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# AN EIGENVALUE PROBLEM FOR THE SCHRÖDINGER–MAXWELL EQUATIONS

Vieri Benci — Donato Fortunato

*Dedicated to Jürgen Moser*

## 1. Introduction

In this paper we study the eigenvalue problem for the Schrödinger operator coupled with the electromagnetic field $\mathbf{E}, \mathbf{H}$. The case in which the electromagnetic field is given has been mainly considered ([1]–[3]).

Here we do not assume that the electromagnetic field is assigned, then we have to study a system of equations whose unknowns are the wave function $\psi = \psi(x,t)$ and the gauge potentials $\mathbf{A} = \mathbf{A}(x,t)$, $\phi = \phi(x,t)$ related to $\mathbf{E}, \mathbf{H}$.

We want to investigate the case in which $\mathbf{A}$ and $\phi$ do not depend on the time $t$ and

$$\psi(x,t) = u(x)e^{i\omega t}, \quad u \text{ real function and } \omega \text{ a real number}$$

In this situation we can assume $\mathbf{A} = 0$ and we are reduced to study the existence of real numbers $\omega$ and real functions $u$, $\phi$ satisfying the system

$$-\frac{1}{2}\Delta u - \phi u = \omega u, \quad \Delta \phi = 4\pi u^2 \tag{1}$$

with the boundary and normalizing conditions

$$u(x) = 0, \quad \phi(x) = g \quad \text{on } \partial\Omega, \quad \|u\|_{L^2} = 1. \tag{2}$$

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Here $g$ is an assigned function and $\partial\Omega$ is the boundary of an open subset $\Omega$ in $\mathbb{R}^3$ (the methods we shall use extend to higher dimensions without any change).

Since the electrostatic potential $\phi$ is not assigned, (1) cannot be reduced to a linear eigenvalue problem. Nevertheless (1) possess an interesting variational structure. In fact, it is not difficult to see that (1) are the Euler–Lagrangian equations of a functional $F$ which is strongly indefinite (see Section 3); this means that $F$ is neither bounded from above nor from below and this indefinitess cannot be removed by a compact perturbation.

We shall prove the following theorem.

Theorem 1. *Let $\Omega$ be a bounded set in $\mathbb{R}^3$ and $g$ a smooth function on the closure $\overline{\Omega}$. Then there is a sequence $(\omega_n, u_n, \phi_n)$, with $\omega_n \subset \mathbb{R}$, $\omega_n \to \infty$ and $u_n, \phi_n$ real functions, solving* (1), (6).

## 2. The Schrödinger–Maxwell equations

In this section we deduce a system of equations describing a quantum particle interacting with a electromagnetic field.

The Schrödinger equation for a particle in a electromagnetic field whose gauge potentials are $\mathbf{A}$, $\phi$ is

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\bigg(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\bigg)^2\psi - e\phi\psi, \tag{3}$$

$\psi(x,t) \in \mathbf{C}$ is the wave function, $m$, $e$ are the mass and the charge of the particle, $\hbar = h/2\pi$, $h$ being the Planck constant.

The Lagrangian density relative to (3) is given by

$$\mathcal{L}_0 = \frac{1}{2}\bigg[i\hbar\frac{\partial\psi}{\partial t}\overline{\psi} + e\phi\,|\psi|^2 - \frac{1}{2m}\bigg|\bigg(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\bigg)\psi\bigg|^2\bigg]. \tag{4}$$

If we set

$$\psi(x,t) = u(x,t)\,e^{iS(x,t)/\hbar}, \quad u, S \in \mathbb{R}.$$

equation (4) takes the following form:

$$\mathcal{L}_0 = \frac{\hbar^2}{2m}\,|\nabla u|^2 - \bigg[S_t - e\phi + \frac{1}{2m}\bigg(\nabla S - \frac{e}{c}\mathbf{A}\bigg)^2\bigg]u^2. \tag{5}$$

Now we consider the lagrangian density of the electromagnetic field $\mathbf{E}, \mathbf{H}$

$$\mathcal{L}_1 = \frac{\mathbf{E}^2 - \mathbf{H}^2}{8\pi}.$$

$\mathbf{E}, \mathbf{H}$ are related to $\mathbf{A}, \phi$ by

$$\mathbf{E} = -\frac{1}{c}\mathbf{A}_t - \nabla\phi, \qquad \mathbf{H} = \nabla \times \mathbf{A}, \tag{6}$$

then

$$\mathcal{L}_1 = \frac{1}{8\pi}\left|\frac{1}{c}\mathbf{A}_t + \nabla\phi\right|^2 - \frac{1}{8\pi}\left|\nabla \times \mathbf{A}\right|^2.$$

Therefore the total action of the system "particle-electromagnetic field" is given by

$$\mathcal{S} = \iint \mathcal{L}_0 + \mathcal{L}_1.$$

Makig the variation of $\mathcal{S}$ with respecto to $\delta u$, $\delta S$, $\delta\phi$ and $\delta\mathbf{A}$ respectively, we get

$$-\frac{\hbar^2}{2m}\Delta u + \left[S_t - e\phi + \frac{1}{2m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right)^2\right] u = 0, \tag{7}$$

$$\frac{\partial}{\partial t}\left(u^2\right) - \frac{1}{m}\nabla\cdot\left[\left(\nabla S - \frac{e}{c}\mathbf{A}\right)u^2\right] = 0, \tag{8}$$

$$\frac{1}{4\pi}\nabla\cdot\left(\frac{1}{c}\mathbf{A}_t + \nabla\phi\right) = eu^2, \tag{9}$$

$$\frac{1}{4\pi}\left[\nabla\times\left(\nabla\times\mathbf{A}\right) + \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{1}{c}\mathbf{A}_t + \nabla\phi\right)\right] = \frac{e}{cm}\left(\nabla S - \frac{e}{c}\mathbf{A}\right)u^2, \tag{10}$$

Using (6) and setting

$$\rho = -eu^2, \quad \mathbf{v} = -\frac{\nabla S - \mathbf{A}e/c}{m}, \quad \mathbf{j} = \frac{e}{m}\left(\nabla S - \frac{e}{c}\mathbf{A}\right)u^2 = \rho\mathbf{v},$$

equations (8)–(10) take the form

$$\frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{j} = 0, \tag{11}$$

$$\nabla\cdot\mathbf{E} = 4\pi\rho, \tag{12}$$

$$\nabla\times\mathbf{H} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}. \tag{13}$$

Equation (11) is a continuity equation and (12), (13) are the Maxwell equations for an electromagnetic field in the presence of a charge and current density given by $\rho$ and $\mathbf{j}$.

## 3. The eigenvalue problem

We look for solutions $u$, $S$, $\mathbf{A}$, $\phi$ of (7)–(10) of the type

$$u = u(x), \quad S = -\omega t, \quad \mathbf{A} = 0, \quad \phi = \phi(x),$$

with this "ansatz", the equations (8) and (10) are identically satisfied, while (7) and (9) become

$$-\frac{\hbar^2}{2m}\Delta u - e\phi u = \omega u, \tag{14}$$

$$\Delta\phi = 4\pi e u^2. \tag{15}$$

We shall assume that the electrostatic potential $\phi$ is assigned on the boundary $\partial\Omega$, namely we assume that

$$\phi = g \quad \text{on } \partial\Omega, \tag{16}$$

where $g$ is a given continuous function on $\overline{\Omega}$. Since $u$ is the amplitude of the wave function representing a particle confined in $\Omega$, we require that $u$ satisfies the normalizing and the boundary conditions

$$\int u^2 = 1, \quad u|_{\partial\Omega} = 0. \tag{17}$$

Constants $\hbar$, $c$, and $m$ are positive so we set for simplicity $\hbar = c = m = 1$. Moreover, $e^2 = +1$, then in (14)–(16) we can rename $e\phi$ again by $\phi$. Then we are reduced to solve the eigenvalue problem (1), (6), namely:
Find $\omega \in \mathbb{R}$, $u \in H_0^1(\Omega)$, $\int u^2 = 1$, and $\phi \in H^1(\Omega)$, $\phi = g$ on $\partial\Omega$ such that

$$\begin{aligned} -\frac{1}{2}\Delta u - \phi u &= \omega u, \\ \Delta\phi &= 4\pi u^2. \end{aligned}$$

Here $H_0^1(\Omega)$ and $H^1(\Omega)$ are the usual Sobolev spaces and the laplacian $\Delta$ is meant in the sense of distributions.

If we set

$$\varphi = \phi - g \in H_0^1(\Omega)$$

the above equations become

$$-\frac{1}{2}\Delta u - (\varphi + g)u = \omega u, \tag{18}$$

$$\Delta\varphi = 4\pi u^2 - g^*, \tag{19}$$

where $g^*$ is the defined by

$$\langle g^*, v\rangle = \int_\Omega g\Delta v \, dx \quad \text{for } v \in C_0^\infty(\Omega).$$

Clearly, $g^*$ can be continuously extended to $H_0^1(\Omega)$.

Now consider the functional

$$F(u,\varphi) = \frac{1}{4}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega (\varphi + g)u^2 - \frac{1}{16\pi}\int_\Omega |\nabla\varphi|^2 + \frac{1}{8\pi}\langle g^*, \varphi\rangle \tag{20}$$

on the manifold

$$M = \{(u,\varphi) \in H_0^1(\Omega) \times H_0^1(\Omega) \mid \|u\|_{L^2(\Omega)} = 1\}.$$

It is easy to verify that $F$ is a $C^1$-functional on $M$. Moreover, the following proposition holds

Proposition 2. *$\omega \in \mathbb{R}$, $(u,\varphi) \in M$ solve the eigenvalue problem* (18), (19) *if and only if $(u,\varphi)$ is a critical point of $F|_M$ having $\omega$ as lagrangian multiplier.*

Proof. $(u,\varphi) \in M$ is a critical point of $F|_M$ with lagrangian multiplier $\omega$ if and only if

$$F'_u(u,\varphi) = \omega u, \qquad F'_\varphi(u,\varphi) = 0, \tag{21}$$

where $F'_u(u,\varphi)$, $F'_\varphi(u,\varphi)$ denote the partial derivatives of $F$ at $(u,\varphi)$, namely, for any $v \in H^1_0(\Omega)$

$$F'_u(u,\varphi)[v] = F'(u,\varphi)[v,0] = \int_\Omega \bigg(\frac{1}{2}(\nabla u \mid \nabla v) - (\varphi + g)uv\bigg)\,dx \tag{22}$$

$$\begin{aligned} F'_\varphi(u,\varphi)[v] &= F'(u,\varphi)[(v,0)] \\ &= \int_\Omega \bigg(-\frac{1}{2}u^2 v - \frac{1}{8\pi}(\nabla\varphi \mid \nabla v)\bigg)\,dx + \frac{1}{8\pi}\langle g^*, v\rangle. \end{aligned} \tag{23}$$

Clearly (21) can be written as (18), (19). □

## 4. Proof of Theorem 1.1

In view of Proposition 2, Theorem 1 is an obvious consequence of the following result

Theorem 3. *Let $\Omega$ be bounded. Then there is a sequence $\{(u_n,\varphi_n)\} \subset M$ of critical points of $F|_M$ whose lagrangian multipliers $\omega_n$ tend to $\infty$.*

The proof of this theorem cannot be achieved directly and it requires some technical preliminaries.

In fact the functional (20) is neither bounded from below nor from above. Moreover, this indefinitess cannot be removed by a compact perturbation. Then the usual methods of the critical point theory cannot be directly used.

To avoid this difficulty we shall reduce the study of (20) to the study of a functional of the only variable $u$. Set

$$\Gamma = \{(u,\varphi) \in H^1_0(\Omega) \times H^1_0(\Omega) \mid F'_\varphi(u,\varphi) = 0\},$$

where $F'_\varphi$, has been defined in (23). Consider the map

$$\Phi : u \in B \to \Phi(u) = \varphi \in H^1_0(\Omega) \text{ solution of (19)}, \tag{24}$$

where $B = \{u \in H^1_0(\Omega) \mid \ \|u\|_{L^2(\Omega)} = 1\}$. Clearly, $\Phi(u) = 4\pi\Delta^{-1}u^2 - g$. Here $\Delta^{-1}$ denotes the inverse of the Riesz isomorphism $\Delta$ between $H^1_0(\Omega)$ and its dual $H^{-1}$ defined by

$$\langle \Delta u, v\rangle = -\int (\nabla u \mid \nabla v)\,dx, \quad u,v \in H^1_0(\Omega).$$

PROPOSITION 4. *The map $\Phi$ is $C^1$ and $\Gamma$ is the graph of $\Phi$.*

PROOF. Since $H_0^1(\Omega)$ is continuously embedded into $L^6$ it is easy to see that the map $u \mapsto u^2$ is $C^1$ from $H_0^1(\Omega)$ into $L^3$ which is continuously embedded into $H^{-1}$. Then, since $\Delta^{-1} : H^{-1} \to H_0^1(\Omega)$ is $C^1$, we easily conclude that $\Phi$ is $C^1$.

Let $G_\Phi$ denote the graph of $\Phi$, then clearly we have

$$(u, \varphi) \in G_\Phi \Leftrightarrow \Delta\varphi = 4\pi u^2 - g^* \Leftrightarrow F'_\varphi(u, \varphi) = 0 \Leftrightarrow (u, \varphi) \in \Gamma. \qquad \square$$

For $u \in B$, $\Phi(u)$ solves (19), then

$$\Delta\Phi(u) = 4\pi u^2 - g^*$$

from which, taking the product with $\Phi(u)$, we have

$$-\frac{1}{8\pi}\int_\Omega |\nabla\Phi(u)|^2\,dx = \frac{1}{2}\int_\Omega u^2\Phi(u)\,dx - \frac{1}{8\pi}\langle g^*, \Phi(u)\rangle. \tag{25}$$

Using (24) and (20) we define the functional $J$ as follows

$$\begin{aligned} J(u) = F(u, \Phi(u)) = {} & \frac{1}{4}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega (g + \Phi(u))u^2 \\ & - \frac{1}{16\pi}\int_\Omega |\nabla\Phi(u)|^2 + \frac{1}{8\pi}\int_\Omega \langle g^*, \Phi(u)\rangle\,dx \end{aligned}$$

for $u \in H_0^1(\Omega)$. Then, inserting (25), we easily get

$$\begin{aligned} J(u) &= \frac{1}{4}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega gu^2 - \frac{1}{16\pi}\int_\Omega |\nabla\Phi(u)|^2 + \frac{1}{8\pi}\int_\Omega |\nabla\Phi(u)|^2\,dx \\ &= \frac{1}{4}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega gu^2 + \frac{1}{16\pi}\int_\Omega |\nabla\Phi(u)|^2 \end{aligned} \tag{26}$$

By Proposition 4 $J|_B$ is $C^1$ and, since $g \in L^\infty$, it is bounded from below.

The following proposition holds

PROPOSITION 5. *Let $(u, \varphi) \in M$ and $\omega \in \mathbb{R}$. The following statements are equivalent*

(a) *$(u, \varphi)$ is a critical point of $F|_M$, having $\omega$ as lagrangian multiplier.*
(b) *$u$ is a critical point of $J|_B$ having $\omega$ as lagrangian multiplier and $\varphi = \Phi(u)$.*

PROOF. Clearly, by Proposition 4, we have

$$\text{(b)} \Leftrightarrow F'_u(u, \varphi) + F'_\varphi(u, \varphi)\Phi'(u) = \omega\ u$$

and

$$(u, \varphi) \in G_\Phi \Leftrightarrow F'_u(u,\varphi) = \omega u F'_\varphi(u, \varphi) = 0 \Leftrightarrow \text{(a)}. \qquad \square$$

By Proposition 5 we are reduced to prove the following result

Theorem 6. *There is a sequence* $\{u_n\}$ *of critical points of* $J|_B$ *having lagrangian multipliers* $\omega_n \to \infty$.

In order to prove this theorem we need some technical lemmas.

Lemma 7. *The functional* $J|_B$ *satisfies the Palais–Smale condition, i.e.*

(27) *any sequence* $\{u_n\} \subset B$ *s.t.* $\{J(u_n)\}$ *is bounded and* $J|'_B(u_n) \to 0$ *contains a convergent subsequence.*

Proof. Let $\{u_n\} \subset B$ s.t. $\{J(u_n)\}$ is bounded and $J|'_B(u_n) \to 0$. Then there are sequences $\{\lambda_n\} \subset \mathbb{R}$, $\{v_n\} \subset H^{-1}$, $v_n \to 0$ in $H^{-1}$ such that

$$F'_u(u_n, \Phi(u_n)) + F'_\varphi(u_n, \Phi(u_n))\Phi'(u_n) = \lambda_n u_n + v_n. \tag{28}$$

By Proposition 4, $(u_n, \Phi(u_n)) \in \Gamma$, then (28) becomes

$$F'_u(u_n, \Phi(u_n)) = \lambda_n u_n + v_n. \tag{29}$$

$\{J(u_n)\}$ is bounded, then by (26) we have that

$$\left\{\frac{1}{4}\int_\Omega |\nabla u_n|^2 - \frac{1}{2}\int_\Omega g u_n^2 + \frac{1}{16\pi}\int_\Omega |\nabla \Phi(u_n)|^2\right\} \tag{30}$$

is bounded. Since $g \in L^\infty$ and $\|u_n\|_{L^2} = 1$, we have

$$\left|\int_\Omega g u_n^2\right| \le \|g\|_{L^\infty} \cdot \|u_n\|_{L^2}^2 = \|g\|_{L^\infty}. \tag{31}$$

From (30) and (31) we deduce that

$$\{u_n\} \text{ and } \{\Phi(u_n)\} \text{ are bounded in } H_0^1(\Omega). \tag{32}$$

Moreover,

$$\{\lambda_n\} \text{ is bounded.} \tag{33}$$

In fact, multiplying (29) by $u_n$ and since $\|u_n\|_{L^2} = 1$, we get

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 - \int_\Omega g u_n^2 + \int_\Omega \Phi(u_n) u_n^2 = \lambda_n + \langle v_n, u_n\rangle. \tag{34}$$

Now

$$\left|\int_\Omega \Phi(u_n) u_n^2\right| \le \|u_n\|_{L^4}^2 \cdot \|\Phi(u_n)\|_{L^2} \le \text{const}\|u_n\|_{H_0^1}^2 \cdot \|\Phi(u_n)\|_{H_0^1}. \tag{35}$$

Then (33) easily follows from (32), (34) and (35). Now (29) can be written as follows

$$-\frac{1}{2}\Delta u_n - \Phi(u_n)u_n - g u_n - \lambda_n u_n = v_n$$

from which we have

$$-\frac{1}{2}u_n - \Delta^{-1}(\Phi(u_n)u_n) - \Delta^{-1}(g u_n) - \lambda_n \Delta^{-1} u_n = \varepsilon_n, \tag{36}$$

where

$$\varepsilon_n = \Delta^{-1} v_n \to 0 \quad \text{in } H^1_0(\Omega). \tag{37}$$

From (32) there are $u, \varphi \in H^1_0(\Omega)$ such that, up to a subsequence,

$$u_n \rightharpoonup u,\ \Phi(u_n) \rightharpoonup \varphi \text{ weakly in } H^1_0(\Omega). \tag{38}$$

Since $H^1_0(\Omega)$ is compactly embedded into $L^p(\Omega)$ for $p < 6$, we deduce that

$$u_n \to u,\ \Phi(u_n) \to \varphi \text{ strongly in } L^p(\Omega),\ p < 6. \tag{39}$$

Clearly, since $g \in L^\infty$, we have

$$g u_n \to g u \text{ strongly in } L^p(\Omega),\ p < 6. \tag{40}$$

Then we have also

$$u_n \to u,\ g u_n \to g u \text{ strongly in } H^{-1}. \tag{41}$$

So, since $\Delta^{-1} : H^{-1} \to H^1_0(\Omega)$ is an isomorphism, (41), (33) and (37) imply that, up to a subsequence,

$$\alpha_n = \Delta^{-1}(g u_n) + \lambda_n \Delta^{-1} u_n + \varepsilon_n \text{ converges strongly in } H^1_0(\Omega). \tag{42}$$

From (36) and (42) we deduce that

$$-\frac{1}{2} u_n - \Delta^{-1}(\Phi(u_n) u_n) = \alpha_n \text{ converges strongly in } H^1_0(\Omega). \tag{43}$$

Then, in order to prove that $u_n$ converges strongly in $H^1_0(\Omega)$, it remains to show that

$$\Phi(u_n) u_n \to \varphi u \text{ strongly in } H^{-1}. \tag{44}$$

Let $6 > p \geq 2$ and consider its conjugate $6/5 < q = p/(p-1) \leq 2$. Clearly,

$$\begin{gathered} \|\Phi(u_n) u_n - \varphi u\|_{L^q} \leq A_n + B_n, \\ A_n = \|\Phi(u_n) u_n - \varphi u_n\|_{L^q}, \quad B_n = \|\varphi u_n + \varphi u\|_{L^q}. \end{gathered} \tag{45}$$

Moreover, by Hölder inequality,

$$A_n \leq \|u_n\|_{L^6} \|\Phi(u_n) - \varphi\|_{L^{6q/(6-q)}}. \tag{46}$$

By (38), $\{\|u_n\|_{L^6}\}$ is bounded. Since $6q/(6-q) \leq 3$, by (39),

$$\|\Phi(u_n) - \varphi\|_{L^{6q/(6-q)}} \to 0.$$

Then, by (46), we deduce that $A_n \to 0$. Analogously, we have $B_n \to 0$. Then, by (45), we deduce

$$\|\Phi(u_n) u_n - \varphi u\|_{L^q} \to 0. \tag{47}$$

Since $L^q$ is continuously embedded into $H^{-1}$, (44) easily follows from (47). $\square$

It is easy to see that the functional $J$ is even and we shall exploit this symmetry property in order to get multiplicity results for the critical points of $J|_B$. To this end we recall the definition of genus. Let $A \subset B$ be a closed subset symmetric with respect to the origin. We say that $A$ has genus $m$ (denoted by $\gamma(A) = m$) if there exists an odd, continuous map $\chi : A \to \mathbb{R}^m \setminus \{0\}$ and $m$ is the smallest integer having this property. If $A = \emptyset$ we write $\gamma(A) = 0$ and if there is no finite such $m$ we set $\gamma(A) = \infty$.

Lemma 8. *For any integer $m$ there exists a compact symmetric subset $K \subset B$ such that $\gamma(K) = m$.*

Proof. Let $H_m$ be an $m$ dimensional subspace of $H_0^1(\Omega)$, and set $K = B \cap H_m$. Then, by a well known property of the genus (see e.g. [4] or [5]) we have $\gamma(K) = m$. □

Lemma 9. *For any $b \in \mathbb{R}$ the sublevel*

$$J^b = \{u \in B \mid J(u) \le b\}$$

*has finite genus.*

Proof. This result is standard in critical point theory, nevertheless, for completeness, we sketch the proof. We argue by contraddiction and assume that

$$D = \{b \in \mathbb{R} \mid \gamma(J^b) = \infty\} \neq \emptyset.$$

Clearly, since $J|_B$ is bounded below, $D$ is bounded below. Then

$$-\infty < \overline{b} = \inf D < \infty. \tag{48}$$

Moreover, since $J|_B$ satisfies the Palais–Smale condition (see Lemma 7), the set

$$Z = \{u \in B \mid J(u) = \overline{b},\ J|'_B(u) = 0\}$$

is compact. Then, by well known properties of the genus (see e.g. Lemma 1.1 in [4]), there exists a closed symmetric neighbourhood $U_Z$ of $Z$ such that $\gamma(U_Z) < \infty$.

Now, by a well known deformation lemma (see e.g. Theorem 1.9 in [4]), there exists $\varepsilon > 0$ such that the sublevel $J^{\overline{b}-\varepsilon}$ includes a strong deformation retract of $J^{\overline{b}+\varepsilon} \setminus U_Z$. Then, by using again the properties of the genus, we get

$$\gamma(J^{\overline{b}+\varepsilon}) \le \gamma(J^{\overline{b}+\varepsilon} \setminus U_Z) + \gamma(U_Z) \le \gamma(J^{\overline{b}-\varepsilon}) + \gamma(U_Z) < \infty$$

and this contradicts (48). □

Now we are ready to complete the proof of Theorem 6. Let $k$ be a positive integer, then, by Lemma 9, there exists an integer $n = n(k)$ such that

$$\gamma(J^k) = n. \tag{49}$$

Now consider the set

$$A_{n+1} = \{A \subset B,\ A \text{ symmetric, closed s.t. } \gamma(A) = n+1\}.$$

By Lemma 8, $A_{n+1} \neq \emptyset$ and, by the monotonicity property of the genus, any $A \in A_{n+1}$ is not contained in $J^k$, then $\sup J(A) > k$ and, consequently,

$$b_k = \inf\{\sup J(A) \mid A \in A_{n+1}\} \geq k. \tag{50}$$

By Lemma 7, $J|_B$ satisfies the Palais–Smale condition, then well known results in critical point theory (see e.g. [4] or [5]) guarantee that $b_k$ is a critical value of $J|_B$. So we conclude that for any integer $k$ there is $\omega_k \in \mathbb{R}$ and $u_k \in B$ such that

$$J'(u_k) = \omega_k u_k \quad \text{and} \quad J(u_k) = b_k \geq k. \tag{51}$$

So we need only to prove that

$$\omega_k \to \infty \quad \text{as } k \to \infty. \tag{52}$$

By (51) and Proposition 5 we have that

$$F'_u(u_k, \varphi_k) = \omega_k u_k, \quad \text{where } \varphi_k = \Phi(u_k).$$

This can be written as follows

$$-\frac{1}{2}\Delta u_k - (\varphi_k + g)u_k = \omega_k u_k$$

from which, we deduce

$$\frac{1}{4}\int_\Omega |\nabla u_k|^2 - \frac{1}{2}\int_\Omega (\varphi_k + g)u_k^2 = \omega_k \int_\Omega u_k^2 = \omega_k. \tag{53}$$

From (20) and (53) we have

$$F(u_k, \varphi_k) = \omega_k - \frac{1}{16\pi}\int_\Omega |\nabla\varphi_k|^2 + \frac{1}{8\pi}\int_\Omega \langle g^*, \varphi_k\rangle\, dx. \tag{54}$$

Now the second equality of (51) can be written

$$F(u_k, \varphi_k) = b_k. \tag{55}$$

From (54) and (55) we get

$$\omega_k = b_k + \frac{1}{16\pi}\int_\Omega |\nabla\varphi_k|^2 - \frac{1}{8\pi}\langle g^*, \varphi_k\rangle. \tag{56}$$

Since $b_k \geq k$ (see (51)), from (56) we have

$$\omega_k \geq k + c_k, \tag{57}$$

where

$$c_k = \frac{1}{16\pi}\int_\Omega |\nabla\varphi_k|^2 - \frac{1}{8\pi}\langle g^*, \varphi_k\rangle.$$

Since

$$\frac{1}{8\pi}\langle g^*, \varphi_k\rangle \leq \mathrm{const}\|g^*\|_{H^{-1}}\|\varphi_k\|_{H_0^1(\Omega)} \tag{58}$$

we have

$$c_k \geq \frac{1}{16\pi}\int_\Omega |\nabla\varphi_k|^2 - \mathrm{const}\|g^*\|_{H^{-1}}\|\varphi_k\|_{H_0^1(\Omega)}.$$

Then we have that $c_k$ is bounded below. So, by (57), we deduce (52) and the proof is complete. □

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*Manuscript received August 4, 1998*


Vieri Benci
Dipartimento di Matematica Applicata "U. Dini"
and
Centro Interdisciplinare per lo Studio dei Sistemi Complessi
Università di Pisa
Via Bonanno 25/b
56100 Pisa, ITALY

*E-mail address*: benci@dma.unipi.it

Donato Fortunato
Dipartimento Interuniversitario di Matematica
Università di Bari
Via E. Orabona 4
70125 Bari, ITALY

*E-mail address*: fortunat@pascal.dm.uniba.it