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ON THE SOLVABILITY OF A RESONANT ELLIPTIC EQUATION WITH ASYMMETRIC NONLINEARITY

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1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$. In this paper we study the existence of the solution for the elliptic equation with Dirichlet boundary condition

$$-\Delta u = \alpha u^+ - \beta u^- + g(x,u), \quad u \in H_0^1(\Omega), \tag{1.1}$$

where α, β are real parameters and $u^+ = \max\{u, 0\}$, $u^- = u^+ - u$. Without loss of generality, we assume $\beta \leq \alpha$. In fact, denoting by (λ_i) the increasing sequence of eigenvalues of $(-\Delta, H_0^1(\Omega))$, we study the case where $\lambda_1 < \beta < \alpha$ and $[\beta, \alpha]$ intersects this linear spectrum. Here $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function with *subcritical growth* at infinity, namely $|g(x,s)| \leq A(|s|^{p-1} + 1)$ with $1 < p < 2N/(N-2)$ if $N \geq 2$. If $N = 1$, we merely suppose that $|g(x,s)| \leq a(x) + b(x)f(s)$ where $a, b \in L^1(\Omega)$, f is continuous and $f(s) = \mathrm{O}(s)$ near 0.

We consider nonlinear terms which are sublinear at infinity, in a sense to be made precise below (see (2.1)). It is well-known that then the existence and multiplicity of solutions of (D) strongly rely on the position of the pair

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$(\alpha,\beta)\in\mathbb{R}^2$ with respect to the so called Fučik spectrum of $(-\Delta, H^1_0(\Omega))$. The latter is defined as

$$\Sigma := \{(\mu,\nu)\in\mathbb{R}^2 : \exists u\in H^1_0(\Omega),\ u\neq 0,\ -\Delta u = \mu u^+ - \nu u^-\}. \tag{1.2}$$

It is clear that Σ contains the lines $\mathbb{R}\times\{\lambda_1\}$ and $\{\lambda_1\}\times\mathbb{R}$ as well as the points (λ_i,λ_i), $i\geq 1$. In the one dimensional case $N=1$, the set Σ can be easily described (see e.g [12]). For higher dimensions, some properties of Σ were obtained by several authors, see [1], [3], [6], [8], [10], [13], [16], [18], [19], [22], [25]. For results concerning the solvability of (1.1) and without being exhaustive, we refer to [3]–[7], [9], [14], [17], [18], [20], [24] and especially to [21]–[23].

In particular, it was first observed by Kavian [16] that Σ contains a global curve C_2 with crosses (λ_2,λ_2). Some qualitative properties of C_2 are also known, see [10]. The first variational characterization of C_2 in terms of the associated energy functional was already presented in [16], through a variant of the well-known mountain pass theorem of Ambrosetti and Rabinowitz. This variational characterization was somewhat clarified in [5, Lemma 4.3] and [11, Proposition 3.2].

The present paper is motivated by a result of Costa and Cuesta [4] where the authors consider (1.1) with $(\alpha,\beta)\in C_2$. As in [4], we find solutions for (1.1) as critical points of the C^1 energy functional defined by

$$E(u) := \frac{1}{2}\int_\Omega [|\nabla u|^2 - \alpha(u^+)^2 - \beta(u^-)^2] - \int_\Omega G(x,u), \quad u\in H^1_0(\Omega),$$

where $G(x,s) := \int_0^s g(x,\xi)\,d\xi$. Due to the resonance of the problem (i.e. the fact that $(\alpha,\beta)\in\Sigma$ and g is sublinear at infinity) the usual Palais–Smale condition is not satisfied. Hence the authors assume that $G(x,s)$ is *nonquadratic at infinity*, in the sense that either $(NQ)_+$ or $(NQ)_-$ below holds:

$$(NQ)_\pm \qquad \lim_{|s|\to\infty}(sg(x,s) - 2G(x,s)) = \pm\infty \quad \text{uniformly for a.e. } x\in\Omega.$$

We refer to [4] for a discussion and examples concerning this kind of nonlinearities. The point is that under $(NQ)_+$ or $(NQ)_-$ the so called *Cerami condition* (cf. [2]) holds for E, namely any sequence $(u_n)\subset H^1_0(\Omega)$ with $(E(u_n))$ bounded and $(1+\|u_n\|)\|\nabla E(u_n)\| = \mathrm{o}\,(1)$ has a convergent subsequence (see [4, Lemma 2.2]). We denote by $\|\cdot\|$ the $H^1_0(\Omega)$-norm. This key observation, together with the above mentioned characterization of C_2, enabled the quoted authors to prove an existence result for (1.1) in case $(NQ)_+$ holds.

Here we concentrate on the case where $(NQ)_-$ holds. The difficulties arising from this assumption, even in the one dimensional case $N=1$, were already pointed out in [4, Section 4]. Roughly speaking, our main assumption concerns the existence of a path $c(t)$ connecting $c(0)=(\alpha,\beta)$ with some eigenpair $c(1)=(\lambda_k,\lambda_k)$ in such a way that a delected "upper neighbourhood" of $c([0,1])$ does not intersect Σ. We stress that we allow $c([0,1]) \subset \Sigma$, see Definition 2.1 and Section 3 for further comments and examples. In this way we are able to refine our previous arguments in [9] and to provide a solution for (1.1).

In Section 2 we state and prove our main result. In Section 3 we discuss three typical situations in which our main assumption holds. We also prove an existence result for (1.1) in case $(NQ)_+$ holds which extends [4, Theorem 1]. Still under assumption $(NQ)_-$, we state in Section 3 an existence theorem for an ordinary differential equation with periodic boundary conditions related to (1.1), which improves [4, Theorem 2].

2. Main result

We consider problem (1.1) with g having subcritical growth at infinity. Moreover, we assume that

$$\lim_{|s|\to\infty} G(x,s)/s^2 = 0 \quad \text{uniformly for a.e. } x \in \Omega. \tag{2.1}$$

Our assumption on (α,β) is expressed in the following definition. Let $(\alpha,\beta) \in \mathbb{R}^2$ be such that $\lambda_1 < \beta < \alpha$.

Definition 2.1. We say that (α,β) is Σ-*connected* to (λ_k,λ_k), $k \geq 2$, if there exist $d > 0$ and a C^1 function $c : [0,1] \to \mathbb{R}^2$ satisfying $c(0) = (\lambda_k,\lambda_k)$, $c(1) = (\alpha,\beta)$ and

$$\xi c([0,1]) \cap \Sigma = \emptyset \qquad \text{for every } \xi \in \,]1, 1+d]\,.$$

We explicitely note that we allow c to intersect Σ. In fact, in a typical situation (see Section 3) we have $c([0,1]) \subset \Sigma$. On the other hand, we suppose that we do not meet Σ when we slightly "lift up" $c([0,1])$. We observe also that despite the fact that we are mostly concerned with the case where $(\alpha,\beta) \in \Sigma$ we do not assume this in Definition 2.1.

Theorem 2.2. *We consider* (1.1) *with g satisfying both* $(NQ)_-$ *and* (2.1). *If (α,β) is Σ-connected to (λ_k,λ_k) for some $k \geq 2$ then* (1.1) *admits a solution.*

The rest of the section is devoted to the proof of Theorem 2.2. Let $c(t) = (\alpha(t),\beta(t))$ be the path given by Definition 2.1. For any $t \in [0,1]$, we introduce the C^1 functionals over $H_0^1(\Omega)$,

$$Q(t,u) := \frac{1}{2}\int_\Omega [|\nabla u|^2 - \alpha(t)(u^+)^2 - \beta(t)(u^-)^2],$$

and

$$E(t,u) := Q(t,u) - \int_\Omega G(x,u), \quad E(u) = E(1,u).$$

It is well-know that critical points of E in $H_0^1(\Omega)$ are weak solutions of problem (1.1). We consider the orthogonal direct sum

$$H_0^1(\Omega) = H_1 \oplus H_2,$$

where H_1 is the finite dimensional eigenspace associated with the eigenvalues $\lambda_1, \ldots, \lambda_k$. Since $c(0) = (\lambda_k, \lambda_k)$, it is clear that

$$Q(0,u) \le 0 \ \forall u \in H_1 \quad \text{and} \quad Q(0,u) \ge \sigma \|u\|^2 \ \forall u \in H_2, \tag{2.2}$$

for some constant $\sigma > 0$. The estimate below describes our assumption on (α, β) in terms of the energy levels of the quadratic forms envolved.

Lemma 2.3. *There exist positive constants η, δ, $\eta < \sigma$, with the following property: for any $t \in [0,1]$ and $u \in H_0^1(\Omega)$, $\|u\| = 1$,*

$$Q(t,u) \in [\eta/2, \eta] \Rightarrow \|\nabla Q(t,u)\|^2 - (\nabla Q(t,u)u)^2 \ge \delta.$$

Proof. Let d be given by definition 2.1 and denote

$$\eta := \min\{d/3(d+1), \sigma/2\}.$$

We suppose by contradiction that for some sequence $(t_n) \subset [0,1]$ and $(u_n) \subset H_0^1(\Omega)$ with $\|u_n\| = 1$ it holds

$$\eta/2 \le Q(t_n, u_n) \le \eta \quad \text{and} \quad \|\nabla Q(t_n,u_n)\|^2 - (\nabla Q(t_n,u_n)u_n)^2 = o(1),$$

as $n \to \infty$. We denote $\mu_n = \nabla Q(t_n,u_n)u_n = 2Q(t_n,u_n) \in [\eta, 2\eta]$. Since

$$\|\nabla Q(t_n,u_n) - \mu_n u_n\|^2 = \|\nabla Q(t_n,u_n)\|^2 - (\nabla Q(t_n,u_n)u_n)^2 = o(1),$$

we have, for every bounded sequence $(v_n) \subset H_0^1(\Omega)$,

$$(1-\mu_n)\int_\Omega \nabla u_n \nabla v_n - \alpha(t_n)\int_\Omega u_n^+ v_n + \int_\Omega \beta(t_n) u_n^- v_n = o(1). \tag{2.3}$$

Up to subsequences, let $\mu = \lim \mu_n \in [\eta, 2\eta]$, $t_0 = \lim t_n \in [0,1]$ and u be a weak limit of (u_n). Using (2.3) with $v_n = u_n$ we see that

$$(1-\mu) = \int_\Omega (\alpha(t_0)(u^+)^2 + \beta(t_0)(u^-)^2).$$

Since $\mu \le 2\eta < 1$, we deduce that $u \ne 0$. By using now (2.3) with arbitrary test functions v, we conclude that u is a nontrivial solution of the problem

$$-\Delta u = \frac{\alpha(t_0)}{1-\mu}u^+ - \frac{\beta(t_0)}{1-\mu}u^-, \quad u \in H_0^1(\Omega).$$

In particular, $(\alpha(t_0), \beta(t_0))/(1-\mu) \in \Sigma$. Since $\mu > 0$, the definition of d implies then that we must have $1/(1-\mu) \ge d+1$, that is $\mu \ge d/(d+1)$. This contradicts the fact that $\mu \le 2d/3(d+1)$. □

We will find a critical point for E through a limit process with an approximate sequence of functionals E_ε, $\varepsilon \to 0$. So let $\varepsilon \in]0, \eta/4[$. Proceeding as in the proof of Lemma 2.3 we see that there exists $\delta_\varepsilon > 0$ such that, for any $t \in [0,1]$ and $u \in H_0^1(\Omega)$, $\|u\| = 1$,

$$Q(t,u) \in [\varepsilon, 2\varepsilon] \Rightarrow \|\nabla Q(t,u)\|^2 - (\nabla Q(t,u)u)^2 \geq \delta_\varepsilon. \tag{2.4}$$

We can of course assume that $\delta_\varepsilon < \delta$. The above conlusions enable us to state a property similar to the one in (2.2) for all quadratic forms $Q(t,\,\cdot\,)$, $t \in [0,1]$, except that we replace the subspaces H_1 and H_2 in (2.2) with some convenient homeomorphic subsets of $H_0^1(\Omega)$. This homeomorphism is in turn given by the flow associated with the ordinary (but non autonomous) differential equation

$$\dot\sigma(t) = h(t,\sigma)\nabla Q(t,\sigma),$$

where $h : [0,1] \times H_0^1(\Omega) \to \mathbb{R}$ is an appropriate cut-off function and $\dot\sigma$ denotes the derivative $d\sigma/dt$. To make this idea precise, we denote by S the unit sphere in $H_0^1(\Omega)$ and introduce the closed disjoint sets

$$\begin{aligned} A_1 &= \{(t,u) \in [0,1] \times S : Q(t,u) \leq \varepsilon\}, \\ A_2 &= \{(t,u) \in [0,1] \times S : Q(t,u) \geq \eta/2\}. \end{aligned}$$

Let $\chi : [0,1] \times S \to [-1,1]$ be a continuous function such that $\chi = -1$ over A_1 and $\chi = 1$ over A_2. Namely, $\chi = \chi_1 - \chi_2$, with $\chi_i : [0,1] \times S \to [0,1]$ defined by

$$\chi_i(t,u) = \frac{\operatorname{dist}((t,u), A_i)}{\operatorname{dist}((t,u), A_1) + \operatorname{dist}((t,u), A_2)},$$

for $i = 1, 2$. It is clear that χ is locally Lipschitz continuous. We need a stronger property of χ.

Lemma 2.4. *Function χ is Lipschitz continous.*

Proof. We observe that in $[0,1] \times S$ both functions $f_i(t,u) = \operatorname{dist}((t,u), A_i)$ are bounded and Lipschitz continuous. Thus the conclusion follows easily once we show that

$$\inf_{[0,1]\times S} (f_1 + f_2) > 0.$$

Arguing by contradiction, if the above does not hold we find sequences $(t_n, u_n) \in A_1$, $(s_n, v_n) \in A_2$ such that $|t_n - s_n| \to 0$ and $\|u_n - v_n\| \to 0$. Passing to a subsequence and using the definitions of A_1 and A_2 together with the weak continuity of Q, we find some $(t, w) \in [0,1] \times H_0^1(\Omega)$ satisfying $\eta/2 \leq 1 - \alpha(t)\int_\Omega (w^+)^2 - \beta(t)\int_\Omega (w^-)^2 \leq \varepsilon$ and this is a contradiction. $\square$

Let $F : [0,1] \times H_0^1(\Omega) \to \mathbb{R}$ be given by

$$F(t,u) = \chi(t, u/\|u\|)\nabla Q(t,u) \text{ if } u \neq 0, \quad F(t,0) = 0.$$

Lemma 2.5. *Function F is locally Lipschitz continuous. Moreover, there exists $L > 0$ such that, for every $(t,u) \in [0,1] \times H_0^1(\Omega)$, $\|F(t,u)\| \le L\|u\|$.*

Proof. Our second statement in the lemma is a direct consequence of the analogous property for ∇Q. Now, let (t,u) and (s,v) be arbitrary in $[0,1]\times H_0^1(\Omega)$ with, say, $0 < \|u\| \le \|v\|$. In particular,

$$\|u/\|u\| - v/\|v\|\|\|u\| \le \|u - v\|. \tag{2.5}$$

It then follows from Lemma 2.4 and (2.5) that, for some $C > 0$,

$$\begin{aligned}\|F(t,u) - F(s,v)\| \le{}& |\chi(t,u/\|u\|) - \chi(s,v/\|v\|)|\|\nabla Q(t,u)\| \\ &+ |\chi(s,v/\|v\|)|\|\nabla Q(t,u) - \nabla Q(s,v)\| \\ \le{}& C(\|u-v\| + |t-s|\,\|u\|) + \|\nabla Q(t,u) - \nabla Q(s,v)\|.\end{aligned}$$

Since ∇Q is locally Lipschitz continuous, the lemma follows. □

Now, let $K = \sup\{|\alpha'(t)| + |\beta'(t)|,\ t \in [0,1]\}$ and S_0 be the Sobolev constant given by the continuous imbedding of $H_0^1(\Omega)$ into $L^2(\Omega)$. We fix any

$$M > KS_0^2\delta_\varepsilon^{-1} \tag{2.6}$$

and consider the Cauchy problem

$$\dot\sigma(t) = MF(t,\sigma(t)), \quad \sigma(0) = u \in H_0^1(\Omega). \tag{2.7}$$

It follows from Lemma 2.5 and standard arguments that (2.7) generates a continuous flow $\sigma : [0,1] \times H_0^1(\Omega) \to H_0^1(\Omega)$. Moreover, for any $t \in [0,1]$, $\sigma(t,\,\cdot\,)$ is a homeomorphism. Since $F(t,0) = 0$, the uniqueness of the Cauchy problem implies also that $\sigma(t,u) \ne 0$ whenever $t \in [0,1]$ and $u \ne 0$. For any non zero function in $H_0^1(\Omega)$, let $\Theta : [0,1] \to \mathbb{R}$ be given by

$$\Theta(t) = \frac{Q(t,\sigma(t,u))}{\|\sigma(t,u)\|^2}.$$

Lemma 2.6. *Function Θ is increasing (resp. decreasing) in any interval $[t_1,t_2]$ such that*

$$\eta/2 \le \Theta(t) \le \eta,\ \forall t \in [t_1,t_2] \quad (\textit{resp. } \varepsilon \le \Theta(t) \le 2\varepsilon,\ \forall t \in [t_1,t_2]).$$

Proof. Let us write $\sigma(t)$ for $\sigma(t,u)$. Since $Q(t,\,\cdot\,)$ is homogeneous we see that, by construction, σ satisfies

$$\dot\sigma(t) = M\nabla Q(t,\sigma(t))$$

over $[t_1, t_2]$. Using Lemma 2.3, (2.6) and the fact that $\nabla Q(t,v)v = 2Q(t,v)$ for any t, v, by a straightforward computation we show then that

$$\begin{aligned}\frac{d\Theta}{dt}(t) =& \|\sigma(t)\|^{-2}\bigg[\frac{\partial Q}{\partial t}(t,\sigma(t)) + \nabla Q(t,\sigma(t))\dot{\sigma}(t)\bigg] + Q(t,\sigma(t))\frac{d}{dt}(\|\sigma(t)\|^{-2}) \\ =& -2^{-1}\|\sigma(t)\|^{-2}\bigg[\alpha'(t)\int_\Omega(\sigma(t)^+)^2 + \beta'(t)\int_\Omega(\sigma(t)^-)^2\bigg] \\ &+ \|\sigma(t)\|^{-2}M\|\nabla Q(t,\sigma(t))\|^2 - M(\nabla Q(t,\sigma(t))\sigma(t))^2\|\sigma(t)\|^{-4} \\ \geq& -KS_0^2 + M(\|\nabla Q(t,v(t))\|^2 - (\nabla Q(t,v(t))v(t))^2) \\ \geq& -KS_0^2 + M\delta > 0,\end{aligned}$$

where we denoted $v(t) = \sigma(t)/\|\sigma(t)\|$. This proves the first statement in the lemma. The case where Θ lies in $[\varepsilon, 2\varepsilon]$ follows from a similar argument by using (2.4) and observing that now $\dot{\sigma}(t) = -M\nabla Q(t,\sigma(t))$. □

Now, let $\gamma_0 : H_0^1(\Omega) \to H_0^1(\Omega)$ be the homeomorphism defined by

$$\gamma_0(u) = \sigma(1,u). \tag{2.8}$$

We observe that γ_0 depends on ε. Let η be as in Lemma 2.3. Taking (2.2) and Lemma 2.6 into account we see that

$$Q(1,\gamma_0(u)) \leq \varepsilon\|\gamma_0(u)\|^2 \ \forall u \in H_1, \quad Q(1,\gamma_0(u)) \geq \eta\|\gamma_0(u)\|^2 \ \forall u \in H_2. \tag{2.9}$$

The above conclusions suggest that we apply the following minimax procedure. For any $R > 0$, we denote

$$S = \gamma_0(H_2), \quad A = R\gamma_0(B_1) \quad \text{and} \quad \partial A = R\gamma_0(\partial B_1) \tag{2.10}$$

where B_1 stands for the unit ball in H_1 with the center at the origin. We denote

$$\Gamma := \{\gamma \in C(A; H_0^1(\Omega)) : \gamma(u) = u \ \forall u \in \partial A\}.$$

Lemma 2.7. *Sets S and ∂A link through A, that is*

$$\partial A \cap S = \emptyset \quad \textit{and} \quad \gamma(A) \cap S \neq \emptyset \ \forall \gamma \in \Gamma.$$

Proof. We first claim that for any $u \in \partial B_1$, $v \in H_2$, $\xi \in \mathbb{R}$, $\xi \neq 0$,

$$\xi\gamma_0(u) \neq \gamma_0(v). \tag{2.11}$$

Indeed, if $\xi\gamma_0(u) = \gamma_0(v)$ then $\xi^2\|\gamma_0(u)\|^2 = \|\gamma_0(v)\|^2$ and (2.9) implies

$$\begin{aligned}\eta\|\gamma_0(v)\|^2 &\leq Q(1,\gamma_0(v)) = Q(1,\xi\gamma_0(u)) \\ &= \xi^2 Q(1,\gamma_0(u)) \leq \varepsilon\xi^2\|\gamma_0(u)\|^2 = \varepsilon\|\gamma_0(v)\|^2,\end{aligned}$$

yielding $\gamma_0(v) = 0$. Thus also $\gamma_0(u) = 0$. By the uniqueness of the Cauchy problem (2.7), $u = 0$. This contradicts $u \in \partial B_1$ and proves (2.11). In particular, this shows that $\partial A \cap S = \emptyset$.

We denote by P the orthogonal projection of $H_0^1(\Omega)$ onto H_1. Again (2.11) implies that for any $t \in [0,1]$ the map $\mathcal{H}_t : B_1 \to H_1$ given by

$$\mathcal{H}_t = P \circ \gamma_0^{-1} \circ (1 + (R-1)t)\gamma_0,$$

has a well-defined Brouwer degree $\deg(\mathcal{H}_t, B_1, 0)$. By the invariance property of the degree,

$$\deg(\mathcal{H}_1, B_1, 0) = \deg(\mathcal{H}_0, B_1, 0) = \deg(P, B_1, 0) = 1.$$

Now, for a given $\gamma \in \Gamma$, the above shows that

$$\deg(P \circ \gamma_0^{-1} \circ \gamma(R\gamma_0), B_1, 0) = \deg(\mathcal{H}_1, B_1, 0) = 1.$$

This implies $\gamma(A) \cap S \neq \emptyset$ and proves the lemma. □

Proof of Theorem 2.2 completed. (1) Let η be given by Lemma 2.3. It follows from (2.1) that there exists $C > 0$ such that, for every $u \in H_0^1(\Omega)$,

$$\eta\|u\|^2 - \int_\Omega G(x,u) \geq \eta\|u\|^2/2 - C. \tag{2.12}$$

On the other hand, it follows easily from (2.1) and $(NQ)_-$ that $G(x,s) \to \infty$ as $|s| \to \infty$, uniformly for a.e. $x \in \Omega$ (see [4, Lemma 2.3]). In particular, there exists $C_1 > 0$ such that, for every $u \in H_0^1(\Omega)$,

$$\int_\Omega G(x,u) \geq -C_1. \tag{2.13}$$

(2) Let's fix any $\varepsilon \in \,]0, \eta/4[$ and consider the homeomorphism γ_0 given in (2.8). Using the compactness of ∂B_1 and the uniqueness of the Cauchy problem (2.7) we see that

$$a_\varepsilon := \inf\{\|\gamma_0(u)\|^2,\ u \in \partial B_1\} > 0.$$

Then we fix $R > 0$ sufficiently large so that

$$-\varepsilon R^2 a_\varepsilon + C_1 < -C. \tag{2.14}$$

For this choice of R, we consider the sets S, A, ∂A as in (2.10). We denote

$$E_\varepsilon(u) := E(u) - 2\varepsilon\|u\|^2, \quad u \in H_0^1(\Omega).$$

It follows from (2.9), (2.13) and (2.14) that for any $v \in \partial A$, say, $v = R\gamma_0(u)$,

$$\begin{aligned} E_\varepsilon(v) &= R^2 Q(1, \gamma_0(u)) - \int_\Omega G(x,v) - 2\varepsilon R^2\|\gamma_0(u)\|^2 \\ &\leq -\varepsilon R^2\|\gamma_0(u)\|^2 + C_1 \leq -\varepsilon R^2 a_\varepsilon + C_1 < -C. \end{aligned}$$

We observe also that $E_\varepsilon(v) \leq C_1$ for any $v \in A$. Similarly, if $v \in S$ (2.9) and (2.12) imply

$$E_\varepsilon(v) \geq \eta\|v\|^2 - \int_\Omega G(x,v) - 2\varepsilon\|v\|^2 \geq (\eta/2 - 2\varepsilon)\|v\|^2 - C \geq -C.$$

We thus conclude that

$$\sup_{\partial A} E_\varepsilon < -C \le \inf_S E_\varepsilon \le \sup_A E_\varepsilon \le C_1. \tag{2.15}$$

In particular,

$$\sup_{\partial A} E_\varepsilon < \inf_S E_\varepsilon. \tag{2.16}$$

(3) It is proved in [4, Lemma 2.2], as a consequence of both (2.1) and $(NQ)_-$, that the Cerami condition (see Section 1) holds for the functional E. In fact, the arguments in [4, Lemma 2.2] show that E_ε also satisfies the Cerami condition, as long as $0 < \varepsilon < 1/4$. This, together with (2.16) implies (see [2]) that E_ε has a critical point u_ε, with a minimax critical level given by

$$E_\varepsilon(u_\varepsilon) = \inf_{\gamma\in\Gamma} \sup_{u\in A} E_\varepsilon.$$

Hence we see that (2.15) implies

$$\nabla E_\varepsilon(u_\varepsilon) = 0 \quad \text{and} \quad -C \le E_\varepsilon(u_\varepsilon) \le C_1.$$

In particular, $(E_\varepsilon(u_\varepsilon))$ is bounded uniformly in ε. Thus again the arguments in [4, Lemma 2.2] imply that $u_{\varepsilon_n} \to u$ in $H_0^1(\Omega)$ along some sequence $\varepsilon_n \to 0$. Clearly,

$$\nabla E(u) = 0 \quad \text{and} \quad -C \le E(u) \le C_1.$$

This completes the proof of Theorem 2.2. □

3. Further results

We start by presenting some situations where Theorem 2.2 applies, namely where the pair (α, β) is Σ-connected to some eigenpair in the sense of Definition 2.1. In the following we let $\lambda_1 < \beta < \alpha$.

Example 3.1. Let's assume $N \ge 2$ and that $\lambda_{k-1} < \beta \le \lambda_k \le \alpha < \lambda_{k+1}$ for some $k \ge 2$. It is known that Σ contains at least two paths $c_i(t)$, $i = 1, 2$, with image in $J := [\lambda_k, \lambda_{k+1}[\times]\lambda_{k-1}, \lambda_k]$ and starting at the point (λ_k, λ_k). Moreover, $\Sigma \cap J$ lies in between the graphs of c_1 and c_2. In fact, if λ_k is a simple eigenvalue then $\Sigma \cap J = \text{range}(c_1) \cup \text{range}(c_2)$. We also recall that it may happen that $c_1 = c_2$. Otherwise, say, the graph of c_1 lies below the graph of c_2. For this and other properties of c_1 and c_2 we refer the reader to [3], [13], [18], [25].

Thus, with the above notation, we see that (α, β) is Σ-connected to (λ_k, λ_k) whenever (α, β) lies in range(c_2) (or above it).

Example 3.2. Let's suppose now $\Omega = B_R(0) \subset \mathbb{R}^N$ is an open ball. Whenever $g(\,\cdot\,,s)$ is radially invariant we may look at the radial solutions of (1.1). In this case Theorem 2.2 also provides a radial solution for (1.1). In fact, the proof remains unchanged except that now we work in the space $H^1_{0,\mathrm{rad}}(\Omega)$ consisting of the radially symmetric functions of $H^1_0(\Omega)$. Indeed, it follows from the principle of symmetric criticality (see e.g. [26, Theorem 1.28]) that a critical point of the restricted functional E is a radial solution of (1.1).

Of course, in this situation we can relax our assumption on (α,β) by merely assuming that (α,β) is Σ_{rad}-connected to some (λ_k,λ_k), in an obvious sense. Here (λ_i) stands for the radial eigenvalues of $(-\Delta, H^1_{0,\mathrm{rad}}(\Omega))$ and Σ_{rad} is given in (1.2) with $H^1_0(\Omega)$ replaced by $H^1_{0,\mathrm{rad}}(\Omega)$. It is proved in [1] that Σ_{rad} consists of the lines $\mathbb{R}\times\{\lambda_1\}$ and $\{\lambda_1\}\times\mathbb{R}$ together with pairs $r_{1,k}$, $r_{2,k}$ $(k\geq 2)$ of (globally defined) curves which cross (λ_k,λ_k). Each set $\mathrm{range}(r_{1,k})\cup\mathrm{range}(r_{2,k})$ is isolated from the rest of Σ_{rad}. We refer to [1] for further regularity, monotonicity and asymptotic properties of these curves.

Let us write $r_{i,k} = (t, s_{i,k}(t))$ for $i = 1,2$, $t\in [\lambda_k,\infty[$ and set $r_k(t) = (t, s_k(t))$, where $s_k = \max\{s_{1,k}, s_{2,k}\}$. It then follows that (α,β) is Σ_{rad}-connected to (λ_k,λ_k) whenever (α,β) lies in $r_k([\lambda_k,\infty[)$.

Example 3.3. We now consider the one dimensional case $N = 1$ with, say, $\Omega =]0,\pi[$. In this case Σ can be computed explicitly (cf. e.g. [4], [12]) and it is precisely the union of the (globally defined) curves $c_{1,k}$, $c_{2,k}$ $(k\geq 2)$ mentioned in Example 3.1 together with the lines $\mathbb{R}\times\{\lambda_1\}$ and $\{\lambda_1\}\times\mathbb{R}$. As in Example 3.1,

Theorem 2.2 applies for any pair $(\alpha,\beta)\in\mathbb{R}^2$ lying in the upper branch $c_{2,k}$.

Next we make some remarks concerning the scalar periodic problem

$$-\ddot{u} = \alpha u^+ - \beta u^- + g(x,u), \quad u(0) - u(2\pi) = 0 = \dot{u}(0) - \dot{u}(2\pi), \tag{3.1}$$

with $0<\beta<\alpha$. Here $\lambda_i = (i-1)^2$ for $i\geq 1$. We refer the reader to [11] and [15] for recent results concerning (3.1). The Fučik spectrum Σ of the associated linear operator is defined as in (1.2) except that now we work in the space $H^1_{\mathrm{per}}(]0,2\pi[)$, consisting of the 2π-periodic functions of the Sobolev space $H^1(]0,2\pi[)$. It is easily seen that Σ consists of the lines $\mathbb{R}\times\{0\}$ and $\{0\}\times\mathbb{R}$ together with the curves defined by

$$C_k = \left\{(\mu,\nu)\in\mathbb{R}^2_+ : \frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\nu}} = \frac{2}{k-1}\right\}, \quad k\geq 2.$$

Assuming (2.1), it is proved in [4, Theorem 2] that (3.1) admits a solution whenever $(\alpha,\beta)\in C_k$ $(k\geq 2)$ and either $(NQ)_+$ holds or else $(NQ)_-$ holds and $\alpha\geq\lambda_{k-1}$, $\beta\geq\lambda_{k-1}$ hold. The latter restriction can in fact be avoided.

Theorem 3.4. *Let* $(\alpha,\beta)\in C_k$, $k\geq 2$, *and assume* (2.1) *and* $(NQ)_-$. *Then* (3.1) *admits at least one solution.*

Proof. We may write the equation in (3.1) as

$$-Lu=\widetilde{\alpha}u^+-\widetilde{\beta}u^-+g(x,u),$$

where $\widetilde{\alpha}=\alpha+1$, $\widetilde{\beta}=\beta+1$ and $Lu=\ddot{u}-u$. With an obvious meaning, let $\widetilde{\Sigma}$ be the Fučik spectrum of $(-L,H^1_{\rm per}(]0,2\pi[))$, that is, $\widetilde{\Sigma}=\Sigma+\{(1,1)\}$. Using the curve C_k we see that $(\widetilde{\alpha},\widetilde{\beta})$ is $\widetilde{\Sigma}$-connected to the eigenpair $(\lambda_k+1,\lambda_k+1)$ of $(-L,H^1_{\rm per}(]0,2\pi[))$. Since L is invertible, the proof of Theorem 2.2 can then be repeated step by step. □

We conclude with a symmetric version of Theorem 2.2, in the sense that we assume that $(NQ)_+$ holds instead of $(NQ)_-$.

Theorem 3.5. *We consider* (1.1) *with* g *satisfying both* $(NQ)_+$ *and* (2.1). *We suppose there exist* $d\in\,]0,1[$ *and a* C^1 *function* $c:[0,1]\to\mathbb{R}^2$ *such that* $c(0)=(\lambda_k,\lambda_k)$ $(k\geq 2)$, $c(1)=(\alpha,\beta)$ *and*

$$\xi c([0,1])\cap\Sigma=\emptyset\quad \textit{for every } \xi\in[1-d,1[\,. \tag{3.2}$$

Then (1.1) *has a solution.*

Sketch of the proof. We follow the steps in the proof of Theorem 2.2. We decompose

$$H^1_0(\Omega)=V_1\oplus V_2,$$

where V_1 is the finite dimensional eigenspace associated to the eigenvalues $\lambda_1,\dots,\lambda_{k-1}$. We use similar notation as in Section 2. Clearly there exists $\sigma>0$ such that

$$Q(0,u)\leq-\sigma\|u\|^2\ \forall u\in V_1\quad\text{and}\quad Q(0,u)\geq 0\ \forall u\in V_2.$$

It follows from (3.2) that a result similar to Lemma 2.3 can be stated, provided we replace the interval $[\eta/2,\eta]$ in that lemma with $[-\eta,-\eta/2]$. As a consequence, for every $\varepsilon>0$ small enough there exists a homeomorphism γ_0 in $H^1_0(\Omega)$ such that (compare with (2.9))

$$\begin{aligned} Q(1,\gamma_0(u))&\leq-\eta\|\gamma_0(u)\|^2\quad\forall u\in V_1,\\ Q(1,\gamma_0(u))&\geq-\varepsilon\|\gamma_0(u)\|^2\quad\forall u\in V_2.\end{aligned}\tag{3.3}$$

For large R (depending on ε), let $S=\gamma_0(V_2)$, $A=R\gamma_0(B_1)$, $\partial A=R\gamma_0(\partial B_1)$ be as in (2.10), where now B_1 stands for the unit ball in V_1 with the center at the origin. Using (2.1) and $(NQ)_+$ we see that there exist positive constants C and C_1 such that, for any $u\in H^1_0(\Omega)$ (compare with (2.12), (2.13)),

$$\int_\Omega G(x,u)\leq C_1\quad\text{and}\quad-\eta\|u\|^2-\int_\Omega G(x,u)\leq-\eta\|u\|^2/2+C. \tag{3.4}$$

Let $E_\varepsilon(u) = E(u) + 2\varepsilon\|u\|^2$. It follows from (3.3) and (3.4) that, provided R is large (compare with (2.15)),

$$\sup_{\partial A} E_\varepsilon < -C_1 \le \inf_S E_\varepsilon \le \sup_A E_\varepsilon \le C.$$

It then follows easily that E admits a critical point u with energy level in $[-C_1, C]$. □

Going through Examples 3.1–3.3 above we see that (3.2) holds when, roughly speaking, (α,β) lies in some "lower branch" of Σ which is isolated from below from the rest of the spectrum Σ. In the particular case where $(\alpha,\beta) \in C_2$ (see Section 1), the variational characterization of C_2 given in [5], [11] implies that (3.2) holds. In this way we obtain [4, Theorem 1] as a corollary of Theorem 3.5. Similar results apply to the periodic problem (3.1).

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