

## ASYMPTOTIC ANALYSIS OF THE NAVIER–STOKES EQUATIONS IN THIN DOMAINS

I. MOISE — R. TEMAM — M. ZIANE

---

*Dedicated to O. A. Ladyzhenskaya*

### 0. Introduction

We are interested in this article with the Navier–Stokes equations of viscous incompressible fluids in three dimensional thin domains. Let  $\Omega_\varepsilon$  be the thin domain  $\Omega_\varepsilon = \omega \times (0, \varepsilon)$ , where  $\omega$  is a suitable domain in  $\mathbb{R}^2$  and  $0 < \varepsilon < 1$ .

Our aim is to derive an asymptotic expansion of the strong solution  $u^\varepsilon$  of the Navier–Stokes equations in the thin domain  $\Omega_\varepsilon$  when  $\varepsilon$  is small, which is valid uniformly in time. This study should give a better understanding of the global existence results in thin domains obtained previously; see [15]–[17] and [23], [22]. We consider in this work two types of boundary conditions: the Dirichlet-periodic boundary condition and the purely periodic condition. For the first type of boundary condition we derive an asymptotic expansion of the solution  $u^\varepsilon$  in terms of the solution of the associated Stokes problem. More precisely, we prove that the solution can be written, for  $\varepsilon$  small, as

$$u^\varepsilon(t) = w^\varepsilon + \bar{u}^\varepsilon \exp\left(-\frac{\nu t}{2\varepsilon^2}\right), \quad \forall t > 0,$$

where  $w^\varepsilon$  is the solution of the associated Stokes problem and  $\bar{u}^\varepsilon$  is a bounded (in time) function depending on the initial data. We also give a new proof and an improvement of the global existence result obtained in [23].

---

1991 *Mathematics Subject Classification.* 34C35, 35Q30, 76D05.

*Key words and phrases.* Navier–Stokes equations, global existence of strong solutions, asymptotic analysis.

For the purely periodic boundary condition case, the asymptotic expansion involves the solution of the 2D-Navier–Stokes equations and a solution of an auxiliary Stokes problem with exterior force

$$f^\varepsilon - \frac{1}{\varepsilon} \int_0^\varepsilon f(x_1, x_2, x_3) dx_3.$$

More precisely, we prove that the solution can be written, as:

$$u^\varepsilon(t) = w^\varepsilon + u_{2D}^\varepsilon(t) + \bar{u}^\varepsilon \exp\left(-\frac{\nu t}{2\varepsilon^2}\right), \quad \forall t > 0 \text{ and } \varepsilon \text{ small,}$$

where  $w^\varepsilon$  is the solution of the auxiliary Stokes problem,  $u_{2D}^\varepsilon(t)$  is the solution of the 2D-Navier–Stokes equations with three components and  $\bar{u}^\varepsilon$  is a bounded (in time) function depending on the initial data. The nondimensionalized form of the Navier–Stokes equations (NSE) reads

$$(0.1) \quad \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f \quad \text{in } \Omega_\varepsilon \times (0, \infty),$$

$$(0.2) \quad \operatorname{div} u = 0 \quad \text{in } \Omega_\varepsilon \times (0, \infty),$$

$$(0.3) \quad u(\cdot, 0) = u_0(\cdot) \quad \text{in } \Omega_\varepsilon.$$

Here  $u = (u_1, u_2, u_3)$  is the velocity vector at point  $x$  and time  $t$ , and  $p = p(x, t)$  is the pressure.

Equations (0.1)–(0.3) are supplemented with boundary conditions. We denote the boundary of  $\Omega_\varepsilon$  by  $\partial\Omega_\varepsilon = \Gamma_t \cup \Gamma_b \cup \Gamma_l$ , where

$$(0.4) \quad \Gamma_t = \omega \times \{\varepsilon\}, \quad \Gamma_b = \omega \times \{0\} \quad \text{and} \quad \Gamma_l = \partial\omega \times (0, \varepsilon).$$

The boundary conditions of interest to us are the mixed Dirichlet-periodic condition, i.e. the Dirichlet boundary condition on  $\Gamma_t \cup \Gamma_b$  and the periodic condition on  $\Gamma_l$ , and the purely periodic boundary condition on  $\partial\Omega_\varepsilon$ , in which case  $\omega = (0, l_1) \times (0, l_2)$  and  $u$  and  $p$  are  $\Omega_\varepsilon$ -periodic, and, for the data

$$\int_{\Omega_\varepsilon} u_0 dx = \int_{\Omega_\varepsilon} f dx = 0.$$

We denote by  $H^s(\Omega_\varepsilon)$ ,  $s \in \mathbb{R}$ , the Sobolev space constructed on  $L^2(\Omega_\varepsilon)$  and  $\mathbb{L}^2(\Omega_\varepsilon) = (L^2(\Omega_\varepsilon))^3$ ,  $\mathbb{H}^s(\Omega_\varepsilon) = (H^s(\Omega_\varepsilon))^3$ . We also denote by  $H_0^s(\Omega_\varepsilon)$  the closure in the space  $H^s(\Omega_\varepsilon)$  of  $C_0^\infty(\Omega_\varepsilon)$ , the space of infinitely differentiable functions with compact support in  $\Omega_\varepsilon$ .

We need also the following spaces:

$$(0.5) \quad \dot{\mathbb{H}}^m(\Omega_\varepsilon) = \left\{ u \in \mathbb{H}^m(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} u dx = 0 \right\},$$

and the spaces  $H_{\text{per}}^m(\Omega_\varepsilon)$ , which are defined with the help of Fourier series; we write

$$(0.6) \quad u(x) = \sum_{k \in \mathbb{Z}^3} u_k \exp\left(2ik \cdot \frac{x}{L}\right),$$

with  $\bar{u}_k = u_{-k}$  (so that  $u$  is real valued) and

$$\frac{x}{L} = \left(\frac{x_1}{l_1}, \frac{x_2}{l_2}, \frac{x_3}{\varepsilon}\right), \quad k \cdot \frac{x}{L} = k_1 \frac{x_1}{l_1} + k_2 \frac{x_2}{l_2} + k_3 \frac{x_3}{\varepsilon}.$$

Then,  $u$  is in  $L^2(\Omega_\varepsilon)$  if and only if

$$|u|_{L^2(\Omega_\varepsilon)}^2 = \varepsilon l_1 l_2 \sum_{k \in \mathbb{Z}^3} |u_k|^2 < \infty,$$

and  $u$  is said to be in  $H_{\text{per}}^s(\Omega_\varepsilon)$ ,  $s \in \mathbb{R}_+$ , if and only if

$$\sum_{k \in \mathbb{Z}^3} (1 + |k|^2)^s |u_k|^2 < \infty.$$

For the mathematical setting of the Navier–Stokes equations, we classically consider a Hilbert space  $H_\varepsilon$ , which is a closed subspace of  $L^2(\Omega_\varepsilon)$ . Depending on the boundary condition, we define the following:

$$H_P = H_P^\varepsilon = \left\{ u \in L^2(\Omega_\varepsilon) : \operatorname{div} u = 0, \int_{\Omega_\varepsilon} u \, dx = 0, \right. \\ \left. u_j \text{ is periodic in the direction } x_j, j = 1, 2, 3 \right\}$$

in the case of the purely periodic boundary condition, and

$$H_{DP} = H_{DP}^\varepsilon = \left\{ u \in L^2(\Omega_\varepsilon) : \operatorname{div} u = 0, u_3 = 0 \text{ on } \Gamma_t \cup \Gamma_b, \int_{\Omega_\varepsilon} u_\alpha \, dx = 0, \right. \\ \left. \text{and } u_\alpha \text{ is periodic in the direction } x_\alpha, \alpha = 1, 2 \right\},$$

in the case of the mixed Dirichlet-periodic boundary condition.

Another useful space is  $V_\varepsilon$ , a closed subspace of  $\mathbb{H}^1(\Omega_\varepsilon)$ , which is defined as follows depending on the boundary condition:

$$V_P = V_P^\varepsilon = \{u \in \mathbb{H}_{\text{per}}^1(\Omega_\varepsilon) : \operatorname{div} u = 0\}, \\ V_{DP} = V_{DP}^\varepsilon = \{u \in \mathbb{H}^1(\Omega_\varepsilon) \cap H_{DP} : u = 0 \text{ on } \Gamma_t \cup \Gamma_b \\ \text{and } u \text{ is periodic in the directions } x_1 \text{ and } x_2\},$$

The scalar product on  $H_\varepsilon$  is denoted by  $(\cdot, \cdot)_\varepsilon$ , the one on  $V_\varepsilon$  is denoted by  $((\cdot, \cdot))_\varepsilon$ , and the associated norms are denoted by  $|\cdot|_\varepsilon$  and  $\|\cdot\|_\varepsilon$  respectively. We denote by  $A_\varepsilon$  the Stokes operator defined as an isomorphism from  $V_\varepsilon$  onto the dual  $V_\varepsilon'$  of  $V_\varepsilon$ , by

$$(0.7) \quad \langle A_\varepsilon u, v \rangle_{V_\varepsilon', V_\varepsilon} = ((u, v))_\varepsilon, \quad \forall v \in V_\varepsilon.$$

The operator  $A_\varepsilon$  is extended to  $H_\varepsilon$  as a linear unbounded operator. The domain of  $A_\varepsilon$  in  $H_\varepsilon$  is denoted by  $D(A_\varepsilon)$ . The space  $D(A_\varepsilon)$  can be fully characterized using the regularity theory. We refer for the study of the regularity of the Stokes operator to [2], [6], [10], [12], [18]–[20] and [24].

Let  $b_\varepsilon$  be the continuous trilinear form on  $V_\varepsilon$  defined by

$$(0.8) \quad b_\varepsilon(u, v, w) = \sum_{i,j=1}^3 \int_{\Omega_\varepsilon} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx, \quad u, v, w \in V_\varepsilon.$$

We denote by  $B_\varepsilon$  the bilinear form on  $V_\varepsilon$  defined for  $(u, v) \in V_\varepsilon \times V_\varepsilon$  by

$$\langle B_\varepsilon(u, v), w \rangle_{V'_\varepsilon, V_\varepsilon} = b_\varepsilon(u, v, w), \quad \forall w \in V_\varepsilon,$$

and we set  $B_\varepsilon(u) = B_\varepsilon(u, u)$ .

We assume in this article that the data  $\nu$ ,  $u_0$  and  $f$  satisfy

$$(0.9) \quad \nu > 0, \quad u_0 \in H_\varepsilon \text{ (or } V_\varepsilon), \quad f \in L^\infty(0, \infty; H_\varepsilon).$$

The system of equations (0.1)–(0.3), with one of the boundary conditions listed above, can be written as a differential equation in  $V'_\varepsilon$

$$(0.10) \quad \begin{cases} u' + \nu A_\varepsilon u + B_\varepsilon(u) = f, \\ u(0) = u_0, \end{cases}$$

where  $u'$  denotes the derivative (in the distribution sense) of the function  $u$  with respect to time. We recall now the classical result of existence of solutions to problem (0.10). See e.g. [4], [9], [10], [14], [19], [20].

**THEOREM 0.1.** *For  $u_0 \in H_\varepsilon$ , there exists a solution (not necessarily unique)  $u = u_\varepsilon$  to problem (0.10) such that*

$$(0.11) \quad u_\varepsilon \in L^2(0, T; V_\varepsilon) \cap L^\infty(0, T; H_\varepsilon), \quad \forall T > 0.$$

*Moreover, if  $u_0 \in V_\varepsilon$ , then there exists  $T_\varepsilon = T_\varepsilon(\Omega_\varepsilon, \nu, u_0, f) > 0$  and a unique solution  $u_\varepsilon$  to problem (0.10) such that*

$$(0.12) \quad u_\varepsilon \in L^2(0, T_\varepsilon; D(A_\varepsilon)) \cap L^\infty(0, T_\varepsilon; V_\varepsilon).$$

The solution  $u_\varepsilon$  which satisfies (0.12) is called the strong solution of (0.10).

## 1. Functional inequalities in thin domains

In this section we present some functional inequalities in thin domains. We will only state the inequalities without proofs and we refer the reader to [23] for a detailed discussion. The functional inequalities considered here are Sobolev-type inequalities and the Cattabriga–Solonnikov regularity inequality for the Stokes operator. We should mention that in the classical Sobolev inequalities, the constants are dilation invariant but do, however, depend on the shape of

the domain, i.e., in our case the thickness  $\varepsilon$ . The significance of the inequalities given below lies in the exact dependence of the constants on  $\varepsilon$ .

First we introduce some notations. For a scalar function  $\varphi \in L^2(\Omega_\varepsilon)$ , we define its average in the thin direction as follows

$$(1.1) \quad (M_\varepsilon \varphi)(x_1, x_2) = \frac{1}{\varepsilon} \int_0^\varepsilon \varphi(x_1, x_2, s) ds,$$

and we set

$$(1.2) \quad N_\varepsilon \varphi = \varphi - M_\varepsilon \varphi, \quad \text{i.e. } M_\varepsilon + N_\varepsilon = I_{L^2(\Omega_\varepsilon)},$$

where  $I_{L^2(\Omega_\varepsilon)}$  is the identity operator on  $L^2(\Omega_\varepsilon)$ . For  $u = (u_1, u_2, u_3) \in \mathbb{L}^2(\Omega_\varepsilon)$ , we write  $M_\varepsilon u = (M_\varepsilon u_1, M_\varepsilon u_2, M_\varepsilon u_3)$  and we set

$$(1.3) \quad N_\varepsilon u = u - M_\varepsilon u, \quad \text{i.e. } M_\varepsilon + N_\varepsilon = I_{\mathbb{L}^2(\Omega_\varepsilon)}.$$

- The Poincaré inequalities:

$$(1.4) \quad \begin{cases} |u|_{L^2(\Omega_\varepsilon)} \leq \varepsilon \left| \frac{\partial u}{\partial x_3} \right|_{L^2(\Omega_\varepsilon)} & \forall u \in V_{DP}^\varepsilon, \\ |u|_{L^2(\Omega_\varepsilon)} \leq \varepsilon^2 |A_\varepsilon u| & \forall u \in D(A_\varepsilon DP), \end{cases}$$

$$(1.5) \quad \begin{cases} |N_\varepsilon u|_{L^2(\Omega_\varepsilon)} \leq \varepsilon \left| \frac{\partial N_\varepsilon u}{\partial x_3} \right|_{L^2(\Omega_\varepsilon)} & \forall u \in V_P^\varepsilon, \\ |N_\varepsilon u|_{L^2(\Omega_\varepsilon)} \leq \varepsilon^2 |A_\varepsilon N_\varepsilon u| & \forall u \in D(A_\varepsilon P). \end{cases}$$

- Ladyzhenskaya's inequalities: There exists a positive constant  $c_0$ , independent of  $\varepsilon$ , such that

$$(1.6) \quad |u|_{L^6(\Omega_\varepsilon)}^2 \leq c_0 \|u\|_\varepsilon^2 \quad \forall u \in V_{DP}^\varepsilon,$$

$$(1.7) \quad |N_\varepsilon u|_{L^6(\Omega_\varepsilon)}^2 \leq c_0 \|N_\varepsilon u\|_\varepsilon^2 \quad \forall u \in V_P^\varepsilon.$$

For  $2 \leq q \leq 6$ , there exists a positive constant  $c(q)$ , independent of  $\varepsilon$ , such that

$$(1.8) \quad |u|_{L^q(\Omega_\varepsilon)}^2 \leq c(q) \varepsilon^{(6-q)/q} \|u\|_\varepsilon^2 \quad \forall u \in V_{DP}^\varepsilon.$$

$$(1.9) \quad |N_\varepsilon u|_{L^q(\Omega_\varepsilon)}^2 \leq c(q) \varepsilon^{(6-q)/q} \|N_\varepsilon u\|_\varepsilon^2 \quad \forall u \in V_P^\varepsilon.$$

- Agmon's inequality: There exists a positive constant  $c_0(\omega)$ , independent of  $\varepsilon$ , such that

$$(1.10) \quad |u|_{L^\infty(\Omega_\varepsilon)} \leq c_0 |u|_{L^2(\Omega_\varepsilon)}^{1/4} \left( \sum_{i,j=1}^3 \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|_{L^2(\Omega_\varepsilon)} \right)^{3/4} \quad \forall u \in D(A_\varepsilon DP),$$

$$(1.11) \quad |N_\varepsilon u|_{L^\infty(\Omega_\varepsilon)} \leq c_0 |N_\varepsilon u|_{L^2(\Omega_\varepsilon)}^{1/4} \left( \sum_{i,j=1}^3 \left| \frac{\partial^2 N_\varepsilon u}{\partial x_i \partial x_j} \right|_{L^2(\Omega_\varepsilon)} \right)^{3/4} \quad \forall u \in D(A_\varepsilon P).$$

- Cattabriga–Solonnikov inequality: There exists a positive constant  $c_0(\omega)$ , independent of  $\varepsilon$ , such that

$$\sum_{i,j=1}^3 \left| \frac{\partial^2 u}{\partial x_i \partial x_j} \right|_{L^2(\Omega_\varepsilon)}^2 \leq c_0 |A_\varepsilon u|_{L^2(\Omega_\varepsilon)}^2, \quad \forall u \in D(A_\varepsilon).$$

## 2. The Dirichlet-periodic boundary condition

In this section we derive an asymptotic expansion of the solution  $u_\varepsilon$  of the Navier–Stokes equations in the thin domains  $\Omega_\varepsilon$ , when  $\varepsilon$  goes to zero. The boundary condition under consideration is the mixed Dirichlet-periodic condition. It is shown in [23] that the  $H^1$ -norm of  $u_\varepsilon$  converges to zero when  $\varepsilon$  goes to zero. Hence, one expects, in this case, a slow motion of the fluid. Our purpose in this section is to establish rigorously that the fluid has slow motion and to find the leading term. For this purpose, we first compare the solution of the nonlinear stationary problem to the solution of the Stokes problem (the linear problem). Then, we compare the solution of the evolutionary problem to the solution of the nonlinear stationary problem. This yields an asymptotic expression of the solution  $u_\varepsilon$  when  $\varepsilon$  is small.

**2.1. Comparison between the nonlinear stationary problem and the Stokes problem.** Consider the steady state Navier–Stokes equations in the thin domain  $\Omega_\varepsilon$

$$(2.1) \quad -\nu \Delta v^\varepsilon + (v^\varepsilon \cdot \nabla)v^\varepsilon + \nabla q^\varepsilon = f^\varepsilon \quad \text{in } \Omega_\varepsilon,$$

$$(2.2) \quad \operatorname{div} v^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon,$$

$$(2.3) \quad v^\varepsilon = 0 \quad \text{on } \omega \times \{0, \varepsilon\},$$

$$(2.4) \quad v^\varepsilon \text{ is periodic in the directions } x_1 \text{ and } x_2.$$

First, note using (1.4), that

$$(2.5) \quad \nu |A_\varepsilon^{1/2} v^\varepsilon|_\varepsilon^2 = (f^\varepsilon, v^\varepsilon) \leq |f^\varepsilon|_\varepsilon |v^\varepsilon|_\varepsilon \leq \varepsilon |f^\varepsilon|_\varepsilon |A_\varepsilon^{1/2} v^\varepsilon|_\varepsilon.$$

Hence

$$(2.6) \quad |A_\varepsilon^{1/2} v^\varepsilon|_\varepsilon^2 \leq \varepsilon^2 |f^\varepsilon|_\varepsilon^2 / \nu^2.$$

Let  $w^\varepsilon$  be the unique solution of the Stokes problem:

$$(2.7) \quad -\nu \Delta w^\varepsilon + \nabla \bar{q}^\varepsilon = f^\varepsilon \quad \text{in } \Omega_\varepsilon,$$

$$(2.8) \quad \operatorname{div} w^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon,$$

$$(2.9) \quad w^\varepsilon = 0 \quad \text{on } \omega \times \{0, \varepsilon\},$$

$$(2.10) \quad w^\varepsilon \text{ is periodic in the directions } x_1 \text{ and } x_2.$$

We note that

$$(2.11) \quad |A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon^2 \leq \varepsilon^2 |f^\varepsilon|_\varepsilon^2 / \nu^2.$$

Now we write the equations satisfied by  $V^\varepsilon = v^\varepsilon - w^\varepsilon$  and  $Q^\varepsilon = q^\varepsilon - \bar{q}^\varepsilon$ . We have

$$(2.12) \quad \begin{cases} -\nu \Delta V^\varepsilon + (V^\varepsilon \cdot \nabla) V^\varepsilon + \nabla Q^\varepsilon \\ \quad = -(w^\varepsilon \cdot \nabla) V^\varepsilon - (V^\varepsilon \cdot \nabla) w^\varepsilon - (w^\varepsilon \cdot \nabla) w^\varepsilon & \text{in } \Omega_\varepsilon, \\ \operatorname{div} V^\varepsilon = 0 & \text{in } \Omega_\varepsilon, \end{cases}$$

and the boundary condition

$$(2.13) \quad \begin{cases} V^\varepsilon = 0 & \text{on } \omega \times \{0, \varepsilon\}, \\ V^\varepsilon \text{ is periodic in the directions } x_1 \text{ and } x_2. \end{cases}$$

We multiply (2.12) with  $V^\varepsilon$ , integrate over  $\Omega_\varepsilon$  and obtain

$$(2.14) \quad \nu |A_\varepsilon^{1/2} V^\varepsilon|_\varepsilon^2 = - \int_{\Omega_\varepsilon} (V^\varepsilon \cdot \nabla) w^\varepsilon \cdot V^\varepsilon dx - \int_{\Omega_\varepsilon} (w^\varepsilon \cdot \nabla) w^\varepsilon \cdot V^\varepsilon dx,$$

and with

$$(2.15) \quad \left| \int_{\Omega_\varepsilon} (V^\varepsilon \cdot \nabla) w^\varepsilon \cdot V^\varepsilon dx \right| \leq |V^\varepsilon|_{\mathbb{L}^4(\Omega_\varepsilon)}^2 |A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon \\ \leq c_0 \varepsilon^{1/2} |A_\varepsilon^{1/2} V^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon$$

and

$$(2.16) \quad \left| \int_{\Omega_\varepsilon} (w^\varepsilon \cdot \nabla) w^\varepsilon \cdot V^\varepsilon dx \right| \leq |w^\varepsilon|_{\mathbb{L}^4(\Omega_\varepsilon)} |A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon |V^\varepsilon|_{\mathbb{L}^4(\Omega_\varepsilon)} \\ \leq c_0 \varepsilon^{1/2} |A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} V^\varepsilon|_\varepsilon,$$

we have

$$(2.17) \quad \nu |A_\varepsilon^{1/2} V^\varepsilon|_\varepsilon^2 \leq c_0 \varepsilon^{1/2} |A_\varepsilon^{1/2} V^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon \\ + c_0 \varepsilon^{1/2} |A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} V^\varepsilon|_\varepsilon \\ \leq c_0 \varepsilon^{1/2} |A_\varepsilon^{1/2} V^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon + \nu |A_\varepsilon^{1/2} V^\varepsilon|_\varepsilon^2 / 2 \\ + c_0^2 \varepsilon |A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon^4 / 2\nu.$$

Let  $R_0$  be a positive function defined on  $\mathbb{R}_+$  and satisfying

$$(2.18) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon R_0^2(\varepsilon) = 0$$

and choose  $\varepsilon_1$  such that, for  $0 < \varepsilon \leq \varepsilon_1$

$$(2.19) \quad c_0 \varepsilon^{1/2} R_0(\varepsilon) \leq \nu / 16.$$

Assume also (see (2.36)) that

$$(2.20) \quad \varepsilon^2 |f^\varepsilon|_\varepsilon^2 \leq R_0^2(\varepsilon) / \nu^2.$$

We then infer from (2.11) and (2.17) that

$$(2.21) \quad |A_\varepsilon^{1/2}V^\varepsilon|_\varepsilon^2 \leq 2c_0^2\varepsilon R_0^2(\varepsilon)|A_\varepsilon^{1/2}w^\varepsilon|_\varepsilon^2/\nu^2.$$

Thanks to (2.18) and (2.21),  $|A_\varepsilon^{1/2}V^\varepsilon|_\varepsilon^2$  is negligible compared to  $|A_\varepsilon^{1/2}w^\varepsilon|_\varepsilon^2$  for  $\varepsilon$  small. We have proved the

LEMMA 2.1. *Let  $w^\varepsilon$  (resp.  $v^\varepsilon$ ) be the solution of the Stokes problem (resp. the nonlinear stationary Navier–Stokes equations) in the thin domain  $\Omega_\varepsilon$ . Assume that (2.18)–(2.20) hold. Then we can write  $v^\varepsilon = w^\varepsilon + V^\varepsilon$ , with  $V^\varepsilon$  small compared to  $w^\varepsilon$ , i.e.,*

$$(2.22) \quad \lim_{\varepsilon \rightarrow 0} \frac{|A_\varepsilon^{1/2}V^\varepsilon|_\varepsilon^2}{|A_\varepsilon^{1/2}w^\varepsilon|_\varepsilon^2} = 0.$$

**2.2. Comparison between the evolutionary and the stationary problems.** In this subsection we prove the global existence of the strong solution  $u^\varepsilon(t)$  for  $\varepsilon$  small and show that up to a time boundary layer near  $t = 0$ , the solution converges exponentially (in time) to a stationary solution of the Navier–Stokes equations. We also show that the convergence, when  $\varepsilon$  goes to zero, is exponential as long as the initial data belongs to a ball in  $H^1$  with radius less than  $\nu/(16c_0\varepsilon^{1/2})$  and center  $v^\varepsilon$ , a solution of the stationary problem.

Let  $U^\varepsilon(t) = u^\varepsilon(t) - v^\varepsilon$ . The equations satisfied by  $U^\varepsilon(t)$  are:

$$(2.23) \quad \begin{cases} \frac{\partial U^\varepsilon}{\partial t} - \nu \Delta U^\varepsilon + (U^\varepsilon \cdot \nabla)U^\varepsilon + (U^\varepsilon \cdot \nabla)v^\varepsilon \\ \quad + (v^\varepsilon \cdot \nabla)U^\varepsilon + \nabla(p^\varepsilon - q^\varepsilon) = 0 & \text{in } \Omega_\varepsilon, \\ \operatorname{div} U^\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ U^\varepsilon = 0 & \text{on } \omega \times \{0, \varepsilon\}, \\ U^\varepsilon \text{ is periodic in the directions } x_1 \text{ and } x_2, \end{cases}$$

and the initial condition reads

$$(2.24) \quad U^\varepsilon(0) = u_0^\varepsilon - v^\varepsilon.$$

Using equations (2.23), we obtain

$$(2.25) \quad \frac{1}{2} \frac{d}{dt} |A_\varepsilon^{1/2}U^\varepsilon(t)|_\varepsilon^2 + \nu |A_\varepsilon U^\varepsilon(t)|_\varepsilon^2 \leq |b(U^\varepsilon, U^\varepsilon, A_\varepsilon U^\varepsilon)| + |b(U^\varepsilon, v^\varepsilon, A_\varepsilon U^\varepsilon)| + |b(v^\varepsilon, U^\varepsilon, A_\varepsilon U^\varepsilon)|,$$

and with inequalities (1.4), (1.8) and (1.10), we can write

$$(2.26) \quad \frac{1}{2} \frac{d}{dt} |A_\varepsilon^{1/2}U^\varepsilon(t)|_\varepsilon^2 + \frac{\nu}{2} |A_\varepsilon U^\varepsilon(t)|_\varepsilon^2 \leq c_0\varepsilon^{1/2} |A_\varepsilon^{1/2}U^\varepsilon(t)|_\varepsilon |A_\varepsilon U^\varepsilon(t)|_\varepsilon^2 + c_0\varepsilon^{1/2} |A_\varepsilon^{1/2}v^\varepsilon|_\varepsilon |A_\varepsilon U^\varepsilon(t)|_\varepsilon^2.$$



Hence,

$$(2.27) \quad \frac{d}{dt} |A_\varepsilon^{1/2} U^\varepsilon(t)|_\varepsilon^2 + [\nu - 2c_0 \varepsilon^{1/2} |A_\varepsilon^{1/2} U^\varepsilon(t)|_\varepsilon - 2c_0 \varepsilon^{1/2} |A_\varepsilon^{1/2} v^\varepsilon|_\varepsilon] |A_\varepsilon U^\varepsilon(t)|_\varepsilon^2 \leq 0.$$

With  $R_0$  defined as in (2.18), (2.19), we supplement (2.20) by assuming that

$$(2.28) \quad |A_\varepsilon^{1/2} U_0^\varepsilon|_\varepsilon^2 + \varepsilon^2 |f^\varepsilon|_\varepsilon^2 / \nu^2 \leq R_0^2(\varepsilon).$$

Then there exists  $T(\varepsilon) > 0$  such that

$$(2.29) \quad |A_\varepsilon^{1/2} U^\varepsilon(t)|_\varepsilon^2 \leq 4R_0^2(\varepsilon) \quad \text{for } 0 \leq t \leq T(\varepsilon).$$

Let  $[0, T(\varepsilon))$  denote the maximal interval on which (2.29) holds. Note that if  $T(\varepsilon) < \infty$ , then

$$(2.30) \quad |A_\varepsilon^{1/2} U^\varepsilon(T(\varepsilon))|_\varepsilon^2 = 4R_0^2(\varepsilon).$$

We infer from (2.6) and (2.29) that

$$(2.31) \quad \frac{d}{dt} |A_\varepsilon^{1/2} U^\varepsilon(t)|_\varepsilon^2 + [\nu - 16c_0 \varepsilon^{1/2} R_0(\varepsilon)] |A_\varepsilon U^\varepsilon(t)|_\varepsilon^2 \leq 0, \quad 0 \leq t \leq T(\varepsilon).$$

Using (2.19) we see that for  $0 < \varepsilon \leq \varepsilon_1$  and  $0 \leq t \leq T(\varepsilon)$ , we have by the Poincaré inequality

$$(2.32) \quad \frac{d}{dt} |A_\varepsilon^{1/2} U^\varepsilon(t)|_\varepsilon^2 + \frac{\nu}{2\varepsilon^2} |A_\varepsilon^{1/2} U^\varepsilon(t)|_\varepsilon^2 \leq 0,$$

which implies that  $|A_\varepsilon^{1/2} U^\varepsilon(t)|_\varepsilon^2$  is decreasing as a function of  $t$  and therefore  $T(\varepsilon) = +\infty$ , for  $\varepsilon \leq \varepsilon_1$ . Moreover, we have

$$(2.33) \quad \begin{aligned} |A_\varepsilon^{1/2} U^\varepsilon(t)|_\varepsilon^2 &\leq |A_\varepsilon^{1/2} U_0^\varepsilon|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) \\ &\leq |A_\varepsilon^{1/2} u_0^\varepsilon - A_\varepsilon^{1/2} v^\varepsilon|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right). \end{aligned}$$

Finally, we write

$$(2.34) \quad w^\varepsilon(t) = v^\varepsilon + U^\varepsilon(t) = w^\varepsilon + V^\varepsilon + U^\varepsilon(t),$$

where  $w^\varepsilon$  is the unique solution of the Stokes problem with exterior force  $f^\varepsilon$ , and  $V^\varepsilon$  and  $U^\varepsilon(t)$  satisfy

$$(2.35) \quad \|V^\varepsilon\|_\varepsilon^2 \leq 2c_0^2 \varepsilon \|w^\varepsilon\|_\varepsilon^4 / \nu^2$$

and

$$\|U^\varepsilon(t)\|_\varepsilon^2 \leq \|u_0^\varepsilon - w^\varepsilon - V^\varepsilon\|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right), \quad t \geq 0$$

**THEOREM 2.2.** *Let  $R_0(\varepsilon)$  be a monotone positive function satisfying condition  $\lim_{\varepsilon \rightarrow 0} \varepsilon R_0^2(\varepsilon) = 0$ . Assume that  $v^\varepsilon$  is a solution of the stationary Navier–Stokes equations with exterior force  $f^\varepsilon$  in the domain  $\Omega_\varepsilon$ , and*

$$(2.36) \quad \|u_0^\varepsilon - v^\varepsilon\|_\varepsilon^2 + \varepsilon^2 |f^\varepsilon|_\varepsilon^2 / \nu^2 \leq R_0^2(\varepsilon).$$

*Then there exists  $\varepsilon_1 = \varepsilon_1(\nu)$  such that for  $0 < \varepsilon \leq \varepsilon_1$ , the maximal time  $T(\varepsilon)$  of existence of the strong solution  $u_\varepsilon(t)$  of the 3D-Navier–Stokes equations in  $\Omega_\varepsilon$  satisfies  $T(\varepsilon) = \infty$ , and for all  $t \geq 0$*

$$(2.37) \quad \|u^\varepsilon(t) - v^\varepsilon\|_\varepsilon^2 \leq \|u_0^\varepsilon - v^\varepsilon\|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right).$$

*Moreover, if  $w^\varepsilon$  is the unique solution of the Stokes problem with exterior force  $f^\varepsilon$ , then*

$$(2.38) \quad u^\varepsilon(t) = w^\varepsilon + V^\varepsilon + U^\varepsilon(t), \quad \forall t \geq 0,$$

*with*

$$(2.39) \quad \|V^\varepsilon\|_\varepsilon^2 \leq \frac{c_0^2}{2\nu} \varepsilon R_0^2(\varepsilon) \|w^\varepsilon\|_\varepsilon^2$$

*and for all  $t \geq 0$*

$$\|U^\varepsilon(t)\|_\varepsilon^2 \leq \|u_0^\varepsilon - v^\varepsilon\|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right).$$

**REMARK 2.1.** (i) We obtained in Theorem 2.1 an improvement for the global regularity result obtained in [23]. Note that the conditions on the data are given in (2.36); in particular, due to (2.19)  $u_0^\varepsilon$  can belong to a ball in  $\mathbb{H}^1(\Omega_\varepsilon)$  of center  $v^\varepsilon$  and radius  $\nu/(16c_0\varepsilon^{1/2})$ .

(ii) We also obtained an asymptotic expansion for the solution  $u^\varepsilon(t)$  for  $\varepsilon$  small which is uniformly valid in time. This asymptotic expansion suggests that the attractor of the dynamical system associated with the Navier–Stokes equation with Dirichlet-periodic boundary condition in the thin domain  $\Omega_\varepsilon$  reduces to the set of stationary solutions, when  $\varepsilon$  is small enough.

(iii) The solution  $w^\varepsilon$  to the stationary problem (2.7)–(2.10) which approximates  $v^\varepsilon$  and hence  $u^\varepsilon$ , can be itself approximated by a simpler expression, possibly an explicit one. For example, in the case of a pressure driven flow,

$$(2.40) \quad f^\varepsilon = P e_1,$$

where  $P$  is constant (the pressure gradient), then  $w^\varepsilon \approx \varphi^\varepsilon e_1$ , with

$$(2.41) \quad \varphi^\varepsilon = P x_3(\varepsilon - x_3)/2\nu.$$

Note that since  $0 < x_3 < \varepsilon$ ,  $\varphi^\varepsilon$  is of order of  $\varepsilon^2$ .

### 3. The purely periodic boundary condition

This section is devoted to the asymptotic study of the solutions  $u^\varepsilon(t)$  of the 3D-Navier–Stokes equations, with the purely periodic boundary condition in the thin domains  $\Omega_\varepsilon$ , when the thickness  $\varepsilon$  goes to zero. We have shown in [23] that the average  $M_\varepsilon u^\varepsilon(t)$  converges to the strong solution of the 2D-Navier–Stokes equations. Therefore, one cannot expect to see the slow motion obtained in the case of the Dirichlet–Periodic condition (see Section 2).

The idea here is to establish some a priori estimates for  $N_\varepsilon u^\varepsilon(t) = u^\varepsilon(t) - M_\varepsilon u^\varepsilon(t)$ , which are similar to those obtained for  $u^\varepsilon(t)$  in the case of the Dirichlet-periodic condition, and to show that the dynamics of the 3D-Navier–Stokes equations is roughly carried by the orbits of a 2D-Navier–Stokes system up to the translation by a 3D-vector function which is independent of time, namely the solution of the Stokes problem with exterior force  $N_\varepsilon f^\varepsilon = f^\varepsilon - M_\varepsilon f^\varepsilon$ .

We recall from [23] the following result: we consider the problem (0.1)–(0.3) with periodic boundary conditions, and we assume that for arbitrary fixed constants  $K_1$  and  $K_2$ ,

$$(3.1) \quad a_0^2(\varepsilon) + \alpha^2(\varepsilon) \leq K_1 \varepsilon \ln |\ln \varepsilon|, \quad b_0^2(\varepsilon) + \beta^2(\varepsilon) \leq K_2 \ln |\ln \varepsilon|,$$

where we have set

$$\begin{aligned} a_0(\varepsilon) &= |A_\varepsilon^{1/2} M_\varepsilon u_0^\varepsilon|_\varepsilon, & b_0(\varepsilon) &= |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|_\varepsilon, \\ \alpha(\varepsilon) &= |M_\varepsilon f^\varepsilon|_\varepsilon, & \beta(\varepsilon) &= |N_\varepsilon f^\varepsilon|_\varepsilon. \end{aligned}$$

Then there exists  $\varepsilon_0 = \varepsilon_0(\nu, K_1, K_2, \omega) > 0$  such that for  $0 < \varepsilon < \varepsilon_0$ , the maximal time of existence  $T(\varepsilon)$  of the strong solution  $u^\varepsilon$  of the 3D-Navier–Stokes equations with periodic boundary conditions satisfies  $T(\varepsilon) = +\infty$ , and

$$u^\varepsilon \in \mathcal{C}([0, \infty); V_P^\varepsilon) \cap L^2(0, T; D(A_{\varepsilon P})) \quad \forall T > 0.$$

Moreover, considering a suitable constant  $K_3(\nu) > K_1 + K_2$  and setting

$$(3.2) \quad R_0^2(\varepsilon) = K_3 \ln |\ln \varepsilon|$$

we have for all  $t \geq 0$

$$(3.3) \quad |A_\varepsilon^{1/2} u^\varepsilon(t)|_\varepsilon^2 \leq \sigma R_0^2(\varepsilon),$$

where  $\sigma$  is constant (depending possibly on  $\nu$ ) such that  $\sigma > 2$ .

**3.1. An auxiliary pseudo-stationary problem.** We consider  $\bar{w}^\varepsilon = N_\varepsilon \bar{w}^\varepsilon$  solution of the following problem

$$(3.4) \quad \nu A_\varepsilon \bar{w}^\varepsilon + N_\varepsilon B_\varepsilon(N_\varepsilon \bar{w}^\varepsilon + M_\varepsilon u^\varepsilon, N_\varepsilon \bar{w}^\varepsilon + M_\varepsilon u^\varepsilon) - N_\varepsilon B_\varepsilon(M_\varepsilon u^\varepsilon, M_\varepsilon u^\varepsilon) = N_\varepsilon f^\varepsilon.$$

Equivalently for all  $v \in V_p$ ,  $\bar{w}^\varepsilon \in N_\varepsilon V_P$  satisfies

$$(3.5) \quad \nu(A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon, A_\varepsilon^{1/2} N_\varepsilon v)_\varepsilon + b_\varepsilon(N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon v) + b_\varepsilon(M_\varepsilon u^\varepsilon, N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon v) \\ + b_\varepsilon(N_\varepsilon \bar{w}^\varepsilon, M_\varepsilon u^\varepsilon, N_\varepsilon v) = (N_\varepsilon f^\varepsilon, N_\varepsilon v)_\varepsilon.$$

Since  $u^\varepsilon = u^\varepsilon(t)$ , we shall consider the time as a parameter.

The proof of the existence and uniqueness of  $\bar{w}^\varepsilon$  is standard (note that the uniqueness holds if we consider  $\varepsilon$  small enough). We omit the details and will only derive the estimates for  $\bar{w}^\varepsilon$ . For this purpose and throughout this section, we use extensively the following estimates on the trilinear form  $b_\varepsilon$

LEMMA 3.1. *Let  $q \in (0, 1/2)$ . There exists a positive constant  $c_1(q)$ , independent of  $\varepsilon$ , such that:*

$$|b_\varepsilon(M_\varepsilon u, N_\varepsilon v, w)| \leq c_1 \varepsilon^q |A_\varepsilon^{1/2} M_\varepsilon u|_\varepsilon |A_\varepsilon N_\varepsilon v|_\varepsilon |w|_\varepsilon \\ |b_\varepsilon(N_\varepsilon v, M_\varepsilon u, w)| \leq c_1 \varepsilon^{1/2} |A_\varepsilon^{1/2} M_\varepsilon u|_\varepsilon |A_\varepsilon N_\varepsilon v|_\varepsilon |w|_\varepsilon$$

for all  $u \in D(A_\varepsilon^{1/2})$ ,  $v \in D(A_\varepsilon)$ ,  $w \in \mathbb{L}^2(\Omega_\varepsilon)$ ,

$$|b_\varepsilon(N_\varepsilon u, N_\varepsilon v, w)| \leq c_1 |A_\varepsilon^{1/2} N_\varepsilon u|_\varepsilon^{1/2} |A_\varepsilon N_\varepsilon u|_\varepsilon^{1/2} |A_\varepsilon^{1/2} N_\varepsilon v|_\varepsilon |w|_\varepsilon \\ \leq c_1 \varepsilon^{1/2} |A_\varepsilon N_\varepsilon u|_\varepsilon |A_\varepsilon^{1/2} N_\varepsilon v|_\varepsilon |w|_\varepsilon$$

for all  $u \in D(A_\varepsilon)$ ,  $v \in D(A_\varepsilon^{1/2})$ ,  $w \in \mathbb{L}^2(\Omega_\varepsilon)$ ,

$$|b_\varepsilon(N_\varepsilon u, N_\varepsilon v, w)| \leq c_1 \varepsilon^{1/2} |A_\varepsilon^{1/2} N_\varepsilon u|_\varepsilon |A_\varepsilon N_\varepsilon v|_\varepsilon |w|_\varepsilon$$

for all  $u \in D(A_\varepsilon^{1/2})$ ,  $v \in D(A_\varepsilon)$ ,  $w \in \mathbb{L}^2(\Omega_\varepsilon)$ .

This lemma is a slight generalization of Lemma 2.7 in [23]; we omit the details of the proof, which essentially relies on the functional inequalities (1.5), (1.7), (1.9) and (1.11).

*Estimates for  $\bar{w}^\varepsilon$ .* We set  $N_\varepsilon v = N_\varepsilon \bar{w}^\varepsilon$  in (3.5) and obtain

$$(3.6) \quad \nu |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon^2 + b_\varepsilon(N_\varepsilon \bar{w}^\varepsilon, M_\varepsilon u^\varepsilon, N_\varepsilon \bar{w}^\varepsilon) = (N_\varepsilon f^\varepsilon, N_\varepsilon \bar{w}^\varepsilon)_\varepsilon,$$

which by (1.9) leads to

$$(3.7) \quad \nu |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon^2 \\ \leq |N_\varepsilon f^\varepsilon|_\varepsilon |N_\varepsilon \bar{w}^\varepsilon|_\varepsilon + c |N_\varepsilon \bar{w}^\varepsilon|_{L^4(\Omega_\varepsilon)}^2 |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon|_\varepsilon \\ \leq \frac{\nu}{4} |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon^2 + \frac{2\varepsilon^2}{\nu} |N_\varepsilon f^\varepsilon|_\varepsilon^2 + c\varepsilon^{1/2} |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon|_\varepsilon,$$

where  $c$  is a constant independent of  $\varepsilon$ .

Now we take into account (3.2) and (3.3) and we obtain the existence of  $\varepsilon_1 = \varepsilon_1(\nu, \omega, K_1, K_2)$  such that for  $0 < \varepsilon \leq \varepsilon_1$ ,

$$(3.8) \quad |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon^2 \leq 4\varepsilon^2 |N_\varepsilon f^\varepsilon|_\varepsilon^2 / \nu^2$$

We observe that  $|\frac{d}{dt}N_\varepsilon\bar{w}^\varepsilon|_\varepsilon$  is small in a sense that we make precise now. Indeed, we differentiate (3.5) with respect to  $t$  and we obtain

$$(3.9) \quad \begin{aligned} & \nu \left( \frac{d}{dt} A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon, A_\varepsilon^{1/2} N_\varepsilon v \right)_\varepsilon + b_\varepsilon \left( \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon v \right) \\ & + b_\varepsilon \left( N_\varepsilon \bar{w}^\varepsilon, \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon v \right) + b_\varepsilon \left( \frac{d}{dt} M_\varepsilon u^\varepsilon, N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon v \right) \\ & + b_\varepsilon \left( M_\varepsilon u^\varepsilon, \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon v \right) + b_\varepsilon \left( \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon, M_\varepsilon u^\varepsilon, N_\varepsilon v \right) \\ & + b_\varepsilon \left( N_\varepsilon \bar{w}^\varepsilon, \frac{d}{dt} M_\varepsilon u^\varepsilon, N_\varepsilon v \right) = 0. \end{aligned}$$

For  $t > 0$  fixed, we set  $v = A_\varepsilon^{-1} \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon$  in (3.9) and we obtain

$$(3.10) \quad \begin{aligned} & \nu \left| \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon \right|_\varepsilon^2 + b_\varepsilon \left( A_\varepsilon N_\varepsilon v, N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon v \right) + b_\varepsilon \left( N_\varepsilon \bar{w}^\varepsilon, A_\varepsilon N_\varepsilon v, N_\varepsilon v \right) \\ & + b_\varepsilon \left( \frac{d}{dt} M_\varepsilon u^\varepsilon, N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon v \right) + b_\varepsilon \left( M_\varepsilon u^\varepsilon, A_\varepsilon N_\varepsilon v, N_\varepsilon v \right) \\ & + b_\varepsilon \left( A_\varepsilon N_\varepsilon v, M_\varepsilon u^\varepsilon, N_\varepsilon v \right) + b_\varepsilon \left( N_\varepsilon \bar{w}^\varepsilon, \frac{d}{dt} M_\varepsilon u^\varepsilon, N_\varepsilon v \right) = 0, \end{aligned}$$

so that, by Lemma 3.1, we obtain

$$(3.11) \quad \begin{aligned} \nu \left| \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon \right|_\varepsilon^2 & \leq 2c_1 \varepsilon^{1/2} |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon v|_\varepsilon^2 \\ & + 2c_1 \varepsilon^{1/2} \left| \frac{d}{dt} M_\varepsilon u^\varepsilon \right|_\varepsilon |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon v|_\varepsilon \\ & + 2c_1 \varepsilon^q |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon v|_\varepsilon^2 \end{aligned}$$

Using (3.8), (3.2) and (3.3), we deduce that there exists  $\varepsilon_2 = \varepsilon_2(\nu, \omega, K_1, K_2)$  such that if  $0 < \varepsilon \leq \varepsilon_2$ , then by (3.8)

$$(3.12) \quad \begin{aligned} \left| \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon \right|_\varepsilon & \leq \frac{c}{\nu} \varepsilon^{1/2} \left| \frac{d}{dt} M_\varepsilon u^\varepsilon \right|_\varepsilon |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon \\ & \leq c(\nu) \varepsilon^{3/2} \left| \frac{d}{dt} M_\varepsilon u^\varepsilon \right|_\varepsilon |N_\varepsilon f^\varepsilon|_\varepsilon. \end{aligned}$$

Now we need to bound  $|\frac{d}{dt}M_\varepsilon u^\varepsilon|_\varepsilon$  in terms of  $R_0^2(\varepsilon)$ . We have

$$(3.13) \quad \frac{1}{2} \frac{d}{dt} |A_\varepsilon^{1/2} u^\varepsilon|_\varepsilon^2 + \nu |A_\varepsilon u^\varepsilon|_\varepsilon^2 + b_\varepsilon(u^\varepsilon, u^\varepsilon, A_\varepsilon u^\varepsilon) = (f^\varepsilon, A_\varepsilon u^\varepsilon)_\varepsilon,$$

and therefore

$$(3.14) \quad \frac{d}{dt} |A_\varepsilon^{1/2} u^\varepsilon|_\varepsilon^2 + \nu |A_\varepsilon u^\varepsilon|_\varepsilon^2 \leq \frac{c}{\nu} |f^\varepsilon|_\varepsilon^2 + \frac{c}{\nu^3} |A_\varepsilon^{1/2} u^\varepsilon|_\varepsilon^6,$$

$c$  being a numerical constant (independent of  $\varepsilon$ ). Let  $t_0 > 0$  be an arbitrarily small time. We deduce from (3.1), (3.3) and (3.14) that

$$(3.15) \quad \nu \int_t^{t+t_0} |A_\varepsilon u^\varepsilon(s)|_\varepsilon^2 ds \leq c(\nu) [R_0^2(\varepsilon) + R_0^6(\varepsilon)] t_0, \quad \forall t \geq 0.$$

Since  $|du^\varepsilon/dt|_\varepsilon \leq \nu |A_\varepsilon u^\varepsilon|_\varepsilon + |B_\varepsilon(u^\varepsilon, u^\varepsilon)|_\varepsilon + |f^\varepsilon|_\varepsilon$ , a simple computation yields

$$(3.16) \quad \int_t^{t+t_0} \left| \frac{du^\varepsilon}{dt} \right|_\varepsilon^2 \leq c(\nu) R_0^2(\varepsilon) (1 + R_0^2(\varepsilon))^2 t_0, \quad \forall t \geq 0.$$

Now we differentiate (0.10) with respect to  $t$  and we obtain

$$(3.17) \quad \frac{d^2 u^\varepsilon}{dt^2} + \nu \frac{d}{dt} A_\varepsilon u^\varepsilon + B_\varepsilon \left( \frac{du^\varepsilon}{dt}, u^\varepsilon \right) + B_\varepsilon \left( u^\varepsilon, \frac{du^\varepsilon}{dt} \right) = 0,$$

which leads then to

$$(3.18) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \left| \frac{du^\varepsilon}{dt} \right|_\varepsilon^2 + \nu \left| A_\varepsilon^{1/2} \left( \frac{du^\varepsilon}{dt} \right) \right|_\varepsilon^2 &\leq \left| b_\varepsilon \left( \frac{du^\varepsilon}{dt}, u^\varepsilon, \frac{du^\varepsilon}{dt} \right) \right| \\ &\leq c \left| \frac{du^\varepsilon}{dt} \right|_\varepsilon^{1/2} \left| A_\varepsilon^{1/2} \left( \frac{du^\varepsilon}{dt} \right) \right|_\varepsilon^{3/2} |A_\varepsilon^{1/2} u^\varepsilon|_\varepsilon, \end{aligned}$$

$c$  being a numerical constant (independent of  $\varepsilon$ ). We infer from (3.18) that

$$(3.19) \quad \frac{d}{dt} \left| \frac{du^\varepsilon}{dt} \right|_\varepsilon^2 \leq \frac{c}{\nu^3} |A_\varepsilon^{1/2} u^\varepsilon|_\varepsilon^4 \left| \frac{du^\varepsilon}{dt} \right|_\varepsilon^2.$$

We apply the uniform Gronwall lemma recalled below (see Lemma 3.2) with

$$y = \left| \frac{du^\varepsilon}{dt} \right|_\varepsilon^2, \quad g = \frac{c}{\nu^3} |A_\varepsilon^{1/2} u^\varepsilon|_\varepsilon^4, \quad h = 0.$$

From (3.16) and (3.3), we infer the following estimates (say  $t_0 \leq 1$ )

$$\begin{aligned} \int_t^{t+t_0} g(s) ds &\leq c(\nu) R_0^4(\varepsilon) t_0 \leq c(\nu) R_0^4(\varepsilon), \\ \int_t^{t+t_0} y(s) ds &\leq c(\nu) R_0^2(\varepsilon) [1 + R_0^2(\varepsilon)]^2 t_0, \end{aligned}$$

so that

$$(3.20) \quad \left| \frac{du^\varepsilon}{dt} \right|_\varepsilon^2 \leq c(\nu) R_0^2(\varepsilon) [1 + R_0^2(\varepsilon)]^2 \exp(c(\nu) R_0^4(\varepsilon))$$

holds for every  $t \geq t_0 > 0$ . Since  $t_0 > 0$  is arbitrarily small and the right hand side of (3.20) is independent of  $t_0$ , (3.20) holds for (almost) every  $t > 0$ . We use (3.20) in (3.12) and we obtain

$$(3.21) \quad \left| \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon \right|_\varepsilon \leq c(\nu) \varepsilon^{3/2} R_0^2(\varepsilon) [1 + R_0^2(\varepsilon)] \exp(c(\nu) R_0^4(\varepsilon)).$$

Taking into account the expression of  $R_0^2(\varepsilon)$  given by (3.2) we conclude that, for any arbitrarily small  $\gamma > 0$ , there exists  $c = c(\nu, q, \gamma)$  such that

$$(3.22) \quad \left| \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon \right|_\varepsilon \leq c(\nu, q, \gamma) \varepsilon^{3/2-\gamma}, \quad \forall t > 0.$$

For the convenience of the reader we recall the uniform Gronwall lemma

LEMMA 3.2. *Let  $g, h, y$  be three positive locally integrable functions on  $(t_0, \infty)$  such that  $y'$  is locally integrable on  $(t_0, \infty)$ , and which satisfy for  $t \geq t_0$*

$$\frac{dy}{dt} \leq gy + h, \quad \int_t^{t+r} g(s) ds \leq a_1, \quad \int_t^{t+r} h(s) ds \leq a_2, \quad \int_t^{t+r} y(s) ds \leq a_3,$$

where  $a_1, a_2, a_3$  and  $r$  are positive constants. Then

$$y(t+r) \leq \left( \frac{a_3}{r} + a_2 \right) \exp(a_1), \quad \forall t \geq t_0.$$

**3.2. An auxiliary two dimensional problem.** We consider first the following evolutionary Navier–Stokes problem in  $\Omega_\varepsilon$ :

$$(3.23) \quad \frac{\partial \bar{u}^\varepsilon}{\partial t} - \nu \Delta \bar{u}^\varepsilon + (\bar{u}^\varepsilon \cdot \nabla) \bar{u}^\varepsilon + \nabla \bar{p}_\varepsilon = M_\varepsilon f^\varepsilon \quad \text{in } \Omega_\varepsilon,$$

$$(3.24) \quad \operatorname{div} \bar{u}^\varepsilon = 0 \quad \text{in } \Omega_\varepsilon,$$

$$(3.25) \quad u^\varepsilon \text{ is periodic in the directions } x_1, x_2 \text{ and } x_3,$$

with the initial condition

$$(3.26) \quad \bar{u}^\varepsilon|_{t=0} = M_\varepsilon u_0^\varepsilon.$$

Since the forcing term  $M_\varepsilon f^\varepsilon$  and the initial data  $M_\varepsilon u_0^\varepsilon$  are independent of  $x_3$ , we can show that there exists a unique global strong solution  $\bar{u}^\varepsilon(t)$  of this three dimensional problem which is independent of  $x_3$ , i.e.  $\bar{u}^\varepsilon = M_\varepsilon \bar{u}^\varepsilon$ . For that purpose we look for  $\bar{u}^\varepsilon = \bar{u}_{2D}^\varepsilon + \bar{u}_v^\varepsilon$ , where  $\bar{u}_{2D}^\varepsilon = (\bar{u}_1^\varepsilon, \bar{u}_2^\varepsilon, 0)$ ,  $\bar{u}_v^\varepsilon = (0, 0, \bar{u}_3^\varepsilon)$ , and  $\bar{u}_{2D}^\varepsilon$  is first defined by the following two dimensional problem:

$$(3.27) \quad \frac{\partial \bar{u}_{2D}^\varepsilon}{\partial t} - \nu \Delta' \bar{u}_{2D}^\varepsilon + (\bar{u}_{2D}^\varepsilon \cdot \nabla') \bar{u}_{2D}^\varepsilon + \nabla' \bar{p}_\varepsilon = M_\varepsilon f_{2D}^\varepsilon \quad \text{in } \omega,$$

$$(3.28) \quad \operatorname{div}' \bar{u}_{2D}^\varepsilon = 0 \quad \text{in } \omega,$$

$$(3.29) \quad \bar{u}_{2D}^\varepsilon \text{ is periodic in the directions } x_1 \text{ and } x_2,$$

with the initial condition

$$(3.30) \quad \bar{u}_{2D}^\varepsilon|_{t=0} = M_\varepsilon (u_{01}^\varepsilon, u_{02}^\varepsilon, 0),$$

where  $\Delta', \nabla', \operatorname{div}'$  are two-dimensional operators,  $f_{2D}^\varepsilon = (f_1^\varepsilon, f_2^\varepsilon, 0)$ . Note that  $\bar{u}_{2D}^\varepsilon$  depends on  $\varepsilon$  only because  $f_{2D}^\varepsilon$  and  $M_\varepsilon(u_{01}^\varepsilon, u_{02}^\varepsilon, 0)$  depend on  $\varepsilon$ . We then define  $\bar{u}_v^\varepsilon$  as the solution of the two-dimensional problem

$$(3.31) \quad \frac{\partial \bar{u}_v^\varepsilon}{\partial t} - \nu \Delta' \bar{u}_v^\varepsilon + (\bar{u}_{2D}^\varepsilon \cdot \nabla') \bar{u}_v^\varepsilon = M_\varepsilon f_3^\varepsilon \vec{e}_3 \quad \text{in } \omega,$$

$$(3.32) \quad \int_\omega \bar{u}_v^\varepsilon \, dx' = 0,$$

$$(3.33) \quad \bar{u}_v^\varepsilon \text{ is periodic in the directions } x_1 \text{ and } x_2,$$

with the initial condition

$$(3.34) \quad \bar{u}_v^\varepsilon|_{t=0} = M_\varepsilon u_{03}^\varepsilon \vec{e}_3.$$

The proof of the existence and uniqueness of  $\bar{u}_{2D}^\varepsilon$  is classical,  $\bar{u}_{2D}^\varepsilon$  is the global strong solution of a 2D-Navier–Stokes problem [9], [10]. Then we solve the linear problem for  $\bar{u}_v^\varepsilon$ ; it is then easy to verify that  $\bar{u}^\varepsilon = \bar{u}_{2D}^\varepsilon + \bar{u}_v^\varepsilon$  is a strong global solution of (3.23)–(3.26).

*Estimates for  $\bar{u}^\varepsilon$  in  $L^2(\omega)$ .* First we multiply (3.27) by  $\bar{u}^\varepsilon$ , integrate over  $\omega$  and obtain

$$(3.35) \quad \frac{1}{2} \frac{d}{dt} |\bar{u}^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A}^{1/2} \bar{u}^\varepsilon|_{L^2(\omega)}^2 = (M_\varepsilon f^\varepsilon, \bar{u}^\varepsilon)_{L^2(\omega)},$$

where  $\tilde{A}$  is the 2D-Stokes operator in  $\omega$ . Thus

$$(3.36) \quad \frac{d}{dt} |\bar{u}^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A}^{1/2} \bar{u}^\varepsilon|_{L^2(\omega)}^2 \leq \frac{1}{\nu \lambda_1} |M_\varepsilon f^\varepsilon|_{L^2(\omega)}^2,$$

$\lambda_1$  being the first eigenvalue of  $\tilde{A}$ . We deduce that for all  $t \geq 0$

$$(3.37) \quad \int_t^{t+1} |\tilde{A}^{1/2} \bar{u}^\varepsilon(s)|_{L^2(\omega)}^2 \, ds \leq \frac{1}{\nu^2 \lambda_1} |M_\varepsilon f^\varepsilon|_{L^2(\omega)}^2 + \frac{1}{\nu \lambda_1} |M_\varepsilon f^\varepsilon|_{L^2(\omega)}^2 + |\bar{u}^\varepsilon(0)|_{L^2(\omega)}^2 \exp(-\nu \lambda_1 t),$$

and taking into account (3.1), we obtain

$$(3.38) \quad \int_t^{t+1} |\tilde{A}^{1/2} \bar{u}^\varepsilon(s)|_{L^2(\omega)}^2 \, ds \leq c(\nu) [|\tilde{A}^{1/2} M_\varepsilon u_0^\varepsilon|_{L^2(\omega)}^2 + |M_\varepsilon f^\varepsilon|_{L^2(\omega)}^2] \leq c(\nu) K_1 \ln |\ln \varepsilon|.$$

*Estimates for  $\bar{u}_{2D}^\varepsilon$  in  $H^1(\omega)$ .* We multiply (3.27) by  $\tilde{A} \bar{u}_{2D}^\varepsilon$ , integrate over  $\omega$  and we obtain

$$(3.39) \quad \frac{1}{2} \frac{d}{dt} |\tilde{A}^{1/2} \bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A} \bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^2 + \tilde{b}(\bar{u}_{2D}^\varepsilon, \bar{u}_{2D}^\varepsilon, \tilde{A} \bar{u}_{2D}^\varepsilon) = (M_\varepsilon f_{2D}^\varepsilon, \tilde{A} \bar{u}_{2D}^\varepsilon)_{L^2(\omega)}$$



Note that  $\tilde{b}(\bar{u}_{2D}^\varepsilon, \bar{u}_{2D}^\varepsilon, \tilde{A}\bar{u}_{2D}^\varepsilon) = 0$  (space periodic case). Thus we deduce:

$$(3.40) \quad \frac{d}{dt} |\tilde{A}^{1/2} \bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A}\bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^2 \leq \frac{1}{\nu} |M_\varepsilon f_{2D}^\varepsilon|_{L^2(\omega)}^2,$$

and consequently for all  $t \geq 0$

$$(3.41) \quad |\tilde{A}^{1/2} \bar{u}_{2D}^\varepsilon(t)|_{L^2(\omega)}^2 \leq |\tilde{A}^{1/2} \bar{u}_{2D}^\varepsilon(0)|_{L^2(\omega)}^2 \exp(-\nu \lambda_1 t) + \frac{1}{\nu^2 \lambda_1} |M_\varepsilon f_{2D}^\varepsilon|_{L^2(\omega)}^2,$$

and also

$$(3.42) \quad \nu \int_t^{t+t_0} |\tilde{A}\bar{u}_{2D}^\varepsilon(s)|_{L^2(\omega)}^2 ds \leq |\tilde{A}^{1/2} \bar{u}_{2D}^\varepsilon(0)|_{L^2(\omega)}^2 \exp(-\nu \lambda_1 t) + \frac{1}{\nu^2 \lambda_1} |M_\varepsilon f_{2D}^\varepsilon|_{L^2(\omega)}^2 + \frac{1}{\nu} |M_\varepsilon f_{2D}^\varepsilon|_{L^2(\omega)}^2.$$

Taking into account the hypothesis (3.1), for all  $t \geq 0$  we obtain from (3.40) and (3.42)

$$(3.43) \quad |\tilde{A}^{1/2} \bar{u}_{2D}^\varepsilon(t)|_{L^2(\omega)}^2 \leq c(\nu) [|\tilde{A}^{1/2} M_\varepsilon u_0^\varepsilon|_{L^2(\omega)}^2 + |M_\varepsilon f^\varepsilon|_{L^2(\omega)}^2] \leq c(\nu) K_1 \ln |\ln \varepsilon|,$$

and also

$$(3.44) \quad \int_t^{t+t_0} |\tilde{A}\bar{u}_{2D}^\varepsilon(s)|_{L^2(\omega)}^2 ds \leq c(\nu) K_1 \ln |\ln \varepsilon|.$$

*Estimates for  $\bar{u}_v^\varepsilon$  in  $H^1(\omega)$ .* Multiply (3.31) by  $\tilde{A}\bar{u}_v^\varepsilon$  and integrate over  $\omega$  to obtain

$$(3.45) \quad \frac{1}{2} \frac{d}{dt} |\tilde{A}^{1/2} \bar{u}_v^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A}\bar{u}_v^\varepsilon|_{L^2(\omega)}^2 + \tilde{b}(\bar{u}_{2D}^\varepsilon, \bar{u}_v^\varepsilon, \tilde{A}\bar{u}_v^\varepsilon) = (M_\varepsilon f_3^\varepsilon \tilde{e}_3, \tilde{A}\bar{u}_v^\varepsilon)_{L^2(\omega)},$$

and therefore with Agmon's inequality,

$$(3.46) \quad \frac{1}{2} \frac{d}{dt} |\tilde{A}^{1/2} \bar{u}_v^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A}\bar{u}_v^\varepsilon|_{L^2(\omega)}^2 \leq |M_\varepsilon f_3^\varepsilon|_{L^2(\omega)} |\tilde{A}\bar{u}_v^\varepsilon|_{L^2(\omega)} + c(\omega) |\bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^{1/2} |\tilde{A}\bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^{1/2} |\tilde{A}^{1/2} \bar{u}_v^\varepsilon|_{L^2(\omega)} |\tilde{A}\bar{u}_v^\varepsilon|_{L^2(\omega)}.$$

We infer from (3.46) that

$$(3.47) \quad \frac{d}{dt} |\tilde{A}^{1/2} \bar{u}_v^\varepsilon|_{L^2(\omega)}^2 \leq \frac{c}{\nu} |M_\varepsilon f_3^\varepsilon|_{L^2(\omega)}^2 + \frac{c}{\nu \lambda_1} |\tilde{A}\bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^2 |\tilde{A}^{1/2} \bar{u}_v^\varepsilon|_{L^2(\omega)}^2.$$

We apply the uniform Gronwall lemma with

$$y = |\tilde{A}^{1/2} \bar{u}_v^\varepsilon|_{L^2(\omega)}^2, \quad g = c |\tilde{A}\bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^2 / \nu \lambda_1, \quad h = c |M_\varepsilon f^\varepsilon|_{L^2(\omega)}^2 / \nu.$$

We use (3.44), (3.49) and (3.1) and for all  $t \geq 0$  we deduce

$$\begin{aligned} \int_t^{t+1} g(s) ds &\leq c(\nu)K_1 \ln |\ln \varepsilon| = a_1, \\ \int_t^{t+1} h(s) ds &\leq c(\nu)K_1 \ln |\ln \varepsilon| = a_2, \\ \int_t^{t+1} y(s) ds &\leq c(\nu)K_1 \ln |\ln \varepsilon| = a_3, \end{aligned}$$

so that

$$(3.48) \quad \begin{aligned} |\tilde{A}^{1/2} \bar{u}_v^\varepsilon(t)|_{L^2(\omega)}^2 &\leq [|\tilde{A}^{1/2} \bar{u}_v^\varepsilon(0)|_{L^2(\omega)}^2 + a_2 + a_3] \exp(a_1) \\ &\leq c(\nu)K_1 \ln |\ln \varepsilon| \exp(c(\nu)K_1 \ln |\ln \varepsilon|) \end{aligned}$$

for all  $t \geq 0$ . We infer from (3.43) and (3.48) that for all  $t \geq 0$

$$(3.49) \quad |\tilde{A}^{1/2} \bar{u}^\varepsilon(t)|_{L^2(\omega)}^2 \leq c(\nu)K_1 \ln |\ln \varepsilon| (1 + \exp(c(\nu)K_1 \ln |\ln \varepsilon|)).$$

Note furthermore, that

$$(3.50) \quad |\tilde{A}^{1/2} \bar{u}^\varepsilon(t)|_\varepsilon^2 = \varepsilon |\tilde{A}^{1/2} \bar{u}^\varepsilon(t)|_{L^2(\omega)}^2 \leq c(\nu, \gamma) \varepsilon^{1-\gamma}$$

for all  $t \geq 0$  and any arbitrarily small  $\gamma > 0$ .

**3.3. The comparison theorem.** Our first result stated at the end of section 3.3 (Theorem 3.3) gives a comparison between  $u^\varepsilon$  and  $\bar{u}^\varepsilon + \bar{w}^\varepsilon$ . We set  $U^\varepsilon = u^\varepsilon - \bar{u}^\varepsilon - \bar{w}^\varepsilon$  and we aim to estimate the  $N_\varepsilon$  and the  $M_\varepsilon$  components of  $U^\varepsilon$ .

*Estimates for  $N_\varepsilon U^\varepsilon = N_\varepsilon u^\varepsilon - N_\varepsilon \bar{w}^\varepsilon$ .* Starting from the weak formulation for the equations defining  $u^\varepsilon$ ,  $\bar{u}^\varepsilon$  and  $\bar{w}^\varepsilon$ , for all  $v \in V_p^\varepsilon$  we obtain

$$(3.51) \quad \begin{aligned} \frac{d}{dt} (N_\varepsilon U^\varepsilon, N_\varepsilon v)_\varepsilon + \nu (A_\varepsilon^{1/2} N_\varepsilon U^\varepsilon, A_\varepsilon^{1/2} N_\varepsilon v)_\varepsilon + b_\varepsilon (N_\varepsilon U^\varepsilon, N_\varepsilon U^\varepsilon, N_\varepsilon v) \\ + b_\varepsilon (N_\varepsilon U^\varepsilon, N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon v) + b_\varepsilon (N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon U^\varepsilon, N_\varepsilon v) \\ + b_\varepsilon (N_\varepsilon U^\varepsilon, M_\varepsilon u^\varepsilon, N_\varepsilon v) + b_\varepsilon (M_\varepsilon u^\varepsilon, N_\varepsilon U^\varepsilon, N_\varepsilon v) \\ + \left( \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon v \right)_\varepsilon = 0, \end{aligned}$$

$$(3.52) \quad N_\varepsilon U^\varepsilon|_{t=0} = N_\varepsilon u_0^\varepsilon - N_\varepsilon \bar{w}^\varepsilon(0).$$

We choose  $v = A_\varepsilon U^\varepsilon(t)$  and we obtain, using Lemma 3.1

$$(3.53) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} |A_\varepsilon^{1/2} N_\varepsilon U^\varepsilon|_\varepsilon^2 + \nu |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon^2 \\ \leq c_1 \varepsilon^{1/2} |A_\varepsilon^{1/2} N_\varepsilon U^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon^2 + 2c_1 \varepsilon^{1/2} |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon^2 \\ + c_1 \varepsilon^{1/2} |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon^2 + c_1 \varepsilon^q |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon^2 \\ + \left| \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon \right|_\varepsilon |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon; \end{aligned}$$

since  $0 < q < 1/2$  and  $0 < \varepsilon < 1$ , we deduce from (3.53)

$$(3.54) \quad \frac{d}{dt} |A_\varepsilon^{1/2} N_\varepsilon U^\varepsilon|_\varepsilon^2 + [\nu - 4c_1 \varepsilon^q |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon|_\varepsilon - 2c_1 \varepsilon^q |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon - 4c_1 \varepsilon^{1/2} |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon] |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon^2 \leq \frac{1}{\nu} \left| \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon \right|_\varepsilon^2.$$

Now using (3.3), (3.8) and (3.2), we deduce that there exists  $\varepsilon_3 = \varepsilon_3(\nu, \omega, K_1, K_2)$  such that if  $0 < \varepsilon \leq \varepsilon_3$ , then by (3.22)

$$(3.55) \quad \frac{d}{dt} |A_\varepsilon^{1/2} N_\varepsilon U^\varepsilon|_\varepsilon^2 + \frac{\nu}{2} |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon^2 \leq \frac{1}{\nu} \left| \frac{d}{dt} N_\varepsilon \bar{w}^\varepsilon \right|_\varepsilon^2 \leq c(\nu, q, \gamma) \varepsilon^{3-\gamma}$$

( $\gamma$  being an arbitrarily small positive number). By the Cauchy–Schwarz inequality we have

$$|A_\varepsilon^{1/2} N_\varepsilon U^\varepsilon|_\varepsilon \leq \varepsilon |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon,$$

which gives together with (3.55)

$$(3.56) \quad |A_\varepsilon^{1/2} N_\varepsilon U^\varepsilon(t)|_\varepsilon^2 \leq |A_\varepsilon^{1/2} N_\varepsilon U^\varepsilon(0)|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + c(\nu, q, \gamma) \varepsilon^{5-\gamma}$$

for all  $t \geq 0$  (we recall that  $q \in (0, 1/2)$  is an arbitrary number and  $\gamma > 0$  is an arbitrarily small number).

*Estimates for  $M_\varepsilon U^\varepsilon = M_\varepsilon u^\varepsilon - M_\varepsilon \bar{u}^\varepsilon$ .* The weak formulation for  $M_\varepsilon U^\varepsilon$  reads:

$$(3.57) \quad \begin{aligned} \frac{d}{dt} (M_\varepsilon U^\varepsilon, M_\varepsilon v)_\varepsilon + \nu (A_\varepsilon^{1/2} M_\varepsilon U^\varepsilon, A_\varepsilon^{1/2} M_\varepsilon v) + b_\varepsilon (M_\varepsilon U^\varepsilon, M_\varepsilon U^\varepsilon, M_\varepsilon v) \\ + b_\varepsilon (M_\varepsilon U^\varepsilon, M_\varepsilon \bar{u}^\varepsilon, M_\varepsilon v) + b_\varepsilon (M_\varepsilon \bar{u}^\varepsilon, M_\varepsilon U^\varepsilon, M_\varepsilon v) \\ + b_\varepsilon (N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, M_\varepsilon v) = 0, \end{aligned}$$

for all  $v \in V_\varepsilon^\varepsilon$  with the initial condition

$$(3.58) \quad M_\varepsilon U^\varepsilon|_{t=0} = 0.$$

*Estimates for  $M_\varepsilon U_{2D}^\varepsilon$  in  $H^1$ .* We choose  $v = A_\varepsilon U_{2D}^\varepsilon$  in (3.57), where  $U_{2D}^\varepsilon = (U_{1D}^\varepsilon, U_{2D}^\varepsilon, 0)$  and we obtain

$$(3.59) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} |A_\varepsilon^{1/2} M_\varepsilon U_{2D}^\varepsilon|_\varepsilon^2 + \nu |A_\varepsilon M_\varepsilon U_{2D}^\varepsilon|_\varepsilon^2 + b_\varepsilon (M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon) \\ + b_\varepsilon (M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon \bar{u}_{2D}^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon) + b_\varepsilon (M_\varepsilon \bar{u}_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon) \\ + b_\varepsilon (N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon) = 0. \end{aligned}$$

Note that  $b_\varepsilon (M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon) = \varepsilon \tilde{b} (M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, \tilde{A} M_\varepsilon U_{2D}^\varepsilon) = 0$ .

Using the  $L^2$ -scalar product and the  $L^2$ -norm on  $\omega$  we rewrite (3.59) as:

$$(3.60) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} |\tilde{A}^{1/2} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}^2 \\ = -\tilde{b} (M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon \bar{u}_{2D}^\varepsilon, \tilde{A} M_\varepsilon U_{2D}^\varepsilon) - \tilde{b} (M_\varepsilon \bar{u}_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, \tilde{A} M_\varepsilon U_{2D}^\varepsilon) \\ - b_\varepsilon (N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon U_{2D}^\varepsilon) / \varepsilon. \end{aligned}$$

We estimate the nonlinear terms as follows:

$$\begin{aligned} & |\tilde{b}(M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon \bar{u}_{2D}^\varepsilon, \tilde{A} M_\varepsilon U_{2D}^\varepsilon)| + |\tilde{b}(M_\varepsilon \bar{u}_{2D}^\varepsilon, M_\varepsilon U_{2D}^\varepsilon, \tilde{A} M_\varepsilon U_{2D}^\varepsilon)| \\ & \leq c\lambda_1^{-1/2} |\tilde{A} M_\varepsilon \bar{u}_{2D}^\varepsilon|_{L^2(\omega)} |\tilde{A}^{1/2} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)} |\tilde{A} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}, \end{aligned}$$

$$\begin{aligned} \frac{1}{\varepsilon} |\tilde{b}_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon U^\varepsilon)| & \leq c_1 \varepsilon^{-1/2} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon U^\varepsilon|_\varepsilon |A_\varepsilon M_\varepsilon U^\varepsilon|_\varepsilon \\ & = c_1 |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon |\tilde{A} M_\varepsilon U^\varepsilon|_{L^2(\omega)}. \end{aligned}$$

We deduce then from (3.60)

$$\begin{aligned} (3.61) \quad & \frac{d}{dt} |\tilde{A}^{1/2} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}^2 \\ & \leq \frac{c}{\nu\lambda_1} |\tilde{A} M_\varepsilon \bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^2 |\tilde{A}^{1/2} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}^2 + \frac{c}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2. \end{aligned}$$

Then we apply the uniform Gronwall lemma with

$$(3.62) \quad y = |\tilde{A}^{1/2} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}^2,$$

$$(3.63) \quad g = \frac{c}{\nu\lambda_1} |\tilde{A} M_\varepsilon \bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^2,$$

$$(3.64) \quad h = \frac{c}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2.$$

We have the following estimate, using (3.44):

$$(3.65) \quad \int_t^{t+1} g(s) ds \leq c(\nu) K_1 \ln |\ln \varepsilon|, \quad \forall t \geq 0.$$

We recall from [23] (formula (3.13)) the following relation

$$\frac{d}{dt} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 + \frac{\nu}{2} |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2 \leq \frac{1}{\nu} |N_\varepsilon f^\varepsilon|_\varepsilon^2,$$

so that for all  $t \geq 0$  we have

$$|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \leq |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2} |N_\varepsilon f^\varepsilon|_\varepsilon^2,$$

and

$$\begin{aligned} (3.66) \quad & \frac{\nu}{2} \int_t^{t+1} |A_\varepsilon N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 ds \leq |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) \\ & + \frac{2\varepsilon^2}{\nu^2} |N_\varepsilon f^\varepsilon|_\varepsilon^2 + \frac{1}{\nu} |N_\varepsilon f^\varepsilon|_\varepsilon^2, \end{aligned}$$

which yield to

$$(3.67) \quad \int_t^{t+1} h(s) ds \leq c(\nu) R_0^4(\varepsilon), \quad \forall t \geq 0,$$

and finally,

$$\begin{aligned} \int_t^{t+1} y(s) ds &= \int_t^{t+1} |\tilde{A}^{1/2} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}^2 ds \\ &\leq 2 \int_t^{t+1} [|\tilde{A}^{1/2} M_\varepsilon u_{2D}^\varepsilon|_{L^2(\omega)}^2 + |\tilde{A}^{1/2} M_\varepsilon \bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^2] ds. \end{aligned}$$

We recall also from [23] (formula (3.27)) the following inequality:

$$\frac{d}{dt} |M_\varepsilon u^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A}^{1/2} M_\varepsilon u^\varepsilon|_{L^2(\omega)}^2 \leq \frac{1}{\nu \lambda_1} |M_\varepsilon f^\varepsilon|_{L^2(\omega)}^2 + \frac{c}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2,$$

which for all  $t \geq 0$  gives

$$\begin{aligned} \nu \int_t^{t+1} |\tilde{A}^{1/2} M_\varepsilon u^\varepsilon(s)|_{L^2(\omega)}^2 ds &\leq |M_\varepsilon u^\varepsilon(t)|_{L^2(\omega)}^2 + \frac{1}{\nu \lambda_1} |M_\varepsilon f^\varepsilon|_{L^2(\omega)}^2 \\ &\quad + \frac{c}{\nu} \int_t^{t+1} |\tilde{A}^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2 ds \\ &\leq c(\nu) [R_0^2(\varepsilon) + R_0^4(\varepsilon)]. \end{aligned}$$

Taking into account (3.39), we conclude

$$(3.68) \quad \int_t^{t+1} y(s) ds \leq c(\nu) [R_0^2(\varepsilon) + R_0^4(\varepsilon)], \quad \forall t \geq 0.$$

Using the usual and uniform Gronwall lemmas, we infer from (3.62), (3.64) and (3.65) that

$$|\tilde{A}^{1/2} M_\varepsilon U_{2D}^\varepsilon(t)|_{L^2(\omega)}^2 \leq c(\nu) [R_0^2(\varepsilon) + R_0^4(\varepsilon)] \exp(c(\nu) K_1 \ln |\ln \varepsilon|), \quad \forall t \geq 0.$$

*Estimates for  $M_\varepsilon U_v^\varepsilon$  in  $H^1$ .* We write  $v = A_\varepsilon U_v^\varepsilon$  in (3.57), where  $U_v^\varepsilon = (0, 0, U_3^\varepsilon)$ , and we find:

$$(3.70) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} |A_\varepsilon^{1/2} M_\varepsilon U_v^\varepsilon|_\varepsilon^2 + \nu |A_\varepsilon M_\varepsilon U_v^\varepsilon|_\varepsilon^2 &+ b_\varepsilon(M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon U_v^\varepsilon, A_\varepsilon M_\varepsilon U_v^\varepsilon) \\ &+ b_\varepsilon(M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon \bar{u}_v^\varepsilon, A_\varepsilon M_\varepsilon U_v^\varepsilon) + b_\varepsilon(M_\varepsilon \bar{u}_{2D}^\varepsilon, M_\varepsilon U_v^\varepsilon, A_\varepsilon M_\varepsilon U_v^\varepsilon) \\ &+ b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon U_v^\varepsilon) = 0. \end{aligned}$$

Using the  $L^2$  scalar product and the  $L^2$  norm on  $\omega$  we rewrite (3.70) as:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\tilde{A}^{1/2} M_\varepsilon U_v^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A} M_\varepsilon U_v^\varepsilon|_{L^2(\omega)}^2 &+ \tilde{b}(M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon U_v^\varepsilon, \tilde{A} M_\varepsilon U_v^\varepsilon) \\ &+ \tilde{b}(M_\varepsilon U_{2D}^\varepsilon, M_\varepsilon \bar{u}_v^\varepsilon, \tilde{A} M_\varepsilon U_v^\varepsilon) + \tilde{b}(M_\varepsilon \bar{u}_{2D}^\varepsilon, M_\varepsilon U_v^\varepsilon, \tilde{A} M_\varepsilon U_v^\varepsilon) \\ &+ b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon U_v^\varepsilon) / \varepsilon = 0, \end{aligned}$$

and then

$$\begin{aligned}
& \frac{d}{dt} |\tilde{A}^{1/2} M_\varepsilon U_v^\varepsilon|_{L^2(\omega)}^2 + \nu |\tilde{A} M_\varepsilon U_v^\varepsilon|_{L^2(\omega)}^2 \\
& \leq \frac{c}{\nu} |M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)} |\tilde{A} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)} |\tilde{A}^{1/2} M_\varepsilon U_v^\varepsilon|_{L^2(\omega)}^2 \\
& \quad + \frac{c}{\nu} |M_\varepsilon \bar{u}_{2D}^\varepsilon|_{L^2(\omega)} |\tilde{A} M_\varepsilon \bar{u}_{2D}^\varepsilon|_{L^2(\omega)} |\tilde{A}^{1/2} M_\varepsilon U_v^\varepsilon|_{L^2(\omega)}^2 \\
& \quad + \frac{c}{\nu} |M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)} |\tilde{A} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)} |\tilde{A}^{1/2} M_\varepsilon \bar{u}_v^\varepsilon|_{L^2(\omega)}^2 \\
& \quad + \frac{c}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u_\varepsilon^\varepsilon|_{L^2(\omega)}^2 |A_\varepsilon N_\varepsilon u_\varepsilon^\varepsilon|_{L^2(\omega)}^2
\end{aligned}$$

We apply again the uniform Gronwall lemma with

$$(3.71) \quad g = \frac{c}{\nu \lambda_1} [|\tilde{A} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)}^2 + |\tilde{A} M_\varepsilon \bar{u}_{2D}^\varepsilon|_{L^2(\omega)}^2],$$

$$(3.72) \quad h = \frac{c}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u_\varepsilon^\varepsilon|_{L^2(\omega)}^2 |A_\varepsilon N_\varepsilon u_\varepsilon^\varepsilon|_{L^2(\omega)}^2 + \frac{c}{\nu} |M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)} |\tilde{A} M_\varepsilon U_{2D}^\varepsilon|_{L^2(\omega)} |\tilde{A}^{1/2} M_\varepsilon \bar{u}_v^\varepsilon|_{L^2(\omega)}^2,$$

$$(3.73) \quad y = |\tilde{A}^{1/2} M_\varepsilon U_v^\varepsilon|_{L^2(\omega)}^2.$$

For all  $t \geq 0$  we have the following estimates:

$$\begin{aligned}
& \int_t^{t+1} g(s) ds, \quad \int_t^{t+1} y(s) ds \leq c(\nu) [R_0^2(\varepsilon) + R_0^4(\varepsilon)], \\
& \int_t^{t+1} h(s) ds \leq c(\nu) [R_0^2(\varepsilon) + R_0^4(\varepsilon)] \exp(c(\nu)(R_0^2(\varepsilon) + R_0^4(\varepsilon)))
\end{aligned}$$

so that, taking into account the fact that  $M_\varepsilon U^\varepsilon(0) = 0$ ,

$$(3.74) \quad |\tilde{A}^{1/2} M_\varepsilon U_v^\varepsilon(t)|_{L^2(\omega)}^2 \leq c(\nu) [R_0^2(\varepsilon) + R_0^4(\varepsilon)] \exp(c(\nu)(R_0^2(\varepsilon) + R_0^4(\varepsilon))).$$

We infer from (3.66) and (3.68) that

$$|\tilde{A}^{1/2} M_\varepsilon U^\varepsilon(t)|_{L^2(\omega)}^2 \leq c(\nu) [R_0^2(\varepsilon) + R_0^4(\varepsilon)] \exp(c(\nu)(R_0^2(\varepsilon) + R_0^4(\varepsilon))), \quad \forall t \geq 0.$$

Note that for all  $t \geq 0$

$$(3.75) \quad |A_\varepsilon^{1/2} M_\varepsilon U^\varepsilon(t)|_{L^2(\omega)}^2 = \varepsilon |\tilde{A}^{1/2} M_\varepsilon U^\varepsilon(t)|_{L^2(\omega)}^2 \leq c(\nu) \varepsilon [R_0^2(\varepsilon) + R_0^4(\varepsilon)] \exp(c(\nu)(R_0^2(\varepsilon) + R_0^4(\varepsilon))), \leq c(\nu, \gamma) \varepsilon^{1-\gamma}.$$

We summarize the previous result in the following Theorem comparing  $u^\varepsilon$  to  $\bar{u}^\varepsilon + \bar{w}^\varepsilon$ .

**THEOREM 3.3.** *In the fully periodical case, we assume that (3.1) holds so that  $u = u^\varepsilon$ , the solution to the Navier–Stokes equations (0.1)–(0.3) is defined and regular for all  $t > 0$ , for  $0 < \varepsilon \leq \varepsilon_1$ , for some  $\varepsilon_1$ . Let  $\bar{w}^\varepsilon$  and  $\bar{u}^\varepsilon$  be the solutions of (3.5) and of the 2D-like Navier–Stokes problem (3.23)–(3.26) (see also (3.27)–(3.34)). Then for  $0 < \varepsilon \leq \varepsilon_3 \leq \varepsilon_1$ , where  $\varepsilon_3$  depends only on the data,  $U^\varepsilon = u^\varepsilon - \bar{u}^\varepsilon - \bar{w}^\varepsilon$  is small in the following sense:*

$$(3.76) \quad \|M_\varepsilon U^\varepsilon(t)\|_\varepsilon^2 \leq c(\nu, \gamma) \varepsilon^{1-\gamma},$$

$$(3.77) \quad \|N_\varepsilon U^\varepsilon(t)\|_\varepsilon^2 \leq \|N_\varepsilon U^\varepsilon(0)\|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + c(\nu, q, \gamma) \varepsilon^{5-\gamma},$$

for all  $t \geq 0$ , some  $q \in (0, 1/2)$  and any  $\gamma > 0$  small.

**REMARK 3.1.** (i) In section 3.4 we will approximate  $\bar{w}^\varepsilon$  by a function  $w^\varepsilon$ , solution of a problem simpler than (3.5) which does not involve  $u^\varepsilon$ . Hence  $w^\varepsilon$  will be “explicit”, and this will make the approximation results above more useful.

(ii) We could also approximate  $\bar{u}^\varepsilon$  by a function  $\bar{u}$  independent of  $\varepsilon$ , solution of a problem similar to (3.23)–(3.26), where  $M_\varepsilon f^\varepsilon$  and  $M_\varepsilon u_0^\varepsilon$  are replaced by their limit as  $\varepsilon \rightarrow 0$ . The estimates on the rest of the expansion depend then of the differences between  $M_\varepsilon f^\varepsilon$  and  $M_\varepsilon u_0^\varepsilon$  and their limit; the details are left to the reader.

**3.4. Comparison between  $N_\varepsilon \bar{w}^\varepsilon$  and  $w^\varepsilon$ .** Let  $w^\varepsilon$  be the unique solution of the Stokes problem:

$$(3.78) \quad \begin{cases} -\nu \Delta w^\varepsilon + \nabla q = N_\varepsilon f^\varepsilon & \text{in } \Omega_\varepsilon, \\ \operatorname{div} w^\varepsilon = 0 & \text{in } \Omega_\varepsilon, \\ w^\varepsilon \text{ is periodic in the directions } x_1, x_2 \text{ and } x_3. \end{cases}$$

We note that  $w^\varepsilon = N_\varepsilon w^\varepsilon$ . Using (1.5) we easily find the following estimates for  $w^\varepsilon$ :

$$(3.79) \quad |A_\varepsilon^{1/2} w^\varepsilon|_\varepsilon \leq \frac{\varepsilon}{\nu} |N_\varepsilon f^\varepsilon|_\varepsilon,$$

$$(3.80) \quad |A_\varepsilon w^\varepsilon|_\varepsilon \leq \frac{1}{\nu} |N_\varepsilon f^\varepsilon|_\varepsilon.$$

Remark also that if  $f^\varepsilon \in H_p$ , then  $\nabla q^\varepsilon = N_\varepsilon f^\varepsilon + \nu \Delta w^\varepsilon \in H_p^\varepsilon$ , which implies  $\nabla q^\varepsilon = 0$ . Consider now the difference

$$N_\varepsilon W^\varepsilon = N_\varepsilon \bar{w}^\varepsilon - N_\varepsilon w^\varepsilon.$$

Using the weak formulation (3.52) for  $N_\varepsilon \bar{w}^\varepsilon$ , we find for  $N_\varepsilon W^\varepsilon$

$$(3.81) \quad \nu |A_\varepsilon^{1/2} N_\varepsilon W^\varepsilon|_\varepsilon^2 + b_\varepsilon(N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon W^\varepsilon) + b_\varepsilon(N_\varepsilon \bar{w}^\varepsilon, M_\varepsilon u^\varepsilon, N_\varepsilon W^\varepsilon) + b_\varepsilon(M_\varepsilon u^\varepsilon, N_\varepsilon \bar{w}^\varepsilon, N_\varepsilon W^\varepsilon) = 0,$$

and thus

$$(3.82) \quad \nu |A_\varepsilon^{1/2} N_\varepsilon W^\varepsilon|_\varepsilon^2 \leq c_1 \varepsilon^{1/2} |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} N_\varepsilon W^\varepsilon|_\varepsilon + 2c_1 \varepsilon^q |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon|_\varepsilon |A_\varepsilon^{1/2} N_\varepsilon \bar{w}^\varepsilon|_\varepsilon |A_\varepsilon^{1/2} N_\varepsilon W^\varepsilon|_\varepsilon,$$

which implies, using (3.3) and (3.8),

$$(3.83) \quad \nu |A_\varepsilon^{1/2} N_\varepsilon W^\varepsilon|_\varepsilon \leq c(\nu) \varepsilon^{5/2} |N_\varepsilon f^\varepsilon|_\varepsilon^2 + c(\nu) \varepsilon^{1+q} R_0(\varepsilon) |N_\varepsilon f^\varepsilon|_\varepsilon \leq c(\nu) \varepsilon^{1+q} R_0^2(\varepsilon).$$

Taking into account (3.2), this then leads to

$$(3.84) \quad |A_\varepsilon^{1/2} N_\varepsilon W^\varepsilon|_\varepsilon \leq c(\nu, q, \gamma) \varepsilon^{1+q-\gamma},$$

for any arbitrarily small  $\gamma > 0$ . Combining (3.84) with Theorem 3.1 we see that (3.70) and (3.71) still hold for  $U^\varepsilon = u^\varepsilon - \bar{u}^\varepsilon - w^\varepsilon$ .

**COROLLARY 3.4.** *Under the hypothesis of Theorem 3.3,  $w^\varepsilon$  being the solution of the Stokes problem (3.72), then for  $0 < \varepsilon \leq \varepsilon_3$ ,  $U^\varepsilon = u^\varepsilon - \bar{u}^\varepsilon - w^\varepsilon$  is small in the sense of (3.70) and ((3.71).*

**REMARK 3.2.** It is easy to see that  $w^\varepsilon$  can be itself approximated by  $\tilde{w}^\varepsilon$ :

$$\tilde{w}^\varepsilon = \sum_{\substack{k \in \mathbb{Z}^3 \\ k \neq 0}} \hat{w}_k e^{ikx},$$

$$\hat{w}_k = \frac{1}{\nu(k_1^2 + k_2^2)} \hat{g}_k \quad \text{if } k_3 = 0, \quad \hat{w}_k = \frac{\varepsilon^2}{\nu k_3^2} \hat{g}_k \quad \text{if } k_3 \neq 0,$$

where  $\hat{g}_k$  are the Fourier coefficients of  $N_\varepsilon f$ .

#### 4. Complements in the space periodic case

In this section we give some complements concerning the purely periodic case. We show how the results of [15], [16] and [23] can be improved, namely that one can obtain, for thin domains, the existence for all time of a smooth solution for a larger set of initial data  $u_0$  and volume forces  $f$ . These results can also be used to improve those of Section 3, but this will not be developed here.

We consider the problem (0.1)–(0.3) with periodic boundary conditions. Let  $R_0(\varepsilon)$  be a positive function satisfying for some  $q \in (0, 1)$

$$(4.1) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^q R_0^2(\varepsilon) = 0.$$

We set

$$(4.2) \quad \begin{cases} R_n^2(\varepsilon) = g_n^2(\varepsilon) R_0^2(\varepsilon), \\ R_m^2(\varepsilon) = g_m^2(\varepsilon) R_0^2(\varepsilon), \end{cases}$$



where

$$(4.3) \quad \begin{cases} g_n^2(\varepsilon) = \frac{\varepsilon^{(5q-1)/6}}{|\ln \varepsilon|}, \\ g_m^2(\varepsilon) = \frac{\varepsilon^{2(q+1)/3}}{|\ln \varepsilon|}. \end{cases}$$

We assume that the data  $u_0^\varepsilon \in V_p^\varepsilon$  and  $f^\varepsilon \in H_p^\varepsilon$  satisfy:

$$(4.4) \quad \begin{cases} |A_\varepsilon^{1/2} M_\varepsilon u_0^\varepsilon|_\varepsilon^2 + |M_\varepsilon f^\varepsilon|_\varepsilon^2 \leq R_m^2(\varepsilon), \\ |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|_\varepsilon^2 + |N_\varepsilon f^\varepsilon|_\varepsilon^2 \leq R_n^2(\varepsilon), \end{cases}$$

and let  $T^\sigma(\varepsilon)$  be the maximal time such that

$$(4.5) \quad |A_\varepsilon^{1/2} u^\varepsilon(t)|_\varepsilon^2 \leq \sigma R_0^2(\varepsilon), \quad 0 \leq t < T^\sigma(\varepsilon).$$

Here  $\sigma > 2$  is a fixed number which will be chosen later on (see (4.46)). Note that if  $T^\sigma(\varepsilon) < \infty$ , then

$$(4.6) \quad |A_\varepsilon^{1/2} u^\varepsilon(T^\sigma(\varepsilon))|_\varepsilon^2 = \sigma R_0^2(\varepsilon).$$

Since  $\lim_{\varepsilon \rightarrow 0} \varepsilon^q R_0^2(\varepsilon) = 0$ , there exists  $\varepsilon_1 = \varepsilon_1(\nu, q)$  (depending also on the function  $R_0$ ) such that

$$(4.7) \quad \varepsilon^q R_0^2(\varepsilon) \leq \nu^2/4 \quad \text{for } 0 < \varepsilon \leq \varepsilon_1.$$

In what follows we restrict ourselves to  $\varepsilon \leq \varepsilon_1$ , and we aim first to derive a number of a priori estimates.

#### A priori estimates.

*Estimates for  $N_\varepsilon u^\varepsilon$ .* We multiply (0.1) with  $A_\varepsilon N_\varepsilon u^\varepsilon$  and we integrate over  $\Omega_\varepsilon$ . We estimate the nonlinear terms using Lemma 3.1, then we take into account (4.5) and (4.7) to obtain for  $0 < \varepsilon \leq \varepsilon_1$ ,  $0 \leq t < T^\sigma(\varepsilon)$

$$(4.8) \quad \frac{d}{dt} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 + \frac{\nu}{2} |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2 \leq \frac{|N_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu},$$

and since by the Cauchy–Schwarz inequality

$$|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 \leq \varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2,$$

we deduce from (4.8) that

$$(4.9) \quad \frac{d}{dt} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 + \frac{\nu}{2\varepsilon^2} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 \leq \frac{|N_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu}.$$

Thus by the Gronwall lemma

$$(4.10) \quad |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \leq |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2} |N_\varepsilon f^\varepsilon|_\varepsilon^2,$$

for  $0 < \varepsilon \leq \varepsilon_1$ ,  $0 \leq t < T^\sigma(\varepsilon)$ . Taking into account (4.4), we deduce

$$(4.11) \quad |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \leq R_n^2 \left[ \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2} \right].$$

We also infer from (4.8) that

$$\frac{\nu}{2} \int_0^t |A_\varepsilon N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 ds \leq |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|_\varepsilon^2 + \frac{|N_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu} t,$$

so that

$$(4.12) \quad \int_0^t |A_\varepsilon N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 ds \leq c(\nu) R_n^2(\varepsilon) (1+t),$$

for  $0 < \varepsilon \leq \varepsilon_1$ ,  $0 \leq t < T^\sigma(\varepsilon)$ .

*Estimates for  $M_\varepsilon u^\varepsilon$ .* We first multiply (0.1) with  $M_\varepsilon u^\varepsilon$  and we integrate over  $\Omega_\varepsilon$ . A simple computation taking into account (4.5) and (4.7) yields:

$$(4.13) \quad \frac{d}{dt} |M_\varepsilon u^\varepsilon|_\varepsilon^2 + \nu |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon|_\varepsilon^2 \leq \frac{|M_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu \lambda_1} + \frac{c\varepsilon}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^4,$$

for  $0 < \varepsilon \leq \varepsilon_1$ ,  $0 \leq t < T^\sigma(\varepsilon)$ , where  $\lambda_1$  is the first eigenvalue of the two-dimensional Stokes operator defined on  $\omega$ . Then (4.13) implies

$$(4.14) \quad \nu \int_0^t |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(s)|_\varepsilon^2 ds \leq \frac{|M_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu \lambda_1} t + \frac{c\varepsilon}{\nu} \int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^4 ds,$$

for  $0 < \varepsilon \leq \varepsilon_1$ ,  $0 \leq t < T^\sigma(\varepsilon)$ .

Using (4.11), (4.2) and (4.7) we estimate

$$\begin{aligned} \frac{c\varepsilon}{\nu} \int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^4 ds &\leq \frac{c\varepsilon}{\nu} \left[ \sup_{0 \leq s \leq t} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 \right] \left( \int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 ds \right) \\ &\leq c(\nu) \varepsilon R_n^4(\varepsilon) \left[ \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2} \right] \left[ \frac{2\varepsilon^2}{\nu} + \frac{2\varepsilon^2}{\nu^2} t \right] \\ &\leq c(\nu) \varepsilon^3 R_n^4(\varepsilon) (1+t) = c(\nu) \varepsilon^3 g_n^4(\varepsilon) R_0^4(\varepsilon) (1+t) \\ &\leq c(\nu) \varepsilon^{3-q} g_n^4(\varepsilon) R_0^2(\varepsilon) (1+t), \end{aligned}$$

so that we deduce from (4.14)

$$(4.15) \quad \begin{aligned} \int_0^t |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(s)|_\varepsilon^2 ds &\leq c(\nu, \lambda_1) R_m^2(\varepsilon) (1+t) \\ &\quad + c(\nu) \varepsilon^{3-q} g_n^4(\varepsilon) R_0^2(\varepsilon) (1+t) \\ &= c(\nu, \lambda_1) g_m^2(\varepsilon) R_0^2(\varepsilon) \left[ 1 + \frac{\varepsilon^{3-q} g_n^4(\varepsilon)}{g_m^2(\varepsilon)} \right] (1+t), \end{aligned}$$

for  $0 < \varepsilon \leq \varepsilon_1, 0 \leq t < T^\sigma(\varepsilon)$ . We set  $u_{2D}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, 0)$ . We multiply (0.1) with  $A_\varepsilon M_\varepsilon u_{2D}^\varepsilon$  and integrate over  $\Omega_\varepsilon$  to obtain:

$$(4.16) \quad \frac{1}{2} \frac{d}{dt} |A_\varepsilon^{1/2} M_\varepsilon u_{2D}^\varepsilon|_\varepsilon^2 + b_\varepsilon(M_\varepsilon u_{2D}^\varepsilon, M_\varepsilon u_{2D}^\varepsilon, A_\varepsilon M_\varepsilon u_{2D}^\varepsilon) + \nu |A_\varepsilon M_\varepsilon u_{2D}^\varepsilon|_\varepsilon^2 \\ + b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon M_\varepsilon u_{2D}^\varepsilon) = (M_\varepsilon f^\varepsilon, A_\varepsilon M_\varepsilon u_{2D}^\varepsilon)_\varepsilon.$$

Note that

$$b_\varepsilon(M_\varepsilon u_{2D}^\varepsilon, M_\varepsilon u_{2D}^\varepsilon, A_\varepsilon M_\varepsilon u_{2D}^\varepsilon) = \varepsilon \tilde{b}(M_\varepsilon u_{2D}^\varepsilon, M_\varepsilon u_{2D}^\varepsilon, \tilde{A} M_\varepsilon u_{2D}^\varepsilon) = 0,$$

due to a well-known orthogonality property in the periodic boundary conditions case; therefore (4.16) becomes

$$(4.17) \quad \frac{d}{dt} |A_\varepsilon^{1/2} M_\varepsilon u_{2D}^\varepsilon|_\varepsilon^2 + \nu |A_\varepsilon M_\varepsilon u_{2D}^\varepsilon|_\varepsilon^2 \leq \frac{c |M_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu} \\ + \frac{c\varepsilon}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2$$

for  $0 < \varepsilon \leq \varepsilon_1, 0 \leq t < T^\sigma(\varepsilon)$ . Hence, with the Gronwall lemma we have:

$$(4.18) \quad |A_\varepsilon^{1/2} M_\varepsilon u_{2D}^\varepsilon(t)|_\varepsilon^2 \leq |A_\varepsilon^{1/2} M_\varepsilon u_{2D}^\varepsilon(0)|_\varepsilon^2 \exp(-\nu \lambda_1 t) \\ + \frac{c\varepsilon}{\nu} \int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 ds + \frac{c |M_\varepsilon f^\varepsilon|_\varepsilon^2}{\nu^2 \lambda_1}$$

for  $0 < \varepsilon \leq \varepsilon_1, 0 \leq t < T^\sigma(\varepsilon)$ .

Using (4.11), (4.12), (4.2) and (4.7) we estimate

$$\frac{c\varepsilon}{\nu} \int_0^t |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 ds \\ \leq \frac{c\varepsilon}{\nu} \left[ \sup_{0 \leq s \leq t} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 \right] \left[ \int_0^t |A_\varepsilon N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 ds \right] \\ \leq c(\nu) \varepsilon R_n^4(\varepsilon) (1+t) = c(\nu) \varepsilon g_n^4 R_n^4(\varepsilon) (1+t) \\ \leq c(\nu) \varepsilon^{1-q} g_n^4(\varepsilon) R_0^2(\varepsilon) (1+t).$$

We deduce from (4.18), (4.2) and the previous estimate that

$$(4.19) \quad |A_\varepsilon^{1/2} M_\varepsilon u_{2D}^\varepsilon(t)|_\varepsilon^2 \leq c(\nu, \lambda_1) R_m^2(\varepsilon) + c(\nu) \varepsilon^{1-q} g_n^4(\varepsilon) R_0^2(\varepsilon) (1+t) \\ = c(\nu, \lambda_1) g_m^2(\varepsilon) \left[ 1 + \frac{\varepsilon^{1-q} g_n^4(\varepsilon)}{g_m^2(\varepsilon)} (1+t) \right] R_0^2(\varepsilon),$$

for  $0 < \varepsilon \leq \varepsilon_1, 0 \leq t < T^\sigma(\varepsilon)$ . Now we set  $v^\varepsilon = (0, 0, M_\varepsilon u_3^\varepsilon)$ . We multiply (0.1) with  $A_\varepsilon M_\varepsilon v^\varepsilon$  and we integrate over  $\Omega_\varepsilon$  to obtain

$$(4.20) \quad \frac{1}{2} \frac{d}{dt} |A_\varepsilon^{1/2} v^\varepsilon|_\varepsilon^2 + \nu |A_\varepsilon v^\varepsilon|_\varepsilon^2 + b_\varepsilon(M_\varepsilon u_{2D}^\varepsilon, v^\varepsilon, A_\varepsilon v^\varepsilon) \\ + b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon v^\varepsilon) = (M_\varepsilon f_3^\varepsilon, A_\varepsilon v^\varepsilon).$$

Note that

$$(4.21) \quad |b_\varepsilon(M_\varepsilon u_{2D}^\varepsilon, v^\varepsilon, A_\varepsilon v^\varepsilon)| \leq c\varepsilon |M_\varepsilon u_{2D}^\varepsilon|_{L^4(\omega)} |\nabla' v^\varepsilon|_{L^4(\omega)} |\tilde{A} v^\varepsilon|_{L^2(\omega)} \\ \leq \frac{1}{8} \nu |A_\varepsilon v^\varepsilon|_\varepsilon^2 + \frac{c}{\nu^3 \varepsilon^2} |M_\varepsilon u_{2D}^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} M_\varepsilon u_{2D}^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} v^\varepsilon|_\varepsilon^2$$

and

$$(4.22) \quad |b_\varepsilon(N_\varepsilon u^\varepsilon, N_\varepsilon u^\varepsilon, A_\varepsilon v^\varepsilon)| \leq \nu |A_\varepsilon v^\varepsilon|_\varepsilon^2 / 8 + c\varepsilon |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2 / \nu.$$

Hence (4.20)–(4.22) for  $0 < \varepsilon \leq \varepsilon_1$ ,  $0 \leq t < T^\sigma(\varepsilon)$  give

$$(4.23) \quad \frac{d}{dt} |A_\varepsilon^{1/2} v^\varepsilon|_\varepsilon^2 + \nu |A_\varepsilon v^\varepsilon|_\varepsilon^2 \leq \frac{c |M_\varepsilon f_3^\varepsilon|_\varepsilon^2}{\nu} + \frac{c\varepsilon}{\nu} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 |A_\varepsilon N_\varepsilon u^\varepsilon|_\varepsilon^2 \\ + \frac{c}{\nu^3 \varepsilon^2} |M_\varepsilon u_{2D}^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} M_\varepsilon u_{2D}^\varepsilon|_\varepsilon^2 |A_\varepsilon^{1/2} v^\varepsilon|_\varepsilon^2,$$

and Gronwall's lemma for  $0 < \varepsilon \leq \varepsilon_1$ ,  $0 \leq t < T^\sigma(\varepsilon)$ , yields

$$(4.24) \quad |A_\varepsilon^{1/2} v^\varepsilon(t)|_\varepsilon^2 \leq |A_\varepsilon^{1/2} v^\varepsilon(0)|_\varepsilon^2 \exp(-\nu \lambda_1 t) + \frac{c |M_\varepsilon f_3^\varepsilon|_\varepsilon^2}{\nu^2 \lambda_1} \\ + c(\nu) \varepsilon \left[ \sup_{0 \leq s \leq t} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(s)|_\varepsilon^2 \right] \left( \int_0^t |A_\varepsilon N_\varepsilon u^\varepsilon(s)|^2 ds \right) \\ + \frac{c(\nu)}{\varepsilon^2} \left[ \sup_{0 \leq s \leq t} \lambda_1^{-1} |A_\varepsilon^{1/2} M_\varepsilon u_{2D}^\varepsilon(s)|_\varepsilon^4 \right] \left( \int_0^t |A_\varepsilon^{1/2} v^\varepsilon(s)|_\varepsilon^2 ds \right),$$

We use (4.11), (4.12), (4.19) and (4.15) in (4.24) and we obtain

$$|A_\varepsilon^{1/2} v^\varepsilon(t)|_\varepsilon^2 \leq c(\nu, \lambda_1) R_m^2(\varepsilon) + c(\nu) \varepsilon R_n^4(\varepsilon) (1+t) + \frac{c(\nu, \lambda_1)}{\varepsilon^2} g_m^6(\varepsilon) \\ \cdot \left[ 1 + \frac{\varepsilon^{1-q} g_n^4(\varepsilon)}{g_m^2(\varepsilon)} (1+t)^2 \right]^2 \left[ 1 + \frac{\varepsilon^{3-q} g_n^4(\varepsilon)}{g_m^2(\varepsilon)} \right] R_0^6(\varepsilon) (1+t).$$

We use (4.7) and we obtain

$$(4.25) \quad |A_\varepsilon^{1/2} v^\varepsilon(t)|_\varepsilon^2 \leq c(\nu, \lambda_1) \left\{ g_m^2(\varepsilon) + \varepsilon^{1-q} g_n^4(\varepsilon) (1+t) + \varepsilon^{-2-2q} g_m^6(\varepsilon) \right. \\ \left. \cdot \left[ 1 + \frac{\varepsilon^{1-q} g_n^4(\varepsilon)}{g_m^2(\varepsilon)} (1+t)^2 \right]^2 \left[ 1 + \frac{\varepsilon^{3-q} g_n^4(\varepsilon)}{g_m^2(\varepsilon)} \right] (1+t) \right\} R_0^2(\varepsilon)$$

for  $0 < \varepsilon \leq \varepsilon_1$ ,  $0 \leq t < T^\sigma(\varepsilon)$ .

Now we take into account the expressions of  $g_m$  and  $g_n$  given by (4.3) and we rewrite (4.19) and (4.25) as

$$(4.26) \quad |A_\varepsilon^{1/2} M_\varepsilon u_{2D}^\varepsilon(t)|_\varepsilon^2 \leq c(\nu, \lambda_1) \frac{\varepsilon^{2(q+1)/3}}{|\ln \varepsilon|} \left[ 1 + \frac{1+t}{|\ln \varepsilon|} \right] R_0^2(\varepsilon),$$

for  $0 < \varepsilon \leq \varepsilon_1$ ,  $0 \leq t < T^\sigma(\varepsilon)$  and

$$(4.27) \quad |A_\varepsilon^{1/2} v^\varepsilon(t)|^2 \varepsilon \leq c(\nu, \lambda_1) \frac{\varepsilon^{2(q+1)/3}}{|\ln \varepsilon|} \left(1 + \frac{1+t}{|\ln \varepsilon|}\right) R_0^2(\varepsilon) \\ + c(\nu, \lambda_1) \frac{1+t}{|\ln \varepsilon|^3} \left(1 + \frac{(1+t)^2}{|\ln \varepsilon|}\right)^2 \left(1 + \frac{\varepsilon^2}{|\ln \varepsilon|}\right) R_0^2(\varepsilon) \\ \leq c(\nu, \lambda_1) \frac{\varepsilon^{2(q+1)/3}}{|\ln \varepsilon|} \left(1 + \frac{1+t}{|\ln \varepsilon|}\right) R_0^2(\varepsilon) \\ + c(\nu, \lambda_1) \left[ \frac{(1+t)^3}{|\ln \varepsilon|^5} + \frac{(1+t)^2}{|\ln \varepsilon|^4} + \frac{(1+t)}{|\ln \varepsilon|^3} \right] R_0^2(\varepsilon).$$

for  $0 < \varepsilon \leq \varepsilon_1$ ,  $0 \leq t < T^\sigma(\varepsilon)$ . At this stage we are able to prove that

$$(4.28) \quad \lim_{\varepsilon \rightarrow 0} \frac{T^\sigma(\varepsilon)}{|\ln \varepsilon|^{1/2}} = \infty.$$

If this were not true, we would have  $(T^\sigma(\varepsilon) < \infty)$ :

$$(4.29) \quad (|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon|_\varepsilon^2 + |A_\varepsilon^{1/2} M_\varepsilon u_{2D}^\varepsilon|_\varepsilon^2 + |A_\varepsilon^{1/2} v^\varepsilon|_\varepsilon^2)(T^\sigma(\varepsilon)) = \sigma R_0^2(\varepsilon),$$

so that, using (4.11), (4.26) and (4.27) we obtain

$$(4.30) \quad \sigma \leq c(\nu) \frac{\varepsilon^{(5q-1)/6}}{|\ln \varepsilon|} + c(\nu, \lambda_1) \frac{\varepsilon^{2(q+1)/3}}{|\ln \varepsilon|} \left[1 + \frac{1+T^\sigma(\varepsilon)}{|\ln \varepsilon|}\right] \\ + c(\nu, \lambda_1) \left[ \frac{(1+T^\sigma(\varepsilon))^3}{|\ln \varepsilon|^5} + \frac{(1+T^\sigma(\varepsilon))^2}{|\ln \varepsilon|^4} + \frac{1+T^\sigma(\varepsilon)}{|\ln \varepsilon|^3} \right]$$

Since the right-hand side of the inequality (4.30) goes to zero as  $\varepsilon$  goes to zero, we find  $\sigma = 0$ , a contradiction. Hence we have proved (4.28).

Now we will prove that  $T^\sigma(\varepsilon) = \infty$ . We use the same notation as in (3.1), namely we set

$$a_0(\varepsilon) = |A_\varepsilon^{1/2} M_\varepsilon u_0^\varepsilon|_\varepsilon, \quad b_0(\varepsilon) = |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|_\varepsilon, \\ \alpha(\varepsilon) = |M_\varepsilon f^\varepsilon|_\varepsilon, \quad \beta(\varepsilon) = |N_\varepsilon f^\varepsilon|_\varepsilon,$$

We also consider:

$$(4.31) \quad K_\varepsilon^2 = |A_\varepsilon^{1/2} M_\varepsilon u_0^\varepsilon|_\varepsilon^2 + \frac{64}{\nu^2 \lambda_1} |M_\varepsilon f^\varepsilon|_\varepsilon^2 + B_\varepsilon^2,$$

where

$$(4.32) \quad B_\varepsilon^2 = |A_\varepsilon^{1/2} N_\varepsilon u_0^\varepsilon|_\varepsilon^2 + |N_\varepsilon f^\varepsilon|_\varepsilon^2$$

Note that  $B_\varepsilon$  and  $K_\varepsilon$  are both bounded by  $c R_0$  and therefore due to (4.1),

$$(4.33) \quad \lim_{\varepsilon \rightarrow 0} \varepsilon^q B_\varepsilon^2 = \lim_{\varepsilon \rightarrow 0} \varepsilon^q K_\varepsilon^2 = 0.$$

We choose  $\varepsilon_4 = \varepsilon_4(\nu, \lambda_1, q) > 0$  satisfying the following conditions, where  $c_{10}(\nu)$  is defined below in (4.37)

$$(4.34) \quad \left\{ \begin{array}{l} \text{(i)} \quad 0 < \varepsilon_4 \leq 1 \\ \text{(ii)} \quad c_{10}(\nu)\varepsilon^q B_\varepsilon^2 \leq 1/32, \quad \varepsilon^{1-q}(1 + |\ln \varepsilon|^{1/2}) \leq 2 \quad \text{for } 0 < \varepsilon \leq \varepsilon_4, \\ \text{(iii)} \quad 2\varepsilon^2/\nu^2 \leq 1/8, \quad \exp(-\nu|\ln \varepsilon|^{1/2}/2\varepsilon^2) \leq 1/4, \\ \quad \quad \exp(-\nu\lambda_1|\ln \varepsilon|^{1/2}) \leq 1/8 \quad \text{for } 0 < \varepsilon \leq \varepsilon_4, \\ \text{(iv)} \quad T^\sigma(\varepsilon)/|\ln \varepsilon|^{1/2} > 4 \quad \text{for } 0 < \varepsilon \leq \varepsilon_4. \end{array} \right.$$

The existence of  $\varepsilon_4$  is obvious, since the left-hand side of the inequalities (ii) and (iii) go to zero as  $\varepsilon$  goes to zero; and by (4.28) the left-hand side of (iv) goes to infinity as  $\varepsilon$  goes to zero.

Using (4.8), (4.10), (4.34) and  $|A_\varepsilon^{1/2}N_\varepsilon^\varepsilon| \leq \varepsilon|A_\varepsilon^1N_\varepsilon^\varepsilon|$ , we easily find

$$(4.35) \quad \int_0^t |A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon(s)|_\varepsilon^3 |A_\varepsilon N_\varepsilon u^\varepsilon(s)|_\varepsilon ds \leq \frac{\nu}{4} \max(1, 1/\nu^3) B_\varepsilon^2(1+t)$$

for  $0 \leq t \leq T^\sigma(\varepsilon)$ . Hence

$$(4.36) \quad |A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \leq b_0^2(\varepsilon) \exp\left(-\frac{\nu t}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2} \beta^2(\varepsilon)$$

and for a suitable constant  $c(\nu)$  which we denote  $c_{10}(\nu)$  and for  $0 \leq t \leq T^\sigma(\varepsilon)$

$$(4.37) \quad |A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \leq a_0^2(\varepsilon) \exp(-\nu\lambda_1 t) + \frac{2}{\nu^2\lambda_1} |M_\varepsilon f^\varepsilon|_\varepsilon^2 + c_{10}(\nu)\varepsilon B_\varepsilon^4(1+t).$$

We set

$$(4.38) \quad t_\varepsilon = |\ln \varepsilon|^{1/2} \quad \text{for } 0 < \varepsilon \leq \varepsilon_4.$$

Observe that by (4.34)(iv),  $t_\varepsilon \leq T^\sigma(\varepsilon)/4$ . According to (4.36) and (4.37), we have

$$\begin{aligned} |A_\varepsilon^{1/2}N_\varepsilon u^\varepsilon(t_\varepsilon)|_\varepsilon^2 &\leq b_0^2(\varepsilon) \exp\left(-\frac{\nu t_\varepsilon}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2} \beta^2(\varepsilon) \\ &\leq b_0^2(\varepsilon) \exp\left(-\frac{\nu|\ln \varepsilon|^{1/2}}{2\varepsilon^2}\right) + \frac{1}{8} \beta^2(\varepsilon) \\ &\leq \frac{1}{4} b_0^2(\varepsilon) + \frac{1}{8} \beta^2(\varepsilon) \leq \frac{1}{4} B_\varepsilon^2, \end{aligned}$$

and

$$\begin{aligned} |A_\varepsilon^{1/2}M_\varepsilon u^\varepsilon(t_\varepsilon)|_\varepsilon^2 &\leq a_0^2(\varepsilon) \exp(-\nu\lambda_1 t_\varepsilon) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} + c_{10}(\nu)(\varepsilon^q B_\varepsilon^2)\varepsilon^{1-q}(1+t_\varepsilon) B_\varepsilon^2 \\ &\leq a_0^2(\varepsilon) \exp(-\nu\lambda_1|\ln \varepsilon|^{1/2}) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} + \frac{\varepsilon^{1-q}}{32}(1+|\ln \varepsilon|^{1/2}) B_\varepsilon^2 \\ &\leq \frac{1}{8} \left( a_0^2(\varepsilon) + \frac{16\alpha^2(\varepsilon)}{\nu^2\lambda_1} \right) + \frac{1}{16} B_\varepsilon^2 \leq \frac{1}{4} K_\varepsilon^2. \end{aligned}$$

Hence, adding the last two relations,

$$(4.39) \quad |A_\varepsilon^{1/2} u^\varepsilon(t_\varepsilon)|_\varepsilon^2 \leq B_\varepsilon^2/4 + K_\varepsilon^2/4 \leq K_\varepsilon^2/2.$$

We claim that for any  $n \geq 1$

$$(4.40) \quad \begin{cases} nt_\varepsilon \leq T^\sigma(\varepsilon) \\ \text{and} \\ |A_\varepsilon^{1/2} N_\varepsilon(nt_\varepsilon)|_\varepsilon^2 \leq B_\varepsilon^2/4, \quad |A_\varepsilon^{1/2} M_\varepsilon(nt_\varepsilon)|_\varepsilon^2 \leq K_\varepsilon^2/4. \end{cases}$$

We have shown that the claim holds for  $n = 1$ . Suppose now that the claim holds for some  $n$ . We want to prove the induction step. For  $nt_\varepsilon \leq t \leq T^\sigma(\varepsilon)$  we obtain the following estimates

$$|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \leq |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(nt_\varepsilon)|_\varepsilon^2 \exp\left(-\frac{\nu(t-nt_\varepsilon)}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2} \beta^2(\varepsilon)$$

and using the induction hypothesis we find

$$(4.41) \quad \begin{aligned} |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 &\leq \frac{1}{4} B_\varepsilon^2 \exp\left(-\frac{\nu(t-nt_\varepsilon)}{2\varepsilon^2}\right) + \frac{2\varepsilon^2}{\nu^2} \beta^2(\varepsilon) \\ &\leq \frac{1}{4} B_\varepsilon^2 \exp\left(-\frac{\nu(t-nt_\varepsilon)}{2\varepsilon^2}\right) + \frac{1}{8} \beta^2(\varepsilon). \end{aligned}$$

Similarly, we have

$$\begin{aligned} |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(t)|_\varepsilon^2 &\leq |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(nt_\varepsilon)|_\varepsilon^2 \exp(-\nu\lambda_1(t-nt_\varepsilon)) + 2\alpha^2(\varepsilon)/\nu^2\lambda_1 \\ &\quad + c_{10}(\nu)\varepsilon[|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(nt_\varepsilon)|_\varepsilon^2 + |M_\varepsilon f^\varepsilon|_\varepsilon^2](1 + (t-nt_\varepsilon)) \end{aligned}$$

for  $nt_\varepsilon \leq t < T^\sigma(\varepsilon)$  and using the induction hypothesis we obtain

$$(4.42) \quad \begin{aligned} |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(t)|_\varepsilon^2 &\leq \frac{1}{4} K_\varepsilon^2 \exp(-\nu\lambda_1(t-nt_\varepsilon)) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} \\ &\quad + c_{10}(\nu)\varepsilon \left[ \frac{1}{4} B_\varepsilon^2 + \beta^2(\varepsilon) \right]^2 (1 + (t-nt_\varepsilon)) \\ &\leq \frac{1}{4} K_\varepsilon^2 \exp(-\nu\lambda_1(t-nt_\varepsilon)) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} \\ &\quad + c_{10}(\nu)\varepsilon \left(\frac{5}{4}\right)^2 B_\varepsilon^4 (1 + (t-nt_\varepsilon)) \\ &= \frac{1}{4} K_\varepsilon^2 \exp(-\nu\lambda_1(t-nt_\varepsilon)) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} \\ &\quad + c_{10}(\nu)(\varepsilon^q B_\varepsilon^2) \left(\frac{5}{4}\right)^2 B_\varepsilon^2 \varepsilon^{1-q} [1 + (t-nt_\varepsilon)] \\ &\leq \frac{1}{4} K_\varepsilon^2 \exp(-\nu\lambda_1(t-nt_\varepsilon)) + \frac{2\alpha^2(\varepsilon)}{\nu^2\lambda_1} \\ &\quad + \frac{1}{32} \left(\frac{5}{4}\right)^2 B_\varepsilon^2 \varepsilon^{1-q} [1 + (t-nt_\varepsilon)]. \end{aligned}$$

Now, if  $nt_\varepsilon \leq t \leq (n+1)t_\varepsilon$ , we obtain from (4.41) and (4.42)

$$(4.43) \quad |A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \leq \frac{1}{4} B_\varepsilon^2 + \frac{1}{8} \beta^2(\varepsilon) \leq \frac{1}{8} 3B_\varepsilon^2$$

$$(4.44) \quad |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon(t)|_\varepsilon^2 \leq \frac{1}{4} K_\varepsilon^2 + \frac{2\alpha^2(\varepsilon)}{\nu^2 \lambda_1} \\ + \frac{1}{32} \left(\frac{5}{4}\right)^2 B_\varepsilon^2 \varepsilon^{1-q} (1 + |\ln \varepsilon|^{1/2}) \\ \leq \frac{1}{4} K_\varepsilon^2 + \frac{1}{32} K_\varepsilon^2 + \frac{1}{32} \left(\frac{5}{4}\right)^2 2B_\varepsilon^2 \leq \frac{1}{8} 3K_\varepsilon^2$$

Hence

$$(4.45) \quad |A_\varepsilon^{1/2} u^\varepsilon(t)|_\varepsilon^2 \leq 3B_\varepsilon^2/8 + 3K_\varepsilon^2/8 \leq K_\varepsilon^2 \quad \text{for } nt_\varepsilon \leq t \leq (n+1)t_\varepsilon$$

and if

$$(4.46) \quad \sigma > \max(1, 16/\nu^2 \lambda_1),$$

we obtain

$$(4.47) \quad |A_\varepsilon^{1/2} u^\varepsilon(t)|_\varepsilon^2 < \sigma R_0^2(\varepsilon) \quad \text{for } nt_\varepsilon \leq t \leq (n+1)t_\varepsilon.$$

In addition, taking  $t = (n+1)t_\varepsilon$  in (4.45) and (4.46), we obtain

$$|A_\varepsilon^{1/2} N_\varepsilon u^\varepsilon((n+1)t_\varepsilon)|_\varepsilon^2 \leq \frac{1}{4} B_\varepsilon^2 \exp\left(-\frac{\nu t_\varepsilon}{2\varepsilon^2}\right) + \frac{1}{8} \beta^2(\varepsilon) \\ \leq \frac{1}{16} B_\varepsilon^2 + \frac{1}{8} \beta^2(\varepsilon) \leq \frac{1}{4} B_\varepsilon^2, \\ |A_\varepsilon^{1/2} M_\varepsilon u^\varepsilon((n+1)t_\varepsilon)|_\varepsilon^2 \leq \frac{1}{4} K_\varepsilon^2 \exp(-\nu \lambda_1 t_\varepsilon) + \frac{2\alpha^2(\varepsilon)}{\nu^2 \lambda_1} \\ + \frac{1}{32} \left(\frac{5}{4}\right)^2 B_\varepsilon^2 \varepsilon^{1-q} (1 + t_\varepsilon) \\ \leq \frac{1}{32} K_\varepsilon^2 + \frac{2\alpha^2(\varepsilon)}{\nu^2 \lambda_1} + \frac{1}{16} \left(\frac{5}{4}\right)^2 B_\varepsilon^2 \\ \leq \frac{1}{32} K_\varepsilon^2 + \frac{1}{8} B_\varepsilon^2 \leq \frac{1}{4} K_\varepsilon^2.$$

This proves the claim for  $n+1$  and proves that  $T^\sigma(\varepsilon) > nt_\varepsilon$  for all  $n$  provided (4.46) is satisfied. Hence  $T^\sigma(\varepsilon) = \infty$  for  $0 < \varepsilon \leq \varepsilon_4$ . We can state the following result

**THEOREM 4.1.** *There exists  $\varepsilon_4 = \varepsilon_4(\nu, q, \sigma)$  such that if  $u_0, f$  are given,  $u_0 \in V_p^\varepsilon$ ,  $f \in H_p^\varepsilon$ ,  $u_0, f$  satisfying (4.1)–(4.4), and  $0 < \varepsilon \leq \varepsilon_4$ , then the strong solution  $u$  of (0.1)–(0.3) with periodic boundary conditions exists for all times, i.e. for all  $T > 0$*

$$u^\varepsilon \in C([0, \infty), V_p^\varepsilon) \cap L^2(0, T; D(A_\varepsilon)).$$



## REFERENCES

- [1] A. ASSEMIEN, G. BAYADA AND M. CHAMBAT, *Inertial effects in the asymptotic behavior of a thin film flow*, *Asymptotic Anal.* **9** (1994), 177–208.
- [2] L. CATTABRIGA, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, *Rend. Sem. Mat. Univ. Padova* **31** (1961), 308–340.
- [3] PH. CIARLET, *Plates and junctions in elastic multi-structures*, *An Asymptotic Analysis*, Masson, Paris and Springer-Verlag, New York, 1990.
- [4] P. CONSTANTIN AND C. FOIAS, *Navier–Stokes equations*, Univ. of Chicago Press, Chicago, 1988.
- [5] H.L. LE DRET, *Problèmes variationnels dans les multi-domaines: Modélisation des jonctions et applications*, Masson, Paris, 1991.
- [6] J. M. GHIDAGLIA, *Régularité des solutions de certains problèmes aux limites linéaires liés aux équations d'Euler*, *Comm. Partial Differential Equations* **9 13** (1984), 1265–1298.
- [7] P. GRISVARD, *Elliptic problems in nonsmooth domains*, *Monographs and Studies in Mathematics*, vol. 24, Pitman, London, 1985.
- [8] J. HALE AND G. RAUGEL, *A damped hyperbolic equation on thin domains*, *Trans. Amer. Math. Soc.* **329** (1992), 185–219.
- [9] O. A. LADYZHENSKAYA, *Solution “in the large” to the boundary-value problem for the Navier–Stokes equations in two space variables*, *Soviet Physics. Dokl.* **123** (1958), 1128–1131.
- [10] ———, *The mathematical theory of viscous incompressible flow*, Gordon and Breach, New York, 1969.
- [11] ———, *Some comments on my papers on the theory of attractors for abstract semi-groups*, *Zap. Nauchn. Sem. LOMI* **182** (1990) (Russian); English translation, *J. Soviet Math.* **62** (1992), 101–147.
- [12] O. A. LADYZHENSKAYA AND V. A. SOLONNIKOV, *On the solutions of the boundary value problems for the Navier–Stokes equations in domains with non compact boundaries*, *Leningrad Universitat Vestnik* **13** (1977), 39–47.
- [13] J. LERAY, *Etude de diverses équations intégrales et non linéaires et de quelques problèmes que pose l'hydrodynamique*, *J. Math. Pures Appl.* **12** (1933), 1–82.
- [14] J. L. LIONS, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Gauthier Villars, Paris, 1969.
- [15] G. RAUGEL AND G. SELL, *Equations de Navier-Stokes dans des domaines minces en dimension trois: régularité globale*, *C.R. Acad. Sci. Paris Sér. I Math.* **309** (1989), 299–303.
- [16] ———, *Navier–Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions*, *J. Amer. Math. Society* **6** (1993), 503–568.
- [17] ———, *Navier–Stokes equations on thin 3D domains. II. Global regularity of spatially periodic conditions*, *Collège de France Proceedings*, *Pitman Res. Notes Math. Ser.*, Pitman, New York and London (1994).
- [18] V. A. SOLONNIKOV, *On general boundary value problems for elliptic systems in the sense of Douglis–Nirenberg*, I, *Izv. Akad. Nauk SSSR, Ser. Mat.* **28** (1964), 665–706; II, *Trudy Mat. Inst. Steklov* **92** (1996), 233–297.
- [19] R. TEMAM, *Navier–Stokes Equations*, *Studies in Mathematics and its Applications* 2, North–Holland, 1984.
- [20] ———, *Navier–Stokes Equations and nonlinear functional analysis*, *CBMS Regional Conference Series*, No. 41, SIAM, Philadelphia, 1995.
- [21] ———, *Infinite dimensional dynamical systems in mechanics and physics*, Springer-Verlag, New York, 1997.

- [22] R. TEMAM AND M. ZIANE, *Navier–Stokes equations in thin spherical domains*, Contemporary Mathematics, AMS (to appear).
- [23] ———, *Navier–Stokes Equations in Three–Dimensional Thin Domains with various boundary conditions*, Adv. Differential Equations **1** (1996), 499–546.
- [24] M. ZIANE, *On the two dimensional Navier–Stokes equations with the free boundary condition*, J. Appl. Math. and Optimization (to appear).
- [25] M. ZIANE, *Optimal bounds on the dimension of attractors for the Navier–Stokes equations in thin spherical domains*, Physica D **105** (1997), 1–19.

*Manuscript received October 28, 1996*

I. MOISE AND R. TEMAM  
The Institute for Scientific  
Computing and Applied Mathematics  
Indiana University  
Bloomington, Indiana 47405, USA  
and  
Laboratoire d'Analyse Numérique  
Université de Paris-Sud  
91405 Orsay, FRANCE  
*E-mail address:* temam@indiana.edu

M. ZIANE  
The Institute for Scientific  
Computing and Applied Mathematics  
Indiana University  
Bloomington, Indiana 47405, USA  
and  
Department of Mathematics  
and Center for Turbulence Research  
Stanford University  
Stanford, California 94309, USA  
*E-mail address:* ziane@leland.stanford.edu