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THE SPACE OF LOOPS ON CONFIGURATION SPACES AND THE MAJER–TERRACINI INDEX

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Dedicated to Jürgen Moser

1. Introduction

The topology of configuration spaces and their free and based loop spaces plays an important role in the study of the existence of periodic solutions of Hamiltonian systems of n-body type (see e.g. [1], [6], [7]). In particular, if $F_n(\mathbb{R}^k)$ is the configuration space of n-particles $u = (u_1, \dots, u_n)$ in $\mathbb{R}^k$, then the free loop space $\Lambda F_n(\mathbb{R}^k)$ is, up to homotopy type, the domain of a functional f whose critical points are solutions of a corresponding n-body problem. Thus, the homotopy-type invariants of $\Lambda F_n(\mathbb{R}^k)$ such as Lusternik–Schnirelmann (LS) category (and its generalizations — commonly called "index theories"), homotopy and homology play an important role in the subject. In some recent work ([15], [16], [17]), P. Majer and S. Terracini introduced an interesting "collision index" in the space $\Lambda F_n(\mathbb{R}^k)$ which allowed contractions of subsets of $\Lambda F_n(\mathbb{R}^k)$ to move through subspaces intermediate to $\Lambda F_n(\mathbb{R}^k)$ and $\Lambda(R^{nk})$, thus allowing a limited number of collisions during the contraction. However, their index is based upon an equivalence relation which results in a generalized notion of collision. Namely, that u_i "collides" with u_j during the deformation h if there is a chain of indices $i_1, \dots, i_s$, such that $u_i = u_{i_1}$, $u_j = u_{i_s}$ and u_{i_r} collides with $u_{i_{r+1}}$, $1 \leq r \leq s$.

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One of our objectives here is to introduce an alternative and more natural approach to that of Majer and Terracini which is based on a modification of the classical concept of LS-category relative to a given open collection $\Gamma = \{\Gamma_\gamma\}$ in $\Lambda(\mathbb{R}^{nk})$ which covers $\Lambda F_n(\mathbb{R}^k)$. This approach substantially simplifies the topology involved as well as providing a useful tool for more general situations, e.g., when $\mathbb{R}^k$ is replaced by an arbitrary manifold. In addition, the computational aspects of this "Γ-category" are more tractable.

In Section 2, we present some of the general theory of the topology of the based loops $\Omega F_n(\mathbb{R}^k)$. In particular, we exhibit the structure of its Hopf algebra. The algebra structure is given in terms of generators and relations, while the primitives in the coalgebra structure are completely determined. The latter is relevant to determining the LS category of subsets A in $\Omega F_n(\mathbb{R}^k)$, which, by the results in [7], are the same when considered as subsets of $\Lambda F_n(\mathbb{R}^k)$. These calculations are made using a coproduct length introduced in [8], [9]. The relations in the Hopf algebra $H_*(\Omega F_n(\mathbb{R}^k))$ are obtained from Whitehead product identities in the homotopy groups $\pi_*(F_n(\mathbb{R}^k))$, which are of Yang–Baxter type. They appear in [10] and a more thorough treatment will be found in [11]. Section 3 contains the general development of Γ-category based on an open cover Γ, while the applications to the Majer–Terracini method are found in Section 4.

2. Topological preliminaries

The (free) loop space $\Lambda X = C^0(S^1, \mathbb{R})$ of a topological space X plays an important role in nonlinear functional analysis especially when X is a subset of Euclidean space such as a finite dimensional manifold. For example, if $\mathcal{H} = W_T^{1,2}(\mathbb{R}^{kn})$ is the Sobolev space of T-periodic (absolutely continuous) functions $u = (u_1, \ldots, u_n)$, $u_i : \mathbb{R} \to \mathbb{R}^k$ and if

$$\Lambda' = \{u = (u_1, \ldots, u_n) \in \mathcal{H} : u_i(t) \neq u_j(t),\ \forall t \in \mathbb{R}\}.$$

is its subspace of non-colliding orbits, then Λ' is the same homotopy type [18] as $\Lambda F_n(\mathbb{R}^k)$ where $F_n(\mathbb{R}^k)$ is the n-th configuration space of $\mathbb{R}^k$. Thus, when employing homotopy-type invariants in Λ' they may be computed in $\Lambda F_n(\mathbb{R}^k)$. In particular, in applying the Lusternik–Schnirelmann (LS) method in Λ', every subset of Λ' of LS-category has its counterpart in Λ of the same LS-category.

On the other hand, the space of based loops $\Omega X \subset \Lambda X$ plays the following role, where recall that $u \in \Omega X$ takes a basepoint of S^1 to a basepoint of X. If A is a subset of ΩX then, under appropriate general conditions [6], we have

$$\operatorname{cat}_{\Omega X} A = \operatorname{cat}_{\Lambda X} A,$$

where cat is LS-category. Since ΩX is an H-space [20] (using multiplication of loops) its homology has the structure of a Hopf Algebra which proves useful in computing in the LS-category of subsets of ΩX. Thus, we are led to a study of the Hopf Algebra $H_*(\Omega F_n(\mathbb{R}^k))$.

We begin with a review of the homotopy groups $\pi_*(F_n(\mathbb{R}^k))$ ([10], [11]). We assume $k \geq 3$ so that $F_n(\mathbb{R}^k)$ is simply connected (=1-connected). The case $F_n(\mathbb{R}^2)$ is special, e.g. $\pi_1(F_n(\mathbb{R}^2))$ is the group of "pure braids" and will not be considered here. We recall that the fibration $p : F_n(\mathbb{R}^k) \to F_m(\mathbb{R}^k)$, $1 \leq m < n$, $p(x_1, \dots, x_n) = (x_1, \dots, x_m)$, always admits a section [5], and the fiber over $(q_1, \dots, q_m) \in F_m(\mathbb{R}^k)$ is $F_{n-m}(\mathbb{R}^k - Q_m)$ where $Q_m = \{q_1, \dots, q_m\}$. For later use we remark that for $m \geq 2$, a section f for p may be chosen so that

$$f(x_1, \dots, x_m) = (x_1, \dots, x_m, y_{m+1}, \dots, y_n)$$

where, for $m+1 \leq i \leq n$, y_i lies on the ray from x_{m-1} through x_m. Thus we have a sequence of fibrations $p_i, 2 \leq i \leq n$,

$$\begin{array}{ccccccccc}
\mathbb{R}^k - Q_{n-1} & & \mathbb{R}^k - Q_{n-2} & & \mathbb{R}^k - Q_{n-2} & \cdots & \mathbb{R}^k - Q_1 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
F_n(\mathbb{R}^k) & \xrightarrow{p_n} & F_{n-1}(\mathbb{R}^k) & \xrightarrow{p_{n-1}} & F_{n-2}(\mathbb{R}^k) & \to \cdots \to & F_2(\mathbb{R}^k) & \xrightarrow{p_2} & \mathbb{R}^k
\end{array}$$

with the various fibers in the first row and where the fiber map p_2 is trivial, i.e. $F_2(\mathbb{R}^k)$ is homeomorphic to $\mathbb{R}^k \times (R^n - Q_1)$. Thus,

$$\pi_* F_n(\mathbb{R}^k) \cong \bigoplus_{r=1}^{n-1} \pi_*(\mathbb{R}^k - Q_r).$$

A convenient set of generators for the homology $H_*(F_n(\mathbb{R}^k))$ which appear in F. Cohen and L. R. Taylor [3] have their following analogues in $\pi_*(F_n(\mathbb{R}^k))$. For $1 \leq s < r \leq n$, define

$$\alpha'_{rs} : S^{k-1} \to F_n(\mathbb{R}^k)$$

by

$$\alpha'_{r,s}(\xi) = (q_1, \dots, q_{r-1}, q_s + \xi, q_r, \dots, q_{n-1}), \quad \xi \in S^{k-1}$$

where $q_j = 4(j-1)e_1, 1 \leq j \leq n-1$, $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^k$. $\alpha'_{r,s}$ may also be viewed as the map $\alpha'_{r,s} : S^{k-1} \to \mathbb{R}^k - Q_{r-1}$, where

$$\alpha'_{r,s}(\xi) = q_s + \xi.$$

Now let $\alpha_{r,s}$ denote the homotopy class of $\alpha'_{r,s}$, so that one may consider

$$\alpha_{r,s} \in \pi_{k-1}(F_n(\mathbb{R}^k)) \quad \text{or} \quad \alpha_{r,s} \in \pi_{k-1}(\mathbb{R}^k - Q_{r-1}).$$

Since $\mathbb{R}^k - Q_{r-1}$ is the homotopy type as the wedge of $r-1$ $(k-1)$-spheres, we write $\mathbb{R}^k - Q_{r-1} \sim S_{r,1} \vee \ldots \vee S_{r,r-1}$ and also denote the generator of $\pi_{k-1}(S_{r,s})$ by $\alpha_{r,s}$. We also note that using the section $f : F_{n-1}(\mathbb{R}^k) \to F_n(\mathbb{R}^k)$, $\alpha_{r,s}$ in $\pi_{k-1}(F_{n-1}(R^k))$ and $\alpha_{r,s}$ in $\pi_{k-1}(F_n(\mathbb{R}^k))$ may be identified for $1 \leq s < r \leq n-1$.

The full symmetric group Σ_n, acts on $\pi_*(F_n(\mathbb{R}^k))$ via the usual free action of Σ_n on $F_n(\mathbb{R}^k)$, i.e. if $\sigma \in \Sigma_n$,

$$\sigma(x_1, \dots , x_n) = (x_{\sigma(1)}, \dots , x_{\sigma(n)})$$

and $\sigma : F_n(\mathbb{R}^k) \to F_n(\mathbb{R}^k)$ induces $\sigma_* : \pi_*(F_n(\mathbb{R}^k)) \to \pi_*(F_n(\mathbb{R}^k)$, so that Σ_n acts on $\pi_*(F_n(\mathbb{R}^k))$ by $\sigma\alpha = \sigma_*\alpha$, $\alpha \in \pi_*(F_n(\mathbb{R}^k))$. The set $\{\alpha_{r,s},\ 1 \le s < r \le n\}$ is fixed (up to sign) under the action of Σ_n.

Proposition 2.1. *If $1 \le s < r \le n$ and $\sigma \in \Sigma_k$, then*

$$\sigma\alpha_{r,s} = \alpha_{\sigma(r),\sigma(s)}$$

with the convention that $\alpha_{\sigma(s),\sigma(r)} = (-1)^k\alpha_{\sigma(r),\sigma(s)}$, if $\sigma(s) < \sigma(r)$.

This proposition is instrumental in proving certain Whitehead products vanish thus producing "commutativity" relations. First, recall that the Whitehead product $[\,\cdot\,,\,\cdot\,]$ is a (natural) operation

$$[\,\cdot\,,\,\cdot\,] : \pi_{p+1}(X) \times \pi_{q+1}(X) \to \pi_{p+q+1}(X)$$

where, when $p = q = 0$, $[\alpha, \beta] = \alpha\beta\alpha^{-1}\beta^{-1}$ and when X is simply connected, $[\alpha, \beta] = 0$ if and only the map $\alpha \vee \beta : S^{p+1} \vee S^{q+1} \to X$ extends to $S^{p+1} \times S^{q+1}$.

Theorem 2.2 ([10], [11]). *For all $\sigma \in \Sigma_k$ the following identities hold:*

(i) $[\alpha_{\sigma(2),\sigma(1)}, \alpha_{\sigma(3),\sigma(1)} + \alpha_{\sigma(3),\sigma(2)}] = 0$,
(ii) $[\alpha_{\sigma(2),\sigma(1)}, \alpha_{\sigma(4),\sigma(3)}] = 0$,

or more simply

(i) $\sigma[\alpha_{2,1}, \alpha_{3,1} + \alpha_{3,2}] = 0$,
(ii) $\sigma[\alpha_{2,1}, \alpha_{4,3}] = 0$.

These identities resemble certain "braid relations" in the work of Baxter and Yang and will be called YB-relations (see, for example [13] and [14]). In the $\mathbb{R}^2$-case (which doesn't concern us here), they yield, when properly modified to to account for a base point, a nice presentation of the pure braid group. In our situation, i.e. $k \ge 3$, they are simply Whitehead product relations in the graded Lie algebra $\pi_*(F_n(\mathbb{R}^k))$. They will be crucial in computing the algebra structure in the Hopf algebra $H_*(\Omega_n(\mathbb{R}^k))$ which follows.

Recall that for any simply connected space X, there is a natural isomorphism $\pi_q(X) \to \pi_{q-1}(\Omega X)$ and the Hurewicz homomorphism $\pi_{q-1}(\Omega X) \to H_{q-1}(\Omega X)$ whose composition we denote by

$$\varphi : \pi_q(X) \to H_{q-1}(\Omega X), \quad q \ge 1.$$

In $H_*(\Omega X)$ we use $\mathbb{Z}$ coefficients.

Samelson's Theorem [19] provides the relation between Whitehead products in $\pi_*(X)$ and multiplication (Pontryagin products) in $H_*(\Omega X)$.

Theorem 2.3. *If $\alpha \in \pi_{p+1}(X)$, $\beta \in \pi_{q+1}(X)$, $p, q \geq 1$, then*

$$\varphi[\alpha, \beta] = (-1)^p(\varphi(\alpha)\varphi(\beta) - (-1)^{pq}\varphi(\beta)\varphi(\alpha)).$$

Definition 2.4. For $\alpha_{r,s} \in \pi_{k-1}(F_n(\mathbb{R}^k)$, $1 \leq s, r \leq k$, $s \neq r$ set

$$\overline{\alpha}_{r,s} = \varphi(\alpha_{r,s})$$

Applying Theorem 2.3 we obtain the following relations in the algebra $H_*(\Omega F_n(\mathbb{R}^k)$.

Theorem 2.5. *The following relations hold in $H_*(\Omega F_n(R^k)$. For $\sigma \in \Sigma^k$,*

(i) $\sigma(\overline{\alpha}_{2,1}(\overline{\alpha}_{3,1} + \overline{\alpha}_{3,2}) + (-1)^k(\overline{\alpha}_{3,1} + \overline{\alpha}_{3,2})\overline{\alpha}_{2,1}) = 0,$
(ii) $\sigma(\overline{\alpha}_{2,1}\overline{\alpha}_{4,3} + (-1)^k\overline{\alpha}_{4,3}\overline{\alpha}_{2,1}) = 0,$

both with the convention that $\overline{\alpha}_{s,r} = (-1)^k\overline{\alpha}_{r,s}$ for $1 \leq s < r \leq k$.

Remark 2.6. We will refer to (i)–(iii) as the SYB-relations (for Samelson–Yang–Baxter). Additional insight into the algebra $H_*(\Omega F_n(\mathbb{R}^k)$ is gained by considering the fibration

$$\Omega(\mathbb{R}^k - Q_{n-1}) \xrightarrow{\Omega j_n} \Omega F_n(\mathbb{R}^k) \xrightarrow{\Omega p_n} F_{n-1}(\mathbb{R}^k)$$

which corresponds to the fibration

$$\mathbb{R}^k - Q_{n-1} \xrightarrow{j_n} F_n(\mathbb{R}^k) \xrightarrow{p_n} F_{n-1}(\mathbb{R}^k).$$

Then Ωf is a section for Ωp_n, where f is a section for p_n. Then we have a map

$$\psi : \Omega F_{n-1}(\mathbb{R}^k) \times \Omega(\mathbb{R}^k - Q_{n-1}) \to \Omega F_n(\mathbb{R}^k)$$

given by $\psi(xy) = \Omega f(x)\Omega j_n(y)$ (loop multiplication). ψ is readily seen to be a homotopy equivalence, (employing, for example, Dold's Theorem [4]) which is fiber-preserving, up-to-homotopy. We stress here that ψ is not an H-space homotopy equivalence. Thus we obtain the following implications.

Proposition 2.7.

(a) *ψ induces*

$$\psi_* : H_*(\Omega F_{n-1}(\mathbb{R}^k)) \otimes H_*(\Omega(\mathbb{R}^k - Q_{n-1})) \to H_*(\Omega F_n(\mathbb{R}^k))$$

which is an isomorphism of $H_*(\Omega F_{n-1}(\mathbb{R}^k))$-modules.

(b) $\psi_0(\overline{\alpha}_{r,s} \otimes \overline{\alpha}_{n,t}) = \overline{\alpha}_{r,s}\overline{\alpha}_{n,t}$, $1 \leq s < r \leq n-1$, $t < n$.

Corollary 2.8.

(a) *$H_*(\Omega F_n(\mathbb{R}^k)$ is generated as an algebra by $\overline{\alpha}_{r,s},\ 1 \le s < r \le k$ which satisfy the* SYB*-relations.*

(b) *Every element $u \in H_*(\Omega F_n(\mathbb{R}^k))$ may be written uniquely in the form*

$$u = x + y,\ x \in H_*(\Omega F_{n-1}(\mathbb{R}^k)) \text{ and } y \in H_*(\Omega F_{n-1}(\mathbb{R}^k)\overline{H}_*(\Omega(\mathbb{R}^k - Q_{n-1})),$$

where $\overline{H}$ is reduced homology.

(c) *By induction on n, there is an isomorphism of $\mathbb{Z}$-modules*

$$\psi' : \bigotimes_{r=2}^{n} H_*(\Omega(\mathbb{R}^k - Q_{r-1}) \to H_*(\Omega F_n R^k))$$

and ψ' restricted to $H_*(\Omega(\mathbb{R}^k - Q_{r-1}))$ is an algebra homomorphism.

(d) *Since each $H_*(\Omega(\mathbb{R}^k - Q_{r-1}))$ is the free associative algebra on $\alpha_{r,s},\ 1 \le s < r$,* ([3])*, $H_*(\Omega F_n(\mathbb{R}^k))$ is a free $\mathbb{Z}$-module.*

Our next result is to show that $H_*(\Omega F_n(\mathbb{R}^k))$ is in fact isomorphic to the free associative algebra generated by the set $A_n = \{\overline{\alpha}_{r,s},\ 1 \le s < r \le n\}$ and $\{1\}$ modulo the SYB-relations. Let $\mathfrak{A}_n$ denote the free, graded, associative algebra generated by A_n modulo the 2-sided ideal I_n generated by the SYB-relations, with the grading corresponding to that of $H_*(\Omega F_n(\mathbb{R}^k))$. Then, there is an algebra homomorphism $g_n : \mathfrak{A}_n \to \mathfrak{A}_{n-1}$ such that

$$g_n(\overline{\alpha}_{r,s}) = \begin{cases} \overline{\alpha}_{r,s} & \text{if } 1 \le s < r \le n-1, \\ 0 & \text{if } 1 \le s < r = n, \end{cases}$$

g_n admits a right inverse $k_n : \mathfrak{A}_{n-1} \to \mathfrak{A}_n$ with $h_n(\overline{\alpha}_{r,s}) = \overline{\alpha}_{r,s},\ 1 \le s < r \le n-1$. Then, we have a split exact sequence

$$0 \to K_n \to \mathfrak{A}_n \to \mathfrak{A}_{n-1} \to 0,$$

where K_n is the kernel of g_n, which is the two-sided ideal generated by $\overline{\alpha}_{n,s} 1 \le s < n$.

Lemma 2.9. *$\mathfrak{A}_n$ is a left module over $\mathfrak{A}_{n-1}$ and $\mathfrak{A}_n = \mathfrak{A}_{n-1} \oplus \mathfrak{A}_{n-1}\mathfrak{A}_{n,n-1}$, where $\mathfrak{A}_{n,n-1}$ is the subalgebra generated by $\overline{\alpha}_{n,s} 1 \le s < n$.*

Proof. It suffices to show that $\overline{\alpha}_{n,t}\overline{\alpha}_{r,s} = \pm\overline{\alpha}_{r,s}\overline{\alpha}_{n,t} + x,\ r < n$, where $x \in \mathfrak{A}_{n,n-1}$. First of all, if n, t, r, s are mutually distinct then using the SYB-relations $\overline{\alpha}_{n,t}\overline{\alpha}_{r,s} = \pm\overline{\alpha}_{n,t}\overline{\alpha}_{r,s}$. Next, we observe that the SYB-relation

$$(1) \qquad (\overline{\alpha}_{2,1}(\overline{\alpha}_{3,1} + \overline{\alpha}_{3,2}) + (-1)^k(\overline{\alpha}_{3,1} + \overline{\alpha}_{3,2})\overline{\alpha}_{2,1}) = 0$$

implies the relations

$$(2) \qquad (\overline{\alpha}_{3,1}(\overline{\alpha}_{2,1} + (-1)^k\overline{\alpha}_{3,2}) + (-1)^k(\overline{\alpha}_{2,1} + (-1)^k\overline{\alpha}_{3,2})\overline{\alpha}_{3,1}) = 0$$

and

$$(3) \qquad (-1)^k\overline{\alpha}_{3,2}(\overline{\alpha}_{2,1}+\overline{\alpha}_{3,1})+(\overline{\alpha}_{2,1}+\overline{\alpha}_{3,1})\overline{\alpha}_{3,2}=0,$$

and hence

$$(4) \qquad \overline{\alpha}_{3,1}\overline{\alpha}_{2,1} = (-1)^{k-1}\overline{\alpha}_{2,1}\overline{\alpha}_{3,1}-\overline{\alpha}_{3,2}\overline{\alpha}_{3,1}+(-1)^{k-1}\overline{\alpha}_{3,1}\overline{\alpha}_{3,2},$$

$$(5) \qquad \overline{\alpha}_{3,2}\overline{\alpha}_{2,1} = (-1)^{k-1}\overline{\alpha}_{2,1}\overline{\alpha}_{3,1}+(-1)^{k-1}\overline{\alpha}_{3,1}\overline{\alpha}_{3,2}+(-1)^{k-1}(\overline{\alpha}_{3,2}\overline{\alpha}_{3,1}).$$

Then let σ denote a permutation such that $\sigma(1)=s,\ \sigma(2)=2,\ \sigma(3)=n$. Then, applying σ to (4)

$$(6) \qquad \overline{\alpha}_{n,s}\overline{\alpha}_{r,s}+\pm\overline{\alpha}_{r,s}\overline{\alpha}_{n,s}-\overline{\alpha}_{n,r}\overline{\alpha}_{n,s}+(-1)^{k-1}\overline{\alpha}_{n,s}\overline{\alpha}_{n,r}.$$

In a similar fashion, using (5) and an appropriate σ,

$$(7) \qquad \overline{\alpha}_{n,r}\overline{\alpha}_{r,s}=(-1)^{k-1}\overline{\alpha}_{r,s}\overline{\alpha}_{n,r}+(-1)^{k-1}\overline{\alpha}_{n,s}\overline{\alpha}_{n,r}+(-1)^{k-1}(\overline{\alpha}_{n,r}\overline{\alpha}_{n,s})$$

which completes the remaining case. □

Corollary 2.10. *The kernel* $K_n=\mathfrak{A}_{n-1}\cdot\mathfrak{A}_{n,n-1}$.

Remark 2.11. The notation $\overline{\alpha}_{r,s}$ represents both an element of $\mathfrak{A}_n$ and $H_*(\Omega F_n(\mathbb{R}^k))$ but the context makes clear wherein the element lies.

Let $\rho_k:\mathfrak{A}_k\to H_*(\Omega F_n(\mathbb{R}^k))$ denote the algebra homomorphism which corresponds $\overline{\alpha}_{r,s}$ in $\mathfrak{A}_k$ to $\overline{\alpha}_{r,s}$ in $H_*(\Omega F_n(\mathbb{R}^k))$.

Theorem 2.12. *ρ_k is an isomorphism, for $n\geq 2$.*

Proof. The proof is by induction on n. The induction easily starts at $n=2$, since $H_*(\Omega F_2(\mathbb{R}^k))\approx H_*(\Omega S^{k-1})$, which is a polynomial algebra on $\overline{\alpha}_{21}$. For $n>2$ we have a diagram of exact sequences,

$$\begin{array}{ccccc} H_*(\Omega F_{n-1}(\mathbb{R}^k))\overline{H}_*(\Omega(\mathbb{R}^k-Q_{n-1})) & \rightarrowtail & H_*(\Omega F_n(\mathbb{R}^k)) & \twoheadrightarrow & H_*(\Omega F_{n-1}(\mathbb{R}^k)) \\ \uparrow{\scriptstyle\rho'_n} & & \uparrow{\scriptstyle\rho_n} & & \uparrow{\scriptstyle\rho_{n-1}} \\ \mathfrak{A}_{n-1}\mathfrak{A}_{n,n-1} & \rightarrowtail & \mathfrak{A}_n & \twoheadrightarrow & \mathfrak{A}_{n-1} \end{array}$$

where ρ_{n-1} is an isomorphism by induction and $\rho_{n,n-1}=\rho'_n|\mathfrak{A}_{n,n-1}:\mathfrak{A}_{n,n-1}\to\overline{H}_*(\Omega(\mathbb{R}^k-Q_{n-1}))$ is also an isomorphism because $\overline{H}_*(\Omega(\mathbb{R}^k-Q_{n-1}))$ is the free associative algebra (without unit) on $A_{n,n-1}$. Let ρ^*_{n-1} and $\rho^*_{n,n-1}$ denote, inverses for ρ_{n-1} and $\rho_{n,n-1}$ respectively, and define (it is well-defined) an inverse ρ^* for ρ'_n using the diagram

$$\begin{array}{ccc} H_*(\Omega F_{n-1}(\mathbb{R}^k))\overline{H}_*(\Omega(\mathbb{R}^k-Q_{n-1})) & \cong & H_*(\Omega F_{n-1}(\mathbb{R}^k))\otimes H_*(\Omega(\mathbb{R}^k-Q_{n-1})) \\ \downarrow & & \downarrow \\ \mathfrak{A}_n\mathfrak{A}_{n,n-1} & \longleftarrow & \mathfrak{A}_{n-1}\otimes\mathfrak{A}_{n,n-1} \end{array}$$

i.e. $\rho^*(\alpha_{r,s}\alpha_{k,t}) = \rho^*_{n-1}(\alpha_{r,s})\rho^*_{n,n-1}(\alpha_{k,t})$. Then, it follows that ρ_n is an isomorphism.

The coalgebra structure in $H^*(\Omega F_n(\mathbb{R}^k))$ is determined inductively using the homotopy equivalence

$$\Omega F_n(\mathbb{R}^k) \sim \Omega F_{n-1}(\mathbb{R}^k) \times \Omega(\mathbb{R}^k - Q_{n-1}).$$

Hence,

$$H_*(\Omega F_n(\mathbb{R}^k) \cong H_*(\Omega F_{n-1}(\mathbb{R}^k) \otimes H_*(\Omega(\mathbb{R}^k - Q_{n-1}))$$

as coalgebras. □

Proposition 2.13. *As coalgebras*

$$H_*(\Omega F_n(\mathbb{R}^k)) \cong \bigotimes_{r=1}^{n-1} H_*(\Omega(\mathbb{R}^k - Q_r)).$$

We note at this point that each $\overline{\alpha}_{r,s}, 1 \leq s < r \leq n$, is a primitive element (being spherical) and that

$$\Delta_{r*} : H_*(\Omega(\mathbb{R}^k - Q_r) \to H_*(\Omega(\mathbb{R}^k - Q_r)) \otimes H_*(\Omega\mathbb{R}^k - Q_r)$$

is completely determined by $\Delta_{r*}(\overline{\alpha}_{r,s}) = 1 \otimes \overline{\alpha}_{r,s} + \overline{\alpha}_{r,s} \otimes 1$, $1 \leq s < r$, where Δ_r is the diagonal map.

Let $\mathcal{P}(A)$ denote the set of primitives in a coalgebra A.

Corollary 2.14.

$$\mathcal{P}(H_*(\Omega F_n(\mathbb{R}^k)) \cong \bigoplus_{r=1}^{n-1} \mathcal{P}(H_*(\Omega(\mathbb{R}^k - Q_r).$$

The set of primitives $\mathcal{P}(H_*(\Omega F_n(\mathbb{R}^k)))$ plays a role in the LS-category of subsets of $\Omega F_n(\mathbb{R}^k)$ as we review shortly. The following result is an exericse.

Proposition 2.15. *Let S^m denote an m-sphere and $H_*(\Omega S^m)$ the corresponding Hopf algebra* (*over $\mathbb{Z}$*) *with generator $\overline{\alpha}$. Then the generators for $\mathcal{P}(H_*(\Omega S^m)$ are $\overline{\alpha}$ when m is odd and $\overline{\alpha}$, $(\overline{\alpha})^2$ when m is even. If we use $\mathbb{Z}_2$ coefficients $\mathcal{P}(H_*(\Omega S^m; \mathbb{Z}_2)$ is generated by $\overline{\alpha}^p$ where p is a power of* 2.

Now, recall the Hilton–Milnor decomposition theorem [12]:

Theorem 2.16.

$$\Omega(\mathbb{R}^k - Q_r) \cong \prod_{w \in W_r} \Omega S_w$$

where W_r is the set of basic elements on the set $\overline{\alpha}_{r,s}$, $1 \leq s < r$, and S_w is a sphere whose dimension depends on the weight of w.

Corollary 2.17.

$$\mathcal{P}(H_*(\Omega(\mathbb{R}^k - Q_r)) \cong \bigoplus_{w \in W_r} \mathcal{P}(H_*(\Omega S_w)).$$

In particular, over $\mathbb{Z}_2, \mathcal{P}(H_*(\Omega(\mathbb{R}^k - Q_r); \mathbb{Z}_2)), r \geq 1$, *has infinite dimension.*

We now review the role that primitives in $H_*(\Omega Y)$ play in computing category in the loop space of a space Y (see [8] and [9]). We note also that $H_*(\Omega Y)$ is a Hopf algebra without the finite type assumption on ΩY required for the corresponding cohomology $H^*(\Omega Y)$ to form a Hopf algebra.

Let Y denote a 0-connected ANR (metric) with a base point $*$ and $A \neq \phi$ a subset of Y. Let Δ_A^n denote the n-fold diagonal map $\Delta_A^m : A \to Y^m$ and let W^m denote the subset of Y (called the fat-wedge) given by

$$W^m = \{(y_1, \dots, y_m) \in Y^m : y_i = * \text{ for some } i,\ 1 \leq i \leq m\}.$$

Then, following G. Whitehead [20], $\operatorname{cat}_Y A \leq m$ if, and only if, Δ_A^m is deformable (in Y^m) into the subset W^m.

On the other hand, for any coefficient ring R, consider the induced homomorphism on reduced homology

$$\overline{\Delta}_{A*}^m : \overline{H}_*(A; R) \to H_*(Y^m, W^m; R),$$

where $\overline{\Delta}_A^m = j\Delta_A^m$ and j is the inclusion $j : Y^m \to (Y^m, W^m)$.

Definition 2.18. $\operatorname{copl}_Y A = \min\{m : \Delta_A^m * = 0\}$, where copl is read "coproduct length". If $\Delta_A^m * \neq 0$ for all m, $\operatorname{copl}_Y A = \infty$.

The following is an immediate consequence.

Proposition 2.19. $\operatorname{copl}_Y A \leq \operatorname{cat}_Y A$.

Among other useful properties of coproduct length are the following (see [8] and [9]).

Proposition 2.20. *If* $A_1 \subset Y_1$ *and* $A_2 \subset Y_2$, *where* Y_1 *and* Y_2 *are* ANR (*metric*),

$$\operatorname{copl}_{Y_1 \times Y_2} A_1 \times A_2 \geq \operatorname{copl}_Y A_1 + \operatorname{copl}_{Y_2} A_2 - 1$$

whenever $\min\{\operatorname{copl}_{Y_1} A_1, \operatorname{copl}_{Y_2} A_2\} \geq 2$.

The following is a modification of a result from [9].

Theorem 2.21. *Let* Y *be an* H-*space* (*e.g. if* $Y = \Omega F_n(\mathbb{R}^k)$) *and* $H_*(Y; R)$ *the corresponding Hopf algebra. We suppose* R *is a* P.I.D. *and* $H_*(Y; R)$ *is a free* R-*module* (*e.g.* R *is a field or* $R = \mathbb{Z}$ *and* $H_*(Y; R)$ *is free abelian*). *Let* $A \neq \phi$ *denote a subset of* Y. *Suppose* $i : A \to Y$ *is the inclusion, then*

$$\operatorname{copl}_Y A \geq \operatorname{rank}_R \mathcal{P}(H_*(Y; R) \cap i_* H_*(A; R)).$$

We now turn to the question of compact subsets of $\Omega F_n(\mathbb{R}^k)$ which have high category which is important in subsequent applications in the next section.

Consider a typical element α of $H_*(\Omega F_n(\mathbb{R}^k); R)$ Then, $\sup(\alpha)$ will denote any choice of compact support for α. Then set

$$A_{r,s}(p) = \bigcup_{p=1}^{m} \sup(\overline{\alpha}_{r,s}^{2^p}).$$

Furthermore, in the representation $\Omega(\mathbb{R}^k - Q_{r-1}) \sim \prod_{w \in W} \Omega S_w,\ r \geq 3$, let $A_r(w_1, \dots, w_p) \subset \Omega F_n(\mathbb{R}^k)$, correspond to $\sup(\overline{\alpha}_{w_1} \times \dots \times \overline{\alpha}_{w_p})$ where w_i is a basic element and $\overline{\alpha}_{w_i}$ is the corresponding generator of ΩS_{w_i}.

Proposition 2.22. *Let $Y = \Omega F_n(\mathbb{R}^k)$. Then*

(a) $\operatorname{cat}_Y A_{r,s}(p) \geq \operatorname{copl}_Y A_{r,s}(p) \geq p$, *when the coefficient ring $R = \mathbb{Z}_2$.*
(b) $\operatorname{cat}_Y A_r(w_1, \dots w_p) \geq p$, *when the coefficient R is $\mathbb{Z}$.*

Corollary 2.23. *For $n \geq 2,\ k \geq 3,\ \Omega F_n(\mathbb{R}^k)$ contains compact subsets of arbitrarily high category.*

Remark 2.24. Proposition 2.22 only illustrates the numerous possibilities.

3. A variant of the Majer–Terracini index

We begin with a brief review of the setting in which Majer and Terracini [15]–[17] introduce a variation of Lusternik–Schnirelmann (LS) category which when applied in the context of the LS min-max method gives an interesting technique for proving the existence of periodic solutions to dynamical systems of the type

$$-\ddot{u}_i = \sum_{i,j=1}^{n} \nabla V_{ij}(u_i - u_j, t), \tag{1}$$

where the V_{ij} are periodic in t of period T and singular at the origin, $u_i \in \mathbb{R}^k,\ i = 1, \dots, n$. As usual, the solutions of (1) are critical points of the functional

$$f(u) = \frac{1}{2} \int_0^T |\dot{u}|^2 - \int_0^T V(u,t) \tag{2}$$

on the domain

$$\Lambda' = \{(u, \dots, u_n) \in \mathcal{H} :\ u_i(t) \neq u_j(t),\ \forall\, t \text{ and } \forall i \neq j\} \tag{3}$$

where $\mathcal{H}$ is the Sobolev space $W_T^{1,2}(\mathbb{R}, \mathbb{R}^{nk})$. Λ' is the same homotopy type as the (topological) free loop space $\Lambda F_n(\mathbb{R}^k)$, where $F_n(\mathbb{R}^k)$ is the nth configuration space of $\mathbb{R}^k$. Λ' will also be designated by $\Lambda' F_n(\mathbb{R}^k)$.

REMARK 3.1. We explicitly note here that the overall assumption on $\mathbb{R}^k$ is always $k \geq 3$, so that Λ is path connected. The methods do not apply in the case of $\mathbb{R}^2$ since then Λ is disconnected, with contractible components.

The Majer–Terracini variation of LS-category goes as follows. Let $D(\mathcal{H})$ denote the set of all deformations of $\mathcal{H}$ into the subspace $\mathbb{R}^{kn}$ of constant functions. Then define a relation $r_{h,u}$ on the set $I = \{1, \dots, n\}$ of indices for each $h \in D(\mathcal{H})$ and $u = (u_1, \dots, u_n) \in \Lambda'$ by

$$(4) \qquad i r_{h,u} j \equiv \exists \lambda_{ij} \in [0,1] \text{ and } t_{i,j} \in \mathbb{R} \quad \text{such that} \quad h_i(u, \lambda_{ij})(t_{ij}) = h_j(u, \lambda_{ij})(t_{ij}),$$

i.e. at some stage λ_{ij} of the deformation parameter, the ith and jth particles collide at time t_{ij}. Let $R_{h,u}$ denote the smallest equivalence relation containing $r_{h,u}$, i.e.,

$$(5) \qquad i R_{h,u} j \equiv \exists \text{ indices } i_1, \dots, i_s \text{ in } I : i r_{h,u} i_1, i_1 r_{h,u} i_2, \dots, i_s r_{h,u} j.$$

Thus, $iR_{h,u}j$ need not mean that at some stage of the deformation the ith and jth particles collide but something more complicated. The relation $R_{h,u}$ is then used to define the class of closed sets A in Λ' (the "admissible sets") over which one min-maxes the functional (2) to find critical values.

DEFINITION 3.2. A closed set $A \subset \Lambda'$ is called *admissible* if for every $h \in D(\mathcal{H})$ there exists a $u \in A$ such that $iR_{h,u}j$ for every $i \neq j$.

Thus, the admissible closed sets play the role of the "non-contractible sets" (category > 1) while a non-admissible set plays the role of a "contractible set" (category $= 1$). A central topological question then becomes the existence of admissible compact sets and indeed the existence of admissible compact sets which require an arbitrarily large number of non-admissible sets to cover them. In the Majer–Terracini method the existence of an admissible set is obtained in an ad hoc manner, relying on [6]. We propose here to give a modification of their concept of admissible set which will allow greater flexibility in "category" computations as well as being conceptually simpler. At the same time, we verify that in their methods the concept of non-admissible may be replaced by a classical notion of "categorical" without sacrificing the validity of their arguments and, indeed, shed more light on the existence of "admissible sets". A concomittant bonus is that the setting can be made quite general.

Let X denote a (separable) metric space and $F_n(X)$ the nth configuration space of X, i.e.

$$F_n(X) = \{(x_1, \dots, x_n) : \; x_i \in X, \; x_i \neq x_j \text{ for } i \neq j\}.$$

Let $\Lambda F_n(X)$ denote the free loop space on $F_n(X)$, i.e., $\Lambda F_n(X) = C^0(S^1, F_n(X))$ with the uniform topology, where S^1 is the unit circle. If base points are chosen in S^1 and X, $\Omega F_n(X)$ is the space of based loops on $F_n(X)$, i.e. those $u \in \Lambda F_n(X)$ which take base point to base point. We then consider subsets $\Lambda_{ij} = \Lambda_{ij}(X)$, $1 \leq i < j \leq n$, of $C^0(S^1, X^n)$ which do not allow i,j collisions, i.e.

$$\Lambda_{ij} = \{u = (u_1, \dots, u_n) \in C^0(S^1, X^n) : \ \forall t \in S^1, \ u_i(t) \neq u_j(t)\}.$$

Similarly, we define $\Omega_{ij} = \Omega_{ij}(X)$, using based loops with base point in $F_n(X)$. Then, set

$$\Lambda^* = \bigcup_{i<j} \Lambda_{ij}, \quad \Omega^* = \bigcup_{i<j} \Omega_{ij}.$$

Observe that $\{\Lambda_{ij}\}$ and $\{\Omega_{ij}\}$ are open covers of Λ^* and Ω^*, respectively, and we now move to a more general setting for the notion of Γ-category where Γ is a collection of open sets in a given space.

Let Y denote an ANR (sep. metric) and $\Gamma = \{\Gamma_\gamma\}$ a fixed collection of open sets in Y. For simplicity, we assume Γ is a finite cover. Now let A denote a subset of Y such that $A \subset \Gamma_\gamma$ for every γ.

Definition 3.4. Γ-cat $A \leq p$ if A may expressed as a union $A = \bigcup_\gamma A_\gamma$ with $\operatorname{cat}_{\Gamma_\gamma} A_\gamma \leq p$ for every γ. If no such p exists, set Γ-cat $A = \infty$. Otherwise, set Γ-cat $A = p$ if p is the minimum integer with $\operatorname{cat}_\Gamma A \leq p$. Here $\operatorname{cat}_{\Gamma_\gamma} A_\gamma$ is the classical category of Lusternik–Schnirelmann.

Proposition 3.5. *The known properties of the classical category imply the following properties of Γ-category. We continue to employ the above setting.*

- Γ.1. (monotone) *If $A_1 \subset A_2$, then* Γ-cat $A_1 \leq \Gamma$-cat A_2.
- Γ.2. (subadditive) Γ-cat $(A_1 \cup A_2) \leq \Gamma$-cat $A_1 + \Gamma$-cat A_2.
- Γ.3. (continuity) Γ-cat $A = p$ *implies that there is an open set U such that* Γ-cat $\overline{U} = p$.
- Γ.4. (finiteness) *If A is compact,* Γ-cat A *is finite.*
- Γ.5. *Let X denote a subset of Y such that $X \subset \Gamma_\gamma$ for every γ and $H : X \times I \to X$ a deformation of X, i.e. $H_0(x) = x$ for $x \in X$. Assume that for each γ, H extends to a deformation $H^\gamma : \Gamma_\gamma \to \Gamma_\gamma$. Then for $A \subset X$,* Γ-cat $A \leq \Gamma$-cat $H_1(A)$.
- Γ.6. *Let $X = \bigcap \Gamma_\gamma$, and $A \subset X$, then* Γ-cat $A \leq \operatorname{cat}_X A$.

We note that if Γ-cat $A = p$, then A admits a decomposition

$$A = \bigcup_\gamma A_\gamma \quad \text{where} \quad A_\gamma = \bigcup_{k=1}^{p} A_{\gamma,k}$$

and A_k^γ is contractible in Γ_γ, i.e. categorical (in the classical sense) in Γ_γ. If we set $B_k = \bigcup_\gamma A_{\gamma,k}$, we are led to the following definition of Γ-categorical.

Definition–Proposition 3.7. Let $B \subset \bigcap_\gamma \Gamma_\gamma$. Then, B is Γ-categorical if B admits a decomposition $B = \bigcup_\gamma B_\gamma$ and each B_γ is categorical in Γ_γ. Thus, $\Gamma\text{-cat}\, A \leq p$ if, and only if, A admits a decomposition $A = \bigcup_{k=1}^p B_k$ such that each B_k is Γ-categorical. In particular, $\Gamma\text{-cat}\, A = 1$ is equivalent to saying that A is Γ-categorical.

As in the case of classical category, cup length in cohomology provides a lower bound for Γ-category.

Definition 3.8. Let A denote a closed subset of X, and write $c\ell_X A$ for the cup length of A in X. Then,

$\Gamma\text{-}c\ell A \leq p \equiv A$ may be written $A = \bigcup_\gamma A_\gamma$ with A_γ closed in Γ_γ and $c\ell_{\Gamma_\gamma} A_\gamma \leq p$. Then, $\Gamma\text{-}c\ell A = p \equiv \Gamma\text{-}c\ell A \leq p$ and p is minimal with this property.

Remark 3.9. As usual, $c\ell_X A$ is classical cup product based on Alexander–Spanier cohomology with suitable coefficients.

We now return to the previous configuration space setting: for examples of Γ-category.

X is a given ANR and $F_n(X)$, the nth configuration space of X. We define the following family of open sets in X^n. Let

$$F_n(i,j) = \{x \in X^n : x_i \neq x_j \text{ for a given } i \neq j\}.$$

Then, $\bigcap_{i<j} F_n(i,j) = F_n(X)$ and if $A \subset F_n(X)$ and $\Gamma = \{F_n(i,j), i < j\}$ we may consider $\Gamma\text{-cat}\, A$ in this case.

On the other hand, we may apply the loop and free loop functors to the $F_n(i,j)$ and we have collections of open sets

$$\Omega\Gamma = \{\Omega F_n(i,j) = \Omega_{ij}\} \quad \text{and} \quad \Lambda\Gamma = \{\Lambda F_n(i,j) = \Lambda_{ij}\}$$

where

$$\bigcap_{i<j} \Omega F_n(i,j) = \Omega F_n(X) \quad \text{and} \quad \bigcap_{i<j} \Lambda F_n(X) = \Lambda F_n(X).$$

Thus, if $A \subset \Omega F_n(X)$ or $B \subset \Lambda F_n(X)$, the concepts $\Omega\Gamma\text{-cat}\, A$ and $\Lambda\Gamma\text{-cat}\, B$ apply.

From the point of view of estimating values of $\Omega\Gamma$- and $\Lambda\Gamma$-category, the following lemma is very useful.

Lemma 3.10. *Suppose* $A \subset \Omega F_n(X)$*, then* $\Lambda\Gamma\text{-cat}\, A \geq \Omega\Gamma\text{-cat}\, A$.

Proof. Consider the fibration $\Omega F_n(i,j) \to \Lambda F_2(i,j) \to F_n(i,j)$. Then, using [6], for every $i < j$ we have

$$\operatorname{cat}_{\Lambda_{ij}} A \geq \operatorname{cat}_{\Omega_{ij}} A. \qquad \square$$

In order to apply the preceding discussion to the Majer–Terracini setting, we consider the configuration space $F_n(\mathbb{R}^k)$ as a subspace of R^{kn} and the Sobolev space of T-periodic functions $W_T^{1,2}(\mathbb{R}, \mathbb{R}^{kn}) = \mathcal{H}$, with typical element $u : t \to (u_1(t), \dots, u_n(t))$, $u_i(t) \in \mathbb{R}^k$, and with inner product

$$u \cdot v = \int_0^T (\dot{u}(t) \cdot \dot{v}(t) + u(t) \cdot v(t))\, dt.$$

$\Lambda' F_n(\mathbb{R}^k)$ is the subset of orbits without collision, i.e.

$$\Lambda' F_n(\mathbb{R}^k) \equiv \{u \in \mathcal{H} :\ u_i(t) \neq u_j(t),\ \text{for all } i \neq j,\ t \in \mathbb{R}\}.$$

$\Lambda' F_n(\mathbb{R}^k)$ has the same homotopy type as $\Lambda F_n(\mathbb{R}^k)$ (which has the uniform or compact-open topology). Indeed the natural inclusion

$$\varphi : \mathcal{H} \to C_T^0(\mathbb{R}, \mathbb{R}^{nk}) \equiv C^0(S^1, \mathbb{R}^{kn})$$

induces

$$\varphi_0 : \Lambda' F_n(\mathbb{R}^k) \to \Lambda(F_n(\mathbb{R}^k))$$

which is the homotopy equivalence (Palais [17]). The analogue in $\mathcal{H}$ of the subspaces $\Lambda_{ij} = \Lambda F_n(i,j)$, $i < j$ fixed, are given by

$$\Lambda'_{ij} = \{u \in \mathcal{H} : u_i(t) \neq u_j(t),\ t \in \mathbb{R}\}$$

and corresponding open collection $\Lambda'\Gamma = \{\Lambda'_{ij}\}$. Note that φ also induces a homotopy equivalence

$$\varphi_{ij} : \Lambda'_{ij} \to \Lambda_{ij}.$$

Furthermore, for $A \subset \Lambda' F_n(\mathbb{R}^k) = \bigcap_{i<j} \Lambda'_{ij}$, the concept $\Lambda'\Gamma'$-cat A is defined.

Recall that a deformation in $\mathcal{H}$ is a homotopy $h : \mathcal{H} \times I \to \mathcal{H}$ such that h_0 is the identity. Let $D(\mathcal{H})$ denote the set of all deformations of $\mathcal{H}$ with h_1 mapping into the space of constant functions (identified with $\mathbb{R}^{kn}$).

Basic Lemma 3.11. *Suppose A is a subset of $\Lambda' F_n(\mathbb{R}^k)$ such that there exists an $h = (h_1, \dots, h_n)$ in $D(\mathcal{H})$ with the property that for every $u \in A$, there exists a pair i, j (depending on u), $i < j$, such that for every $s \in [0,1]$ and $t \in \mathbb{R}$*

$$h_i(u,s)(t) \neq h_j(u,s)(t).$$

Then, if $\Lambda'\Gamma = \{\Lambda'_{ij}\}$, we have $\Lambda'\Gamma$-cat $A \leq 2$.

Proof. Let $V_{i,j} = \{u \in \mathcal{H} : h_i(u,s)(t) \neq h_j(u,s)(t),\ \forall s, t\}$. Then, $V_{i,j}$ is deformable into $F_2(i,j)$ which has the same homotopy type as $F_2(\mathbb{R}^k)$ which, in turn, has the same homotopy type as $\mathbb{R}^k - 0$ which has category 2. Hence, each V_{ij} decomposes into 2 open subsets which are categorical in Λ_{ij}. Thus, $\Lambda'\Gamma$-cat$|, A \leq 2$ □

Corollary 3.12. *Suppose $A \subset \Lambda' F_n(\mathbb{R}^k)$, $\Lambda\Gamma'$- $cat\, A > 2$. Then for every $h \in D(\mathcal{H})$, there is a $u \in A$ such that for every i, j, there exists $s_{ij} \in [0,1]$, $t_{ij} \in \mathbb{R}$, such that*

$$h_i(u, s_{ij}(t_{ij}) = h_j(u, s_{ij})t_{ij})$$

i.e., if $u = (u_1, \dots, u_n)$, every pair of particles u_i, u_j collides during the deformation h.

Remark 3.13. Thus, every closed subset $A \subset \Lambda' F_n(\mathbb{R}^k)$ is admissible in the sense of Majer–Terracini [16] if it has $\Lambda\Gamma'$-category greater than 2.

We will apply the above corollary to the simple deformation $h = (h_1, \dots, h_n)$: $\mathcal{H} \times I \to \mathcal{H}$ defined by

$$(*) \qquad h_i(u, s)(t) = u_i(t) - s(u_i(t) - [u_i]), \quad s \in I,\ t \in \mathbb{R},$$

where

$$[u_i] = \frac{1}{T} \int_0^T u_i(t)\, dt,$$

to prove an inequality which is basic for the Majer–Terracini method.

Proposition 3.14. *Let E denote the set*

$$E = \left\{ u \in \Lambda' F_n(\mathbb{R}^k) : \frac{1}{2n} \sum_{i \neq j} [u_i - u_j]^2 \geq \frac{T}{6} \|\dot{u}\|_2^2 \right\}.$$

Then, $\Lambda'\Gamma$-cat $E \leq 2$.

Proof. Suppose $\Lambda'\Gamma$-cat $E > 2$. Then, using the deformation $(*)$ above, for some $u \in E$ and every pair i, j, there is a pair (s_{ij}, t_{ij}) such that

$$u_i(t_{ij}) - s_{ij}(u_i(t_{ij}) - [u_i]) = u_j(t_{ij}) - s_{ij}(u_j(t_{ij}) - [u_j]).$$

This implies

$$(1 - s_{ij})|u_i(t_{ij}) - u_j(t_{ij}) - [u_i - u_j]| = |[u_i - u_j]|$$

and hence

$$|[u_i - u_j]| < \|u_i - u_j - [u_i - u_j]\|_\infty.$$

Appling the Sobolev inequality we obtain

$$|[u_i - u_j]|^2 < \|u_i - u_j - [u_i - u_j]\|_\infty^2 \leq \frac{T}{12} \|\dot{u}_i - \dot{u}_j\|_2^2.$$

Therefore,

$$\sum_{i \neq j} [u_i - u_j]^2 < \frac{T}{6} \left(\sum_{i,j} \|\dot{u}_i\|_2^2 + \|\dot{u}_j\|_2^2 \right) \leq \frac{T}{6} 2n \sum_i \|\dot{u}_i\|_2^2$$

and hence

$$\frac{1}{2n}\sum_{i\neq j}[u_i - u_j]^2 < \frac{T}{6}\|\dot{u}\|_2^2. \qquad \square$$

The corresponding inequalities in Majer–Terracini [15]–[17] are not as sharp because the authors employed the relation $i\mathbb{R}_{h,u}j$ described earlier in this paper rather than the simpler concept of Γ'-category. Following Majer–Terracini we use as "control function" $g(u)$ the function

$$g(u) = \frac{1}{2n}\sum_{i\neq j}[u_i - u_j]^2$$

to obtain the following restatement of Proposition 3.14.

Proposition 3.15. *If* $E = \{u \in \Lambda' F_n(\mathbb{R}^k) : g(u) \geq (T/6)\|\dot{u}\|_2^2\}$ *then* $\Lambda'\Gamma\text{-cat}\, E \leq 2$.

Corollary 3.16. *Let* $f : \Lambda' F_n(\mathbb{R}^k) \to \mathbb{R}$ *denote a* C^1*-functional of the form*

$$f(u) = \frac{1}{2}\int_0^T |\dot{u}|^2 - \int_0^T V(u,t), \quad u \in \Lambda' F_n(\mathbb{R}^k),$$

where $V(u,t) \leq 0$ *for all* u *and* $0 \leq t \leq 1$, *then*

$$\Lambda'\Gamma\text{-cat}\,\{u \in \Lambda' F_n(\mathbb{R}^k) : g(u) \geq (T/3) f(u)\} \leq 2.$$

Corollary 3.17. *For any* $c \in \mathbb{R}$ *and any* $b \geq (T/3)c$ *we have*

$$\Lambda'\Gamma\text{-cat}\{u \in \Lambda' F_n(\mathbb{R}^k) : f(u) \leq c \text{ and } g(u) \geq b\} \leq 2.$$

Another important step in the Majer–Terracini method is to verify the existence of compact sets in $\Lambda' F_n(\mathbb{R}^k)$ of arbitrarily high $\Lambda'\Gamma$-category.

Let $p_n : F_n(\mathbb{R}^k) \to F_2(\mathbb{R}^k)$, $n \geq 3$, denote the projection

$$p_n(x_1, \ldots, x_n) = (x_1, x_2).$$

A section for p_n is defined by

$$\sigma_n(x_1, x_2) = (x_1, \ldots x_n)$$

where for $j \geq 3$

$$x_j = x_1 + 4(j-1)(x_2 - x_1).$$

Observe that, $\sigma_n(x_1, x_2)$ belongs to $F_n(L(x, x_2))$, where $L(x_1, x_2)$ is the half-line from x_1 through x_2. Also, σ_n imbeds $F_2(\mathbb{R}^k)$ in $F_n(\mathbb{R}^k)$. Let $F_2^*(\mathbb{R}^k) = \sigma_n(F_2(\mathbb{R}^k))$, so that $\sigma_n : F_2(\mathbb{R}^k) \to F_2^*(\mathbb{R}^k)$ is a homeomorphism with inverse $p_n | F_2^*(\mathbb{R}^k)$.

LEMMA 3.18. *Let C^* denote a set in $F_2^*(\mathbb{R}^k)$ which is contractible in $F_n(i,j)$. Let C denote the corresponding set in $F_2(\mathbb{R}^k)$, i.e. $C = \sigma_n^{-1}(C^*)$. Then, C is contractible in $F_2(\mathbb{R}^k)$.*

PROOF. Let $G : C^* \times I \to F_n(i,j)$ denote a contraction, i.e. $G_0(x) = x$, $x \in C^*$ and $G_1(x)$ is a point in $F_n(i,j)$. Let $p_{ij} : F_n(i,j) \to F_2(\mathbb{R}^k)$ denote projection on the ith and jth coordinates. We define a homotopy $H : C \times I \to F_2(\mathbb{R}^k)$ by $H = p_{ij} \circ G \circ (\sigma_n \times \mathrm{id})$. Note that $H_0(y) = p_{ij}(G_0(\sigma_n(\mu))$, and $H_1(y)$ is a point in $F_2(\mathbb{R}^k)$, $y \in C$. Furthermore, if $y = (y_1, y_2), G_0(r_n(y)) = (y_1, \ldots, y_n)$ and y_i and y_j lie on the half-line $L(y_1, y_2)$ from y_1 to y_2, where $H_0(y_1, y_2) = (y_i, y_j)$, $i < j$. Using a simple homotopy K taking (y_1, y_2) along $L(y_1, y_2)$ to (y_i, y_j), we see that K followed by H, gives a contraction of C in $F_2(\mathbb{R}^k)$. □

PROPOSITION 3.19. *Let Γ denote collection $\{F_n(i,j),\ 1 \le i < j \le n\}$ and A a subset of $F_2(\mathbb{R}^k)$. Then, if $A^* = \sigma_n(A)$,*

$$\mathrm{cat}_{F_2(\mathbb{R}^k)} A \le \frac{n(n-1)}{2} \Gamma\text{-cat}\, A^*.$$

PROOF. Suppose $\Gamma\text{-cat}\, A^* = \beta < \infty$. Then, $A^* = \bigcup_{i<j} B_{ij}$ and $\mathrm{cat}_{F_2(i,j)} B_{ij} \le \beta$. Then, each $B_{ij} = \bigcup_{\lambda=1}^{\beta} B_{ij}^*(\lambda)$ where $B_{ij}^*(\lambda)$ is contractible in $F_2(i,j)$. Then, $A = \bigcup_{i,j,\lambda} B_{ij}(\lambda)$, where $B_{ij}(\lambda) = \sigma_n^{-1}(B_{ij}^*)$ which is contractible in $F_2(\mathbb{R}^k)$. □

REMARK 3.20. Our overall assumption is that $k \ge 3$, so that $F_n(\mathbb{R}^k)$ is 0-connected and contractions may always be assumed to be to any point of $F_n(\mathbb{R}^k)$.

Now, the arguments used above apply to the following situation. Using the notation above, the section $\sigma_n : F_2(\mathbb{R}^k) \to F_n(\mathbb{R}^k)$ induces sections

$$\begin{aligned}
\Lambda\sigma_n &: \Lambda F_2(\mathbb{R}^k) \to \Lambda F_n(\mathbb{R}^k),\\
\Lambda'\sigma_n &: \Lambda' F_2(\mathbb{R}^k) \to \Lambda' F_n(\mathbb{R}^k),\\
\Omega\sigma_n &: \Omega F_2(\mathbb{R}^k) \to F_n(\mathbb{R}^k),\\
\Omega'\sigma_n &: \Omega' F_2(\mathbb{R}^k) \to \Omega' F_n(\mathbb{R}^k),
\end{aligned}$$

where in the last item above $\Omega' F_n(\mathbb{R}^k)$ is the subset of $\Lambda' F_n(\mathbb{R}^k)$ consisting of loops based at a particular point.

THEOREM 3.20. *If $A \subset \Lambda' F_2(\mathbb{R}^k)$ and $A^* = \Lambda'\sigma_n(A)$, then for $\Lambda'\Gamma = \Lambda' F_n(i,j)$*

$$\mathrm{cat}_{\Lambda' F_2(\mathbb{R}^k)} A \le \frac{n(n-1)}{2} \Lambda'\Gamma\text{-cat}\, A^*.$$

The corresponding inequalities are valid for subsets of $\Lambda F_2(\mathbb{R}^k)$, $\Omega F_2(\mathbb{R}^k)$ and $\Omega' F_2(\mathbb{R}^k)$ using the open covers $\Lambda\Gamma$, $\Omega\Gamma$ and $\Omega'\Gamma$, respectively.

COROLLARY 3.21. *$\Lambda' F_n(\mathbb{R}^k)$, $n \ge 2$, contains compact subsets of arbitrarily high $\Lambda'\Gamma$-category, where $\Lambda'\Gamma = \{\Lambda_{ij}\}$.*

4. The Majer–Terracini method

We now illustrate how the general concept of Γ-category and, in particular, Theorem 3.20 play a role in the Majer–Terriacini method by proving one of their main results.

First of all, the method requires the use of several results which are only valid modulo translation by $\mathbb{R}^k$. More precisely let $\mathcal{H}_0$ denote the subspace of $\mathcal{H}$ defined by

$$\mathcal{H}_0 = \left\{ u \in \mathcal{H} : \sum_{i=1}^{n} [u_i] = 0 \right\},$$

and $\pi : \mathcal{H} \to H_0$, the orthogonal projection given by $\pi u = u - \overline{x}$, where

$$\overline{x}_i = x = \frac{1}{n} \sum_{i=1}^{n} [u_i], \quad 1 \le i \le n, \ x \in \mathbb{R}^k.$$

Let $\Lambda_0' F_n(\mathbb{R}^k) = \mathcal{H}_0 \cap \Lambda' F_n(\mathbb{R}^k)$ and one easily verifies that we have a homeomorphism ψ

$$\begin{array}{ccc}
\Gamma' F_n(\mathbb{R}^k) & \xleftarrow{\ \psi\ } & \Lambda_0' F_n(\mathbb{R}^k) \times \mathbb{R}^k \\
{\scriptstyle \pi_0}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{proj}} \\
\Lambda_0' F_n(\mathbb{R}^k) & \xleftarrow[\mathrm{id}]{} & \Lambda_0' F_n(\mathbb{R}^k)
\end{array}$$

where π_0 is the restriction of π and $\psi(u,x) = u + \overline{x}$, and $\overline{x}_i = x \in \mathbb{R}^k$. On the other hand, let $\mathbb{R}^k$ act on $\mathcal{H}$ by translation, i.e. $xu = u + \overline{x}$, where $\overline{x}_i = x$, $1 \le i \le n$, $x \in \mathbb{R}^k$. Then, the orbit space $(\Lambda' F_n(\mathbb{R}^k))/\mathbb{R}^k$ is homeomorphic to $\Lambda_0' F_n(\mathbb{R}^k)$. Thus, the statement that statements made in the context of $\Lambda F_n(\mathbb{R}^k)$ are valid "modulo translations by $\mathbb{R}^k$" is equivalent to asserting validity in the orbit space $\Lambda_0' F_n(\mathbb{R}^k)$. In short, the Majer–Terracini method has as its natural habitat $\Lambda_0' F_n(\mathbb{R}^k) \subset \mathcal{H}_0$.

In order to carry over the results of Section 3 to $\mathcal{H}_0$ we employ the following

PROPOSITION 4.1. *Let A denote an $\mathbb{R}^k$-invariant subset of $\Lambda' F_n(\mathbb{R}^k)$ and $\Lambda'\Gamma = \{\Lambda_{ij}'\}$ as in Section 3. Let $A_0 = \pi(A)$ and $\Lambda_0'\Gamma = \{\pi(\Lambda_{ij}')\}$. Then*

$$\Lambda_0'\Gamma\text{-cat}\, A_0 = \Lambda'\Gamma\text{-cat}\, A.$$

PROOF. Since A and Λ_{ij}' are $\mathbb{R}^k$-invariant, π defines a homotopy equivalence from Λ_{ij}' to $\pi(\Lambda_{ij}')$ which carries A to A_0. Since $\operatorname{cat}_{\Lambda_{ij}} A = \operatorname{cat}_{\pi(\Lambda_{ij})} A_0$ for every i,j, $1 \le i < j \le n$, the result follows. □

COROLLARY 4.2. *$\Lambda_0' F_n(\mathbb{R}^k)$ admits compact subsets of arbitrarily high $\Lambda_0'\Gamma$-category.*

Our next objective is to illustrate how to apply the properties of Γ-category in the proof of the following theorem of Majer–Terracini ([16], [17]).

The setting is the following dynamical system of n-body type

$$-m_i\ddot{u}_i = \sum_{i\neq j} \nabla V_{ij}(u_i - u_j, t), \quad 1 \le i, j \le u, \tag{H}$$

where $u = (u_1, \dots, u_n) \in \mathcal{H} = W_T^{1,2}(\mathbb{R}, \mathbb{R}^{k_n})$, $t \in \mathbb{R}$. The assumptions on the potentials V_{ij} are the following:

- (V_1) $V_{ij} \in C_T^1(\mathbb{R}^k - 0; (-\infty, 0])$ for $1 \le i \ne j \le n$.
- (V_2) There exists $U \in C^1(\mathbb{R}^k - 0; \mathbb{R})$ such that $\lim_{x\to 0} U(x) = \infty$ and there exists a ρ_0 such that for $0 < |x| < \rho_0$, $|\nabla U(x)|^2 + U(x) \le -V_{ij}(x,t)$ for $i \ne j$.
- (V_3) There exists $\rho > 0$ and $\theta, 0 \le \theta < \pi/2$ such that $\operatorname{ang}(\nabla V_{ij}(x,t), x) \le \theta$ for all x with $|x| > \rho$ and $i \ne j$.

Remark 4.3. (V_2) is a "Strong Force" condition and (V_3) is an "angle" condition, where

$$\operatorname{ang}(y,x) = \begin{cases} \cos^{-1} \dfrac{x \cdot y}{|x||y|} & \text{if } |x||y| \ne 0, \\ 0, & \text{otherwise.} \end{cases}$$

As usual, the (non-collision) T-periodic solutions of (H) are critical points of the associated functional

$$f(u) = \int_0^T \frac{|\dot{u}|^2}{2} - \sum_{i\ne j} V_{ij}(u_i - u_j)\, dt. \tag{f}$$

Theorem 4.4 (Majer–Terracini, [17]). *Under the assumptions* (V_1)–(V_3), *the functional f has an unbounded sequence of critical values.*

Before proceeding with the proof we state without proof the necessary analytic prerequisites (Propositions 4.7, 4.8, 4.10, 4.11) whose proofs may be found in [15]–[17]. Following Majer–Terracini let

$$g(u) = \frac{1}{2n} \sum_{i\ne j} ([u_i] - [u_j])^2. \tag{g}$$

Then, both f and g, which are defined on $\Lambda' F_n(\mathbb{R}^k)$ and for the remainder of this section we assume that f satisfies properties (V_1)–(V_3).

The motivation for the function g is the following simple purely algebraic lemma and its corollary.

Lemma 4.5. *If $x_i \in \mathbb{R}^k$ and $\sum x_i = 0$, then*

$$\frac{1}{2n} \sum_{i\ne j} (x_i - x_j)^2 = \sum_{i=1}^n x_i^2.$$

Corollary 4.6. *If* $u \in \Lambda_0' F_n(\mathbb{R}^k)$

$$g(u) = \frac{1}{2n}([u_i] - [u_j])^2 = \sum_{i=1}^{n} [u_i]^2$$

and (*using the* $W_T^{1,2}$*-norm*) $\|u\| \leq M(g(u) + 2f(u))$ *for some* $M \geq 0$.

The proof of Corollary 4.6 is made simpler if one uses the equivalent norm

$$\|u\|' = \int_0^T |\dot{u}|^2 \, dt + [u]^2.$$

Corollary 4.6 leads to the following Palais–Smale condition.

Proposition 4.7. *Every sequence* $\{u_j\}$ *in* $\Lambda_0' F_n(\mathbb{R}^k)$ *such that for* $c, b \in \mathbb{R}$, $f(u_j) \to c, \nabla f(u_j) \to 0$ *and* $g(u_j) \leq b$, *has a convergent subsequence.*

Proposition 4.8. *For every* $c > 0$, *there exists* $b = b(c)$ *such that the nonlinear eigenvalue problem*

$$\nabla f(u) = \lambda \nabla g(u), \quad \lambda \geq 0$$

has no solution in the range $\{u \in \Lambda' : f(u) \leq c \text{ and } g(u) \geq b\}$.

Remark 4.9. A precise value which works for $b(c)$ in Proposition 4.8 is given in [16].

Proposition 4.10. *For* c *and* $b > 0$ *in* $\mathbb{R}$, *every sequence in* $\Lambda_0' F_n(\mathbb{R}^k)$ *such that* $f(u_j) \to c$, $g(u_j) \to b$ *and* $\nabla f(u_j) - \lambda_j \nabla g(u_j) \to 0$ *with* $\lambda_j \geq 0$, *possesses a convergent subsequence.*

For the next proposition recall the following conventional notations. If $f \in C^1(\Lambda; \mathbb{R})$, where Λ is an open in a Hilbert space, then

$$\begin{aligned} K_c &= \{u \in \Lambda : f(u) = c \text{ and } \nabla f(u) = 0\}, \\ f^b &= \{u \in \Lambda : f(u) \leq b\}, \\ f_b &= \{u \in \Lambda : f(u) \geq b\}. \end{aligned}$$

Also recall that a deformation $\eta \in C^0(\Lambda \times I, \Lambda)$ is a homotopy such that for $u \in \Lambda$, $\eta_0(u) = \eta(u, 0) = u$.

Proposition 4.11 (Abstract Deformation Lemma). *Let* Λ *denote an open set of any Hilbert space,* $f \in C^1(\Lambda; \mathbb{R})$ *and* $g \in C^2(\Lambda; R)$.

Furthermore, assume that for c, b *in* $\mathbb{R}$ *the following conditions are satisfied:*

- (H_1) *If* $u_j \to u_0 \in \partial\Lambda$ *and* $g(u_j)$ *is bounded, then* $f(u_j) \to \infty$.
- (H_2) *If* $f(u) = c$, $g(u) = b$, *then* $\nabla g(u) \neq 0$.
- (H_3) *If* $f(u_j) \to c$, $g(u_j) \leq b$ *and* $\nabla f(u_j) \to 0$, *then* $\{u_j\}$ *has a convergent subsequence.*

(H$_4$) *If* $f(u_j) \to c$, $g(u_j) \to b$ *and* $\nabla f(u_j) - \lambda_j g(u_j) \to 0$ *where* $\lambda_j \geq 0$, *then* $\{u_j\}$ *has a convergent subsequence.*

(H$_5$) *If* $f(u) = c$, $g(u) = b$, *then* $\nabla f(u) \neq \lambda \nabla g(u)$ *for every* $\lambda > 0$.

Then, for every $\overline{\varepsilon} > 0$ *and every neighbourhood* N *of* $K_c \cap g^b$ *there is a deformation* $\eta : \Lambda \times I \to \Lambda$ *and an* ε, $0 < \varepsilon < \overline{\varepsilon}$, *such that:*

(1) $\eta(u,s) = u$, *for any* s, $0 \leq s \leq 1$, *if* $|f(u) - c| \geq \overline{\varepsilon}$ *or* $g(u) \geq b$.
(2) η *preserves sublevels of* f, *i.e.* $\eta(f^a, s) \subset f^a$ *for* $a \in \mathbb{R}$.
(3) $\eta_1(f^{c+\varepsilon} \cap g^b) \subset (f^{c-\varepsilon} \cap g^b) \cup N$.
(4) $\eta_1((f^{c+\varepsilon} \cap g^b) - N) \subset f^{c-\varepsilon} \cap g^b$.

Remark 4.12. As is customary, when $K_c \cap g^b = \phi$, we take $N = \phi$. The above deformation result will be applied in the special case of $\Lambda = \Lambda_0' F_n(\mathbb{R}^k)$ and the restriction of the functional f of Theorem 4.4 to $\Lambda_0' F_n(\mathbb{R}^k)$.

Now, we turn to the proof of Theorem 4.4 which is a variation of the argument in [17] with some essential changes since we will be employing $\Lambda_0'\Gamma$-category in the space $\Lambda_0' F_n(\mathbb{R}^k)$.

Proof of Theorem 4.4. First we introduce the simplified notation $\Lambda' = \Lambda' F_n(\mathbb{R}^k)$, $\Lambda_0' = \Lambda_0' F_n(\mathbb{R}^k)$ and recall the open family $\Lambda_0'\Gamma = \{\Lambda_{ij} \cap \mathcal{H}_0\}$ of Proposition 4.1. Let

$$\mathfrak{A}_m = \{A = \Lambda_0 : A \text{ is compact and } \Lambda_0'\Gamma\text{-cat}\, A \geq 2m\}$$

and $c_m^* = \inf_{A \in \mathfrak{A}_m} \sup_A f$, for $m \geq 1$.

Note that an application of Corollary 4.2 implies that $\mathfrak{A}_m$ contains non-empty compact subsets for every m, so that $0 \leq c_m^* < \infty$. Note, also that c_m^* depends only on the restriction of f to Λ_0' and for this and the need for the deformation lemma (Proposition 4.11) in the Hilbert space $\mathcal{H}_0$, we will use the notation f to mean this restriction to Λ_0' in the remainder of the proof.

Now choose an $a_m c_m^*$ and apply Corollary 3.17 and Proposition 4.8 to obtain $b_m = b(a_m)$ so that

$$\Lambda_0'\Gamma\text{-cat}\{u \in \Lambda_0' : f(u) \leq a_m \text{ and } g(u) \geq b_m\} \leq 2,$$

and the conditions of Proposition 4.11 are satisfied (with $c = a_m$ and $b = b_m$).

Let $c_m = \inf\{c : f^c \cup (f^{a_m} \cap g_{b_m}) \supset A, \text{ for some } A \in \mathfrak{A}_m\}$. Then, $c_m \leq c_m^* < a_m$. Also, for $\varepsilon > 0$,

$$\Lambda_0'\Gamma\text{-cat}(f^{c_m+\varepsilon} \cup (f^{a_m} \cap g_{b_m})) \geq 2m$$

and hence

$$\Lambda_0'\Gamma\text{-cat}\, f^{3_m+2} \geq 2m - 2 = 2(m-1).$$

Hence $c_{m-1}^* \leq c_m$.

We now are in a position to prove the following multiplicity result which implies in particular, that for $c = c_m$ and $b = b(a_m)$, $K_c \cap g^b$ is non-empty. First, we may choose the sequences $\{a_m\}$ and $\{b_m\}$ to be monotone increasing. Now suppose,

$$c_m = c_{m+1} = \ldots = c_{m+s} = c, \quad s \geq 0.$$

Then, we assert that for $b = b_{m+s}$

$$\Lambda_0'\Gamma\text{-cat}\,(K_c \cap g^b) \geq 2s + 1.$$

To verify this, let $B = K_c \cap g^b$ and suppose $\Lambda_0'\Gamma\text{-cat}\,B \leq 2s$. Choose a neighbourhood N of B such that $\Lambda_0'\Gamma\text{-cat}\,N = \Lambda_0'\Gamma\text{-cat}\,B$. Now, choose ε depending on $\overline{\varepsilon} = 1$ in Proposition 4.11 (Deformation Lemma) and choose a set A in $\mathfrak{A}_{m+s}$ such that $A \subset f^{c+\varepsilon} \cup (f^a \cap g_b)$, where $a = a_{m+s}$ and $\Lambda_0'\Gamma\text{-cat}\,A \geq 2(m+s)$. Then, if η is the deformation in Proposition 4.11, $A - N$ is compact,

$$\Lambda_0'\Gamma\text{-cat}\ \eta_1(A - N) \geq 2m$$

and

$$\eta_1(f^{c+\varepsilon} \cup (f^a \cap g_b) - N) \subset f^{c-\varepsilon} \cup (f^a \cap g_b).$$

This, contradicts the definition of c_m. Thus, in this case $\Lambda_0'\Gamma\text{-cat}(K_c \cap g^b) \geq 2s + 1$.

To prove that the sequence $\{c_m\}$ is unbounded, suppose $\lim c_m^* = c^* < \infty$. Fix $a > c^*$ and set $b = b(a)$. Assume $a_m = a$ and $b_m = b$ for all m in the preceding. Then $K_{c^*} \cap g^b$ is compact (and non-empty) and set $p = \Lambda_0'\Gamma\text{-cat}(K_{c^*} \cap g^b) < \infty$. Let $\overline{\varepsilon} = a - c^*$ and choose ε, N and η as in Proposition 4.11 with $\Lambda_0'\Gamma\text{-cat}\,N = p$. Note that for every m, there exists a compact set A_m such that $\Lambda_0'\Gamma\text{-cat}\,A_m \geq 2m$ and $A_m \subset f^{c^*+\varepsilon}$. Furthermore,

$$A_m \subset ((A_m \cap g^b) - N) \cup (A_m \cap g_b) \cup N$$

where $\Lambda_0'\Gamma\text{-cat}\,N = p$ and $\Lambda_0'\Gamma\text{-cat}(A_m \cap g_b) \leq 2$ and hence, $\Lambda_0'\Gamma\text{-cat}((A_m \cap g^b) - N)) \geq 2m - (p+2)$ for all m. Applying the deformation η, we have $\eta_1((A_m \cap g^b) - N) \subset f^{c^*-\varepsilon} \cap g^b$ and $\Lambda_0'\Lambda\text{-cat}(\eta_1((A_m \cap b^p) - N)) \geq 2m - (p+2)$ for all m. But the definition of c_m^* preclude the existence of compact subsets in $f^{c^*-\varepsilon}$ of $\Lambda_0'\Gamma$-category $\geq 2m$ which gives a contradiction. Thus, the sequence $\{c_m\}$ and $\{c_m^*\}$ must be unbounded which completes the proof of Theorem 4.4.□

References

[1] A. Bahri and P. H. Rabinowitz, *Periodic solutions of Hamiltonian systems of n-body type*, J. Analyse Linéaire **8** (1991), 561–649.

[2] R. Bott and H. Samelson, *On the Pontryagin product in spaces of paths*, Comment. Math. Helv. **27** (1953), 320–337.

[3] F. COHEN AND L. R. TAYLOR, *Configuration spaces; Applications to Gelfand–Fuks cohomology*, Bull. Amer. Math. Soc. **84** (1978), 134–136.

[4] A. DOLD, *Über fasernweise Homotopieäquivalenz von Fäserräumen*, Math. Z. **62** (1955), 111–136.

[5] E. FADELL, *On homotopy groups of configuration spaces and the string problem of Dirac*, Duke Math. J. **29**, 231–242.

[6] E. FADELL AND S. Y. HUSSEINI, *A note on the category of the free loop space*, Proc. Amer. Math. Soc. **107** (1989), 527–536.

[7] ———, *Infinite cup length in free loop spaces with an application to a problem of N-body type*, Annales de l'Institut Poincaré, Analyse Non Linéaire **9** (1992), 305–319.

[8] ———, *Category of loop spaces of open subsets in Euclidean space*, J. Nonlinear Anal. **17** (1991), 1153–1161.

[9] ———, *Relative category, products and coproducts*, Rend. Sem. Mat. Fis. Milano **64** (1994), 99–115.

[10] ———, *On the growth properties of the homology of free loop spaces of configuration spaces*, Proceedings of the First World Congress of Nonlinear Analysts, Walter de Gruyter, Berlin and New York, 1996, pp. 2127–2134.

[11] ———, Geometry and topology of configuration spaces (to appear).

[12] P. J. HILTON, *On the homotopy groups of the union of spheres*, J. London Math. Soc. **30** (1955), 154–172.

[13] T. KOHNO, *Monodromy repesentations of braid groups and Yang–Baxter equations*, Ann. Inst. Fourier **37** (1987), 139–160.

[14] ———, *Linear representations of braid groups and classical Yang–Baxter equations*, Contemp. Math. **78** (1988), 339–363.

[15] P. MAJER AND S. TERRACINI, *Periodic solutions to some N-body type problems: the fixed energy case*, Duke Math. J. **69** (1993), 683-697.

[16] ———, *Periodic solutions to some problems of N-body type*, Arch. Rational Mech. Anal. **124** (1993), 381-404.

[17] ———, *Multiple periodic solutions to come N-body type problems via a collision index*, Dipart. Mat. Politec. Milano, Preprint **64** (1962), 1–16.

[18] R. S. PALAIS, *Homotopy theory of infinite dimensional manifolds*, Topology **5** (1966), 1–16.

[19] H. SAMELSON, *A connection between the Whitehead and the Pontryagin product*, Amer. J. Math. **75** (1953), 744-772.

[20] G. W. WHITEHEAD, *Elements of Homotopy Theory*, Springer-Verlag, Berlin, New York, 1978.

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