Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 11, 1998, 187–197

ON ORBITS OF THE SAME TYPE

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1. Introduction

It is well-known that a homeomorphism of S^1 onto itself with a dense orbit is topologically conjugate to an irrational rotation (cf. [7], [8], [11]). We study homeomorphisms of $(S^1)^n$ with a dense orbit under an additional assumption of equicontinuity related to the rational independence.

In this paper, we concentrate on orbits in a discrete dynamical system induced by a homeomorphism of a Hausdorff uniform space whose closure is homeomorphic to S^1. We consider a discrete dynamical system that exhibits a connected set A filled by embedded circles such that each circle is topologically conjugate to an irrational rotation. The hypothesis is that all dynamics within the set are Lyapunov stable, that is, if any two points are close to each other they stay close under the iteration. The main result shows that the rotation numbers of two circles are in $(1, q)$ resonances and if no twisting occurs the rotation number is constant on A. To illustrate this fact, we give some examples.

Thus the investigation of the possible behaviour of orbits and of its complexity in a discrete dynamical system yields an index theory for $\mathbb{Z}$-actions in connection with S^1-index. To this end, we show that the existence theorem about continuous extensions holds for $\mathbb{Z}$-actions. Moreover, an index theory for group actions is important from the viewpoint of applications to differential equations (cf. [1], [2], [6], [12]).

1991 *Mathematics Subject Classification.* 54H20, 34D20, 58G10, 11T72.
Key words and phrases. Discretedynamical systems.

2. Preliminaries

Throughout this paper, we consider a discrete dynamical system

$$\pi : \mathbb{Z} \times X \to X, \quad (n, x) \mapsto f^n(x),$$

induced by a homeomorphism $f : X \to X$ of a Hausdorff space X. This dynamical system will be denoted by (X, f).

Let (X, f) and (Y, g) be discrete dynamical systems. A map $\Phi : X \to Y$ is said to be *equivariant*, denoted by $\Phi : (X, f) \to (Y, g)$, if $\Phi \circ f = g \circ \Phi$. Two dynamical systems (X, f) and (Y, g) are said to be *isomorphic*, denoted by $(X, f) \simeq (Y, g)$, if there is a homeomorphism $\Phi : (X, f) \to (Y, g)$. In this case we say that f is *topologically conjugate* to g.

We introduce the concept of Lyapunov stability on uniform spaces $(X, \mathcal{U})$, where $\mathcal{U}$ is a uniformity on X (cf. [16, IV(2.1)]).

Definition 1. Let $f : X \to X$ be a bijection, and let $A \subset X$ be a set such that $f(A) = A$. The set A is said to be *Lyapunov stable* with respect to f if for every $M \in \mathcal{U}$, there exists an $N \in \mathcal{U}$ such that for all $x, y \in A$ and for all $n \in \mathbb{Z}$

$$(x, y) \in N \text{ implies } (f^n(x), f^n(y)) \in M.$$

Definition 2. The set of n real numbers $\alpha_1, \dots, \alpha_n$ is said to be *rationally dependent* if the relation $c_1\alpha_1 + \ldots + c_n\alpha_n = 0$ holds for some rational numbers $c_1, \dots, c_n$, not all equal zero. Otherwise, $\{\alpha_1, \dots, \alpha_n\}$ is said to be *rationally independent*.

The following result, which is based on the main theorem of Kronecker (cf. [14, Satz 65]), is useful for the classification of the type of orbits.

Proposition 3. *If $\{1, \alpha_1, \dots, \alpha_n\}$ is rationally independent, then*

$$\overline{\{(e^{2\pi i m\alpha_1}, \dots, e^{2\pi i m\alpha_n}) : m \in \mathbb{Z}\}} = (S^1)^n.$$

Here the bar denotes the closure.

We consider a map $f : S^1 \times S^1 \to S^1 \times S^1$, $(z_1, z_2) \mapsto (e^{2\pi i\alpha_1} z_1, e^{2\pi i\alpha_2} z_2)$, $\alpha_1, \alpha_2 \in \mathbb{R}$ and an orbit $O(1, 1) = \{(e^{2\pi i n\alpha_1}, e^{2\pi i n\alpha_2}) : n \in \mathbb{Z}\}$. We make a classification by α_1, α_2:

(i) If $\alpha_1 = p_1/q_1$, $\alpha_2 = p_2/q_2 \in \mathbb{Q}$, $q_1, q_2 \in \mathbb{N}$, $q := \operatorname{lcm}(q_1, q_2)$, where lcm denotes the least common multiple, then $\overline{O(1, 1)} \simeq \{e^{2\pi i j/q} : j = 1, \dots, q\}$.
(ii) If $\alpha_1, \alpha_2 \in \mathbb{R} \setminus \mathbb{Q}$, $\alpha_1/\alpha_2 \in \mathbb{Q}$, then $\overline{O(1, 1)} \simeq S^1$.
(iii) If $\alpha_1 \in \mathbb{R}\setminus\mathbb{Q}$, $\alpha_2 = p/q \in \mathbb{Q}$, then $\overline{O(1, 1)} = S^1 \times \{e^{2\pi i jp/q} : j = 1, \dots, q\}$.
(iv) If $\{1, \alpha_1, \alpha_2\}$ is rationally independent, then $\overline{O(1, 1)} = S^1 \times S^1$.

We recall that two topological spaces X and Y are *homeomorphic*, denoted by $X \simeq Y$, if there is a homeomorphism from X to Y.

3. The main result

In this section we discuss a connected set A filled by embedded circles with the restricted dynamics on each circle conjugate to an irrational rotation under the condition that all dynamics within the set are Lyapunov stable. We show that the rotation numbers of two circles are in $(1, q)$ resonances and that if no twisting occurs the rotation number is constant on A. To obtain this conclusion some assumptions are necessary, as we illustrate later with examples.

For $\alpha \in \mathbb{R}$, let $\widetilde{\alpha} : S^1 \to S^1$ be defined by

$$\widetilde{\alpha}(z) := e^{2\pi i\alpha} z \quad \text{for } z \in S^1.$$

From the theory of a homeomorphism of S^1 onto itself due to H. Poincaré there are various methods of approach, see [7], [8], [11]. All what is needed for the main result is the fact that a homeomorphism of the unit circle with a dense orbit is topologically conjugate to an irrational rotation. For the method of topological dynamics, see [11], [16], [17].

Proposition 4. *Let $g : S^1 \to S^1$ be a homeomorphism with a dense orbit. Then there exists an $\alpha \in [0, 1] \cap (\mathbb{R} \setminus \mathbb{Q})$ such that $(S^1, g) \simeq (S^1, \widetilde{\alpha})$.*

In connection with Proposition 4, homeomorphisms of $(S^1)^n, n \geq 2$ with a dense orbit can be studied under an additional assumption of equicontinuity, where the rational independence plays an important role. For the necessity of equicontinuity in continuous dynamical systems, see [9]. In fact, there is a homeomorphism $f : (S^1)^2 \to (S^1)^2$ such that the orbit of each point is dense in $(S^1)^2$ and $\{f^n : n \in \mathbb{Z}\}$ is not equicontinuous, see [16, III(1.18), IV(2.2)].

Proposition 5. *For an $n \in \mathbb{N} \setminus \{1\}$, let $g : (S^1)^n \to (S^1)^n$ be a homeomorphism with a dense orbit such that $\{g^m : m \in \mathbb{Z}\}$ is equicontinuous. Then there exist $\alpha_1, \dots, \alpha_n \in [0, 1]$ such that $\{1, \alpha_1, \dots, \alpha_n\}$ is rationally independent and g is topologically conjugate to $\widetilde{\alpha}$, where $\widetilde{\alpha} : (S^1)^n \to (S^1)^n$ is defined by $\widetilde{\alpha}(z_1, \dots, z_n) := (e^{2\pi i\alpha_1} z_1, \dots, e^{2\pi i\alpha_n} z_n)$ for any $(z_1, \dots, z_n) \in (S^1)^n$.*

Proof. It follows from the equicontinuity of $\{g^m : m \in \mathbb{Z}\}$ that for any $x \in (S^1)^n$, $\{g^m(x) : m \in \mathbb{Z}\}$ is dense in $(S^1)^n$. Hence, $(S^1)^n$ has the structure of a compact topological group such that $g(x) = a \cdot x$ for all $x \in (S^1)^n$ and an $a \in (S^1)^n$ and $\{a^m : m \in \mathbb{Z}\}$ is dense in $(S^1)^n$ (cf. [16, IV(3.42)]). Therefore there exists a homeomorphism $\Psi : (S^1)^n \to (S^1)^n$ that is a morphism of groups because $(S^1)^n$ is a connected compact Lie group (cf. [3]). Thus there are $\alpha_1, \dots, \alpha_n \in [0, 1]$ such that $\Psi(a) = (e^{2\pi i\alpha_1}, \dots, e^{2\pi i\alpha_n})$. For any $x \in (S^1)^n$, we have

$$\Psi(g(x)) = \Psi(a \cdot x) = \Psi(a)\Psi(x) = \widetilde{\alpha}(\Psi(x)).$$

Since $\{a^m : m \in \mathbb{Z}\}$ is dense in $(S^1)^n$ and Ψ is a homeomorphism, $\{1, \alpha_1, \dots, \alpha_n\}$ is rationally independent (cf. [16, III(1.14)]). $\square$

The proof of the following main result uses connectedness, where we consider $f \times f$ a translation by $(\alpha(C_0), \alpha(C_1))$ of rotation numbers on the two-torus $C_0 \times C_1$. The Lyapunov stability implies that the translation vector has $(1, q)$ resonance, which finishes the proof.

Theorem 6. *Let $f : X \to X$ be a continuous bijection on a Hausdorff uniform space $(X, \mathcal{U})$. Let $A \subset X$ be connected and Lyapunov stable with respect to f such that $f(A) = A$ and $\overline{O(x)}$ is homeomorphic to S^1 for all $x \in A$. Then the following statements hold:*

(a) *For every $x \in A$, there exist an $\alpha_x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$ such that $(\overline{O(x)}, f) \simeq (S^1, \widetilde{\alpha}_x)$.*
(b) *If $a \in A$, then there exists an $N \in \mathcal{U}$ such that for every $u \in N(a) \cap A$ there are integers q, r with $\alpha_a + q\alpha_u + r = 0$, where $N(a) := \{x \in X : (a, x) \in N\}$ for $N \subset X \times X$.*
(c) *For $x, y \in A$, the set $\{1, \alpha_x, \alpha_y\}$ is rationally dependent.*

Proof. (a) For each $x \in A$, let $\Phi_x : \overline{O(x)} \to S^1$ be a homeomorphism, and define a map $g_x : S^1 \to S^1$ by $g_x(y) := \Phi_x \circ f \circ \Phi_x^{-1}(y)$ for $y \in S^1$. Since all assumptions of Proposition 4 are satisfied, there exist a homeomorphism $\Gamma_x : S^1 \to S^1$ and an $\alpha_x \in [0,1] \cap (\mathbb{R} \setminus \mathbb{Q})$ such that $\Gamma_x \circ g_x = \widetilde{\alpha}_x \circ \Gamma_x$. Hence, we have, with $\Psi_x := \Gamma_x \circ \Phi_x$

$$\Psi_x \circ f|_{\overline{O(x)}} = \widetilde{\alpha}_x \circ \Psi_x. \tag{1}$$

This proves statement (a). We first provide two claims for statements (b) and (c).

For $M \subset X \times X$, we denote $M^{-1} := \{(x, y) \in X \times X : (y, x) \in M\}$, $M \circ M := \{(x, y) \in X \times X : (x, z) \in M \text{ and } (z, y) \in M \text{ for some } z \in X\}$.

Claim 1. Let $a \in A$. Then there exists an $N \in \mathcal{U}$ such that for every $u \in N(a) \cap A$, the set $\{1, \alpha_a, \alpha_u\}$ is rationally dependent.

Proof. Let $a \in A$. Let $M, W \in \mathcal{U}$, $M = M^{-1}$ and $M \circ M \circ M \circ M \subset W$ such that $(a, w) \notin W$ for a $w \in O(a)$. By the Lyapunov stability of A with respect to f, there is an $N \in \mathcal{U}, N \subset M$ such that for all $x \in A$ and for all $n \in \mathbb{Z}$

$$(a, x) \in N \text{ implies } (f^n(a), f^n(x)) \in M. \tag{2}$$

Let $u \in N(a) \cap A$. Assume that $\{1, \alpha_a, \alpha_u\}$ is rationally independent. Then, for all $(z_1, z_2) \in (S^1)^2$, the set $\{(e^{2\pi i n\alpha_a} z_1, e^{2\pi i n\alpha_u} z_2) : n \in \mathbb{Z}\}$ is dense in $(S^1)^2$, since $\{(e^{2\pi i n\alpha_a}, e^{2\pi i n\alpha_u}) : n \in \mathbb{Z}\}$ is dense in $(S^1)^2$ by Proposition 3. Hence, it follows from (1) that there exists an $n_0 \in \mathbb{Z}$ such that $(w, f^{n_0}(a)) \in M$ and $(u, f^{n_0}(u)) \in M$. Since $(a, u) \in N$, we have $(a, w) \in N \circ M \circ M^{-1} \circ M^{-1} \subset W$, which leads to a contradiction. This proves Claim 1.

Claim 2. Let $a \in A$. Then there exists an $N \in \mathcal{U}$ such that for any $u \in N(a) \cap A$ there exist integers q, r with $\alpha_a + q\alpha_u + r = 0$.

Proof. Let $a \in A$. Let $M, W \in \mathcal{U}, M = M^{-1}$ and $M \circ M \circ M \subset W$ such that $(a, y) \notin W$ for all $y \in R$, where

$$R := \Psi_a^{-1}(\{e^{2\pi i s}\Psi_a(a) : 1/4 \leq s \leq 3/4\}).$$

Then there exists an $N \in \mathcal{U}$, $N \subset M$ such that the condition (2) is satisfied. Let $u \in N(a) \cap A$. By Claim 1, there exist relatively prime numbers $p, q \in \mathbb{Z}$ and an $r \in \mathbb{Q}$ such that $p\alpha_a + q\alpha_u + r = 0$. There are two steps to consider.

(i) To show that $r \in \mathbb{Z}$, we assume that $r = m/n \in \mathbb{Q} \setminus \mathbb{Z}$ for relatively prime numbers $m \in \mathbb{Z}$, $n \in \mathbb{N}$. Then there exist an $s \in \mathbb{N}$ such that

$$\operatorname{dist}(rs/p, \mathbb{Z}) \geq 1/3,$$

where we denote $\operatorname{dist}(b, \mathbb{Z}) := \inf\{|b - n| : n \in \mathbb{Z}\}$ for $b \in \mathbb{R}$. Since α_u is irrational, there exists a sequence (v_k) in $s + np\mathbb{Z}$ such that $\operatorname{dist}(v_k\alpha_u/p, \mathbb{Z}) \to 0$. In particular, we can choose a $k_0 \in \mathbb{N}$ such that $(u, f^{v_{k_0}}(u)) \in M$ and $f^{v_{k_0}}(a) \in R$ because $\operatorname{dist}(v_k\alpha_u, p\mathbb{Z}) \to 0$ and $v_k\alpha_a = (-r/p - (q/p)\alpha_u)v_k \to -rs/p$ modulo $\mathbb{Z}$. This is a contradiction to the choice of N, M and W.

(ii) To show that $|p| = 1$, we assume that $|p| \geq 2$ and $r \in \mathbb{Z}$. Since p, q are relatively prime, there exist $\beta_1, \beta_2 \in \mathbb{Z}$ such that $r = p\beta_1 + q\beta_2$, hence we have $p(\alpha_a + \beta_1) + q(\alpha_u + \beta_2) = 0$. Similarly as in the proof of (i), we can choose an $s \in \mathbb{N}$ such that

$$\operatorname{dist}(qs/p, \mathbb{Z}) \geq 1/3,$$

and a sequence (v_k) in $\mathbb{Z}$ such that

$$\operatorname{dist}(v_k(\alpha_u + \beta_2), s + p\mathbb{Z}) \to 0,$$

showing $(u, f^{v_{k_0}}(u)) \in M$ and $f^{v_{k_0}}(a) \in R$ for some $k_0 \in \mathbb{N}$. We have again a contradiction. Consequently, Claim 2 is proved, that is, statement (b) is complete.

(c) For arbitrary $a \in A$, let

$$B := \{x \in A : \{1, \alpha_a, \alpha_x\} \text{ is rationally dependent}\}.$$

Then B is nonempty. We shall show that B is open. Let $b \in B$. By Claim 1, there exists an $N \in \mathcal{U}$ such that for every $u \in N(b) \cap A =: U$, the set $\{1, \alpha_b, \alpha_u\}$ is rationally dependent. As $\{1, \alpha_a, \alpha_b\}$ is rationally dependent, it is clear that for every $u \in U, \{1, \alpha_a, \alpha_u\}$ is rationally dependent, hence we have $U \subset B$. A similar argument establishes the closedness of B. Since A is connected, we obtain that $B = A$, and thus statement (c) follows. □

The following example shows that Theorem 6 is false if the assumption of connectedness is dropped.

EXAMPLE 7. Let $f : [0,1] \times S^1 \to [0,1] \times S^1$ be defined by $f(\alpha, x) := (\alpha, e^{2\pi i\alpha}x)$ and $A := \{\alpha_1, \alpha_2\} \times S^1$ be the set such that $\{1, \alpha_1, \alpha_2\}$ is rationally independent. Then A is Lyapunov stable with respect to f but not connected.

We now make the following observation for examples below.

REMARK 8. For $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $q_1, \dots, q_n \in \mathbb{N}$, $q := \gcd(q_1, \dots, q_n)$ where gcd denotes the greatest common divisor, we have

$$\overline{\{e^{2\pi imq\alpha} : m \in \mathbb{Z}\}} \simeq \overline{\{(e^{2\pi imq_1\alpha}, \dots, e^{2\pi imq_n\alpha}) : m \in \mathbb{Z}\}},$$

since $r_1, \dots, r_n (q_j = r_j q)$ are relatively prime and $\overline{\{e^{2\pi imq\alpha} : m \in \mathbb{Z}\}} = S^1$ is compact. For the extension of uniformly continuous functions, see [15, II.3.3, Satz 2].

From the following examples we see explicitly that for $x, y \in X$ the set $\{1, \alpha_x, \alpha_y\}$ is rationally dependent.

EXAMPLE 9. Let $X := \bigcup_{t\in[0,1]}\{t\} \times S^1 \times tS^1$, and let $f : X \to X$ be defined by

$$f(t, x, ty) := (t, e^{2\pi i2\alpha}x, e^{2\pi i\alpha}ty), \quad \alpha \in [0, 1/2] \cap (\mathbb{R} \setminus \mathbb{Q}).$$

Then X is compact, connected and Lyapunov stable with respect to f. For any $(t, x, ty) \in X$, we have, by Remark 8,

$$(\overline{O(t,x,ty)}, f) \simeq \begin{cases} (S^1, \widetilde{\alpha}) & \text{for } t \neq 0, \\ (S^1, \widetilde{2\alpha}) & \text{for } t = 0. \end{cases}$$

EXAMPLE 10. Let $X := \bigcup_{t\in[0,1]}\{t\} \times S^1 \times tS^1$, and let $f : X \to X$ be defined by

$$f(t, x, ty) := (t, e^{2\pi i2\alpha}x, e^{2\pi i(\alpha+1/2)}ty), \quad \alpha \in [0, 1/4] \cap (\mathbb{R} \setminus \mathbb{Q}).$$

Then X is connected and Lyapunov stable with respect to f. For any $(t, x, ty) \in X$ we have

$$(\overline{O(t,x,ty)}, f) \simeq \begin{cases} (S^1, \widetilde{\alpha + 1/2}) & \text{for } t \in (0,1], \\ (S^1, \widetilde{2\alpha}) & \text{for } t = 0. \end{cases}$$

EXAMPLE 11. Let $X := \bigcup_{t\in[0,1]}\{t\} \times (1-t)S^1 \times tS^1$, and let $f : X \to X$ be defined by

$$f(t, (1-t)x, ty) := (t, e^{2\pi i2\alpha}(1-t)x, e^{2\pi i3\alpha}ty), \quad \alpha \in [0, 1/3] \cap (\mathbb{R} \setminus \mathbb{Q}).$$

Then X is connected and Lyapunov stable with respect to f. For any $(t, (1-t)x, ty) \in X$ we have

$$(\overline{O(t,(1-t)x,ty)}, f) \simeq \begin{cases} (S^1, \widetilde{\alpha}) & \text{for } t \in (0,1), \\ (S^1, \widetilde{2\alpha}) & \text{for } t = 0, \\ (S^1, \widetilde{3\alpha}) & \text{for } t = 1. \end{cases}$$

The following result for orbits of the same type is an immediate consequence of Theorem 6 if no twisting occurs.

Corollary 12. *Let all hypotheses be as in Theorem* 6. *We suppose that for any $x \in A$ and for any $u \in N(x) \cap A$ with some $N \in \mathcal{U}$, there exist no $q \in \mathbb{Z} \setminus \{\pm 1\}$ such that $\alpha_x + q\alpha_u + r = 0$ for some $r \in \mathbb{Z}$. If we choose all $\alpha_x \in [0, 1/2]$, then the α_x is constant on A.*

Proof. Let

$$H : A \to \mathbb{R}, \quad x \mapsto \beta_x, \quad \beta_x := \begin{cases} \alpha_x & \text{for } 0 \le \alpha_x \le 1/2, \\ 1 - \alpha_x & \text{for } 1/2 \le \alpha_x \le 1. \end{cases}$$

By Theorem 6(a), it is clear that $(\overline{O(x)}, f) \simeq (S^1, \widetilde{\beta}_x)$. Let $a \in A$. By assumption, for any $u \in N(a) \cap A$, we have $\alpha_a + q\alpha_u + r = 0$ with $q = \pm 1$, $r \in \mathbb{Z}$ and hence $\beta_a + q\beta_u + r' = 0$, $q = \pm 1$, $r' \in \mathbb{Z}$. It follows that $\beta_a = \beta_u$ for all $u \in N(a) \cap A$. Since A is connected, we conclude that $\beta_a = \beta_x$ for all $x \in A$. This completes the proof. $\square$

4. A $\mathbb{Z}$-index

Considering the rotation number stated in the last section, we give a definition of $\mathbb{Z}$-index induced by a homeomorphism of a compact space. Then a $\mathbb{Z}$-version of the Borsuk–Ulam theorem and the existence theorem of a continuous equivariant extension are fundamental in an index theory for $\mathbb{Z}$-actions.

Given a continuous action $\pi : G \times X \to X$ of a topological group G on a Hausdorff space X, we denote

$$\Sigma(X, G) := \{A \subset X : A \text{ is } G\text{-invariant}\}.$$

A *G-index* is a mapping

$$i : \Sigma(X, G) \to \mathbb{N} \cup \{0, \infty\},$$

which has the following properties:

(a) $i(A) = 0$ if and only if $A = \emptyset$.
(b) If $A, B \in \Sigma(X, G)$ and $\Phi : A \to B$ is a continuous equivariant map, then $i(A) \le i(B)$.
(c) If $A \in \Sigma(X, G)$ is a closed set, then there exists an open neighbourhood $U \in \Sigma(X, G)$ of A such that $i(A) = i(U)$. (Continuity)
(d) If $A, B \in \Sigma(X, G)$ are closed sets, then $i(A \cup B) \le i(A) + i(B)$. (Subadditivity)

In the following we always assume that $f : X \to X$ is a homeomorphism on a compact space X such that

(a) $\{f^n : n \in \mathbb{Z}\}$ is equicontinuous; and

(b) there exists an irrational number α in $[0,1]$ such that for every $x \in X$ there exists a homeomorphism $\Phi : \overline{O(x)} \to S^1$ with the property

$$\Phi(f(u)) = e^{2\pi i\alpha}\Phi(u) \quad \text{for all } u \in \overline{O(x)}.$$

In this framework we define the $\mathbb{Z}$-index $i(A)$ of an invariant subset A of X as the smallest integer k such that there exist an $m \in \mathbb{N}$ and a continuous map $\Phi : A \to S^{2k-1}$ satisfying the following equivariance property

$$\Phi(f(u)) = e^{2\pi i m\alpha}\Phi(u) \quad \text{for all } u \in A.$$

A $\mathbb{Z}$-version of the Borsuk–Ulam theorem which states that for $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ and $k, l \in \mathbb{N}$ with $k > l$ there exists no continuous map $(S^{2k-1}, \widetilde{\alpha}) \to (S^{2l-1}, \widetilde{\alpha})$ implies the dimension property for an index (cf. [10]). We can show that this is an index in the sense of the above definition.

The main tool for showing the continuity and the subadditivity for an index is the theorem about the existence of a continuous equivariant extension. The existence theorem for compact Lie groups has been proved by using Bochner integrals, see Tietze–Gleason Theorem [13]. In general this does not hold for $\mathbb{Z}$-actions, but the almost periodicity allows us to do that.

Theorem 13. *For any closed set $A \in \Sigma(X, f)$ and for any continuous map $\varphi : (A, f) \to (\mathbb{C}^k, \widetilde{\alpha})$, φ has a continuous extension*

$$\Phi : (X, f) \to (\mathbb{C}^k, \widetilde{\alpha}).$$

Proof. Let $A \in \Sigma(X, f)$ be a closed set and let $\varphi : (A, f) \to (\mathbb{C}^k, \widetilde{\alpha})$ be a continuous map. Since A is a closed subset of normal space X, there exists, by the extension theorem of Tietze–Urysohn, a continuous map $\widehat{\varphi} : X \to \mathbb{C}^k$ such that $\widehat{\varphi}|_A = \varphi$. Let us define $\Phi : X \to \mathbb{C}^k$ by

$$\Phi(x) := \lim_{k\to\infty} \frac{1}{2k+1} \sum_{|n|\le k} e^{-2\pi i n\alpha}\widehat{\varphi}(f^n(x)) \quad \text{for each } x \in X.$$

Then Φ is well-defined because $F : \mathbb{Z} \to \mathbb{C}^k$, $n \mapsto e^{-2\pi i n\alpha}\widehat{\varphi}(f^n(x))$ is an almost periodic map implying the existence of the mean value of F (cf. [5]).

It is easy to verify that Φ is an equivariant map and $\Phi|_A = \varphi$.

To show that Φ is continuous, let $\varepsilon > 0$ and $V_\varepsilon := \{(y, y') \in \mathbb{C}^k \times \mathbb{C}^k : \|y - y'\| < \varepsilon\}$. As $\widehat{\varphi}$ is uniformly continuous, there exists an $M \in \mathcal{U}$ such that for all $z, z' \in X$

$$(z, z') \in M \text{ implies } (\widehat{\varphi}(z), \widehat{\varphi}(z')) \in V_\varepsilon.$$

Since $\{f^n : n \in \mathbb{Z}\}$ is equicontinuous on the compact space X, there exists an $N \in \mathcal{U}$ such that for all $x, y \in X$ and for all $n \in \mathbb{Z}$

$$(x, y) \in N \text{ implies } (f^n(x), f^n(y)) \in M,$$

hence we have

$$(\widehat{\varphi}(f^n(x)), \widehat{\varphi}(f^n(y))) \in V_\varepsilon.$$

Consequently, we obtain that for all $x, y \in X$ with $(x, y) \in N$

$$\begin{aligned}
\|\Phi(x) - \Phi(y)\| &= \left\| \lim_{k\to\infty} \frac{1}{2k+1} \sum_{|n|\le k} e^{-2\pi i n\alpha}(\widehat{\varphi}(f^n(x)) - \widehat{\varphi}(f^n(y))) \right\| \\
&\le \limsup_{k\to\infty} \frac{1}{2k+1} \sum_{|n|\le k} \|e^{-2\pi i n\alpha}(\widehat{\varphi}(f^n(x)) - \widehat{\varphi}(f^n(y)))\| \\
&\le \lim_{k\to\infty} \frac{1}{2k+1}(2k+1)\varepsilon = \varepsilon,
\end{aligned}$$

that is, $(\Phi(x), \Phi(y)) \in V_\varepsilon$. This completes the proof. □

For the proof of the following theorem we use Theorem 13 and the fact that S^{2k-1} is a neighbourhood retract of $\mathbb{C}^k$. See [13] for compact Lie groups.

Theorem 14. *For any closed set $A \in \Sigma(X, f)$ and for any continuous map $\varphi : (A, f) \to (S^{2k-1}, \widetilde{\alpha})$, there exist an open neighbourhood $U \in \Sigma(X, f)$ of A and a continuous extension of φ*

$$\Phi : (U, f) \to (S^{2k-1}, \widetilde{\alpha}).$$

Finally, we obtain the following result by using the concept of join and its topological properties (cf. [4]).

Theorem 15. *For $j = 1, 2$, let $A_j \in \Sigma(X, f)$ be a closed set and let $\varphi_j : (A_j, f) \to (S^{2k_j-1}, \widetilde{\alpha})$ be a continuous map. Then there exists a continuous map*

$$\Phi : (A_1 \cup A_2, f) \to (S^{2k_1-1} * S^{2k_2-1}, \widetilde{\alpha}),$$

where $S^{2k_1-1} * S^{2k_2-1}$ has the initial topology with respect to $q : S^{2k_1-1} * S^{2k_2-1} \to \triangle^1$ and partial functions $p_j : S^{2k_1-1} * S^{2k_2-1} \rightarrowtail S^{2k_j-1}$ $(j = 1, 2)$. Furthermore, there exists a continuous map

$$\Psi : (A_1 \cup A_2, f) \to (S^{2(k_1+k_2)-1}, \widetilde{\alpha}).$$

Proof. For $j = 1, 2$, by Theorem 14, there exist an open neighbourhood $U_j \in \Sigma(X, f)$ of A_j and a continuous extension $\Phi_j : (U_j, f) \to (S^{2k_j-1}, \widetilde{\alpha})$ of φ_j. Since X is normal, there exists a continuous function $\gamma'_j : X \to [0, 1]$ such that

$$\gamma'_j(x) = \begin{cases} 1 & \text{for } x \in A_j, \\ 0 & \text{for } x \in X \setminus U_j. \end{cases}$$

Let $\gamma_j : X \to \mathbb{R}$, $x \mapsto \lim_{k\to\infty} 1/(2k+1) \sum_{|n|\le k} \gamma'_j(f^n(x))$. Then γ_j is well-defined, continuous and invariant, that is, $\gamma_j \circ f = \gamma_j$. It follows that $\gamma_j|_{A_j} = 1, \gamma_j|_{X\setminus U_j} = 0$ and $\gamma_j(X) \subset [0, 1]$.

Let $\widetilde{\gamma} : A_1 \cup A_2 \to \Delta^1 = \{(t_1, t_2) \in [0,1]^2 : t_1 + t_2 = 1\}$,

$$\widetilde{\gamma}(x) := (\widetilde{\gamma}_1(x), \widetilde{\gamma}_2(x)), \quad \widetilde{\gamma}_j(x) := \frac{\gamma_j(x)}{\gamma_1(x) + \gamma_2(x)} \quad \text{for } j = 1, 2.$$

Then $\widetilde{\gamma}$ is continuous, invariant and $\widetilde{\gamma}_j|_{(A_1 \cup A_2) \setminus U_j} = 0$.

We define $\Phi : A_1 \cup A_2 \to S^{2k_1-1} * S^{2k_2-1}$ by

$$\Phi(x) := [(\widetilde{\gamma}_1(x), \widetilde{\gamma}_2(x), \widetilde{\Phi}_1(x), \widetilde{\Phi}_2(x))],$$

where $\widetilde{\Phi}_j : X \to S^{2k_j-1}$ is an arbitrary extension of Φ_j. Then Φ is well-defined, continuous and equivariant, since $\widetilde{\Phi}_j|_{U_j} = \Phi_j$ is continuous, equivariant and since $\widetilde{\gamma}_j$ is continuous, invariant and $\widetilde{\gamma}_j|_{(A_1 \cup A_2) \setminus U_j} = 0$. Thus the first part is proved. The second part is clear because $S^{2k_1-1} * S^{2k_2-1}$ is homeomorphic to $S^{2(k_1+k_2)-1}$. This completes the proof. $\square$

It is elementary to observe from Theorems 14 and 15 that the mapping i defined above is an index (cf. [10]).

References

[1] V. Benci, *A geometrical index for the group S^1 and some applications to the study of periodic solutions of ordinary differential equations*, Comm. Pure Appl. Math. **34** (1981), 393–432.

[2] H. Berestycki, J. M. Lasry, G. Mancini and B. Ruf, *Existence of multiple periodic orbits on star-shaped Hamiltonian surfaces*, Comm. Pure Appl. Math. **38** (1985), 253–289.

[3] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972.

[4] R. Brown, *Topology: a Geometric Account of General Topology, Homotopy Types and the Fundamental Groupoid*, Ellis Horwood Ltd., Chichester, 1988.

[5] C. Corduneanu, *Almost Periodic Functions*, Chelsea, New York, 1989.

[6] E. R. Fadell and P. H. Rabinowitz, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math. **45** (1978), 139–174.

[7] H. Furstenberg, *Strict ergodicity and transformations of the torus*, Amer. J. Math. **83** (1961), 573–601.

[8] E. R. van Kampen, *The topological transformations of a simple closed curve into itself*, Amer. J. Math. **57** (1935), 142–152.

[9] J. F. Kent, *Locally connected almost periodic minimal sets: recent advances in topological dynamics*, Lecture Notes in Math. **318** (1973), Springer, Berlin, 150–166.

[10] I. S. Kim, *Zu einer Indextheorie für fastperiodische Aktionen*, Doctoral Thesis, LMU München, 1995.

[11] N. G. Markley, *Transitive homeomorphisms of the circle*, Math. Systems Theory **2** (1968), 247–249.

[12] J. Mawhin and M. Willem, *Critical Point Theory and Hamiltonian Systems*, Springer, New York, 1989.

[13] R. S. Palais, *The Classification of G-spaces*, Mem. Amer. Math. Soc. **36**, Providence, R. I., 1960.

[14] O. Perron, *Irrationalzahlen*, 4. Aufl., De Gruyter, Berlin, 1960.

[15] H. Schubert, *Topologie*, 4. Aufl., B. G. Teubner, Stuttgart, 1975.

[16] J. de Vries, *Elements of Topological Dynamics*, Kluwer Academic Publishers, Dordrecht, 1993.

[17] P. Walters, *An Introduction to Ergodic Theory*, Springer, Berlin, 1982.

Manuscript received October 9, 1997

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