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ON THE EXISTENCE OF POSITIVE SOLUTIONS OF HIGHER ORDER DIFFERENCE EQUATIONS

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1. Introduction

Let a, b $(b > a)$ be integers. We shall denote $[a,b] = \{a, a+1, \dots, b\}$. The notation of all other intervals will carry its standard meaning, e.g. $[0,\infty)$ denotes the set of nonnegative real numbers. Also, the symbol Δ^i denotes the ith forward difference operator with stepsize 1.

In this paper we shall consider the n-th order difference equation

$$\Delta^n y + Q(k, y, \Delta y, \dots, \Delta^{n-2} y) = P(k, y, \Delta y, \dots, \Delta^{n-1} y), \quad k \in [0, N] \tag{1.1}$$

satisfying the boundary conditions

$$\Delta^i y(0) = 0, \quad 0 \le i \le n-3, \tag{1.2}$$

$$\alpha \Delta^{n-2} y(0) - \beta \Delta^{n-1} y(0) = 0, \tag{1.3}$$

$$\gamma \Delta^{n-2} y(N+1) + \delta \Delta^{n-1} y(N+1) = 0, \tag{1.4}$$

where $n \ge 2$, $N(\ge n-1)$ is a fixed positive integer, $\alpha,\ \beta,\ \gamma$ and δ are constants so that

$$\rho = \alpha\gamma(N+1) + \alpha\delta + \beta\gamma > 0 \tag{1.5}$$

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and

$$\alpha > 0, \quad \gamma > 0, \quad \beta \ge 0, \quad \delta \ge \gamma. \tag{1.6}$$

Further, we assume that there exist functions $f : [0, \infty) \to [0, \infty)$ and p, p_1, $q, q_1 : [0, N] \to \Re$ such that

(i) $uf(u) \neq 0$ for all $u \neq 0$,

(ii) for $u \neq 0$,

$$q(k) \le \frac{Q(k, u, u_1, \dots, u_{n-2})}{f(u)} \le q_1(k),$$

$$p(k) \le \frac{P(k, u, u_1, \dots, u_{n-1})}{f(u)} \le p_1(k),$$

(iii) $p_1(k)$ is not identical to $q(k)$ and $p_1(k) \le q(k)$, $k \in [0, N]$.

We shall give an existence result for positive solutions of the boundary value problem (1.1)–(1.4), assuming that f is either superlinear or sublinear. No monotonicity assumption on f is required. To be precise, we introduce the notation

$$f_0 = \lim_{u \to 0} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Function f is said to be *superlinear* if $f_0 = 0$, $f_\infty = \infty$, and f is *sublinear* provided $f_0 = \infty$, $f_\infty = 0$. By a *positive solution* y of (1.1)–(1.4), we mean $y : [0, N+n] \to \Re$, y satisfies (1.1) on $[0, N]$, y fulfills (1.2)–(1.4), and y is nonnegative on $[0, N+n]$, positive on $[n-1, N+n-2]$.

The motivation for the present work stems from many recent investigations. In fact, applications of (1.1)–(1.4) and their continuous version have been made to singular boundary value problems by Agarwal and Wong [2], [15]. Other particular cases of (1.1)–(1.4) and their continuous analogs have also been the subject matter of several recent publications on singular boundary value problems (e.g. see [1], [5], [10]–[12] and the references cited therein). In the special case where $n = 2$, the continuous version of (1.1)–(1.4) arises in applications involving nonlinear elliptic problems in annular regions, for this we refer to [3], [4], [9], [14]. In all these applications, it is frequent that only positive solutions are useful. We are particularly motivated by the work of [6]–[8], and our result is a generalization and extension of theirs to a discrete case.

The plan of this paper is as follows. In Section 2 we shall state a fixed point theorem due to Krasnosel'skiĭ [13], and present some properties of certain Green's function which will be used later. In Section 3, we provide an appropriate Banach space and a cone so that the fixed point theorem from [13] may be applied to yield a positive solution for (1.1)–(1.4).

2. Preliminaries

Theorem 2.1. ([13]) *Let B be a Banach space, and let $C \subset B$ be a cone in B. Assume that Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$S : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$$

be a completely continuous operator such that, either

(a) $\|Sy\| \le \|y\|$, $y \in C \cap \partial\Omega_1$ *and* $\|Sy\| \ge \|y\|$, $y \in C \cap \partial\Omega_2$ *or*
(b) $\|Sy\| \ge \|y\|$, $y \in C \cap \partial\Omega_1$ *and* $\|Sy\| \le \|y\|$, $y \in C \cap \partial\Omega_2$.

Then S has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

To apply Theorem 2.1 in Section 3, we need a mapping whose kernel $g(i,j)$ is the Green's function of the boundary value problem

$$\begin{gathered} -\Delta^n y = 0, \\ \Delta^i y(0) = 0, \quad 0 \le i \le n-3, \\ \alpha\Delta^{n-2}y(0) - \beta\Delta^{n-1}y(0) = 0, \\ \gamma\Delta^{n-2}y(N+1) + \delta\Delta^{n-1}y(N+1) = 0. \end{gathered}$$

It can be verified that

$$G(i,j) = \Delta^{n-2}g(i,j) \quad \text{(w.r.t. i)}$$

is the Green's function of the boundary value problem

$$\begin{gathered} -\Delta^2 w = 0, \\ \alpha w(0) - \beta\Delta w(0) = 0, \\ \gamma w(N+1) + \delta\Delta w(N+1) = 0. \end{gathered}$$

Further, we have

$$G(i,j) = \frac{1}{\rho}\begin{cases} [\beta + \alpha(j+1)][\delta + \gamma(N+1-i)] & j \in [0, i-1], \\ (\beta + \alpha i)[\delta + \gamma(N-j)] & j \in [i, N]. \end{cases} \tag{2.1}$$

We observe that conditions (1.5) and (1.6) imply that $G(i,j)$ is nonnegative on $[0, N+2] \times [0, N]$, and positive on $[1, N+1] \times [0, N]$.

Lemma 2.1. *For $(i,j) \in [1, N] \times [0, N]$, we find that*

$$G(i,j) \ge K\ G(j,j), \tag{2.2}$$

where $0 < K < 1$ is given by

$$K = \frac{(\beta + \alpha)(\delta + \gamma)}{(\beta + \alpha N)(\delta + \gamma N)}. \tag{2.3}$$

Proof. For $j \in [0, i-1]$, using (2.1), we reduce inequality (2.2) to

$$[\beta + \alpha(j+1)][\delta + \gamma(N+1-i)] \geq K(\beta + \alpha j)[\delta + \gamma(N-j)]. \tag{2.4}$$

For (2.4) to hold true, it is sufficient that K satisfies

$$\min_{(i,j)\in[1,N]\times[0,N]} [\beta + \alpha(j+1)][\delta + \gamma(N+1-i)] \geq K \max_{j\in[0,N]} (\beta + \alpha j)[\delta + \gamma(N-j)],$$

which gives

$$(\beta + \alpha)[\delta + \gamma(N+1-N)] \geq K(\beta + \alpha N)(\delta + \gamma N),$$

or

$$K \leq \frac{(\beta+\alpha)(\delta+\gamma)}{(\beta+\alpha N)(\delta+\gamma N)}. \tag{2.5}$$

For $j \in [i, N]$, inequality (2.2) becomes

$$(\beta + \alpha i)[\delta + \gamma(N-j)] \geq K(\beta + \alpha j)[\delta + \gamma(N-j)],$$

or

$$\beta + \alpha i \geq K(\beta + \alpha j).$$

Again, it suffices to find K such that

$$\min_{i\in[1,N]} (\beta + \alpha i) \geq K \max_{j\in[0,N]} (\beta + \alpha j),$$

which provides

$$K \leq \frac{\beta+\alpha}{\beta+\alpha N}. \tag{2.6}$$

Taking the intersection of (2.5) and (2.6), we immediately get (2.3). □

Lemma 2.2. *For $(i,j) \in [0, N+2] \times [0, N]$, we find that*

$$G(i,j) \leq L\, G(j,j), \tag{2.7}$$

where $L > 1$ is given by

$$L = \begin{cases} (\beta+\alpha)/\beta & \beta > 0, \\ 2 & \beta = 0. \end{cases} \tag{2.8}$$

Proof. In the case where $j \in [i, N]$, from (2.1) it is clear that we may take $L = 1$ in (2.7). For $j \in [0, i-1]$, (2.7) is the same as

$$[\beta + \alpha(j+1)][\delta + \gamma(N+1-i)] \leq L(\beta + \alpha j)[\delta + \gamma(N-j)]. \tag{2.9}$$

For (2.9) to hold true, it is sufficient that L satisfies

$$[\beta + \alpha(j+1)][\delta + \gamma(N-j)] \leq L(\beta + \alpha j)[\delta + \gamma(N-j)] \tag{2.10}$$

where we have used the fact that $1 - i \le -j$. If $\beta \ne 0$, (2.10) leads to

$$L \ge \max_{j\in[0,N]} \frac{\beta + \alpha(j+1)}{\beta + \alpha j} = \frac{\beta+\alpha}{\beta}. \tag{2.11}$$

If $\beta = 0$, (2.10) provides

$$L \ge \max_{j\in[1,N]} \ (j+1)/j = 2. \tag{2.12}$$

Expression (2.8) follows immediately from (2.11) and (2.12). □

We shall need the following notations in Section 3. For a nonnegative $y(\in B)$ which is not identically zero on $[0, N]$, we denote

$$\theta = \sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell) - p(\ell)]f(y(\ell))$$

and

$$\Gamma = \sum_{\ell=0}^{N} G(\ell,\ell)[q(\ell) - p_1(\ell)]f(y(\ell)).$$

In view of (i)–(iii), it is clear that $\theta \ge \Gamma > 0$. Further, we define the constant

$$\xi = K\Gamma/L\theta.$$

It is noted that $0 < \xi < 1$.

3. Main results

Let

$$B = \{y : [0, N+n] \to \Re \mid \Delta^i y(0) = 0,\ 0 \le i \le n-3\},$$

be the Banach space with norm $\|y\| = \max_{k\in[0,N+2]} |\Delta^{n-2}y(k)|$ and let

$$C = \Big\{y \in B \ \Big|\ \Delta^{n-2}y(k) \text{ be nonnegative and is not identically zero on } [0,N+2];\ \min_{k\in[1,N]} \Delta^{n-2}y(k) \ge \xi\|y\|\Big\}.$$

We note that C is a cone in B.

Lemma 3.1. *Let $y \in B$. For $0 \le i \le n-3$, we find that*

$$|\Delta^i y(k)| \le \frac{k^{(n-2-i)}}{(n-2-i)!}\ \|y\|, \quad k \in [0, N+n-i]. \tag{3.1}$$

In particular,

$$|y(k)| \le \frac{(N+n)^{(n-2)}}{(n-2)!}\ \|y\|, \quad k \in [0, N+n]. \tag{3.2}$$

Proof. For $y \in B$, we see that

$$\Delta^{n-3}y(k) = \sum_{\ell=0}^{k-1} \Delta^{n-2}y(\ell), \quad k \in [0, N+3],$$

which implies

$$|\Delta^{n-3}y(k)| \le k\|y\|, \quad k \in [0, N+3]. \tag{3.3}$$

Next, since

$$\Delta^{n-4}y(k) = \sum_{\ell=0}^{k-1} \Delta^{n-3}y(\ell), \quad k \in [0, N+4],$$

on using (3.3) we get

$$|\Delta^{n-4}y(k)| \le \sum_{\ell=0}^{k-1} \ell\|y\| = \frac{k^{(2)}}{2!}\,\|y\|, \quad k \in [0, N+4].$$

Continuing in the same manner we obtain (3.2). □

Lemma 3.2. *Let $y \in C$. For $0 \le i \le n-3$, we find that*

$$\Delta^{i}y(k) \ge 0, \quad k \in [0, N+n-i], \tag{3.4}$$

and

$$\Delta^{i}y(k) \ge \frac{(k-1)^{(n-2-i)}}{(n-2-i)!}\,\xi\|y\|, \quad k \in [1, N+n-2-i]. \tag{3.5}$$

In particular,

$$y(k) \ge \xi\|y\|, \quad k \in [n-1, N+n-2]. \tag{3.6}$$

Proof. Inequality (3.4) is obvious because of the fact that

$$\Delta^{i}y(k) = \sum_{\ell=0}^{k-1} \Delta^{i+1}y(\ell), \quad k \in [0, N+n-i],\ 0 \le i \le n-3.$$

To prove (3.5), we note that

$$\Delta^{n-3}y(k) = \sum_{\ell=0}^{k-1} \Delta^{n-2}y(\ell) \ge \sum_{\ell=1}^{k-1} \xi\|y\| = (k-1)\xi\|y\|, \quad k \in [1, N+1]. \tag{3.7}$$

Next, using (3.7) we find that

$$\Delta^{n-4}y(k) = \sum_{\ell=0}^{k-1} \Delta^{n-3}y(\ell) \ge \sum_{\ell=1}^{k-1} (\ell-1)\xi\|y\| = \frac{(k-1)^{(2)}}{2!}\,\xi\|y\|,$$

for $k \in [1, N+2]$. Continuing the process we obtain (3.5). Inequality (3.6) follows immediately from (3.5) when we take $i = 0$ and substitute $k = n-1$ in the right hand side of (3.5). □

Remark 3.1. If $y \in C$ is a solution of (1.1)–(1.4), then (3.4) and (3.6) imply that y is a positive solution of (1.1)–(1.4).

To obtain a positive solution of (1.1)–(1.4), we shall seek a fixed point of an operator $S : C \to B$

$$Sy(k) = \sum_{\ell=0}^{N} g(k,\ell)[Q(\ell, y, \Delta y, \dots, \Delta^{n-2}y) - P(\ell, y, \Delta y, \dots, \Delta^{n-1}y)], \tag{3.8}$$

for $k \in [0, N+n]$ in the cone C. It follows that

$$\Delta^{n-2}Sy(k) = \sum_{\ell=0}^{N} G(k,\ell)[Q(\ell, y, \Delta y, \dots, \Delta^{n-2}y) - P(\ell, y, \Delta y, \dots, \Delta^{n-1}y)],$$

for $k \in [0, N+2]$ and, in view of condition (ii), we get for $k \in [0, N+2]$,

$$\begin{aligned}\sum_{\ell=0}^{N} G(k,\ell)[q(\ell) - p_1(\ell)]f(y(\ell)) \\ \le \Delta^{n-2}Sy(k) \le \sum_{\ell=0}^{N} G(k,\ell)[q_1(\ell) - p(\ell)]f(y(\ell)).\end{aligned} \tag{3.9}$$

Theorem 3.1. *Suppose that* (i)–(iii) *hold. If*

(a) f *is superlinear, i.e.,* $f_0 = 0$, $f_\infty = \infty$ *or*

(b) f *is sublinear, i.e.,* $f_0 = \infty$, $f_\infty = 0$,

then (1.1)–(1.4) *has a solution in* C.

Proof. First we shall show that the operator $S : C \to B$ defined in (3.8) maps C into itself. For this, let $y \in C$. Then, from (3.9) and (iii) we find

$$\Delta^{n-2}Sy(k) \ge \sum_{\ell=0}^{N} G(k,\ell)[q(\ell) - p_1(\ell)]f(y(\ell)) \ge 0, \quad k \in [0, N+2]. \tag{3.10}$$

Further, it follows from (3.9) and Lemma 2.2 that

$$\begin{aligned}\Delta^{n-2}Sy(k) &\le \sum_{\ell=0}^{N} G(k,\ell)[q_1(\ell) - p(\ell)]f(y(\ell)) \\ &\le L\sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell) - p(\ell)]f(y(\ell)),\end{aligned}$$

for $k \in [0, N+2]$. Therefore,

$$\|Sy\| \le L\sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell) - p(\ell)]f(y(\ell)) = L\theta. \tag{3.11}$$

Now, using (3.9), Lemma 2.1 and (3.11) we find for $k \in [1, N]$,

$$\Delta^{n-2} Sy(k) \geq \sum_{\ell=0}^{N} G(k,\ell)[q(\ell) - p_1(\ell)]f(y(\ell))$$
$$\geq K \sum_{\ell=0}^{N} G(\ell,\ell)[q(\ell) - p_1(\ell)]f(y(\ell)) = K\Gamma \geq \xi \|Sy\|.$$

Hence,

$$\min_{k\in[1,N]} \Delta^{n-2} Sy(k) \geq \xi \|Sy\|. \tag{3.12}$$

It follows from (3.10) and (3.12) that $S(C) \subseteq C$. Also, standard arguments yield that S is completely continuous.

(a) Suppose that f is superlinear. Since $f_0 = 0$, we may choose $a_1 > 0$ such that $f(u) \leq \varepsilon u$ for $0 < u \leq a_1$, where $\varepsilon > 0$ satisfies

$$\frac{L\varepsilon (N+n)^{(n-2)}}{(n-2)!} \sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell) - p(\ell)] \leq 1. \tag{3.13}$$

Let $y \in C$ be such that $\|y\| = a_1 (n-2)!/(N+n)^{(n-2)}$. Then, from (3.2), we have $|y(k)| \leq a_1,\ k \in [0, N+n]$. Hence, applying (3.9), Lemma 2.2, (3.2) and (3.13) successively gives for $k \in [0, N+2]$,

$$\Delta^{n-2} Sy(k) \leq \sum_{\ell=0}^{N} G(k,\ell)[q_1(\ell) - p(\ell)]f(y(\ell))$$
$$\leq L \sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell) - p(\ell)]f(y(\ell))$$
$$\leq L\varepsilon \sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell) - p(\ell)]y(\ell)$$
$$\leq L\varepsilon \sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell) - p(\ell)]\frac{(N+n)^{(n-2)}}{(n-2)!}\|y\| \leq \|y\|.$$

Consequently,

$$\|Sy\| \leq \|y\|. \tag{3.14}$$

If we set

$$\Omega_1 = \left\{ y \in B \;\middle|\; \|y\| < \frac{a_1(n-2)!}{(N+n)^{(n-2)}} \right\},$$

then (3.14) holds for $y \in C \cap \partial\Omega_1$.

Next, since $f_\infty = \infty$, we may choose $\overline{a}_2 > 0$ such that $f(u) \geq Mu$ for $u \geq \overline{a}_2$, where $M > 0$ satisfies

$$\xi M \sum_{\ell=n-1}^{N} G(n-1,\ell)[q(\ell) - p_1(\ell)] \geq 1. \tag{3.15}$$

Let

$$a_2 = \max\left\{2\frac{a_1(n-1)!}{(N+n)^{(n-2)}}, \frac{1}{\xi}\overline{a}_2\right\},$$

and let $y \in C$ be such that $\|y\| = a_2$. Then, from (3.6) we have

$$y(k) \geq \varepsilon\|y\| \geq \xi \cdot \frac{1}{\xi}\overline{a}_2 = \overline{a}_2, \quad k \in [n-1, N+n-2].$$

Hence, $f(y(k)) \geq My(k)$ for $k \in [n-1, N+n-2]$. In view of (3.9), (3.6) and (3.15), we find

$$\begin{aligned}
\Delta^{n-2}Sy(n-1) &\geq \sum_{\ell=0}^{N} G(n-1,\ell)[q(\ell) - p_1(\ell)]f(y(\ell)) \\
&\geq \sum_{\ell=n-1}^{N} G(n-1,\ell)[q(\ell) - p_1(\ell)]f(y(\ell)) \\
&\geq M \sum_{\ell=n-1}^{N} G(n-1,\ell)[q(\ell) - p_1(\ell)]y(\ell) \\
&\geq M \sum_{\ell=n-1}^{N} G(n-1,\ell)[q(\ell) - p_1(\ell)]\xi\|y\| \geq \|y\|.
\end{aligned}$$

Therefore,

$$\|Sy\| \geq \|y\|. \tag{3.16}$$

If we set

$$\Omega_2 = \left\{y \in B \;\middle|\; \|y\| < a_2\right\},$$

then (3.16) holds for $y \in C \cap \partial\Omega_2$.

In view of (3.14) and (3.16), it follows from Theorem 2.1 that S has fixed point $y \in C \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that

$$\frac{a_1(n-2)!}{(N+n)^{(n-2)}} \leq \|y\| \leq a_2.$$

This y is a positive solution of (1.1)–(1.4).

(b) Suppose that f is sublinear. Since $f_0 = \infty$, there exists $a_3 > 0$ such that $f(u) \geq \overline{M}u$ for $0 < u \leq a_3$, where $\overline{M} > 0$ satisfies

$$\xi\overline{M} \sum_{\ell=n-1}^{N} G(n-1,\ell)[q(\ell) - p_1(\ell)] \geq 1. \tag{3.17}$$

Let $y \in C$ be such that $\|y\| = a_3(n-2)!/(N+n)^{(n-2)}$. Then, from (3.2), we have $|y(k)| \le a_3,\ k \in [0, N+n]$. Hence, using (3.9), (3.6) and (3.17) successively, we get

$$\begin{aligned}\Delta^{n-2}Sy(n-1) &\ge \sum_{\ell=0}^{N} G(n-1,\ell)[q(\ell)-p_1(\ell)]f(y(\ell)) \\ &\ge \sum_{\ell=n-1}^{N} G(n-1,\ell)[q(\ell)-p_1(\ell)]f(y(\ell)) \\ &\ge \overline{M} \sum_{\ell=n-1}^{N} G(n-1,\ell)[q(\ell)-p_1(\ell)]y(\ell) \\ &\ge \overline{M} \sum_{\ell=n-1}^{N} G(n-1,\ell)[q(\ell)-p_1(\ell)]\xi\|y\| \ge \|y\|,\end{aligned}$$

from which inequality (3.16) follows immediately. If we set

$$\Omega_1 = \left\{ y \in B \;\middle|\; \|y\| < \frac{a_3(n-2)!}{(N+n)^{(n-2)}} \right\},$$

then (3.16) holds for $y \in C \cap \partial\Omega_1$. Next, in view of $f_\infty = 0$, we may choose $\overline{a}_4 > 0$ such that $f(u) \le \overline{\varepsilon}\, u$ for $u \ge \overline{a}_4$, where $\overline{\varepsilon} > 0$ satisfies

$$\frac{L\overline{\varepsilon}(N+n)^{(n-2)}}{(n-2)!} \sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell)-p(\ell)] \le 1. \tag{3.18}$$

There are two cases to consider, namely, f is bounded and f is unbounded.

Case 1. Suppose that f is bounded, i.e., $f(u) \le R,\ u \in [0,\infty)$ for some $R > 0$. Let

$$a_4 = \max\left\{ 2a_3, \frac{LR(N+n)^{(n-2)}}{(n-2)!} \sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell)-p(\ell)] \right\},$$

and let $y \in C$ be such that $\|y\| = a_4(n-2)!/(N+n)^{(n-2)}$. For $k \in [0, N+2]$, from (3.9) and Lemma 2.2 we find

$$\begin{aligned}\Delta^{n-2}Sy(k) &\le \sum_{\ell=0}^{N} G(k,\ell)[q_1(\ell)-p(\ell)]f(y(\ell)) \le R\sum_{\ell=0}^{N} G(k,\ell)[q_1(\ell)-p(\ell)] \\ &\le LR\sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell)-p(\ell)] \le \frac{a_4(n-2)!}{(N+n)^{(n-2)}} = \|y\|.\end{aligned}$$

Hence, (3.14) holds.

Case 2. Suppose that f is unbounded, i.e., there exists

$$a_4 > \max\left\{ 2\, \frac{a_3(n-2)!}{(N+n)^{(n-2)}}, \overline{a}_4 \right\}$$

such that $f(u) \le f(a_4)$ for $0 < u \le a_4$. Let $y \in C$ be such that $\|y\| = a_4(n-2)!/(N+n)^{(n-2)}$. Then, from (3.2) we have $|y(k)| \le a_4,\ k \in [0, N+n]$. Hence, applying (3.9), we successively get from Lemma 2.2 and (3.18) for $k \in [0, N+2]$,

$$\begin{aligned}\Delta^{n-2}Sy(k) &\le \sum_{\ell=0}^{N} G(k,\ell)[q_1(\ell)-p(\ell)]f(y(\ell)) \le L\sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell)-p(\ell)]f(y(\ell))\\ &\le L\sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell)-p(\ell)]f(a_4) \le L\sum_{\ell=0}^{N} G(\ell,\ell)[q_1(\ell)-p(\ell)]\bar{\varepsilon}a_4\\ &\le \frac{a_4(n-2)!}{(N+n)^{(n-2)}} = \|y\|,\end{aligned}$$

from which (3.14) follows immediately.

In both Cases 1 and 2, if we set

$$\Omega_2 = \left\{ y \in B \;\middle|\; \|y\| < \frac{a_4(n-2)!}{(N+n)^{(n-2)}} \right\},$$

then (3.14) holds for $y \in C \cap \partial\Omega_2$. Now that we have obtained (3.14) and (3.16), it follows from Theorem 2.1 that S has a fixed point $y \in C \cap (\overline{\Omega}_2 \backslash \Omega_1)$ such that

$$\frac{a_3(n-2)!}{(N+n)^{(n-2)}} \le \|y\| \le \frac{a_4(n-2)!}{(N+n)^{(n-2)}}.$$

This y is a positive solution of (1.1)–(1.4). The proof of the theorem is complete. □

The following two examples illustrate Theorem 3.1.

Example 3.1. We consider the boundary value problem

$$\begin{aligned}\Delta^2 y + \frac{2}{[k(13-k)+1]^r}\, y^r &= 0, \quad k \in [0, 11],\\ 12y(0) - \Delta y(0) &= 0,\\ 12y(12) + 13\Delta y(12) &= 0,\end{aligned}$$

where $r \ne 1$. Taking $f(y) = y^r$ (which is superlinear if $r > 1$, and sublinear if $r < 1$), we find

$$\frac{Q(k,y)}{f(y)} = \frac{2}{[k(13-k)+1]^r} \quad \text{and} \quad \frac{P(k,y,\Delta y)}{f(y)} = 0.$$

Hence, we may choose

$$q(k) = \frac{1}{[k(13-k)+1]^r}, \quad q_1(k) = \frac{2}{[k(13-k)+1]^r},$$

and

$$p(k) = p_1(k) = 0.$$

All conditions of Theorem 3.1 are fulfilled and therefore the boundary value problem has a positive solution. One such solution is given by $y(k) = k(13-k)+1$.

Example 3.2. We consider the boundary value problem

$$\Delta^3 y + \frac{24k}{[k(5000-(k-1)(k-6)(k+1))+1]^r}\,(y+1)^r = 0, \quad k \in [0,10],$$
$$y(0) = 0,$$
$$3\Delta y(0) - 625\Delta^2 y(0) = 0,$$
$$162\Delta y(11) + 163\Delta^2 y(11) = 0,$$

where $r < 1$. Taking $f(y) = (y+1)^r$ (which is sublinear if $r < 1$), we find

$$\frac{Q(k,y,\Delta y)}{f(y)} = \frac{24k}{[k(5000-(k-1)(k-6)(k+1))+1]^r},$$

and

$$\frac{P(k,y,\Delta y,\Delta^2 y)}{f(y)} = 0.$$

Hence, we may take

$$q(k) = \frac{k}{[k(5000-(k-1)(k-6)(k+1))+1]^r},$$
$$q_1(k) = \frac{24k}{[k(5000-(k-1)(k-6)(k+1))+1]^r}$$

and

$$p(k) = p_1(k) = 0.$$

Again, all conditions of Theorem 3.1 are satisfied and so the boundary value problem has a positive solution. Indeed, $y(k) = k[5000-(k-1)(k-6)(k+1)]$ is one such solution.

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