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DIFFERENTIAL EQUATIONS AND IMPLICIT FUNCTION: A GENERALIZATION OF THE NEAR OPERATORS THEOREM

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1. Introduction

Many extensions of Implicit Function Theorem have been proposed for studying non linear differential equations and systems as the already classic Hildebrandt and Graves Theorem [7]. The global invertibility problem has been considered in several forms (see for example [2]), and the differentiability hypothesis has been weakened in various ways to face up different problems connected with differential equations.

S. Campanato in [3] has introduced the notion of "near operators" for studying the existence of solutions of elliptic differential equations and systems.

Definition 1.1 (near operators). Let $\mathcal{X}$ be a set, $\mathcal{B}$ a Banach space with norm $\|\cdot\|$, $A, B : \mathcal{X} \to \mathcal{B}$. We say that A is *near* B in $\mathcal{X}$ if there exist two real and positive constants $\alpha, k, \in (0,1)$, such that for all $x_1, x_2 \in X$

$$\|B(x_1) - B(x_2) - \alpha[A(x_1) - A(x_2)]\| \leq k\|B(x_1) - B(x_2)\|. \tag{1.1}$$

The main result on this operators is the following global invertibility theorem (see [3]).

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THEOREM 1.1. *Let $\mathcal{X}$ be a set, $\mathcal{B}$ a Banach space, A, $B : \mathcal{X} \to \mathcal{B}$ such that A is near B in $\mathcal{X}$. If B is bijective between $\mathcal{X}$ and $\mathcal{B}$ then A is bijective $\mathcal{X}$ and $\mathcal{B}$.*

If we take away the injectivity hypothesis on B we obtain a surjectivity theorem: *if B is surjective then $\mathcal{A}$ is surjective* (it follows from Theorem 1.1 by replacing set $\mathcal{X}$ with the quotient set $\mathcal{X}|_{\sim}$, where $\sim$ is the equivalence: $x \sim y$ if and only if $B(x) = B(y)$.)

Moreover, we remind that if $\mathcal{B}$ is a Hilbert space with the scalar product $(\cdot\,,\,\cdot)$, then A is near B in $\mathcal{X}$ if and only if A is strictly monotone with respect to B (see [4]), i.e. there exist two positive constants M and ν with $M \geq \nu > 0$, such that for all $u, v \in \mathcal{X}$:

$$\|A(u) - A(v)\| \leq M\|B(u) - B(v)\|,$$
$$\nu\|B(u) - B(v)\|^2 \leq \big(A(u) - A(v) \,\big|\, B(u) - B(v)\big)\,.$$

This theory has been first applied to a class of systems of differential equations satisfying a special ellipticity condition, Condition A, which we state below. Let Ω be a bounded convex open set in $\mathbb{R}^n$, with C^2 boundary.

Let $x = (x_1, \ldots, x_n) \in \Omega$, $\xi = \{\xi_{ij}\}_{i,j=1,\ldots,n}$, $\xi_{ij} \in \mathbb{R}^N$. Let $a(x, \xi)$ be a map $\Omega \times \mathbb{R}^{n^2N} \to \mathbb{R}^N$, measurable in x, continuous in ξ, such that:

$$a(x, 0) = 0. \tag{1.2}$$

CONDITION A. *There exist three positive constants α, β, γ, with $\gamma + \delta < 1$, such that*[1]*:*

$$\Big\|\sum_{i=1}^{n} \xi_{ii} - \alpha[a(x, \xi + \eta) - a(x, \eta)]\Big\|_N \leq \gamma\|\xi\|_{n^2N} + \delta\Big\|\sum_{i=1}^{n} \xi_{ii}\Big\|_N \tag{1.3}$$

a.e. in Ω, for all $\xi, \eta \in \mathbb{R}^{n^2N}$.

If $u = (u_1, \ldots, u_N)$ is a map, $\Omega \to \mathbb{R}^N$, we set:

$$D_i u = \frac{\partial u}{\partial x_i} = \left(\frac{\partial u_1}{\partial x_i}, \ldots, \frac{\partial u_N}{\partial x_i}\right),$$
$$Du = (D_1 u, \ldots, D_n u),$$
$$H(u) = \{D_i D_j u\}_{i,j=1,\ldots,n}.$$

In particular if Δ is the Laplace operator then Δu is the N-vector $(\Delta u_1, \ldots, \Delta u_N)$. In [3] the following system is considered

$$a(x, H(u)) = f(x),$$

and the following theorem is proved:

[1] If $m \in \mathbb{N}$, $\|\cdot\|_m$ and $(\cdot\,,\,\cdot)_m$ are respectively norm and scalar product in $\mathbb{R}^m$.

Theorem 1.2. *If a satisfies hypotheses* (1.2) *and* (1.3)*, so that $A(u) = a(x, H(u))$ is a operator between $H^2 \cap H_0^1(\Omega, \mathbb{R}^N)^2$ and $L^2(\Omega, \mathbb{R}^N)$, then*

(i) *A is near Δ in $H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$, and consequently,*
(ii) *A is bijective between $H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ and $L^2(\Omega, \mathbb{R}^N)$.*

This result makes important progress in the study of non variational elliptic systems. We remark that in the case of a linear equation such as $\sum_{i,j} a_{ij}(x) \cdot D_{ij}u = f$, with $a_{ij} \in L^\infty(\Omega)$, Condition A is equivalent to ellipticity hypothesis: $M\|\xi\|_n^2 \geq \sum_{i,j} a_{ij}(x)\xi_i\xi_j \geq \nu\|\xi\|_n^2$, for all $\xi \in \mathbb{R}^n$ (see [4]). Moreover, in [13] it is proved that Condition A is stronger than the following condition: there exists $\varepsilon > 0$ such that (when $n > 1$)

$$\Big(\sum_{i=1}^n a_{ii}(x)\Big)^2 \geq (n - 1 + \varepsilon) \sum_{i,j=1}^n a_{ij}^2(x), \quad \text{a.e. in } \Omega.$$

This is a generalized form of the Cordes condition (see [6] and [10]).

The notion of near operator and Theorem 1.1 with a suitable version of Condition A have also permitted to consider some problems about parabolic systems, see [5] and [11]. While the following property proved in [12] has permitted to study the existence of solutions of a class of non linear hyperbolic problems: "*if A is near B and $B(\mathcal{X})$ is dense in $\mathcal{B}$ then $A(\mathcal{X})$ is dense in $B(\mathcal{X})$.*

We consider now the contents of this paper. Our main theorem, Theorem 2.1, is an Implicit Function Theorem: indeed we study the existence of a function implicitly defined by an equation of the type $F(x, y) = 0$, where $F(x, \cdot\,)$ is "near" an injective and open operator.

The features of Theorem 2.1 are: generality of the domain of the function (it is a Cartesian product between a topological space and a set), and the low regularity of the function. Moreover, the hypothesis of bijectivity of the Fréchet differential of the function in the classic Hildebrandt–Graves Theorem (see [7]) is replaced by the hypothesis of nearness between the function and an open and injective operator. Indeed we prove that the hypotheses of Hildebrandt–Graves Theorem are a particular case of that of Theorem 2.1: *if A is defined on a Banach space, if its differential B in a point x_0 is bijective, then a neighbourhood of x_0 exists where A is near B* (see Lemma 2.1 and Proposition 3.1). On the other hand many of the F-differential generalizations in the literature makes possible to prove an Implicit Function Theorem. For example, in [9], there is a survey of these subjects and it is proved a generalization of Implicit Function Theorem.

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[2] If m is a non negative integer, $H^m(\Omega, \mathbb{R}^N)$ is the Sobolev space of functions $v : \Omega \to \mathbb{R}^N$ having finite norm:

$$\|v\|_{H^m(\Omega,\mathbb{R}^N)} = \Big\{ \int_\Omega \sum_{|\beta|\leq m} \|D^\beta v\|_N^2 \, dx \Big\}.$$

In Section 3 it is proved that the hypotheses of the Implicit Function Theorem of [9] also are a special case of Theorem 2.1 (see Theorem 3.2).

In Section 4 some examples of applications of the results of Section 2 are given to solve two problems. The first problem concerns the existence and uniqueness of the solution to the following system of differential equations

$$a(x, H(u)) + g(x, u)) = f.$$

The second one is an open mapping problem:

Let $\mathcal{X}$ be a set, $\mathcal{B}$ be a Banach space and $A, B : \mathcal{X} \to \mathcal{B}$. If A is near B on $\mathcal{X}$ and if $B(\mathcal{X})$ is a neighbourhood of $B(x_0)$ then $A(\mathcal{X})$ is a neighbourhood of $A(x_0)$.

The last proposition is also proved in [12] without using Implicit Function Theorem. Finally, a simple example of operator between $L^2(\Omega)$ and $L^2(\Omega)$ that is near the Identity map on $L^2(\Omega)$ but not F-differentiable is given.

2. Generalizations of Implicit Function Theorem

Let X be a topological space, Z a Banach space normed with $\|\cdot\|$, Ω a neighbourhood of $z_0 \in Z$, $\Phi : X \times \Omega \to Z$.

Lemma 2.1. *Let us suppose that*

(2.1) $(x_0, z_0) \in X \times \Omega$ *exists such that* $\Phi(x_0, z_0) = 0$,

(2.2) *the map* $x \to \Phi(x, z_0)$ *is continuous at* x_0,

(2.3) *there exist positive numbers* α, k, *with* $k \in (0, 1)$, *and a neighbourhood of* x_0, $U(x_0) \subseteq X$, *such that:*

$$\|z_1 - z_2 - \alpha[\Phi(x, z_1) - \Phi(x, z_2)]\| \le k\|z_1 - z_2\|, \quad \forall x \in U(x_0),\ \forall z_1, z_2 \in \Omega.$$

Then the following are true: there exists a ball $S(z_0, \sigma) = \{z \in Z : \|z - z_0\| < \sigma\} \subset \Omega$, and a neighbourhood of x_0, $V(x_0) \subset U(x_0)$, such that there is exactly one solution $z = z(x) : V(x_0) \to S(z_0, \sigma)$ of the following problem:

$$\begin{cases} \Phi(x, z(x)) = 0 & \text{for all } x \in V(x_0), \\ z(x_0) = z_0. \end{cases} \tag{2.4}$$

Moreover, function $z = z(x)$ is continuous in x_0.

Proof. Existence: let $\sigma > 0$ be such that $S(z_0, \sigma) \subset \Omega$. We set

$$\mathcal{I}_x(z) = z - \alpha\Phi(x, z), \quad \forall x \in U(x_0). \tag{2.5}$$

We prove that exists a neighbourhood $V(x_0) \subset U(x_0)$ of x_0, such that for all $x \in V(x_0)$ the following are true:

(i) $\mathcal{I}_x : S(z_0, \sigma) \to S(z_0, \sigma)$.

(ii) $\mathcal{I}_x$ is a contraction.

Indeed, (i) follows from the next inequalities by (2.3) and $\Phi(x_0, z_0) = 0$

$$\begin{aligned}\|\mathcal{I}_x(z) - z_0\| &= \|z - \alpha\Phi(x, z) - z_0\| \\ &\le \|z - z_0 - \alpha[\Phi(x, z) - \Phi(x, z_0)]\| + \alpha\|\Phi(x, z_0)\| \\ &\le k\|z - z_0\| + \alpha\|\Phi(x, z_0) - \Phi(x_0, z_0)\|.\end{aligned}$$

We obtain from these inequalities and from (2.2) that for all $\varepsilon > 0$ there exists $V(x_0) \subseteq U(x_0)$ such that:

$$\|\mathcal{I}_x(z) - z_0\| \le k\|z - z_0\| + \alpha\varepsilon, \quad \forall x \in V(x_0).$$

From this $\mathcal{I}_x(z) \in S(z_0, \sigma)$, for all $z \in S(z_0, \sigma)$ if $\varepsilon < (1 - k)\sigma/\alpha$ and $x \in V(x_0)$. Proposition (ii) follows from (2.3): for all $x \in U(x_0)$ and all $z_1,\ z_2 \in \Omega$ we have

$$\|\mathcal{I}_x(z_1) - \mathcal{I}_x(z_2)\| = \|z_1 - z_2 - \alpha[\Phi(x, z_1) - \Phi(x, z_2)]\| \le k\|z_1 - z_2\|.$$

Therefore, it follows from (i) and (ii), by the fixed point theorem, that for all $x \in V(x_0)$ exists exactly one $z = z(x) \in S(z_0, \sigma)$ such that $z(x) = \mathcal{I}_x(z(x))$, that is, from (2.5):

$$\Phi(x, z(x)) = 0, \quad \forall x \in V(x_0).$$

On the other hand $z \to \Phi(x, z)$ is a injective map in Ω, for all $x \in U(x_0)$, because (2.3) implies that:

$$\|z_1 - z_2\| \le \frac{\alpha}{1 - k}\|\Phi(x, z_1) - \Phi(x, z_2)]\|, \quad \forall z_1, z_2 \in \Omega,\ \forall x \in U(x_0).$$

Since $\Phi(x_0, z(x_0)) = 0 = \Phi(x_0, z_0)$ we have $z(x_0) = z_0$, which completes the proof of the existence of a solution to problem (2.4).

Uniqueness: it is a trivial consequence of the fact that $z \to \Phi(x, z)$ is injective.

Continuity of $z = z(x)$ in x_0: it follows from (2.2) and from the inequality (obtained from (2.3)):

$$\begin{aligned}\|z(x) - z(x_0)\| &\le \frac{\alpha}{1 - k}\|\Phi(x, z(x)) - \Phi(x, z(x_0))\| \\ &= \frac{\alpha}{1 - k}\|\Phi(x_0, z_0) - \Phi(x, z_0))\|.\end{aligned}$$ □

Remark 2.1. If a map $\Phi : X \times Z \to Z$ satisfies the hypotheses of Lemma 2.1, and the hypothesis (2.3) holds for all $z_1, z_2 \in Z$ and all $x \in U(x_0)$, then similarly to what was previously done, we can prove that for all $x \in U(x_0)$ there exists only one solution $z : U(x_0) \to Z$ of problem (2.4). In particular, if (2.3) holds for all $x \in X$, then we obtain a solution of the problem (2.4) defined on the whole X.

Now we prove the following generalization of Implicit Functions Theorem. Let X be a topological space, Y a set, Z a Banach space $F : X \times Y \to Z$, $B : Y \to Z$.

THEOREM 2.1. *Let us suppose that:*

(2.6) *there exists* $(x_0, y_0) \in X \times Y$ *such that* $F(x_0, y_0) = 0$,

(2.7) *the map* $x \to F(x, y_0)$ *is continuous in* $x = x_0$,

(2.8) *there exist positive numbers* α, k, *with* $k \in (0, 1)$, *and a neighbourhood of* x_0, $U(x_0) \subset X$, *such that for all* $y_1, y_2 \in Y$ *and all* $x \in U(x_0)$

$$\|B(y_1) - B(y_2) - \alpha[F(x, y_1) - F(x, y_2)]\| \le k\|B(y_1) - B(y_2)\|,$$

(2.9) B *is injective,*

(2.10) $B(Y)$ *is a neighbourhood of* $z_0 = B(y_0)$.

Then the following are true: there exists a ball $S(z_0, \sigma) \subset B(Y)$ *and a neighbourhood of* x_0, $V(x_0) \subset U(x_0)$, *such that there is exactly one solution* $y = y(x) : V(x_0) \to B^{-1}(S(z_0, \sigma))$ *of the following problem:*

$$\begin{cases} F(x, y(x)) = 0 \quad \forall x \in V(x_0), \\ y(x_0) = y_0. \end{cases} \tag{2.11}$$

PROOF. Existence: we set

$$\Phi(x, z) = F(x, B^{-1}(z)). \tag{2.12}$$

The map Φ satisfies the hypotheses of Lemma 2.1, with $\Omega = B(Y), z_0 = B(y_0)$, $\Phi(x_0, z_0) = F(x_0, y_0) = 0$ and $x \to \Phi(x, z_0) = F(x, y_0)$ continuous in x_0. Moreover, if α and k are as in the hypothesis (2.8), setting $z_1 = B(y_1)$ and $z_2 = B(y_2)$ we obtain that:

$$\begin{aligned} \|z_1 - z_2 - \alpha[\Phi(x, z_1) - \Phi(x, z_2)]\| &= \|B(y_1) - B(y_2) - \alpha[F(x, y_1) - F(x, y_2)]\| \\ &\le k\|B(y_1) - B(y_2)\| = \|z_1 - z_2\|, \end{aligned}$$

for all z_1, $z_2 \in B(Y)$ and all $x \in U(x_0)$. Hence Φ also satisfies hypothesis 2.3, from this, as consequence of Lemma 2.1, we obtain that $S(z_0, \sigma) \subset \Omega = B(Y)$ and there exists $V(x_0) \subset U(x_0)$ such that there is exactly one solution $z = z(x) \in S(x_0, \sigma)$ of the following problem

$$\begin{cases} \Phi(x, z(x)) = 0 \quad \forall x \in V(x_0), \\ z(x_0) = z_0. \end{cases}$$

From this and from (2.12), setting $y(x) = B^{-1}(z(x))$ we obtain the proof of existence.

Uniqueness: we observe that function $y \to F(x, y)$ is injective for all $x \in U(x_0)$ and all $y \in Y$, consequently to (2.9) and to the following inequality (obtained from (2.8)):

$$\|B(y_1) - B(y_2)\| \le \frac{\alpha}{1 - k}\|F(x, y_1) - F(x, y_2)\|, \quad \forall x \in U(x_0),\ \forall y_1, y_2 \in Y.$$

Hence, if $y_1 = y_1(x)$ is another solution of the problem (2.9), and $F(x, y_1(x)) = 0 = F(x, y(x))$, for all $x \in V(x_0)$, it follows that $y_1(x) = y(x)$, for all $x \in V(x_0)$.□

REMARK 2.2. Let $C : Y \to Z$ be another map that satisfies hypotheses (2.8)–(2.10). Then from Theorem 2.1 it follows that there exist $S(z_0, \sigma_1) \subset C(Y)$ and $V_1(x_0) \subset U(x_0)$ such that exactly one solution $y_1 = y_1(x) \in C^{-1}(S(z_0, \sigma_1))$ of the problem exists:

$$\begin{cases} F(x, y_1(x)) = 0 \quad \forall x \in V_1(x_0), \\ y_1(x_0) = y_0. \end{cases}$$

Therefore the injectivity of $y \to F(x, y)$ (see the proof of uniqueness in the Theorem 2.1) implies that $y_1(x) = y(x)$, for all $x \in V_1(x_0) \cap V(x_0)$.

REMARK 2.3. If $B(Y) = Z$, from the Remark 2.1, we obtain that the solution of the problem (2.11) is defined on the whole $U(x_0)$. In particular, if for all $x \in X$ (by (2.8)) holds then $y = y(x)$ is defined on the whole X.

REMARK 2.4. (Approximating functions of the solution of problem (2.11)). Let us assume the notations and the hypotheses of the Theorem 2.1. We can find a sequence of approximating functions of the solution $y = y(x)$ of problem 2.1, in a suitable neighbourhood of x_0, by simplified Newton's method, as it happens in the classic Implicit Funtions Theorem. Indeed, if we define $\{y(x)\}_{n\in\mathbb{N}} \subset Y$ in the following way:

$$\begin{cases} y_0(x) = y_0, \\ y_n(x) = B^{-1}[By_{n-1}(x) - \alpha F(x, y_{n-1}(x))] \quad \forall x \in U(x_0), \end{cases}$$

then $\lim_{n\to\infty} B(y_n(x)) = B(y(x))$ in Z, for all $x \in U_1(x_0) \cap V(x_0)$, where $U_1(x_0) \subset U(x_0)$.

PROOF. Let $\varepsilon \in (0, \sigma(1-k)/\alpha)$, there exist $U_1(x_0) \subset U(x_0)$ and $\sigma > 0$ such that the sequence $\{By_n(x)\}_{n\in\mathbb{N}}$ is in $S(z_0, \sigma) \subset B(Y)$, indeed:

$$\begin{aligned} &\|B(y_n(x)) - B(y_0)\| \\ &\leq \|B(y_{n-1}(x)) - B(y_0) - \alpha[F(x, y_{n-1}(x)) - F(x, y_0(x))]\| + \alpha\|F(x, y_0(x))\| \\ &\leq k\|B(y_{n-1}(x)) - B(y_0)\| + \alpha\|F(x, y_0)\| \\ &\leq \left(\sum_{i=0}^{n-1} k^i\right)\alpha\|F(x, y_0)\| \leq \frac{\alpha}{1-k}\|F(x, y_0) - F(x_0, y_0)\| \leq \frac{\varepsilon\alpha}{1-k} \leq \sigma, \end{aligned}$$

for all $x \in U_1(x_0) \subset U(x_0)$. Moreover, for all $x \in U_1(x_0)$, $\{z_n(x)\}_{n\in\mathbb{N}} = B\{(y_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Z, because, by (2.18), (if $n > m$) we have

$$\begin{aligned} &\|B(y_n(x)) - B(y_m(x))\| \\ &\quad\leq \|B(y_{n-1}(x)) - B(y_{m-1}(x)) - \alpha[F(x, y_{n-1}(x)) - F(x, y_{m-1}(x))]\| \\ &\quad\leq k\|B(y_{n-1}(x)) - B(y_{m-1}(x))\| \leq k^m\|B(y_{n-m}(x)) - B(y_0(x)) \leq k^m\sigma, \end{aligned}$$

for all $x \in U_1(x_0)$. Let $z_\infty(x) \in S(z_0, \sigma)$ be the limit of $B(y_n(x))$ in Z and $y_\infty(x) \in Y$ such that $B(y_\infty(x)) = z_\infty(x)$. We prove that the solution $y(x)$ of problem (2.11) coincides with $y_\infty(x)$, for all $x \in U_1(x_0) \cap V(x_0)$. In fact, by (2.8), for all $x \in U_1(x_0)$ it follows that

$$\|F(x, y_n(x)) - F(x, y_\infty(x))]\| \leq \frac{k+1}{\alpha} \|B(y_n(x)) - B(y_\infty(x))\|.$$

Taking limits as $n \to \infty$ we have $F(x, y_\infty(x)) = 0$, for all $x \in U_1(x_0)$ (because $\lim_{n\to\infty} F(x, y_n(x)) = 0$, for all $x \in U_1(x_0)$). Hence the uniqueness of the solution of the problem (2.11) implies that $y(x) = y_\infty(x)$, for all $x \in U_1(x_0) \cap V(x_0)$. In particular the definition of the sequence $\{y_n(x)\}_{n\in\mathbb{N}}$ implies $y_\infty(x_0) = y_n(x_0) = y_0$ for all $n \in \mathbb{N}$. □

We prove the following lemma about the regularity of the solution of problem (2.4).

Lemma 2.2. *Let us assume the hypotheses of Lemma* 2.1*, if* $z : V(x_0) \to S(z_0, \sigma)$ *is the solution of problem* (2.4)*, then the following are true:*

(i) *if* $x \to \Phi(x, z)$ *is injective on* $V(x_0)$*, for all* $z \in S(z_0, \sigma)$*, then* $x \to z(x)$ *is injective on* $V(x_0)$*,*

(ii) *if* $x \to \Phi(x, z)$ *is continuous on* $V(x_0)$*, for all* $z \in S(z_0, \sigma)$*, then* $x \to z(x)$ *is continuous on* $V(x_0)$*,*

(iii) *if* (X, d) *is a metric space and if there exists* $M > 0$ *and* $\alpha \in (0, 1]$ *such that*

$$\|\Phi(x_1, z) - \Phi(x_2, z)\| \leq M[d(x_1, x_2)]^\alpha,$$

for all $z \in S(z_0, \sigma)$*, and all* $x_1, x_2 \in V(x_0)$*, then the solution of the problem* (2.4) *is* α*-Holder continuous on* $V(x_0)$*.*

Proof. (i) We know that $\Phi(x_1, z(x_1)) = \Phi(x_2, z(x_2)) = 0$ for all x_1, $x_2 \in V(x_0)$. Then (2.3) implies the following:

$$\begin{aligned}\alpha\|\Phi(x_1, z(x_1)) - \Phi(x_2, z(x_1))\| &= \alpha\|\Phi(x_2, z(x_2)) - \Phi(x_2, z(x_1))\| \\ &\leq \|z(x_1) - z(x_2) - \alpha[\Phi(x_2, z(x_1)) - \Phi(x_2, z(x_2))]\| + \|z(x_1) - z(x_2)\| \\ &\leq (k+1)\|z(x_1) - z(x_2)\|.\end{aligned}$$

Hence, if $z(x_1) = z(x_2)$ then $\Phi(x_1, z(x_1)) = \Phi(x_2, z(x_2))$, which yields $x_1 = x_2$, because $x \to \Phi(x, z)$ is one-to-one.

(ii) and (iii). Condition (2.3) implies the following:

$$\begin{aligned}\|z(x_1) - z(x_2)\| &\leq \frac{\alpha}{1-k}\|\Phi(x_1, z(x_1)) - \Phi(x_1, z(x_2))\| \\ &= \frac{\alpha}{1-k}\|\Phi(x_2, z(x_2)) - \Phi(x_1, z(x_2))\|, \quad \forall x_1, x_2 \in V(x_0).\end{aligned}$$

Both results then follow easily. □

We obtain the following regularity results of the solution of problem (2.11) by the above Lemma.

THEOREM 2.2 (Regularity of the solution). *Let us assume the hypotheses of Theorem* 2.1. *Let* $y = y(x)$*:* $V(x_0) \to B^{-1}(S(z_0, \sigma))$ *be the solution of problem* (2.11)*. The following are true:*

(i) *if* $x \to F(x, y)$ *is injective on* $V(x_0)$ *then also* $x \to y(x)$ *is injective on* $V(x_0)$*,*
(ii) *if* Y *is a topological space and* B^{-1} *is continuous in* z_0 *then* $y = y(x)$ *is continuous in* x_0*,*
(iii) *let* Y *be a topological space,* B^{-1} *continuous on* $S(z_0, \sigma)$*, if for all* $y \in B^{-1}(S(z_0, \sigma))$ $x \to F(x, y)$ *is continuous in* $V(x_0)$ *then* $y \to y(x)$ *is continuous in* $V(x_0)$*,*
(iv) *if* (X, d_1) *and* (Y, d_2) *are metric spaces,* B^{-1} *is Holder continuous with exponent* $\beta \in (0, 1]$ *on* $S(z_0, \sigma)$ *and* $x \to F(x, y)$ *is Holder continuous with exponent* $\alpha \in (0, 1]$*, on* $V(x_0)$*, then* $y = y(x)$ *is Holder continuous with exponent* $\alpha\beta$ *on* $V(x_0)$*.*

PROOF. (i) Let us assume the notation of the proof of Theorem 2.1. If we set $\Phi(x, z) = F(x, B^{-1}(z))$, then $\Phi(x, z)$ satisfies the hypothesis (i) of Lemma 2.2, consequently $x \to z(x)$ is injective, and so it is also $y = y(x) = B^{-1}(z(x))$.

(ii) Let us assume the notation of Theorem 2.1. By Lemma 2.2 we know that $x \to z(x)$ is continuous in x_0, hence $y(x) = B^{-1}(z(x))$ is continuous in x_0 $(B(y_0) = z_0 = z(x_0))$.

(iii) $\Phi(x, z) = F(x, B^{-1}(z))$ satisfies the hypothesis (ii) of Lemma 2.2, this implies that $x \to z(x)$ is continuous in $V(x_0)$ hence, by continuity of B^{-1} in $S(z_0, \sigma)$, it follows that also $y = y(x) = B^{-1}(z(x))$ is continuous in $V(x_0)$.

(iv) $\Phi(x, z) = F(x, B^{-1}(z))$ verifies the hypothesis (iii) of Lemma 2.2, this implies that $x \to z(x)$ is α-Holder continuous in $V(x_0)$, hence we have $y(x) = B^{-1}(z(x))$ is $\alpha\beta$-Holder continuous on $V(x_0)$ because B^{-1} is β-Holder continuous on $S(z_0, \sigma)$. □

If we remove hypothesis (2.9), injectivity of B, from the Theorem 2.1, we obtain a similar theorem, which however cannot be properly called "Implicit Functions Theorem" because there is no uniqueness of the solution of problem (2.11).

THEOREM 2.3. *Let us suppose that*

(2.14) *there exists* $(x_0, y_0) \in X \times Y$ *such that* $F(x_0, y_0) = 0$*,*

(2.15) *the map* $x \to F(x, y_0)$ *is continuous in* $x = x_0$*,* (2.16) *there exist positive numbers* α, k, *with* $k \in (0,1)$, *and a neighbourhood of* x_0, $U(x_0) \subset X$, *such that for all* $y_1, y_2 \in Y$ *and all* $x \in U(x_0)$

$$\|B(y_1) - B(y_2) - \alpha[F(x,y_1) - F(x,y_2)]\| \leq k\|B(y_1) - B(y_2)\|,$$

(2.17) $B(Y)$ *is a neighbourhood of* $z_0 = B(y_0)$.

Then the following are true: there exist a ball $S(z_0,\sigma) \subset B(Y)$ *and a neighbourhood of* x_0, $V(x_0) \subset U(x_0)$, *such that for all* $x \in V(x_0)$ *there exists a subset* $G(x) \subset B^{-1}(S(z_0,\sigma))$ *where* $F(x,y) = 0$, *for all* $y \in G(x)$.

Proof. Let us set, as in the proof of Theorem 2.1, $\Phi(x,z) = F(x, B^{-1}(z))$, for all $z \in B(Y)$ and all $x \in U(x_0)$. Φ is well defined even if B is not invertible, in fact we observe that if $B(y_1) = B(y_2) = z$ then $F(x,y_1) = F(x,y_2)$, because (2.21) implies the following

$$\begin{aligned}\alpha\|F(x,y_1) - F(x,y_2)]\| &= \|B(y_1) - B(y_2) - \alpha[F(x,y_1) - F(x,y_2)]\| \\ &\leq k\|B(y_1) - B(y_2)\| = 0, \quad \forall x \in U(x_0).\end{aligned}$$

By proceeding as in the proof of Theorem 2.1 we can easily prove that Φ satisfies the hypotheses of Lemma 2.1; in particular, concerning hypothesis (2.3), by setting $z_1 = B(y_1)$ and $z_2 = B(y_2)$ we have

$$\begin{aligned}\|z_1 - z_2 - \alpha[\Phi(x,z_1) - \Phi(x,z_2)]\| &= \|B(y_1) - B(y_2) - \alpha[F(x,y_1) - F(x,y_2)]\| \\ &\leq k\|B(y_1) - B(y_2)\| = k\|z_1 - z_2\|,\end{aligned}$$

for all $x \in U(x_0)$ and all $z_1, z_2 \in B(Y)$. It follows that there exist $S(z_0,\sigma) \subset \Omega = B(Y)$ and $V(x_0) \subset U(x_0)$ such that for all $x \in V(x_0)$ there exists exactly one solution $z = z(x) \in S(z_0,\sigma)$ of the following

$$F(x, B^{-1}(z(x))) = \Phi(x, z(x)) = 0,$$

we set $G(x) = B^{-1}(z(x))$ and obtain the thesis. □

3. Comparison with other Implicit Function Theorems

Now let us compare Theorem 2.1 with two known Implicit Function Theorems: the classic Hildebrandt and Graves Theorem [7], and the recent Robinson Theorem [9]. We are going to prove that these theorems are particular cases of Theorem 2.1.

Lemma 3.1. *Let* X, Y, Z *be Banach spaces normed with* $\|\cdot\|_X$, $\|\cdot\|_Y$, $\|\cdot\|_Z$, *and* $F : U(x_0,y_0) \to Z$ *a function defined in a neighbourhood* $U(x_0,y_0) \subset X \times Y$ *of* (x_0,y_0), *which satisfies the following*

- (i) *there exists a partial* $\mathcal{F}$*-derivative* $F_y(x,y)$, *with respect to the second variable* y *in* $U(x_0,y_0)$, *continuous in* (x_0,y_0),
- (ii) $F_y(x_0,y_0) : Y \to Z$ *is bijective.*

Then there exists a neighbourhood of (x_0, y_0), $W(x_0, y_0) \subset U(x_0, y_0)$ *such that* $F_y(x, y) : Y \to Z$ *is bijective for all* $(x, y) \in W(x_0, y_0)$.

Proof. By Banach open mapping Theorem, the hypothesis (i) above on $F_y(x_0, y_0)$ implies that

$$\exists \delta > 0 : \ \|v\|_Y \le \frac{\|F_y(x_0, y_0)v\|_Z}{\delta}, \quad \forall v \in Y. \tag{3.1}$$

Moreover, from the continuity of $F_y(x, y)$ in (x_0, y_0), it follows that, for $\varepsilon \in (0, \delta)$, there exists $W(x_0, y_0)$ such that for all $(x, y) \in W(x_0, y_0)$ we have[3]

$$\|F_y(x_0, y_0)v - F_y(x, y)v\|_Z \le \|F_y(x_0, y_0) - F_y(x, y)\|_{\mathcal{L}(Y,Z)}\|v\|_Y \le \varepsilon\|v\|_Y. \tag{3.2}$$

From (3.2), (3.3), for $k = \varepsilon/\delta$, it follows that

$$\|F_y(x_0, y_0)v - F_y(x, y)v\|_Z \le k\|F_y(x_0, y_0)v\|_Z \quad \forall (x, y) \in W(x_0, y_0),\ \forall v \in Y.$$

Hence $F_y(x, y)$ is near $F_y(x_0, y_0)$, for all $(x, y) \in W(x_0, y_0)$, in Y (see Definition 1.1), with $\alpha = 1$. It follows that, for all $(x, y) \in W(x_0, y_0)$, $F_y(x, y)$ is bijective between Y and Z because so $F_y(x_0, y_0)$ is (see Theorem 1.1). □

Lemma 3.2. *Let us assume for* $F : U(x_0, y_0) \to Z$ *the hypotheses of Lemma* 3.1. *Moreover, let us suppose that there exists a neighbourhood of* (x_0, y_0), $U_1(x_0, y_0) \subset U(x_0, y_0)$, *where* $y \to F_y(x, y)$ *is continuous. Then there exist* r_1, $r_2 > 0$ *and* $k \in (0, 1)$ *such that* $S(x_0, r_1) \times S(y_0, r_2) \subset U_1(x_0, y_0)$ *and*

$$\|F_y(x, y_0)(y_1 - y_2) - [F(x, y_1) - F(x, y_2)]\|_Z \le k\|F_y(x, y_0)(y_1 - y_2)\|_Z,$$

for all $x \in S(x_0, r_1)$ *and all* $y_1, y_2 \in S(y_0, r_2)$.

Proof. By Lemma 3.1 there exists a neighbourhood of (x_0, y_0), $W(x_0, y_0) \subset U(x_0, y_0)$ where $F_y(x, y)$ is bijective. We set $W_1(x_0, y_0) = W(x_0, y_0) \cap U_1(x_0, y_0)$. Let $S(x_0, \sigma_1)$ and $S(y_0, \sigma_2)$ be such that $S(x_0, \sigma_1) \times S(y_0, \sigma_2) \subset W_1(x_0, y_0)$. Then $F_y(x, y_0)$ is bijective for all $x \in S(x_0, \sigma_1)$, while $t \to F_y(x, y_1 + t(y_2 - y_1))$ is continuous in $[0, 1]$, for all $x \in S(x_0, \sigma_1)$ and all y_1, $y_2 \in S(y_0, \sigma_2)$. Then we can consider, for all $x \in S(x_0, \sigma_1)$ and for all y_1, $y_2 \in S(y_0, \sigma_2)$, the following[4]

$$\begin{aligned} &\|F_y(x, y_0)(y_1 - y_2) - [F(x, y_1) - F(x, y_2)]\|_z \\ &\quad = \left\|F_y(x, y_0)(y_1 - y_2) - \left[\int_0^1 F_y(x, y_2 + t(y_1 - y_2))dt\right](y_1 - y_2)\right\|_Z \end{aligned} \tag{3.3}$$

[3] $\|\cdot\|_{\mathcal{L}(Y,Z)}$ is the norm in the space of linear operators between Y and Z.
[4] I_z is the identity function on Z.

$$= \left\| \left(I_Z - \left[\int_0^1 F_y(x, y_2 + t(y_1 - y_2)) dt \right] [F_y(x, y_0)]^{-1} \right) \cdot F_y(x, y_0)(y_1 - y_2) \right\|_Z$$

$$\leq \left\| I_Z - \left[\int_0^1 F_y(x, y_2 + t(y_1 - y_2)) dt \right] [F_y(x, y_0)]^{-1} \right\|_{\mathcal{L}(Z,Z)} \cdot \|F_y(x, y_0)(y_1 - y_2)\|_Z .$$

The above inequality implies the thesis of the lemma if we find $k \in (0, 1)$ such that for all x and for all y_1, y_2 belonging to suitable neighbourhoods, respectively, of x_0 and y_0, it yields

$$\mathcal{M}(x, y_1, y_2) = \left\| I_Z - \left[\int_0^1 F_y(x, y_2 + t(y_1 - y_2)) dt \right] [F_y(x, y_0)]^{-1} \right\|_{\mathcal{L}(Z,Z)} \leq k. \tag{3.4}$$

Set $y = [F_y(x, y_0)]^{-1} z$; (3.4) is equivalent to the following

$$\mathcal{M}(x, y_1, y_2) = \sup_{\substack{y \in Y \\ y \neq 0}} \frac{\left\| F_y(x, y_0) y - \left[\int_0^1 F_y(x, y_2 + t(y_1 - y_2)) dt \right] y \right\|_Z}{\|F_y(x, y_0) y\|_Z} \leq k.$$

We observe that for all $\varepsilon > 0$ there exist $\rho \in (0, \sigma_1)$ and $r_2 \in (0, \sigma_2)$ such that:

$$\begin{aligned} &\left\| F_y(x, y_0) - \int_0^1 F_y(x, y_2 + t(y_1 - y_2))\, dt \right\|_{\mathcal{L}(Y,Z)} \\ &= \left\| \int_0^1 [F_y(x, y_2 + t(y_1 - y_2)) - F_y(x, y_0)]\, dt \right\|_{\mathcal{L}(Y,Z)} \\ &\leq \int_0^1 \left\| [F_y(x, y_2 + t(y_1 - y_2)) - F_y(x, y_0)] \right\|_{\mathcal{L}(Y,Z)} dt \\ &\leq \int_0^1 \left\| [F_y(x, y_2 + t(y_1 - y_2)) - F_y(x_0, y_0)] \right\|_{\mathcal{L}(Y,Z)} dt \\ &\quad + \left\| F_y(x_0, y_0) - F_y(x, y_0) \right\|_{\mathcal{L}(Y,Z)} < \varepsilon, \end{aligned} \tag{3.5}$$

for all $x \in S(x_0, \rho)$ and for all $y \in S(y_0, r_2)$ the last inequality follows from continuity of $F_y(x, y)$ in (x_0, y_0). By Banach open mapping Theorem, the given hypothesis on $F_y(x_0, y_0)$ implies that there exist $\delta > 0$, $\varepsilon_1 \in (0, \delta)$ and $r \in (0, \sigma_1)$ such that

$$\begin{aligned} \delta \|y\|_Y &\leq \|F_y(x_0, y_0) y\|_Z \\ &\leq \|F_y(x_0, y_0) - F_y(x, y_0)\|_{\mathcal{L}(Y,Z)} \|y\|_Y + \|F_y(x, y_0) y\|_Z \\ &\leq \varepsilon_1 \|y\|_Y + \|F_y(x, y_0) y\|_Z, \quad \forall y \in Y,\ \forall x \in S(x_0, r). \end{aligned} \tag{3.6}$$

From (3.4), (3.5) and (3.6), choosing $\varepsilon \in (0, \delta - \varepsilon_1)$, we obtain that there exist ρ, r, $r_2 > 0$ such that:

$$\mathcal{M}(x, y_1, y_2) \leq \varepsilon \sup_{\substack{y \in Y \\ y \neq 0}} \frac{\|y\|_Y}{\|F_y(x, y_0)y\|_Z} \leq \frac{\varepsilon}{\delta - \varepsilon_1} < 1, \tag{3.7}$$

for all $x \in S(x_0, r_1)$ (with $r_1 = \min(r, \rho)$), for all y_1, $y_2 \in S(y_0, r_2)$. Thus the proof is completed. □

We obtain the following result as a particular case of Lemma 3.2.

PROPOSITION 3.1. *Let $A : V(y_0) \to Z$, where $V(y_0) \subset Y$ is a neighbourhood of y_0. We assume that $A \in C^1(V(y_0))$ and $A'(y_0)$ is bijective between Y and Z. Then there exists $\sigma > 0$ such that $S(y_0, \sigma) \subset V(y_0)$ and A is near $A'(y_0)$ in $S(y_0, \sigma)$ (see the Definition 1.1).*

LEMMA 3.3. *Let $F : U(x_0, y_0) \to Z$ be such that*

(i) *there exists the partial $\mathcal{F}$-derivative $F_y(x, y)$, with respect to the second variable y in $U(x_0, y_0)$, and it is continuous in (x_0, y_0),*
(ii) *$F_y(x_0, y_0) : Y \to Z$ is bijective,*
(iii) *there exists a neighbourhood of (x_0, y_0), $U_1(x_0, y_0) \subset U(x_0, y_0)$, where $y \to F_y(x, y)$ is continuous.*

Then there exists ρ_1, $\rho_2 > 0$ and $k \in (0, 1)$ such that $S(x_0, \rho_1) \times S(y_0, \rho_2) \subset U_1(x_0, y_0)$ and for all $x \in S(x_0, \rho_1)$ and all $y_1, y_2 \in S(y_0, \rho_2)$

$$\|F_y(x_0, y_0)(y_1 - y_2) - [F(x, y_1) - F(x, y_2)]\|_Z \leq k\|F_y(x_0, y_0)(y_1 - y_2)\|_Z.$$

PROOF.

$$\begin{aligned} &\|F_y(x_0, y_0)(y_1 - y_2) - [F(x, y_1) - F(x, y_2)]\|_Z \\ &\qquad \leq \|[F_y(x_0, y_0) - F_y(x, y_0)](y_1 - y_2)\|_Z \\ &\qquad\quad + \|F_y(x, y_0)(y_1 - y_2) - [F(x, y_1) - F(x, y_2)]\|_Z. \end{aligned}$$

Hence, by Lemma 3.2, there exist $r_1, r_2 > 0$ and $k_1 \in (0, 1)$ such that

$$S(x_0, r_1) \times S(y_0, r_2) \subset U_1(x_0, y_0)$$

and

$$\begin{aligned} &\|F_y(x_0, y_0)(y_1 - y_2) - [F(x, y_1) - F(x, y_2)]\|_Z \\ &\leq \|[F_y(x_0, y_0) - F_y(x, y_0)](y_1 - y_2)\|_Z + k_1\|F_y(x, y_0)(y_1 - y_2)\|_Z \\ &\leq (k_1 + 1)\|[F_y(x_0, y_0) - F_y(x, y_0)](y_1 - y_2)\|_Z \\ &\quad + k_1\|F_y(x_0, y_0)(y_1 - y_2)\|_Z, \quad \forall x \in S(x_0, r_1),\ \forall y_1, y_2 \in S(y_0, r_2). \end{aligned} \tag{3.8}$$

From (ii), by Banach open mapping Theorem, there exists $\delta > 0$ such that:

$$\|y\|_Y \leq \delta\|F_y(x_0, y_0)y\|_Y, \quad \forall y \in Y. \tag{3.9}$$

From (3.8) and (3.9), choosing $\varepsilon < 1 - k_1/\delta(1+k_1)$ and using (i), we know that there exist ρ_1 and $\rho_2 > 0$, with $\rho_1 \le r_1$, $\rho_2 \le r_2$, such that

$$\begin{aligned}\|F_y(x_0,y_0)(y_1-y_2) - [F(x,y_1)-F(x,y_2)]\|_Z \\ &\le \varepsilon(k_1+1)\|y_1-y_2\|_Y + k_1\|F_y(x_0,y_0)(y_1-y_2)\|_Z \\ &\le [\varepsilon\delta(k_1+1)+k_1]\|F_y(x_0,y_0)(y_1-y_2)\|_Z,\end{aligned}$$

for all $x \in S(x_0,\rho_1)$ and all $y_1,\ y_2 \in S(y_0,\rho_2)$. Hence we complete the proof choosing $k = \varepsilon\delta(k_1+1)+k_1$. □

We prove the following Hildebrandt–Graves Theorem by means of Theorem 2.1.

Theorem 3.1. *Let $F : U(x_0,y_0) \to Z$ be defined in an open neighbourhood of (x_0,y_0), $U(x_0,y_0) \subset X \times Y$, with $F(x_0,y_0) = 0$, which satisfies the following hypotheses*
(3.10) *F is continuous in (x_0,y_0),*
(3.11) *there exists the partial $\mathcal{F}$-derivative F_y in $U(x_0,y_0)$,*
(3.12) *$F_y(x_0,y_0) : Y \to Z$ is bijective,*
(3.13) *$y \to F_y(x,y)$ is continuous in a open neighbourhood of (x_0,y_0), $U_1(x_0,y_0) \subset U(x_0,y_0)$.*
Then there exist $\sigma_1,\ \sigma_2 > 0$, such that there is exactly one solution $y = y(x) : S(x_0,\sigma_1) \to S(y_0,\sigma_2)$ of the following problem

$$\begin{cases} F(x,y(x)) = 0 \quad \forall x \in S(x_0,\sigma_1), \\ y(x_0) = y_0. \end{cases} \tag{3.14}$$

Moreover, the solution of problem (3.14) is continuous in $S(x_0,\sigma_1)$.

Proof. It follows by proving that the hypotheses of Theorem 2.1 hold true. We set $B = F_y(x_0,y_0)$. Lemma 3.3 implies that there exist $\rho_1,\ \rho_2 > 0$ and $k \in (0,1)$ such that $S(x_0,\rho_1) \times S(y_0,\rho_2) \subset U_1(x_0,y_0)$ and

$$\begin{aligned}\|B(y_1)-B(y_2) - [F(x,y_1)-F(x,y_2)]\|_Z \\ &\le \|F_y(x_0,y_0)(y_1-y_2) - [F(x,y_1)-F(x,y_2)]\|_Z \\ &\le k\|F_y(x_0,y_0)(y_1-y_2)\|_Z \\ &= \|B(y_1)-B(y_2)\|_Z, \quad \forall x \in S(x_0,\rho_1),\ \forall y_1,y_2 \in S(y_0,\rho_2).\end{aligned}$$

Hence the hypothesis (2.8) is verified by setting $Y = S(y_0,\rho_2)$. Moreover, (3.12) implies that B is injective. Finally, the Banach open mapping theorem and (3.12) imply that $B(Y)$ is a neighbourhood of $z_0 = 0$. To sum up, (iii) and Theorem 2.2 imply that the solution $y = y(x)$ is continuous. □

Finally, we deduce also the Robinson Theorem (see [9], Theorem 3.2) from Theorem 2.1.

THEOREM 3.2[5]. *Let X, Y be normed spaces, Z be a Banach space, $(x_0, y_0) \in X \times Y$, $U(x_0)$ be a neighbourhood of x_0 in X, $V(y_0)$ be a neighbourhood of y_0 in Y.*

Let $F : U(x_0) \times V(y_0) \to Z$ be such that $F(x_0, y_0) = 0$, and $f : V(y_0) \to Z$ be such that $f(y_0) = 0$. Moreover, we suppose that
(3.15) *$f \approx_y F$ in (x_0, y_0)*[6].
(3.16) *For all $y \in V(y_0)$, $x \to F(x, y)$ is Lipschitzian in $U(x_0)$ with modulus ϕ.*
(3.17) *$f(V(y_0))$ is a neighbourhood of 0 in Z.*
(3.18) *$\delta(f, V(y_0)) = d_0 > 0$*[7].
Then there exist two neighbourhoods of x_0 and y_0, respectively, $U_1(x_0) \subset U(x_0)$ and $V_1(y_0) \subset V(y_0)$, such that there exist only one solution $y = y(x) : U_1(x_0) \to V_1(y_0)$ of the following problem:

$$\begin{cases} F(x, y(x)) = 0 \quad \forall x \in U_1(x_0), \\ y(x_0) = y_0. \end{cases} \tag{3.19}$$

Moreover, for all $\lambda > \phi/d_0$, there exists a neighbourhood x_0, $U_2(x_0) \subset U_1(x_0)$, such that y is Lipschitzian on $U_2(x_0)$ with modulus λ.

PROOF. It follows by verifying in turn each of the hypotheses of Theorem 2.1. Setting $B(y) = f(y)$, we observe that (3.17) above implies that $B(Y)$ is a neighbourhood of 0 in Z. Moreover, (3.18) above implies that

$$\|f(y_1) - f(y_2)\|_Z \geq d_0 \|y_1 - y_2\|_Y, \quad \forall y_1, y_2 \in V(y_0).$$

Hence f is injective and therefore B is injective on $V(y_0)$.

It remains to prove that B verifies the hypothesis (2.8). If we choose $\varepsilon \in (0, d_0)$, by (3.15) above, there exist $\mathcal{U}(x_0)$ and $\mathcal{V}(y_0)$ such that, for all $x \in \mathcal{U}(x_0)$ and for all $y \in \mathcal{V}(y_0)$, by (3.18) we have:

$$\begin{aligned} \|B(y_1) - B(y_2) &- [F(x, y_1) - F(x, y_2)]\|_Z \\ &= \|f(y_1) - f(y_2) - [F(x, y_1) - F(x, y_2)]\|_Z \\ &\leq \varepsilon \|y_1 - y_2\|_Y \leq \frac{\varepsilon}{d_0} \|f(y_1) - f(y_2)\|_Z \\ &= \frac{\varepsilon}{d_0} \|B(y_1) - B(y_2)\|_Z. \end{aligned}$$

Setting $Y = \mathcal{V}(y_0)$, we verify hypothesis (2.8) with $k = \varepsilon/d_0$. Thus Theorem 2.1 implies the existence and uniqueness of the solution of problem (3.19). From

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[5]We remark that in the Theorem proved in [9] Y is a Banach space and Z is a normed space.

[6]We say that f strongly approximates F, with respect to y, at (x_0, y_0) (written: $f \approx_y F$ in (x_0, y_0)) if for all $\varepsilon > 0$ there exist two neighbourhoods of x_0 and y_0, respectively, $\mathcal{U}(x_0)$ and $\mathcal{V}(y_0)$, such that: $\|f(y_1) - f(y_2) - [F(x, y_1) - F(x, y_2)]\|_Z \leq \varepsilon \|y_1 - y_2\|_Y$, for all $x \in \mathcal{U}(x_0)$ and all $y_1, y_2 \in \mathcal{V}(y_0)$.

[7]$\delta(f, V(y_0)) = \inf\{\|f(y_1) - f(y_2)\|_Z / \|y_1 - y_2\|_Y,\ y_1 \neq y_2,\ y_1, y_2 \in V(y_0)\}$

the hypothesis (3.18) above, it follows that $B^{-1} = f^{-1}$ is Lipschitzian, while for (3.16) above also $x \to F(x, y)$ is Lipschitzian. Hence by (iv) of Theorem 2.2 we obtain that $y = y(x)$ is Lipschitzian in a neighbourhood of x_0. We make the calculation of Lipschitz modulus λ by using hypotheses (3.15), (3.16) and (3.18) and by proceeding in the same way as in the proof of Theorem 3.2 of [9]. □

4. Some examples

Example 4.1. (An application of Implicit Function Theorem to a class of non variational elliptic systems).

Let $g(x, v)$ be a map, $\Omega \times \mathbb{R}^N \to \mathbb{R}^N$, measurable in x and continuous in v with the following properties

(4.1) $g(x, 0) = 0$ a.e. in Ω.

(4.2) There exists a real constant c, with $c < \lambda_0$ (where λ_0 is the first eigenvalue of the Laplace operator $-\Delta$) such that, for all $v, w \in \mathbb{R}^N$

$$0 \leq (g(x,v) - g(x,w) | v - w)_N \quad \text{a.e. in } \Omega,$$
$$\|g(x,v) - g(x,w)\|_N \leq c\|v - w\|_N \quad \text{a.e. in } \Omega.$$

We consider the following problem: given $f : \Omega \to \mathbb{R}^N$, find u such that

$$\begin{cases} u \in H^2 \cap H^1_0(\Omega, \mathbb{R}^N), \\ a(x, H(u)) + g(x, u) = f(x), \quad \text{a.e. in } \Omega. \end{cases} \tag{4.3}$$

We use Theorem 2.1 for solving this problem and prove the following

Proposition 4.1. *Let us assume that conditions* (1.2)–(1.3) *on* a *and* (4.1)–(4.2) *on* g *hold, with* $c < 1 - (\gamma + \delta)/\alpha\lambda_0$, *if* $f \in L^2(\Omega, \mathbb{R}^N)$ *then problem* (4.3) *has one and only one solution.*

We are going to use the notations of Theorem 2.1 and set

$$\begin{aligned} F(f,u) &= a(x, H(u)) + g(x,u) - f, & X &= L^2(\Omega, \mathbb{R}^N), \\ B(u) &= \Delta u + \alpha g(x,u), & Y &= H^2 \cap H^1_0(\Omega, \mathbb{R}^N), \\ C(u) &= \Delta u, & \mathcal{B} &= L^2(\Omega, \mathbb{R}^N). \end{aligned}$$

The proof of Proposition 4.1 is preceeded by the following Lemmas.

Lemma 4.1. *If* $\alpha c < \lambda_0$ *then* B *is near to* C *in* X, *i.e.*

$$\|C(u) - C(v) - [B(u) - B(v)]\|_{\mathcal{B}} \leq k_1 \|C(u) - C(v)\|_{\mathcal{B}}, \quad \forall u, v \in X, \tag{4.4}$$

where $k_1 = \alpha c / \lambda_0 < 1$.

PROOF. By (4.2) we have

$$\|C(u) - C(v) - [B(u) - B(v)]\|_{\mathcal{B}}^2 = \alpha^2 \int_\Omega \|g(x,u) - g(x,v)\|_N^2 \, dx \\ \le \alpha^2 c^2 \int_\Omega \|u - v\|_N^2 \, dx.$$

From the following known inequalities and from (4.5) the results follow if $\alpha c < \lambda_0$:

$$\lambda_0 \int_\Omega \|u\|_N^2 dx \le \int_\Omega \|Du\|_{Nn}^2 \, dx, \quad \lambda_0 \int_\Omega \|Du\|_{Nn}^2 dx \le \int_\Omega \|\Delta u\|_N^2 dx. \qquad \square$$

LEMMA 4.2. *If $\alpha c < \lambda_0$ and $u, v \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ then:*

$$\int_\Omega \|\Delta(u - v)\|_N^2 dx \le k_2(c) \int_\Omega \|\Delta(u - v) + \alpha[g(x,u) - g(x,v)]\|_N^2 \, dx \tag{4.6}$$

where $k_2(c) = \lambda_0^2/(\lambda_0 - \alpha c)^2$.

PROOF. From (4.4) we obtain:

$$\|C(u) - C(v)\|_{\mathcal{B}} \le \|C(u) - C(v) - [B(u) - B(v)]\|_{\mathcal{B}} + \|B(u) - B(v)\|_{\mathcal{B}} \\ \le k_1 \|C(u) - C(v)\|_{\mathcal{B}} + \|B(u) - B(v)\|_{\mathcal{B}}.$$

From this the result follows easily. $\square$

LEMMA 4.3. *If $u \in H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ then (see* [9], [11])

$$\int_0 \|H(u)\|_{n^2 N}^2 \, dx \le \int_\Omega \|\Delta u\|_N^2 \, dx. \tag{4.7}$$

PROOF OF PROPOSITION 4.1. $F(f,u)$ and B satisfy the assumptions of Theorem 2.1. In particular by Theorem 1.1, B is bijective between Y and $\mathcal{B}$. Indeed, by Lemma 4.1, B is near C, which is bijective between Y and $\mathcal{B}$. It remains to prove the nearness hypothesis (2.8). If $f \in L^2(\Omega, \mathbb{R}^N)$ and $u \in H^2(\Omega) \cap H_0^1(\Omega, \mathbb{R}^N)$ then by Condition A and Lemmas 4.3[8] and 4.2 we have

$$\begin{aligned}
&\|B(u) - B(v) - \alpha[F(f,u) - F(f,v)\|_{\mathcal{B}}^2 \\
&\quad = \int_\Omega \|\Delta(u - v) - \alpha[a(x, H(u)) - a(x, H(v))]\|_N^2 \, dx \\
&\quad \le \int_\Omega (\gamma \|H(u) - H(v)\|_{n^2 N} + \delta \|\Delta(u - v)\|_N)^2 \, dx \\
&\quad \le \gamma(\gamma + \delta) \int_\Omega \|H(u) - H(v)\|_{n^2 N} dx + \delta(\gamma + \delta) \int_\Omega \|\Delta(u - v)\|_N^2 \, dx \\
&\quad \le (\gamma + \delta)^2 \int_\Omega \|\Delta(u - v)\|_N^2 \, dx \\
&\quad \le (\gamma + \delta)^2 k_2(c) \int_\Omega \|\Delta(u - v) + \alpha[g(x,u) - g(x,v)]\|_N^2 \, dx.
\end{aligned}$$

[8]It follows from: $(\gamma G + \delta D)^2 \le \gamma(\gamma + \delta) G^2 + \delta(\gamma + \delta) D^2$ for all $G, D \in \mathbb{R}$, for all $\gamma, \delta \in \mathbb{R}^+$.

If $c < \lambda_0 1 - (\gamma+\delta)/\alpha$ then $(\gamma+\delta)^2 k_2(c) = (\gamma+\delta)^2\lambda_0^2/(\lambda_0 - \alpha c)^2 < 1$. The thesis of Proposition follows from Theorem 2.1 and Remark 2.3. $\square$

Remark 4.1. We also obtain the following known proposition by Lemma 4.1 and Theorem 2.1 (or Theorem 1.1): if (4.1), (4.2) hold in g, then $\Delta u + g(x,u)$ is bijective between $H^2 \cap H_0^1(\Omega, \mathbb{R}^N)$ and $L^2(\Omega, \mathbb{R}^N)$.

Remark 4.2. From Proposition 4.1 it also follows: if $c < 1 - (\gamma+\delta)/\alpha\lambda_0$ then there are no bifurcation points for the operator $G(\lambda, u) = a(x, H(u)) + \lambda u$ when $\lambda < c$.

Example 4.2 (An open mapping Theorem). Let $\mathcal{X}$ be a set and $\mathcal{B}$ be a Banach space with norm $\|\cdot\|$; let $A, B : \mathcal{X} \to \mathcal{B}$. We prove the following open mapping theorem.

Theorem 4.1. *Let A be near B in $\mathcal{X}$. Let $y_0 \in \mathcal{X}$ such that $B(\mathcal{X})$ is a neighbourhood of $B(y_0)$ then $A(\mathcal{X})$ is a neighbourhood of $A(y_0)$.*

Proof. With the notation of Section 2 we are going to apply Theorem 2.3:

$$\begin{gathered} X = \mathcal{B}, \quad Y = \mathcal{X}, \quad Z = \mathcal{B}, \\ F(f,y) = A(y) - f, \quad f \in X,\ y \in Y, \\ A(y_0) = f_0,\ (\text{then } F(f_0,y_0) = 0), \quad B(y_0) = z_0, \\ f \to F(f, y_0) \text{ is continuous.} \end{gathered}$$

It remains to prove hypothesis (2.8):

$$\begin{aligned} &\|B(y_1) - B(y_2) - \alpha[F(f,y_1) - F(f,y_2)]\| \\ &\qquad = \|B(y_1) - B(y_2) - \alpha[A(y_1) - A(y_2)]\| \le k\|B(y_1) - B(y_2)\|. \end{aligned}$$

The hypotheses of Theorem 2.3 are proved. Hence there exist $S(z_0,\sigma) \subseteq B(\mathcal{X})$ and a neighbourhood $V(f_0) \subseteq \mathcal{B}$ such that for all $f \in V(f_0)$ there exists a subset $G(f) \subseteq B^{-1}S(z_0,\sigma)$, where $F(f,y) = 0$, for all $y \in G(f)$. Thus: $A(y) - f = 0$. So the neighbourhood $V(f_0) \subseteq A(\mathcal{X})$. $\square$

Example 4.3. (A operator near to identity on $L^2(\Omega)$ but not F-differentiable). Let $A : L^2(\Omega) \to L^2(\Omega)$ be such that $A(u) = f(u(x))$, where $f(t) = t(1 + \operatorname{arctg} t^2/2)$, $t \in \mathbb{R}$. It is trivial to prove that there exists $k \in (0,1)$ such that: $\|u - v - [A(u) - A(v)]\|_{L^2(\Omega)} \le k\|u - v\|_{L^2(\Omega)}$, for all $u, v \in L^2(\Omega)$. A is near identity on $L^2(\Omega)$ but is not F-differentiable on $L^2(\Omega)$, because:

$$|f(t)| \le \left(1 + \frac{\pi}{4}\right)|t|, \quad |f'(t)| \le 2. \tag{4.8}$$

Indeed, if f satisfies (4.8) and A is F-differentiable then A must be linear (see, for example, [1, Theorem 3.6]).

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