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M-PERIODIC PROBLEM OF ORDER $2k$

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1. Introduction

In monograph [2] the Du Bois–Reymond lemma (fundamental lemma) for periodic functions of order 1 is proved. Next, using the variational method, the authors prove an existence theorem for the periodic problem

$$\begin{aligned} &\ddot{u}(t) = \nabla F(t, u(t)), \quad && t \in [0, T] \text{ a.e.}, \\ &u(0) = u(T), && \dot{u}(0) = \dot{u}(T), \end{aligned}$$

in the case when a coercivity condition for the average of F is satisfied and the nonlinearity ∇F is bounded by an integrable function.

In our paper we prove a generalization of the fundamental lemma and then, using the variational method, we give sufficient conditions for the existence of a solution to the following M-periodic problem (matrix-periodic problem)

$$\begin{aligned} (1.1)\quad &\frac{d}{dt}\bigg(\dots \bigg(\frac{d}{dt}\bigg(\frac{d}{dt} u^{(k)} - F_{u_{k-1}}(t, u, \dots, u^{(k-1)}) \bigg) \\ &\quad + F_{u_{k-2}}(t, u, \dots, u^{(k-1)}) \bigg) + \dots + (-1)^{k-1} F_{u_1}(t, u, \dots, u^{(k-1)}) \bigg) \\ &\quad + (-1)^k F_{u_0}(t, u, \dots, u^{(k-1)}) \bigg) = 0, \quad t \in [0, T] \text{ a.e.}, \end{aligned}$$

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$$\begin{bmatrix} u(0) \\ u'(0) \\ \vdots \\ u^{(k-1)}(0) \end{bmatrix} = A \begin{bmatrix} u(T) \\ u'(T) \\ \vdots \\ u^{(k-1)}(T) \end{bmatrix},$$

$$\begin{bmatrix} u^{(k)}|_{t=0} \\ \left(\frac{d}{dt}u^{(k)} - F_{u_{k-1}}\right)\Big|_{t=0} \\ \vdots \\ \Big(\underbrace{\frac{d}{dt}\Big(\ldots\Big(\frac{d}{dt}\Big(\frac{d}{dt}}_{k-1 \text{ times}} u^{(k)} - F_{u_{k-1}}\Big) + F_{u_{k-2}}\Big) \\ + \ldots + (-1)^{k-2}F_{u_2}\Big) + (-1)^{k-1}F_{u_1}\Big)\Big|_{t=0} \end{bmatrix} \tag{1.2}$$

$$= B \begin{bmatrix} u^{(k)}|_{t=T} \\ \left(\frac{d}{dt}u^{(k)} - F_{u_{k-1}}\right)\Big|_{t=T} \\ \vdots \\ \Big(\underbrace{\frac{d}{dt}\Big(\ldots\Big(\frac{d}{dt}\Big(\frac{d}{dt}}_{k-1 \text{ times}} u^{(k)} - F_{u_{k-1}}\Big) + F_{u_{k-2}}\Big) \\ + \ldots + (-1)^{k-2}F_{u_2}\Big) + (-1)^{k-1}F_{u_1}\Big)\Big|_{t=T} \end{bmatrix},$$

where $F : [0,T] \times (\mathbb{R}^n)^k \ni (t, u_0, u_1, \ldots, u_{k-1}) \mapsto F(t, u_0, u_1, \ldots, u_{k-1}) \in \mathbb{R}$, $A = [a_{i,l}]_{i,l=0,\ldots,k-1}$ is a nonsingular matrix such that $A^{-1} = A'$ (A' — transposed matrix) and

$$B = \begin{bmatrix} a_{k-1,k-1} & -a_{k-2,k-1} & \ldots & (-1)^{k-1}a_{0,k-1} \\ -a_{k-1,k-2} & a_{k-2,k-2} & \ldots & (-1)^{k}a_{0,k-2} \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{k-1}a_{k-1,0} & (-1)^{k}a_{k-2,0} & \ldots & a_{0,0} \end{bmatrix}.$$

If $k = 3$, then equation (1.1) and boundary conditions (1.2) have the form

$$\frac{d}{dt}\left(\frac{d}{dt}\left(\frac{d}{dt}u'''(t) - F_{u_2}(t, u(t), u'(t), u''(t))\right) + F_{u_1}(t, u(t), u'(t), u''(t))\right) \\ - F_{u_0}(t, u(t), u'(t), u''(t)) = 0, \quad t \in [0,T] \text{ a.e.},$$

$$\begin{bmatrix} u(0) \\ u'(0) \\ u''(0) \end{bmatrix} = A \begin{bmatrix} u(T) \\ u'(T) \\ u''(T) \end{bmatrix},$$

$$\begin{bmatrix} u'''|_{t=0} \\ \left(\dfrac{d}{dt}u''' - F_{u_2}\right)\Big|_{t=0} \\ \left(\dfrac{d}{dt}\left(\dfrac{d}{dt}u''' - F_{u_2}\right) + F_{u_1}\right)\Big|_{t=0} \end{bmatrix} = B \begin{bmatrix} u'''|_{t=T} \\ \left(\dfrac{d}{dt}u''' - F_{u_2}\right)\Big|_{t=T} \\ \left(\dfrac{d}{dt}\left(\dfrac{d}{dt}u''' - F_{u_2}\right) + F_{u_1}\right)\Big|_{t=T} \end{bmatrix},$$

respectively.

In the case of $A = I$ and F not depending on $u_1, \dots, u_{k-1}$ (i.e. $F = F(t, u)$), the above boundary conditions and equation (1.1) are reduced to the periodic problem of type

$$\begin{aligned} &u^{(2k)}(t) + (-1)^k \nabla F(t, u(t)) = 0, && t \in [0, T] \text{ a.e.}, \\ &u^{(i)}(0) = u^{(i)}(T), && i = 0, \dots, 2k - 1. \end{aligned}$$

When $A = -I$ and $F = F(t, u)$, we obtain the antiperiodic problem

$$\begin{aligned} &u^{(2k)}(t) + (-1)^k \nabla F(t, u(t)) = 0, && t \in [0, T] \text{ a.e.}, \\ &u^{(i)}(0) = -u^{(i)}(T), && i = 0, \dots, 2k - 1. \end{aligned}$$

Moreover, in the case of $k = 1$ and $A = I$, the results obtained are reduced to those proved in [2].

2. Fundamental lemma

Let $n \ge 1$, $k \ge 2$ be some fixed positive integers, A — a $k \times k$-dimensional nonsingular real matrix with $A^{-1} = A'$, $T > 0$ — a fixed positive number and $I = [0, T]$. We define

$$\begin{aligned} H_0^{k,n} = \{h : I \to \mathbb{R}^n;\ \ & h^{(i)} \text{ is absolutely continuous on } I \\ & \text{and } h^{(i)}(0) = h^{(i)}(T) = 0 \text{ for } 0 \le i \le k-1,\ h^{(k)} \in L^2(I, \mathbb{R}^n)\}, \\ H_A^{k,n} = \{h : I \to \mathbb{R}^n;\ \ & h^{(i)} \text{ is absolutely continuous on } I \\ & \text{for } 1 \le i \le k-1,\ [h(0), h'(0), \dots, h^{(k-1)}(0)]' \\ & = A \circ [h(1), h'(1), \dots, h^{(k-1)}(1)]',\ \ h^{(k)} \in L^2(I, \mathbb{R}^n)\}. \end{aligned}$$

In the proof of the fundamental lemma we shall use the following classical result concerning a moments problem (see, for example [3, Section 5.8]).

Lemma 2.1. *If $a_0, a_1, \dots, a_{k-1} \in \mathbb{R}^n$, then there exists a function $l \in L^2(I, \mathbb{R}^n)$ such that*

$$\int_I 1 \cdot l(t)\, dt = a_0, \quad \int_I (T - t) l(t)\, dt = a_1, \quad \dots, \quad \int_I (T - t)^{k-1} l(t)\, dt = a_{k-1}.$$

We have

THEOREM 2.1 (the fundamental lemma). *If* $v \in L^2(I,\mathbb{R})$, $w \in L^1(I,\mathbb{R})$, $\alpha_0,\dots,\alpha_{k-1} \in \mathbb{R}$ *and*

$$\int_I v(t)h^{(k)}(t)\,dt = (-1)^k \int_I w(t)h(t)\,dt + \sum_{i=0}^{k-1} (-1)^{k-1-i}\alpha_{k-1-i}h^{(i)}(T) \tag{2.1}$$

for any $h \in H_A^{k,1}$, *then there exist constants* $c_0,\dots,c_{k-1} \in \mathbb{R}$ *such that*

$$v(t) = \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} w(s)\,ds\,dt_{k-1}\dots dt_1 + c_{k-1}t^{k-1} + \dots + c_1 t + c_0, \tag{2.2}$$

for $t \in I$ *a.e. and* (*after identifying* v *with the above right-hand side*)

$$\begin{bmatrix} v(0) \\ v'(0) \\ \vdots \\ v^{(k-1)}(0) \end{bmatrix} = B \circ \begin{bmatrix} v(T)-\alpha_0 \\ v'(T)-\alpha_1 \\ \vdots \\ v^{(k-1)}(T)-\alpha_{k-1} \end{bmatrix},$$

where $B = [b_{i,l}]_{i,l=0,\dots,k-1}$, $b_{i,l} = (-1)^{l+i}a_{k-1-i,k-1}$.

PROOF. The form (2.2) of v follows immediately from the fact that $H_0^{k,1} \subset H_A^{k,1}$ and from the generalization of the Du Bois–Reymond lemma to the case of derivatives of order k and the Dirichlet boundary conditions, proved in [4] (cf. also [1]). So, let us identify v with the function

$$I \ni t \mapsto \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} w(s)\,ds\,dt_{k-1}\dots dt_1 + c_{k-1}t^{k-1} + \dots + c_1 t + c_0.$$

Integrating by parts, we obtain

$$\begin{aligned}\int_I v(t)h^{(k)}(t)\,dt &= v(t)h^{(k-1)}(t)|_{t=0}^{t=T} - \int_I v'(t)h^{(k-1)}(t)\,dt \\ &= v(t)h^{(k-1)}(t)|_{t=0}^{t=T} - v'(t)h^{(k-2)}(t)|_{t=0}^{t=T} + \int_I v''(t)h^{(k-2)}(t)\,dt \\ &= \dots = v(t)h^{(k-1)}(t)|_{t=0}^{t=T} - v'(t)h^{(k-2)}(t)|_{t=0}^{t=T} \\ &\quad + \dots + (-1)^{k-1}v^{(k-1)}(t)h(t)|_{t=0}^{t=T} + (-1)^k \int_I v^{(k)}(t)h(t)\,dt.\end{aligned}$$

In view of the above, from assumption (2.1) we have

$$\begin{aligned}&v(t)h^{(k-1)}(t)|_{t=0}^{t=T} - v'(t)h^{(k-2)}(t)|_{t=0}^{t=T} \\ &\qquad + \dots + (-1)^{k-1}v^{(k-1)}(t)h(t)|_{t=0}^{t=T} = \sum_{i=0}^{k-1}(-1)^{k-1-i}\alpha_{k-1-i}h^{(i)}(T),\end{aligned}$$

for any $h \in H_A^{k,1}$, i.e.

$$
\begin{aligned}
(2.3)\quad & (v(T)-\alpha_0)h^{(k-1)}(T) - v(0)h^{(k-1)}(0) \\
& - [(v'(T)-\alpha_1)h^{(k-2)}(T) - v'(0)h^{(k-2)}(0)] \\
& + \ldots + (-1)^{k-1}[(v^{(k-1)}(T)-\alpha_{k-1})h(T) - v^{(k-1)}(0)h(0)] = 0,
\end{aligned}
$$

for any $h \in H_A^{k,1}$.

Now, let us fix $i \in \{0,\ldots,k-1\}$ and define

$$h_i : [0,T] \ni t \mapsto \int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-1}} l(s)\,ds\,dt_{k-1}\ldots dt_1 + \frac{1}{i!}t^i$$

where $l \in L^2(I,\mathbb{R})$ is such that

$$
\begin{aligned}
\int_I 1 \cdot l(t)\,dt &= a_{i,k-1}, \\
\int_I (T-t)l(t)\,dt &= a_{i,k-2}, \\
&\vdots \\
\int_I (T-t)^{k-2-i}l(t)\,dt &= a_{i,i+1}(k-2-i)!, \\
\int_I (T-t)^{k-1-i}l(t)\,dt &= (a_{i,i}-1)(k-1-i)!, \\
\int_I (T-t)^{k-i}l(t)\,dt &= \left(a_{i,i-1} - \frac{T}{1}\right)(k-i)!, \\
&\vdots \\
\int_I (T-t)^{k-2}l(t)\,dt &= \left(a_{i,1} - \frac{T^{i-1}}{(i-1)!}\right)(k-2)!, \\
\int_I (T-t)^{k-1}l(t)\,dt &= \left(a_{i,0} - \frac{T^i}{i!}\right)(k-1)!\,.
\end{aligned}
$$

It is easy to see that

$$
\begin{aligned}
h_i(t) &= \int_0^t \frac{(T-s)^{k-1}}{(k-1)!}l(s)\,ds + \frac{1}{i!}t^i, \\
h_i'(t) &= \int_0^t \frac{(T-s)^{k-2}}{(k-2)!}l(s)\,ds + i\frac{1}{i!}t^{i-1}, \\
&\vdots \\
h_i^{(i-1)}(t) &= \int_0^t \frac{(T-s)^{k-i}}{(k-i)!}l(s)\,ds + i(i-1)\cdot\ldots\cdot 2\,\frac{1}{i!}t,
\end{aligned}
$$

$$h_i^{(i)}(t) = \int_0^t \frac{(T-s)^{k-1-i}}{(k-1-i)!} l(s)\,ds + i!\frac{1}{i!},$$
$$h_i^{(i+1)}(t) = \int_0^t \frac{(T-s)^{k-2-i}}{(k-2-i)!} l(s)\,ds,$$
$$\vdots$$
$$h_i^{(k-2)}(t) = \int_0^t (T-s) l(s)\,ds,$$
$$h_i^{(k-1)}(t) = \int_0^t l(s)\,ds.$$

Consequently, $h_i^{(j)}(0) = 0$ for $j \in \{0, \dots, k-1\}$, $j \neq i$, $h_i^{(i)}(0) = 1$ and $h_i^{(j)}(T) = a_{i,j}$ for $j \in \{0, \dots, k-1\}$.

This implies, in view of $I = A \circ A'$, that $h_i \in H_A^{k,1}$.

Now, let us observe that from (2.3) we have

$$\begin{aligned}(-1)^i (h^{(i)}(T)(v^{(k-1-i)}(T) - \alpha_{k-1-i}) - h^{(i)}(0) v^{(k-1-i)}(0))\\ = \sum_{\substack{l=0\\ l\neq i}}^{k-1} (-1)^{l+1} (h^{(l)}(T)(v^{(k-1-l)}(T) - \alpha_{k-1-l}) - h^{(l)}(0) v^{(k-1-l)}(0)),\end{aligned}$$

for any $h \in H_A^{k,1}$, i.e.

$$\begin{aligned}h^{(i)}(0) v^{(k-i-1)}(0) = \sum_{l=0}^{k-1} (-1)^{l+i} h^{(l)}(T)(v^{(k-1-l)}(T) - \alpha_{k-1-l})\\ - \sum_{\substack{l=0\\ l\neq i}}^{k-1} (-1)^{l+i} (h^{(l)}(0) v^{(k-1-l)}(0),\end{aligned}$$

for any $h \in H_A^{k,1}$. Substituting h_i in the above equality, we have

$$v^{(k-i-1)}(0) = \sum_{l=0}^{k-1} (-1)^{l+i} a_{i,l} (v^{(k-1-l)}(T) - \alpha_{k-1-l}).$$

Finally, from the arbitrariness of $i \in \{0, 1, \dots, k-1\}$ we get

$$\begin{aligned}v^{(i)}(0) &= \sum_{l=0}^{k-1} (-1)^{l+k-1-i} a_{k-1-i,l} (v^{k-1-l}(T) - \alpha_{k-1-l})\\ &= \sum_{l=0}^{k-1} (-1)^{k-1-l+k-1-i} a_{k-1-i,k-1-l} (v^{(k-1-k+1+l)}(T) - \alpha_{k-1-k+1+l})\\ &= \sum_{l=0}^{k-1} (-1)^{l+i} a_{k-1-i,k-1-l} (v^{(l)}(T) - \alpha_l) = \sum_{l=0}^{k-1} b_{i,l} (v^{(l)}(T) - \alpha_l),\end{aligned}$$

for $i = 0, 1, \dots, k-1$. The proof is completed. □

From the above theorem we immediately obtain the following

Corollary 2.1. *If* $v = (v_1, \ldots, v_n) \in L^2(I, \mathbb{R}^n)$, $w = (w_1, \ldots, w_n) \in L^1(I, \mathbb{R}^n)$, $\alpha_0 = (\alpha_0^1, \ldots, \alpha_0^n), \ldots, \alpha_{k-1} = (\alpha_{k-1}^1, \ldots, \alpha_{k-1}^n) \in \mathbb{R}^n$ *and equality* (2.1) *holds for any* $h \in H_A^{k,n}$, *then there exist constants* $c_0, c_1, \ldots, c_{k-1} \in \mathbb{R}^n$ *such that formula* (2.2) *holds for* $t \in I$ *a.e. and* (*after identifying* v *with the right-hand side of* (2.2))

$$\begin{bmatrix} v_1(0) & v_2(0) & \ldots & v_n(0) \\ v_1'(0) & v_2'(0) & \ldots & v_n'(0) \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{(k-1)}(0) & v_2^{(k-1)}(0) & \ldots & v_n^{(k-1)}(0) \end{bmatrix}$$
$$= B \circ \begin{bmatrix} v_1(T) - \alpha_0^1 & v_2(T) - \alpha_0^2 & \ldots & v_n(T) - \alpha_0^n \\ v_1'(T) - \alpha_1^1 & v_2'(T) - \alpha_1^2 & \ldots & v_n'(T) - \alpha_1^n \\ \vdots & \vdots & \ddots & \vdots \\ v_1^{(k-1)}(T) - \alpha_{k-1}^1 & v_2^{(k-1)}(T) - \alpha_{k-1}^2 & \ldots & v_n^{(k-1)}(T) - \alpha_{k-1}^n \end{bmatrix},$$

where the matrix B *is as in theorem* (2.1).

Proof. It suffices to consider the functions $h \in H_A^{k,n}$ of the form $h = (0, \ldots, 0, h_i, 0, \ldots, 0)$ with $h_i \in H_A^{k,1}$ and use the previous theorem. □

3. Some properties of the space $H_A^{k,n}$

Let us define the following inner product in the space $H_A^{k,n}$

$$(g, h) = \int_I g(t)h(t)\,dt + \int_I g'(t)h'(t)\,dt + \ldots + \int_I g^{(k)}(t)h^{(k)}(t)\,dt.$$

The norm generated by this product is as follows:

$$\|h\| = \Big(\int_I |h(t)|^2\,dt + \int_I |h'(t)|^2\,dt + \ldots + \int_I |h^{(k)}(t)|^2\,dt\Big)^{1/2}. \tag{3.1}$$

In the same way as in [2, Proposition 1.1] one can obtain

Lemma 3.1. *For any* $i \in \{0, \ldots, k-1\}$, *there exists a constant* e_i *such that*

(a) *if* $h \in H_A^{k,n}$, *then*

$$\max_{t \in [0,T]} |h^{(i)}(t)| \le e_i \|h\|,$$

(b) *if* $h \in H_A^{k,n}$ *and* $\int_I h^{(i)}(t)\,dt = 0$, *then*

$$\max_{t \in [0,T]} |h^{(i)}(t)| \le e_i \|h^{(i+1)}\|_{L^2(I,\mathbb{R}^n)}.$$

From (b) of the above lemma we immediately get

Lemma 3.2. *For any $i \in \{0, \ldots, k-1\}$, there exists a constant d_i such that if $h \in H_A^{k,n}$ and $\int_I h^{(i)}(t)\,dt = 0$, then*

$$\int_I |h^{(i)}(t)|^2\,dt \le d_i \int_I |h^{(i+1)}(t)|^2\,dt.$$

This lemma implies

Lemma 3.3. *There exists a constant d such that if $h \in H_A^{k,n}$ and $\int_I h^{(i)}(t)\,dt = 0$ for $i = 0, \ldots, k-1$, then, for any $i = 0, \ldots, k-1$*

$$\int_I |h^{(i)}(t)|^2\,dt \le d \int_I |h^{(k)}(t)|^2\,dt.$$

Moreover, we have

Lemma 3.4. *The space $H_A^{k,n}$ with norm (3.1) is complete.*

Proof. Let $(h_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $H_A^{k,n}$. From the completeness of $L^2(I,\mathbb{R}^n)$ it follows that, for any $i \in \{0, \ldots, k\}$, there exists a function $l_i \in L^2(I,\mathbb{R}^n)$ such that

$$h_n^{(i)} \xrightarrow[n\to\infty]{} l_i \quad \in L^2(I,\mathbb{R}^n).$$

Moreover, for any $i \in \{0, \ldots, k-1\}$ and $0 \le s \le t \le T$, $n \in \mathbb{N}$, we have

$$\begin{aligned} |h_n^{(i)}(t) - h_n^{(i)}(s)| &\le \int_s^t |h_n^{(i+1)}(\tau)|\,d\tau \\ &\le (t-s)^{1/2}\biggl(\int_s^t |h_n^{(i+1)}(\tau)|^2\,d\tau\biggr)^{1/2} \\ &\le (t-s)^{1/2}\|h_n^{(i+1)}\|_{L^2(I,\mathbb{R}^n)} \le M_i(t-s)^{1/2}, \end{aligned} \tag{3.2}$$

where M_i is such that $\|h_n^{(i+1)}\|_{L^2(I,\mathbb{R}^n)} \le M_i$ for $n \in \mathbb{N}$. This means that the sequence $(h_n^{(i)})_{n\in\mathbb{N}}$ is equi-uniformly continuous.

From Lemma 3.1(a) we get

$$\max_{t\in[0,T]} |h_n^{(i)}(t)| \le e_i\|h_n\|.$$

This means, in view of the boundedness of the sequence $(h_n)_{n\in\mathbb{N}}$ in $H_A^{k,n}$, that the sequence $(h_n^{(i)})_{n\in\mathbb{N}}$ is equi-bounded.

So, from the Arzela–Ascoli theorem it follows that s subsequence of $(h_n^{(i)})_{n\in\mathbb{N}}$ is uniformly convergent to a continuous function. The uniqueness of the limit in $L^2(I,\mathbb{R}^n)$ implies that this continuous limit is l_i. It is easy to see that the sequence $(h_n^{(i)})_{n\in\mathbb{N}}$ converges uniformly to l_i (it suffices to contradict this assertion and repeat the above reasoning).

Thus, for any $i \in \{0,\dots,k-1\}$, $h_n^{(i)} \underset{n\to\infty}{\rightrightarrows} l_i$ uniformly on I and l_i is continuous on I. From this fact it follows that

$$\begin{bmatrix} l_0(0) \\ l_1(0) \\ \vdots \\ l_{k-1}(0) \end{bmatrix} = A \circ \begin{bmatrix} l_0(T) \\ l_1(T) \\ \vdots \\ l_{k-1}(T) \end{bmatrix}. \tag{3.3}$$

Now, let us observe that, for any $t \in I$,

$$h_n^{(k-1)}(t) = \int_0^t h_n^{(k)}(s)\,ds + h_n^{(k-1)}(0), \quad n = 1,2,\dots,$$

and

$$h_n^{(k-1)}(t) \xrightarrow[n\to\infty]{} l_{k-1}(t), \quad h_n^{(k-1)}(0) \xrightarrow[n\to\infty]{} l_{k-1}(0),$$

$$\int_0^t h_n^{(k)}(s)\,ds = \int_0^t (h_n^{(k)}(s) - l_k(s))\,ds + \int_0^t l_k(s)\,ds \xrightarrow[n\to\infty]{} \int_0^t l_k(s))\,ds$$

(the last convergence follows from the estimates

$$\left| \int_0^t (h_n^{(k)}(s) - l_k(s))\,ds \right| \le \int_0^T |h_n^{(k)}(s) - l_k(s)|\,ds \le \|h_n^{(k)} - l_k\|_{L^2(I,\mathbb{R}^n)} T^{\frac{1}{2}} \Big).$$

So, for $t \in I$,

$$\begin{aligned} l_{k-1}(t) &= \lim_{n\to\infty} h_n^{(k-1)}(t) = \lim_{n\to\infty} \left(\int_0^t (h_n^{(k)}(s)\,ds + h_n^{(k-1)}(0) \right) \\ &= \int_0^t l_k(s)\,ds + l_{k-1}(0). \end{aligned}$$

In an analogous way, for any $i = 0,\dots,k-2$,

$$l_i(t) = \int_0^t l_{i+1}(s)\,ds + l_i(0) \quad \text{for } t \in I.$$

This means that function l_0 is such that $l_0^{(i)}$ is absolutely continuous on I for $i = 0,\dots,k-1$, and $l_0^{(i)} = l_i$ for $i = 0,\dots,k$. Consequently, $l_0^{(k)} \in L^2(I,\mathbb{R}^n)$ and, in view of equality (3.3),

$$\begin{bmatrix} l_0(0) \\ l_0'(0) \\ \vdots \\ l_0^{(k-1)}(0) \end{bmatrix} = A \circ \begin{bmatrix} l_0(T) \\ l_0'(T) \\ \vdots \\ l_0^{(k-1)}(T) \end{bmatrix}.$$

So, $l_0 \in H_A^{k,n}$ and, of course, $h_n \xrightarrow[n\to\infty]{} l_0$ in $H_A^{k,n}$. The proof is completed. □

Lemma 3.5. *If* $h_n \xrightarrow[n\to\infty]{} h_0$ *weakly in* $H_A^{k,n}$, *then* $h_n^{(i)} \underset{n\to\infty}{\rightrightarrows} h_0^{(i)}$ *uniformly on* I *for any* $i \in \{0,\dots,k-1\}$.

Proof. Let a sequence $(h_n)_{n\in\mathbb{N}}$ be weakly convergent to h_0 in $H_A^{k,n}$. So, it is bounded in $H_A^{k,n}$. Let us fix any number $i \in \{0,\dots,k-1\}$. From Lemma 3.1(a) it follows that $(h_n^{(i)})_{n\in\mathbb{N}}$ is equi-bounded on I. In an analogous way as in the proof of Lemma 3.4 (see inequality (3.2)) one can show that this sequence is equi-uniformly continuous on I. Then, from the Arzela–Ascoli theorem it follows that a subsequence $(h_{n_k}^{(i)})_{k\in\mathbb{N}}$ of $(h_n^{(i)})_{n\in\mathbb{N}}$ is uniformly convergent on I to some continuous function $\overline{h_0^i}$. Of course, $h_{n_k}^{(i)} \xrightarrow[k\to\infty]{} \overline{h_0^i}$ weakly in the space of continuous functions on I. On the other hand, since $h_{n_k} \xrightarrow[k\to\infty]{} h_0$ weakly in $H_A^{k,n}$, Lemma 3.1(a) holds and the linear continuous operator preserves a weak convergence, therefore $h_{n_k}^{(i)} \xrightarrow[k\to\infty]{} h_0^{(i)}$ weakly in the space of continuous functions on I.

Thus $h_0^{(i)} = \overline{h_0^i}$ on I and, consequently, $h_{n_k}^{(i)} \underset{k\to\infty}{\rightrightarrows} h_0^{(i)}$ uniformly on I. To assert that $h_n^{(i)} \underset{n\to\infty}{\rightrightarrows} h_0^{(i)}$ uniformly on I, it suffices to contradict this assertion and repeat the above reasoning. The proof is completed. □

4. Existence of a solution to M-periodic problem of order $2k$

Let us consider the following functional

$$\varphi : H_A^{k,n} \ni u \mapsto \int_I f(t, u(t), u'(t), \dots, u^{(k)}(t))\, dt. \tag{4.1}$$

Using the same method as in [2, Theorem 1.4], one can prove

Theorem 4.1. *Let* $f : I \times (\mathbb{R}^n)^{k+1} \ni (t, u_0, \dots, u_k) \mapsto f(t, u_0, \dots, u_k) \in \mathbb{R}$ *be measurable in* t *for each* $u = (u_0,\dots,u_k) \in (\mathbb{R}^n)^{k+1}$ *and continuously differentiable in* $u = (u_0,\dots,u_k)$ *for* $t \in I$ *a.e. If there exist* $a \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$, $b \in L^1(I, \mathbb{R}_0^+)$ *and* $c \in L^2(I, \mathbb{R}_0^+)$, *such that, for* $t \in I$ *a.e.*, $u = (u_0,\dots,u_k) \in (\mathbb{R}^n)^{k+1}$, *one has*

$$\begin{aligned}
&|f(t, u_0, \dots, u_k)| \le a(|(u_0,\dots,u_{k-1})|)(b(t) + |u_k|^2),\\
&|f_{u_i}(t, u_0, \dots, u_k)| \le a(|(u_0,\dots,u_{k-1})|)(b(t) + |u_k|^2), \quad i = 0,\dots,k-1,\\
&|f_{u_k}(t, u_0, \dots, u_k)| \le a(|(u_0,\dots,u_{k-1})|)(c(t) + |u_k|),
\end{aligned}$$

then the functional φ *given by* (4.1) *is continuously differentiable on* $H_A^{k,n}$, *and*

$$\langle \varphi'(u), h\rangle = \int_I \sum_{i=0}^{k} f_{u_i}(t, u(t), u'(t), \dots, u^{(k)}(t)) h^{(i)}(t)\, dt \quad \textit{for } u, h \in H_A^{k,n}.$$

Now, let $f : I \times (\mathbb{R}^n)^{k+1} \to \mathbb{R}$ be defined by

$$f(t, u_0, u_1, \dots, u_k) = \frac{1}{2}|u_k|^2 + F(t, u_0, u_1, \dots, u_{k-1}), \tag{4.2}$$

and let the following assumption be satisfied

(A) $F : I \times (\mathbb{R}^n)^k \to \mathbb{R}$ is measurable in t for $(u_0, \ldots, u_{k-1}) \in (\mathbb{R}^n)^k$, continuously differentiable in $(u_0, \ldots, u_{k-1})$ for $t \in I$ a.e. and satisfies the conditions

$$|F(t, u_0, \ldots, u_{k-1})| \le a(|(u_0, \ldots, u_{k-1})|)b(t),$$
$$|F_{u_i}(t, u_0, \ldots, u_{k-1})| \le a(|(u_0, \ldots, u_{k-1})|)b(t), \quad i = 0, \ldots, k-1,$$

for $t \in I$ a.e., $(u_0, \ldots, u_{k-1}) \in (\mathbb{R}^n)^k$ and an $a \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$, $b \in L^1(I, \mathbb{R}_0^+)$.

It is easy to see that function (4.2) satisfies the assumptions of Theorem 4.1. Consequently, the functional

$$\varphi : H_A^{k,n} \ni u \mapsto \int_I \left(\frac{1}{2}|u^{(k)}(t)|^2 + F(t, u(t), u'(t), \ldots, u^{(k-1)}(t))\right) dt \in \mathbb{R}, \tag{4.3}$$

is continuously differentiable on $H_A^{k,n}$, and, for $u, h \in H_A^{k,n}$,

$$\langle \varphi'(u), h \rangle = \int_I \left(\sum_{i=0}^{k} F_{u_i}(t, u(t), u'(t), \ldots, u^{(k-1)}(t)) h^{(i)}(t) + u^{(k)}(t) h^{(k)}(t) \right) dt,$$

Moreover, since the functional

$$H_A^{k,n} \ni u \mapsto \int_I \frac{1}{2}|u^{(k)}(t)|^2 \, dt \in \mathbb{R},$$

being convex and continuous, is weakly l.s.c. and the functional

$$H_A^{k,n} \ni u \mapsto \int_I F(t, u(t), u'(t), \ldots, u^{(k-1)}(t)) \, dt \in \mathbb{R},$$

being weakly continuous (see Lemma 3.5), is weakly l.s.c., therefore the functional φ given by (4.3) is weakly l.s.c.

Theorem 4.2. *If F satisfies* (A) *and*

(B) *there exists $g \in L^1(I, \mathbb{R}_0^+)$ such that*

$$|F_{u_i}(t, u_0, \ldots, u_{k-1})| \le g(t),$$

for $t \in I$ a.e., $u \in \mathbb{R}^n$, $i = 0, \ldots, k-1$,

(C) $\int_I F(t, W(t), W'(t), \ldots, W^{(k-1)}(t)) \, dt \to \infty$ *as* $\sum_{i=0}^{k-1} |c_1| \to \infty$ *with* $W(t) = c_0 + c_1 t + \ldots + c_{k-1} t^{k-1}$,

then the functional φ given by (4.3) *is coercive, i.e.*

$$\varphi(u) \to \infty \quad \text{as } \|u\| \to \infty.$$

Proof. It is easy to check that any function $u \in H_A^{k,n}$ can be represented in the form

$$u(t) = \widetilde{u}(t) + \overline{u}(t) = \widetilde{u}(t) + c_{k-1} t^{k-1} + c_{k-2} t^{k-2} + \ldots + c_1 t + c_0, \quad t \in I,$$

with $c_0, \ldots, c_{k-1} \in \mathbb{R}^n$ and

$$\int_I \widetilde{u}(t)\,dt = 0, \quad \int_I \widetilde{u}'(t)\,dt = 0, \quad \ldots, \quad \int_I \widetilde{u}^{(k-1)}(t)\,dt = 0.$$

Indeed, it suffices to choose the vectors $c_0, \ldots, c_{k-1} \in \mathbb{R}^n$ for which

$$\begin{aligned}
\int_I (c_{k-1}t^{k-1} + \ldots + c_1 t + c_0)\,dt &= \int_I u(t)\,dt,\\
\int_I ((k-1)c_{k-1}t^{k-2} + \ldots + c_1)\,dt &= \int_I u'(t)\,dt,\\
&\ \vdots\\
\int_I ((k-1)\ldots 2c_{k-1}t + (k-2)!c_{k-2})\,dt &= \int_I u^{(k-2)}(t)\,dt,\\
\int_I (k-1)!c_{k-1}\,dt &= \int_I u^{(k-1)}(t)\,dt.
\end{aligned}$$

Now, let us notice that

$$\|u\| \to \infty \Rightarrow \sum_{i=0}^{k-1} |c_i| + \int_I |u^{(k)}(t)|^2\,dt \to \infty. \tag{4.4}$$

Indeed, if we denote $\overline{u}(t) = c_{k-1}t^{k-1} + \ldots + c_1 t + c_0$, we have

$$\begin{aligned}
\|u\|^2 &= \sum_{i=0}^{k-1} \int_I |u^{(i)}(t)|^2\,dt + \int_I |u^{(k)}(t)|^2\,dt\\
&= \sum_{i=0}^{k-1} \int_I |\widetilde{u}^{(i)}(t) + \overline{u}^{(i)}(t)|^2\,dt + \int_I |u^{(k)}(t)|^2\,dt\\
&= \sum_{i=0}^{k-1} \int_I |\widetilde{u}^{(i)}(t)|^2\,dt + 2\sum_{i=0}^{k-1} \int_I \widetilde{u}^{(i)}(t)\overline{u}^{(i)}(t)\,dt\\
&\quad + \sum_{i=0}^{k-1} \int_I |\overline{u}^{(i)}(t)|^2\,dt + \int_I |u^{(k)}(t)|^2\,dt.
\end{aligned}$$

From Lemma 3.3 we have

$$\sum_{i=0}^{k-1} \int_I |\widetilde{u}^{(i)}(t)|^2\,dt \le k \cdot d \int_I |\widetilde{u}^{(k)}(t)|^2\,dt = k \cdot d \int_I |u^{(k)}(t)|^2\,dt,$$

$$\begin{aligned}
\sum_{i=0}^{k-1}\int_I \widetilde{u}^{(i)}(t)\overline{u}^{(i)}(t)\,dt &\le \sum_{i=0}^{k-1}\int_I |\widetilde{u}^{(i)}(t)|\cdot|\overline{u}^{(i)}(t)|\,dt \\
&\le \sum_{i=0}^{k-1}\max_{t\in I}|\overline{u}^{(i)}(t)|T^{1/2}\left(\int_I |\widetilde{u}^{(i)}(t)|^2\,dt\right)^{1/2} \\
&\le T^{1/2}\cdot\sum_{i=0}^{k-1}\max_{t\in I}|\overline{u}^{(i)}(t)|d^{1/2}\left(\int_I |\widetilde{u}^{(k)}(t)|^2\,dt\right)^{1/2} \\
&= T^{1/2}\cdot d^{1/2}\left(\int_I |u^{(k)}(t)|^2\,dt\right)^{1/2}\cdot\sum_{i=0}^{k-1}\max_{t\in I}|\overline{u}^{(i)}(t)| \\
&\le T^{1/2}\cdot d^{1/2}\left(\int_I |u^{(k)}(t)|^2\,dt\right)^{1/2} \\
&\quad\cdot\sum_{i=0}^{k-1}\left[(k-1)!\max\{T^0,\dots,T^{k-1}\}\sum_{j=0}^{k-1}|c_j|\right] \\
&= T^{1/2}\cdot d^{1/2}\cdot k!\max\{T^0,\dots,T^{k-1}\}\sum_{i=0}^{k-1}|c_i|\left(\int_I |u^{(k)}(t)|^2\,dt\right)^{1/2},
\end{aligned}$$

$$\begin{aligned}
\sum_{i=0}^{k-1}\int_I |\overline{u}^{(i)}(t)|^2\,dt &\le T\sum_{i=0}^{k-1}(\max_{t\in I}|\overline{u}^{(i)}(t)|)^2 \le T\left(\sum_{i=0}^{k-1}\max_{t\in I}|\overline{u}^{(i)}(t)|\right)^2 \\
&\le T\left(\sum_{i=0}^{k-1}\left[(k-1)!\max\{T^0,\dots,T^{k-1}\}\sum_{j=0}^{k-1}|c_j|\right]\right)^2 \\
&= T\left(k!\max\{T^0,\dots,T^{k-1}\}\sum_{i=0}^{k-1}|c_j|\right)^2 \\
&= T(k!)^2(\max\{T^0,\dots,T^{k-1}\})^2\left(\sum_{i=0}^{k-1}|c_i|\right)^2.
\end{aligned}$$

So,

$$\begin{aligned}
\|u\|^2 \le k\cdot d\int_I |u^{(k)}(t)|^2\,dt& \\
&+ 2\cdot T^{1/2}\cdot d^{1/2}k!\max\{T^0,\dots,T^{k-1}\}\sum_{i=0}^{k-1}|c_i|\left(\int_I |u^{(k)}(t)|^2\,dt\right)^{1/2} \\
&+ T(k!)^2(\max\{T^0,\dots,T^{k-1}\})^2\left(\sum_{i=0}^{k-1}|c_i|\right)^2.
\end{aligned}$$

The above means that (4.4) is true. Now, we have

$$\varphi(u) = \int_I \frac{1}{2}|u^{(k)}(t)|^2\,dt + \int_I F(t,u(t),\dots,u^{(k-1)}(t))\,dt$$

$$
\begin{aligned}
&= \int_I \frac{1}{2}|u^{(k)}(t)|^2\,dt + \int_I F(t,\overline{u}(t),\ldots,\overline{u}^{(k-1)}(t)\,dt \\
&\quad + \int_I [F(t,u(t),\ldots,u^{(k-1)}(t)) - F(t,\overline{u}(t),\ldots,\overline{u}^{(k-1)}(t))]\,dt \\
&= \int_I \frac{1}{2}|u^{(k)}(t)|^2\,dt + \int_I F(t,\overline{u}(t),\ldots,\overline{u}^{(k-1)}(t))\,dt \\
&\quad + \int_I \int_0^1 \sum_{i=0}^{k-1} F_{u^i}(t,\overline{u}(t)+s\widetilde{u}(t),\ldots,\overline{u}^{(k-1)}(t)+s\widetilde{u}^{(k-1)}(t))\widetilde{u}^{(i)}(t)\,ds\,dt \\
&= \int_I \frac{1}{2}|u^{(k)}(t)|^2\,dt + \int_I F(t,\overline{u}(t),\ldots,\overline{u}^{(k-1)}(t)) \\
&\quad + \sum_{i=0}^{k-1} \int_I \widetilde{u}^{(i)}(t) \int_0^1 F_{u^i}(t,\overline{u}(t)+s\widetilde{u}(t)),\ldots,\overline{u}^{(k-1)}(t)+s\widetilde{u}^{(k-1)}(t))\,ds)dt \\
&\geq \int_I \frac{1}{2}|u^{(k)}(t)|^2\,dt + \int_I F(t,\overline{u}(t),\ldots,\overline{u}^{(k-1)}(t))\,dt \\
&\quad - \sum_{i=0}^{k-1} \max\{|\widetilde{u}^{(i)}(t)|;\ t\in I\} \int_I g(t)\,dt \\
&\geq \int_I \frac{1}{2}|u^{(k)}(t)|^2\,dt + \int_I F(t,\overline{u}(t),\ldots,\overline{u}^{(k-1)}(t))\,dt \\
&\quad - \Big(\sum_{i=0}^{k-1} e_i\Big)\|\widetilde{u}\| \int_I g(t)\,dt \\
&\geq \int_I \frac{1}{2}|u^{(k)}(t)|^2\,dt + \int_I F(t,\overline{u}(t),\ldots,\overline{u}^{(k-1)}(t))\,dt \\
&\quad - \Big(\sum_{i=0}^{k-1} e_i\Big)\Big(k\,d\int_I |u^{(k)}(t)|^2\,dt\Big)^{1/2} \int_I g(t)\,dt,
\end{aligned}
$$

where e_0 is the constant from Lemma 3.1(a), d is the constant from Lemma 3.3 and $c_0,\ldots,c_{k-1}$ are such that

$$u(t) = \widetilde{u}(t) + c_{k-1}t^{k-1} + \ldots + c_1 t + c_0,$$

with

$$\int_I \widetilde{u}(t)\,dt = 0, \quad \int_I \widetilde{u}'(t)\,dt = 0, \quad \ldots, \quad \int_I \widetilde{u}^{(k-1)}(t)\,dt = 0.$$

Consequently, using (4.4) we assert that

$$\varphi(u) \to \infty \quad \text{as } \|u\| \to \infty.$$

The proof is concluded. □

From the above theorem it follows that any minimizing sequence of φ is bounded. This means, in view of the reflexivity of $H_A^{k,n}$ and the weak lower semicontinuity of φ, that φ has its minimum on $H_A^{k,n}$. Let us denote the minimum point of φ on $H_A^{k,n}$ as u_*. The differentiability of φ on $H_A^{k,n}$ implies, for $H_A^{k,n}$, $\langle \varphi'(u_*), h\rangle = 0$, i.e.

$$\int_I u_*^{(k)}(t)h^{(k)}(t)\,dt + \int_I \sum_{i=0}^{k-1} F_{u_i}(t, u_*(t), u_*'(t), \dots, u_*^{(k-1)}(t))h^{(i)}(t)\,dt = 0,$$

for $h \in H_A^{k,n}$. Integrating by parts we obtain

$$\begin{aligned}
&\int_I F_{u_{k-1}}(t, u_*(t), u_*'(t), \dots, u_*^{(k-1)}(t))h^{(k-1)}(t)\,dt \\
&\quad= \int_I \bigg(\int_0^t F_{u_{k-1}}(s, u_*(s), u_*'(s), \dots, u_*^{(k-1)}(s))\,ds\bigg)' h^{(k-1)}(t)\,dt \\
&\quad= \int_0^t F_{u_{k-1}}(s, u_*(s), u_*'(s), \dots, u_*^{(k-1)}(s))\,ds\, h^{(k-1)}(t)\bigg|_{t=0}^{t=T} \\
&\qquad- \int_I \bigg(\int_0^t F_{u_{k-1}}(s, u_*(s), u_*'(s), \dots, u_*^{(k-1)}(s))\,ds\bigg) h^{(k)}(t)\,dt \\
&\quad= \int_I F_{u_{k-1}}(t, u_*(t), u_*'(t), \dots, u_*^{(k-1)}(t))\,dt\, h^{(k-1)}(T) \\
&\qquad- \int_I \bigg(\int_0^t F_{u_{k-1}}(s, u_*(s), u_*'(s), \dots, u_*^{(k-1)}(s))\,ds\bigg) h^{(k)}(t)\,dt,
\end{aligned}$$

and analogously,

$$\begin{aligned}
\int_I F_{u_{k-2}} h^{(k-2)} &= \int_I F_{u_{k-2}} h^{(k-2)}(T) - \int_I \bigg(\int_0^t F_{u_{k-2}}\bigg) h^{(k-1)}(T) \\
&\quad+ \int_I \bigg(\int_0^t \int_0^{t_1} F_{u_{k-2}}\bigg) h^{(k)}(t)\,dt, \\
&\;\;\vdots \\
\int_I F_{u_0} h &= \int_I F_{u_0} h(T) - \bigg(\int_0^t F_{u_0}\bigg) h'(T) \\
&\quad+ \dots + (-1)^{k-1} \int_I \bigg(\int_0^t \int_0^{t_1} \dots \int^{t_{k-2}} F_{u_0}\bigg) h^{(k-1)}(T) \\
&\quad+ (-1)^k \int_I \bigg(\int_0^t \int_0^{t_1} \dots \int^{t_{k-1}} F_{u_0}\bigg) h^{(k)}(t)\,dt.
\end{aligned}$$

So, using Corollary (2.1), we assert that there exist constants $c_0, \dots, c_{k-1} \in \mathbb{R}^n$ such that

$$u_*^{(k)}(t) - \int_0^t F_{u_{k-1}} + \int_0^t \int_0^{t_1} F_{u_{k-2}} + \dots + (-1)^k \int_0^t \int_0^{t_1} \dots \int_0^{t_{k-1}} F_{u_0} = c_0 + c_1 t + \dots + c_{k-1} t^{k-1}, \tag{4.5}$$

for $t \in I$ a.e. and (after identifying $\psi(t) = u_*^{(k)}(t) - \int_0^t F_{u_{k-1}} + \int_0^t \int_0^{t_1} F_{u_{k-2}} + \ldots + (-1)^k \int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-1}} F_{u_0}$ with the above right-hand side)

$$
\tag{4.6}
\begin{bmatrix} \psi(0) \\ \psi'(0) \\ \vdots \\ \psi^{(k-2)}(0) \\ \psi^{(k-1)}(0) \end{bmatrix}
= B \begin{bmatrix}
\psi(T) - (-1)^k \Big[\int_I \Big(\int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-2}} F_{u_0} \Big) - \int_I \Big(\int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-3}} F_{u_1} \Big) \\
\qquad + \ldots + (-1)^{k-1} \int_I F_{u_{k-1}} \Big] \\
\psi'(T) - (-1)^k \Big[\int_I \Big(\int_0^t \int_0^{t_1} \ldots \int_0^{t_{k-3}} F_{u_0} \Big) + \ldots + (-1)^{k-2} \int_I F_{u_{k-2}} \Big] \\
\vdots \\
\psi^{(k-2)}(T) - (-1)^k \Big[\int_I \Big(\int_0^t F_{u_0} \Big) - \int_I F_{u_1} \Big] \\
\psi^{(k-1)}(T) - (-1)^k \int_I F_{u_0}
\end{bmatrix},
$$

where B is as in Theorem (2.1).

As usual, we say that an integrable function $l : [0,T] \to \mathbb{R}^n$ has a weak derivative if l possesses an absolutely continuous representant (in the sense of the measure theory) that is differentiable a.e. on $[0,T]$ with the derivative integrable on $[0,T]$. This derivative is called a weak derivative of l and denoted as $\frac{d}{dt} l$.

In the case when an integrable function $l : [0,T] \to \mathbb{R}^n$ has a continuous representant, we write $l|_{t=0}$, $l|_{t=T}$ for the values of this representant at 0, T, respectively.

So, from formula (4.5) it follows that the function u_* satisfies equation (1.1) a.e. on $[0,T]$ and from (4.6) it follows that u_* satisfies the boundary conditions (1.2).

On the account of the above identifying of an integrable function with their absolutely continuous representant we say that u_* is a weak solution of problem (1.1)–(1.2). We have thus proved,

THEOREM 4.3. *If a function $F : I \times (\mathbb{R}^n)^k \to \mathbb{R}$ satisfies conditions* (A)–(C), *then there exists a function $u \in H_A^{k,n}$ being a weak solution of equation* (1.1) *and satisfying boundary conditions* (1.2).

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