g1) and (g2) that G is completely continuous. We will show that the index $\operatorname{ind}(G, P_r)$ takes different values for small r and for large r.

First, we claim that $J(u) = tG(u)$ ($0 \le t \le 1$) has no solutions on ∂P_r for small $r > 0$. Otherwise we can find $\{u_n\}$ and $\{t_n\}$ with $u_n \to 0, u_n \neq 0$, $t_n \to \bar{t} \in [0,1]$ such that $J(u_n) = t_n G(u_n)$. Let $v_n = u_n/\|u_n\|$, then we have

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla\varphi = t_n \int_\Omega \frac{g(x,u_n)}{\|u_n\|^{p-1}}\varphi, \quad \forall \varphi \in X. \tag{4.9}$$

According to condition (g3), we can write g as

$$g(x,s) = \underline{\lambda} a(x) s^{p-1} + f(x,s), \tag{4.10}$$

where f satisfies

$$\lim_{s\to 0} \frac{f(x,s)}{a(x)s^{p-1}} = 0 \quad \text{uniformly with respect to } x \in \Omega. \tag{4.11}$$

Then (4.9) can be written as

$$\int_\Omega |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla\varphi = t_n \underline{\lambda} \int_\Omega a v_n^{p-1}\varphi + t_n \int_\Omega \frac{f(x,u_n)}{\|u_n\|^{p-1}}\varphi. \tag{4.12}$$

We may assume without any loss of generality that $v_n \rightharpoonup v_0$ in X. By (4.10) and (4.11), similarly as in the proof of Lemma 3.6, we find that v_0 satisfies $-\Delta_p u = \bar{t}\underline{\lambda} a(x) u^{p-1}$.

Taking $\varphi = v_n$ in (4.12) and letting $n \to \infty$, we obtain

$$1 = \bar{t}\underline{\lambda} \int_\Omega a v_0^p$$

which yields that $v_0 \neq 0$ and $\lambda_1 = \bar{t}\underline{\lambda}$. Since $\underline{\lambda} < \lambda_1$, this is a contradiction. Hence

$$\operatorname{ind}(G, P_r) = \operatorname{ind}(0, P_r) = 1. \tag{4.13}$$

Let $Q(t,u) = t\overline{\lambda} G_1(u) + (1-t)G(u)$ $(0 \le t \le 1)$, where G_1 and G are as in (3.3) and (4.8), respectively. Then Q maps $[0,1] \times P$ to P^* and Q is completely continuous. We claim that $J(u) = Q(t,u)$ has no solution on ∂P_R for large R. Arguing by contradiction, we can find $\{u_n\}$ and $\{t_n\}$ such that $\|u_n\| \to \infty$, $t_n \to t_0 \in [0,1]$ satisfying $J(u_n) = Q(t_n, u_n)$.

Let $v_n = u_n/\|u_n\|$. Without loss of generality we may assume that $v_n \rightharpoonup \overline{v}$ in X, and $\{v_n\}$ satisfies, for all $\varphi \in X$,

$$\begin{aligned} \int_\Omega |\nabla v_n|^{p-2}\nabla v_n \cdot \nabla\varphi &= t_n \overline{\lambda} \int_\Omega a(x)\, v_n^{p-1}\varphi + (1-t_n) \int_\Omega \frac{g(x,u_n)}{\|u_n\|^{p-1}}\varphi \\ &= \overline{\lambda} \int_\Omega a(x) v_n^{p-1}\varphi + (1-t_n) \int_\Omega \frac{g(x,u_n) - \overline{\lambda} a(x) u_n^{p-1}}{\|u_n\|^{p-1}}\varphi. \end{aligned} \tag{4.14}$$

By the assumptions on g and Lemma 4.2 (with $g(x,u_n)-\overline{\lambda}a(x)u_n^{p-1}$ replacing f), we have

$$\lim_{\|u_n\|\to\infty}\sup_{\|\varphi\|\le 1}\int_\Omega \frac{g(x,u_n)-\overline{\lambda}a(x)u_n^{p-1}}{\|u_n\|^{p-1}}|\varphi| = 0.$$

Similarly as in the proof of Lemma 3.6, we can get from (4.14) that $\overline{v}$ ($\overline{v}\neq 0$) satisfies the equation $-\Delta_p u = \overline{\lambda}a(x)u^{p-1}$, which is impossible because $\overline{\lambda}>\lambda_1$ and λ_1 is the only eigenvalue of equation (1.2) having a positive eigenfunction. Therefore it follows as in (3.21) that

$$\operatorname{ind}(G,P_R)=\operatorname{ind}(Q(0,u),P_R)=\operatorname{ind}((Q(1,u),P_R)=\operatorname{ind}(\overline{\lambda}G_1,P_R)=0.$$

This, (4.13) and (iv) of Proposition 2.3 imply that

$$\operatorname{ind}(G,P_R\backslash\overline{P_r})=\operatorname{ind}(G,P_R)-\operatorname{ind}(G,P_r)=-1.$$

Hence (4.7) has a nontrivial solution. □

Theorem 4.5. *Suppose that g satisfies* (g1), (g2), $0<a(x)\in L^\infty(\Omega)\cap L^1(\Omega)$ *and the following limits exist uniformly with respect to $x\in\Omega$:*

(g3)′ $\displaystyle\lim_{s\to 0}\frac{g(x,s)}{a(x)s^{p-1}}=\overline{\beta}>\lambda_1$,

(g4)′ $\displaystyle\lim_{s\to\infty}\frac{g(x,s)}{a(x)s^{p-1}}=\underline{\beta}<\lambda_1$.

Then (4.7) *has a nontrivial solution.*

Proof. By the argument of the preceding theorem we show that $\operatorname{ind}(G,P_r)=0$ for small r and $\operatorname{ind}(G,P_R)=1$ for large R. Hence the conclusion. □

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YISHENG HUANG
Department of Mathematics
Stockholm University
S-106 91 Stockholm, SWEDEN

E-mail address: yisheng@matematik.su.se