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ON LOCAL MOTION OF A COMPRESSIBLE BAROTROPIC VISCOUS FLUID WITH THE BOUNDARY SLIP CONDITION

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Dedicated to O. A. Ladyzhenskaya

1. Introduction

We consider the motion of a compressible barotropic viscous fluid in a bounded domain $\Omega \subset \mathbb{R}^3$ with the boundary slip condition. Let $\rho = \rho(x,t)$ be the density of the fluid, $v = v(x,t)$ the velocity, $p = p(\rho(x,t))$ the pressure, $f = f(x,t)$ the external force field per unit mass. Then the motion is described by the following problem (see [3]):

$$
\begin{aligned}
&\rho(v_t + v\cdot\nabla v) - \operatorname{div}\mathbf{T}(v,p) = \rho f && \text{in } \Omega^T = \Omega\times(0,T),\\
&\rho_t + \operatorname{div}(\rho v) = 0 && \text{in } \Omega^T,\\
&\rho|_{t=0} = \rho_0 \quad v|_{t=0} = v_0 && \text{in } \Omega,\\
&\overline{\tau}_\alpha\cdot\mathbf{T}(v,p)\cdot\overline{n} + \gamma v\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T = S\times(0,T),\\
&v\cdot\overline{n} = 0 && \text{on } S^T,
\end{aligned}
\tag{1.1}
$$

where $\mathbf{T}(v,p)$ is the stress tensor of the form

$$
\begin{aligned}
\mathbf{T}(v,p) &= \{T_{ij}(v,p)\}_{i,j=1,2,3}\\
&= \{\mu(\partial_{x_i}v_j + \partial_{x_j}v_i) + (\nu-\mu)\operatorname{div} v\,\delta_{ij} - p\delta_{ij}\}_{i,j=1,2,3},
\end{aligned}
\tag{1.2}
$$

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$\overline{n}$, $\overline{\tau}_\alpha$, $\alpha = 1, 2$, are unit orthonormal vectors such that $\overline{n}$ is the outward normal vector and $\overline{\tau}_1$, $\overline{\tau}_2$ are tangent to S. Finally, γ is a positive constant and μ, ν are constant viscosity coefficients.

By virtue of $(1.1)_2$ and $(1.1)_5$ the total mass is conserved, so

$$\int_\Omega \rho_0(x)\,dx = \int_\Omega \rho(x,t)\,dx = M. \tag{1.3}$$

Moreover, from thermodynamic considerations we have

$$\nu > \mu/3. \tag{1.4}$$

Let us introduce the quantity for vectors u, $v \in H^1(\Omega)$

$$E_\Omega(u,v) = \int_\Omega (\partial_{x_i} u_j + \partial_{x_j} u_i)(\partial_{x_i} v_j + \partial_{x_j} v_i)\,dx, \tag{1.5}$$

where the summation convention over the repeated indices is used. We assume that $E_\Omega(u) = E_\Omega(u,u)$.

We recall from [7] that the vectors for which $E_\Omega(u) = 0$ form a finite dimensional affine space of vectors such that

$$u = A + B \times x,$$

where A and B are constant vectors.

We define $H(\widetilde{\Omega}) = \{u : E_\Omega(u) < \infty,\ u\cdot\overline{n} = 0 \text{ on } S\}$. If Ω is a region obtained by rotation about a vector B, we denote by $H(\Omega)$ the space of functions in $H(\widetilde{\Omega})$ satisfying the condition

$$\int_\Omega u(x)u_0(x)\,dx = 0,$$

where $u_0 = B \times x$; otherwise we set $H(\Omega) = \widetilde{H}(\Omega)$ (see [7]).

From Lemmas 2.1, 2.2 from [4] (see also Lemma 4 in [7]) we infer

Lemma 1.1 (Korn inequality). *Let $S \in H^{3+\alpha}$, $\alpha \in (1/2, 1)$. Then for any $u \in H(\Omega)$,*

$$\|u\|^2_{1,\Omega} \le cE_\Omega(u), \tag{1.6}$$

where c is a positive constant.

We introduce Lagrangian coordinates as the initial data for the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi, \tag{1.7}$$

then

$$x = \xi + \int_0^t \overline{v}(\xi,\tau)\,d\tau \equiv x_{\overline{v}}(\xi,t), \tag{1.8}$$

and $\overline{v}(\xi,t) = v(x_{\overline{v}}(\xi,t),t)$. We sometimes omit the index $\overline{v}$ in $x_{\overline{v}}$.

To prove the existence of solutions to problem (1.1) we integrate the equation of continuity using the Lagrangian coordinates (1.8). Therefore we have

$$\rho(x,t) = \rho_0(\xi)\exp\bigg(-\int_0^t (\nabla_{\overline{v}}\cdot\overline{v})(\xi,\tau)\,d\tau\bigg), \tag{1.9}$$

where $\nabla_{\overline{v}} = (\partial\xi_k/\partial x_i)\overline{v}_{i,\xi_k}$.

Assuming that $v = u$ in (1.9) and assuming that transformation (1.8) is generated by vector u, we also consider instead of (1.1) the following linearized problem

$$\begin{aligned} &\rho_{\overline{u}}\overline{v}_t - \mathrm{div}_{\overline{u}}\mathbf{T}_{\overline{u}}(\overline{v},p(\rho_{\overline{u}})) = \rho_{\overline{u}}f_{\overline{u}} && \text{in } \Omega^T,\\ &\overline{\tau}_{\overline{u}\alpha}\cdot\mathbf{T}_{\overline{u}}(\overline{v},p(\rho_{\overline{u}}))\cdot\overline{n}_{\overline{u}} + \gamma\overline{v}\cdot\overline{\tau}_{\overline{u}\alpha} = 0,\ \alpha=1,2, && \text{on } S^T,\\ &\overline{v}\cdot\overline{n}_{\overline{u}} = 0 && \text{on } S^T,\\ &\overline{v}|_{t=0} = v_0 && \text{in } \Omega, \end{aligned} \tag{1.10}$$

where $\rho_{\overline{u}}(\xi,t) = \rho(x_{\overline{u}}(\xi,t),t)$, $f_{\overline{u}}(\xi,t) = f(x_{\overline{u}}(\xi,t),t)$ and see also (6.3), which assigns the transformation

$$v = \Phi(u). \tag{1.11}$$

A fixed point of (1.11) is a solution to problem (1.1).

To prove the existence of solutions to problem (1.10) we first consider the following problem

$$\begin{aligned} &\omega_t - \mathrm{div}\,\mathbb{D}(\omega) = F && \text{in } \Omega^T,\\ &\overline{\tau}_\alpha\cdot\mathbb{D}(\omega)\cdot\overline{n} = G_\alpha,\ \alpha=1,2, && \text{on } S^T,\\ &\omega\cdot\overline{n} = G_3 && \text{on } S^T,\\ &\omega|_{t=0} = \omega_0 && \text{in } \Omega. \end{aligned} \tag{1.12}$$

To consider this problem we use the potential technique developed by V. A. Solonnikov. The existence will be proved in Sobolev–Slobodetskiĭ spaces.

To prove the existence of solutions to problem (1.10) we need also examine the following problem

$$\begin{aligned} &\eta\omega_t - \mathrm{div}\,\mathbb{D}(\omega) = F_1 && \text{in } \Omega^T,\\ &\overline{\tau}_\alpha\cdot\mathbb{D}(\omega)\cdot\overline{n} = G_{1\alpha},\ \alpha=1,2, && \text{on } S^T,\\ &\omega\cdot\overline{n} = G_{13} && \text{on } S^T,\\ &\omega|_{t=0} = \omega_0 && \text{in } \Omega, \end{aligned} \tag{1.13}$$

where $\eta \ge \eta_0 > 0$, η_0 is a constant and η is a given function.

2. Notation and auxiliary results

We use the anisotropic Sobolev–Slobodetskiĭ spaces $W_2^{l,l/2}(Q^T)$, $l \in \mathbb{R}_+$, $Q^T = Q \times (0,T)$, where Q is either Ω (a domain in $\mathbb{R}^3$) or S (the boundary of Ω), with the norm

$$\begin{aligned}\|u\|^2_{W_2^{l,l/2}(Q^T)} &= \int_0^T \|u\|^2_{W_2^l(Q)}\,dt + \int_Q \|u\|^2_{W_2^{l/2}(0,T)}\,dx\\ &\equiv \|u\|^2_{W_2^{l,0}(Q^T)} + \|u\|^2_{W_2^{0,l/2}(Q^T)},\end{aligned}$$

where

$$\|u\|^2_{W_2^l(Q)} = \sum_{|\alpha|\le l} \|D^\alpha u\|^2_{L_2(Q)},$$

for integer l, and

$$\|u\|^2_{W_2^l(Q)} = \sum_{|\alpha|\le [l]} \|D^\alpha u\|^2_{L_2(Q)} + \sum_{|\alpha|=[l]} \int_Q\int_Q \frac{|D_x^\alpha u(x) - D_{x'}^\alpha u(x')|^2}{|x-x'|^{s+2(l-[l])}}\,dx\,dx',$$

for noninteger l, where $s = \dim Q$, $D_x^\alpha = \partial_{x_1}^{\alpha_1}\ldots\partial_{x_s}^{\alpha_s}$, $\alpha = (\alpha_1,\ldots,\alpha_s)$ is a multiindex, $[l]$ is the integer part of l. For $Q = S$ the above norm is introduced by using local mappings and a partition of unity.

Finally,

$$\|u\|^2_{W_2^{l/2}(0,T)} = \sum_{i\le l/2} \|\partial_t^i u\|^2_{L_2(0,T)},$$

for integer $l/2$, and

$$\|u\|^2_{W_2^{l/2}(0,T)} = \sum_{i\le [l/2]} \|\partial_t^i u\|^2_{L_2(0,T)} + \sum_{i=[l/2]} \int_0^T\int_0^T \frac{|\partial_t^i u(t) - \partial_{t'}^i u(t')|^2}{|t-t'|^{1+2(l/2-[l/2])}}\,dt\,dt'.$$

Moreover,

$$\begin{aligned}|u|_{p,\Omega} &= \|u\|_{L_p(\Omega)}, \quad p \in [1,\infty],\\ [u]_{\alpha,\Omega} &= \left(\int_\Omega\int_\Omega \frac{|u(x)-u(x')|^2}{|x-x'|^{3+2\alpha}}\,dx\,dx'\right)^{1/2},\\ [u]_{\alpha,(0,T)} &= \left(\int_0^T\int_0^T \frac{|u(t)-u(t')|^2}{|t-t'|^{1+2\alpha}}\,dt\,dt'\right)^{1/2},\end{aligned}$$

and

$$\begin{aligned}[u]_{\alpha,\Omega^T,x} &= \left(\int_0^T dt \int_\Omega\int_\Omega \frac{|u(x,t)-u(x',t)|^2}{|x-x'|^{3+2\alpha}}\,dx\,dx'\right)^{1/2},\\ [u]_{\alpha,\Omega^T,t} &= \left(\int_\Omega dx \int_0^T\int_0^T \frac{|u(x,t)-u(x,t')|^2}{|t-t'|^{1+2\alpha}}\,dt\,dt'\right)^{1/2}.\end{aligned}$$

To consider the problems with vanishing initial conditions we need a space of functions which admit a zero extension to $t<0$. Therefore, for every $\gamma\ge 0$, we introduce the space $H^{l,l/2}_\gamma(Q^T)$ with the norm (see [1], [5], [8])

$$\|u\|^2_{H^{l,l/2}_\gamma(Q^T)}=\int_0^T e^{-2\gamma t}\|u\|^2_{W^l_2(Q)}\,dt+\|u\|^2_{H^{0,l/2}_\gamma(Q^T)}.$$

For $l/2\notin\mathbb{Z}$,

$$\begin{aligned}\|u\|^2_{H^{0,l/2}_\gamma(Q^T)}=&\gamma^l\int_0^T e^{-2\gamma t}\|u\|^2_{L_2(Q)}\,dt\\&+\int_0^T e^{-2\gamma t}\,dt\int_0^\infty\frac{\|\partial_t^k u_0(\,\cdot\,,t-\tau)-\partial_t^k u_0(\,\cdot\,,t)\|^2_{L_2(Q)}}{\tau^{1+2(l/2-k)}}\,d\tau,\end{aligned}$$

where $k=[l/2]<l/2$, and $u_0(x,t)=u(x,t)$ for $t>0$, $u_0(x,t)=0$ for $t<0$. For $l/2\in\mathbb{Z}$,

$$\|u\|^2_{H^{0,l/2}_\gamma(Q^T)}=\int_0^T e^{-2\gamma t}(\gamma^l\|u\|^2_{L_2(Q)}+\|\partial_t^{l/2}u\|^2_{L_2(Q)})\,dt,$$

and we assume that $\partial_t^j u|_{t=0}=0$, $j=0,\dots,l/2-1$, so $u_0(x,t)$ has a generalized derivative $\partial_t^{l/2}u_0$ in $Q\times(-\infty,T)$.

For simplicity we write

$$\begin{aligned}\|u\|_{l,Q^T}&=\|u\|_{W_2^{l,l/2}(Q^T)}, &\|u\|_{l,Q}&=\|u\|_{W^l_2(Q)},\\ \|u\|_{l,\gamma,Q^T}&=\|u\|_{H_\gamma^{l,l/2}(Q^T)}, &\|u\|_{p,Q}&=\|u\|_{L_p(Q)}.\end{aligned}$$

In the above definitions we can use the notation

$$\|u\|_{W^l_2(Q)}=\bigg(\sum_{|\alpha|\le[l]}|D^\alpha_x u|^2_{2,Q}+\sum_{|\alpha|=[l]}[D^\alpha_x u]^2_{l-[l],Q}\bigg)^{l/2}.$$

We set $\mathbb{R}^n_+=\{x\in\mathbb{R}^n : x_n>0\}$, $\mathbb{R}^{n+1}_T=\mathbb{R}^n\times(0,T)$, $\mathbb{D}^{n+1}_T=\mathbb{R}^n_+\times(0,T)$, $n=2,3$. For functions defined in $\mathbb{R}^{n+1}_\infty$ and vanishing sufficiently fast at infinity we define the Fourier transform with respect to x and the Laplace transform with respect to t by the formula

$$\widetilde{f}(\xi,s)=\int_0^\infty e^{-st}\,dt\int_{\mathbb{R}^n}f(x,t)e^{-ix\cdot\xi}\,dx, \tag{2.1}$$

where $x\cdot\xi=\sum_{i=1}^n x_i\xi_i$. Hence we define the norm

$$|||u|||^2_{l,\gamma,\mathbb{R}^{n+1}_\infty}=\int_{\mathbb{R}^n}d\xi\int_{-\infty}^\infty|\widetilde{u}(\xi,s)|^2(|s|+|\xi|^2)^l\,d\xi_0,\quad s=\gamma+i\xi_0\,,\ \gamma\in\mathbb{R}_+.$$

Similarly, for functions defined in $\mathbb{D}^{n+1}_\infty$, we have

$$\widetilde{f}(\xi',x_n,s)=\int_0^\infty e^{-st}\,dt\int_{\mathbb{R}^{n-1}}f(x,t)e^{-ix'\cdot\xi'}\,dx',$$

where $x'=(x_1,\dots,x_{n-1})$, $\xi'=(\xi_1,\dots,\xi_{n-1})$, and introduce the norm

$$|||u|||^2_{l,\gamma,\mathbb{D}^{n+1}_\infty}=\sum_{j\le[l]}\int_{\mathbb{R}^{n-1}}d\xi'\int_{-\infty}^{\infty}\|\partial^j_{x_n}\widetilde u(\xi',x_n,s)\|^2_{0,\mathbb{R}^1_+}(|s|+|\xi'|^2)^{l-j}\,d\xi_0$$
$$+\int_{\mathbb{R}^{n-1}}d\xi'\int_{-\infty}^{\infty}\|\widetilde u(\xi',\,\cdot\,,s)\|^2_{l,\mathbb{R}^1_+}\,d\xi_0\,,\quad s=\gamma+i\xi_0,\quad \gamma\in\mathbb{R}_+.$$

We introduce a partition of unity. Let us define two collections of open subsets $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$, $k\in\mathcal{M}\cup\mathcal{N}$, such that $\overline\omega^{(k)}\subset\Omega^{(k)}\subset\Omega$, $\bigcup_k\omega^{(k)}=\bigcup_k\Omega^{(k)}=\Omega$, $\overline\Omega^{(k)}\cap S=\emptyset$ for $k\in\mathcal{M}$ and $\overline\Omega^{(k)}\cap S\ne\emptyset$ for $k\in\mathcal{N}$. We assume that at most N_0 of the $\Omega^{(k)}$ have nonempty intersection, and $\sup_k\operatorname{diam}\Omega^{(k)}\le 2\lambda$ for some $\lambda>0$. Let $\zeta^{(k)}(x)$ be a smooth function such that $0\le\zeta^{(k)}(x)\le 1$, $\zeta^{(k)}(x)=1$ for $x\in\omega^{(k)}$, $\zeta^{(k)}(x)=0$ for $x\in\Omega\backslash\Omega^{(k)}$ and $|D^\nu_x\zeta^{(k)}(x)|\le c/\lambda^{|\nu|}$. Then $1\le\sum_k(\zeta^{(k)}(x))^2\le N_0$. Introducing the function

$$\eta^{(k)}(x)=\frac{\zeta^{(k)}(x)}{\sum_l(\zeta^{(l)}(x))^2},$$

we have $\eta^{(k)}(x)=0$ for $x\in\Omega\backslash\Omega^{(k)}$, $\sum_k\eta^{(k)}(x)\zeta^{(k)}(x)=1$ and $|D^\nu_x\eta^{(k)}(x)|\le c/\lambda^{|\nu|}$. By $\xi^{(k)}$ we denote the center of $\omega^{(k)}$ and $\Omega^{(k)}$ for $k\in\mathcal{M}$ and the center of $\overline\omega^{(k)}\cap S$ and $\overline\Omega^{(k)}\cap S$ for $k\in\mathcal{N}$.

Considering problems invariant with respect to translations and rotations we can introduce a local coordinates system $y=(y_1,y_2,y_3)$ with the center at $\xi^{(k)}$ such that the part $\widetilde S^{(k)}=S\cap\overline\Omega^{(k)}$ of the boundary is described by $y_3=F(y_1,y_2)$. Then we consider new coordinates defined by

$$z_i=y_i,\quad i=1,2,\quad z_3=y_3-F(y_1,y_2).$$

We will denote this transformation by $\widehat\Omega^{(k)}\supset\widehat\omega^{(k)}\ni z=\Phi_k(y)$, where $y\in\omega^{(k)}\subset\Omega^{(k)}$; we assume that the sets $\widehat\omega^{(k)}$, $\widehat\Omega^{(k)}$ are described in local coordinates at $\xi^{(k)}$ by the inequalities

$$\begin{aligned}&|y_i|<\lambda,\quad i=1,2,\qquad 0<y_3-F(y_1,y_2)<\lambda,\\&|y_i|<2\lambda,\quad i=1,2,\qquad 0<y_3-F(y_1,y_2)<2\lambda,\end{aligned}$$

respectively.

Let $y=Y_k(x)$ be a transformation from coordinates x to local coordinates y, which is a composition of a translation and a rotation. Then we set

$$\widehat u^{(k)}(z,t)=u(\Phi_k^{-1}\circ Y_k^{-1}(z),t),\quad \widetilde u^{(k)}(z,t)=\widehat u^{(k)}(z,t)\widehat\zeta^{(k)}(z,t).$$

From [5] we recall the necessary for us properties of spaces $H^{l,l/2}_\gamma(\mathbb{R}^{n+1}_T)$ and $H^{l,l/2}_\gamma(\mathbb{D}^{n+1}_T)$, $l\in\mathbb{R}_+$.

Lemma 2.1. *Any function* $u \in H_\gamma^{l,l/2}(\mathbb{R}_T^{n+1})$, $T < \infty$, *can be extended to* $\mathbb{R}_\infty^{n+1}$ *in such a way that the extended function* $u' \in H_\gamma^{l,l/2}(\mathbb{R}_\infty^{n+1})$ *and*

$$\|u'\|_{l,\gamma,\mathbb{R}_\infty^{n+1}} \le c\|u\|_{l,\gamma,\mathbb{R}_T^{n+1}}. \tag{2.2}$$

Any function $u \in H_\gamma^{l,l/2}(\mathbb{D}_T^{n+1})$, $T < \infty$, *can be extended to* $\mathbb{R}_T^{n+1}$ *in such a way that the extended function* $u' \in H_\gamma^{l,l/2}(\mathbb{R}_T^{n+1})$ *and*

$$\|u'\|_{l,\gamma,\mathbb{R}_T^{n+1}} \le c\|u\|_{l,\gamma,\mathbb{D}_T^{n+1}}. \tag{2.3}$$

Lemma 2.2. *There exist constants* c_1 *and* c_2, *which do not depend on* u *and* γ, *such that*

$$c_1|||u|||_{l,\gamma,\mathbb{R}_\infty^{n+1}} \le \|u\|_{l,\gamma,\mathbb{R}_\infty^{n+1}} \le c_2|||u|||_{l,\gamma,\mathbb{R}_\infty^{n+1}}. \tag{2.4}$$

Lemma 2.3. *There exist constants* c_3 *and* c_4, *which do not depend on* u *and* γ, *such that*

$$c_3|||u|||_{l,\gamma,\mathbb{D}_\infty^{n+1}} \le \|u\|_{l,\gamma,\mathbb{D}_\infty^{n+1}} \le c_4|||u|||_{l,\gamma,\mathbb{D}_\infty^{n+1}}. \tag{2.5}$$

From [5] we recall

Lemma 2.4. *Let* $u \in H_\gamma^{r,r/2}(\Omega^T)$. *Then for every* $\varepsilon \in (0,1)$ *and* $0 \le q < r - |\alpha|$,

$$\begin{aligned} \|D_x^\alpha u\|_{q,\gamma,\Omega^T} &\le \varepsilon^{r-|\alpha|-q}\|u\|_{r,\gamma,\Omega^T} + c\varepsilon^{-q-|\alpha|}\|e^{-\gamma t}u\|_{0,\Omega^T} \\ &\le (\varepsilon^{r-|\alpha|-q} + c\gamma^{-r/2}\varepsilon^{-q-|\alpha|})\|u\|_{r,\gamma,\Omega^T}. \end{aligned} \tag{2.6}$$

Lemma 2.5 (see [5]). *Let* $u \in H_\gamma^{l,l/2}(\mathbb{R}_T^{n+1})$ *and* $0 < 2m + |\alpha| < l$. *Then* $\partial_t^m D_x^\alpha u \in H_\gamma^{l_1,l_1/2}(\mathbb{R}_T^{n+1})$, *where* $l_1 = l - 2m - |\alpha|$ *and*

$$\|\partial_t^m D_x^\alpha u\|_{l_1,\gamma,\mathbb{R}_T^{n+1}} \le c\|u\|_{l,\gamma,\mathbb{R}_T^{n+1}}. \tag{2.7}$$

Moreover, for $\rho \in (0, l_1)$ *and* $\varepsilon > 0$,

$$\|\partial_t^m D_x^\alpha u\|_{\rho,\gamma,\mathbb{R}_T^{n+1}} \le \varepsilon^{l_1-\rho}\|u\|_{l,\gamma,\mathbb{R}_T^{n+1}} + c\varepsilon^{-h}\|e^{-\gamma t}u\|_{0,\mathbb{R}_T^{n+1}}, \tag{2.8}$$

where $h = \rho + 2m + |\alpha|$.

Let $u \in H_\gamma^{l,l/2}(\mathbb{D}_T^{n+1})$ *and* $0 \le 2m + |\alpha| < l - 1/2$. *Then* $\partial_t^m D_x^\alpha u|_{x_n=0} \in H_\gamma^{l_2,l_2/2}(\mathbb{R}_T^n)$, *where* $l_2 = l - 2m - |\alpha| - 1/2$ *and*

$$\|\partial_t^m D_x^\alpha u|_{x_n=0}\|_{l_2,\gamma,\mathbb{R}_T^n} \le c\|u\|_{l,\gamma,\mathbb{D}_T^{n+1}}. \tag{2.9}$$

To obtain the results from Section 6 we need

Lemma 2.6 (see [5]). *Let's assume that* $a \in W_2^{l+1}(\Omega)$, $f \in W_2^l(\Omega)$, $g \in W_2^{l+1}(\Omega)$, $l > 1/2$, $\Omega \subset \mathbb{R}^3$. *Then the following estimates hold*

$$
\begin{aligned}
\|af\|_{l,\Omega} &\le c_1|a|_{\infty,\Omega}\|f\|_{l,\Omega} + \|a\|_{l+1,\Omega}(\varepsilon\|f\|_{l,\Omega} + c_2(\varepsilon)\|f\|_{0,\Omega}),\\
\|af\|_{l,\Omega} &\le c_3\|f\|_{l,\Omega}(\varepsilon\|a\|_{l+1,\Omega} + c_4(\varepsilon)\|a\|_{0,\Omega}),\\
\|ag\|_{l+1,\Omega} &\le c_5|a|_{\infty,\Omega}\|g\|_{l+1,\Omega} + \|a\|_{l+1,\Omega}(\varepsilon\|g\|_{l+1,\Omega} + c_6(\varepsilon)\|g\|_{0,\Omega}),\\
\|ag\|_{l+1,\Omega} &\le c_7\|a\|_{l+1,\Omega}\|g\|_{l+1,\Omega},
\end{aligned}
\tag{2.10}
$$

where $\varepsilon \in (0,1)$.

3. Existence of solutions to problem (1.12) with vanishing initial data and in the half space

Considering the problem (1.12) in the half–space $x_3 > 0$ and with vanishing initial conditions we have

$$
\begin{aligned}
&\omega_t - \operatorname{div}\mathbb{D}(\omega) = 0 && x_3 > 0,\\
&\mu\bigg(\frac{\partial\omega_i}{\partial x_3} + \frac{\partial\omega_3}{\partial x_i}\bigg) = b_i,\ i = 1,2, && x_3 = 0,\\
&\omega_3 = b_3 && x_3 = 0,\\
&\omega|_{t=0} = 0 && x_3 > 0.
\end{aligned}
\tag{3.1}
$$

By applying the Fourier–Laplace transformation

$$
\begin{aligned}
&\widetilde{f}(\xi', x_3, s) = \int_0^\infty e^{-st}\,dt\int_{\mathbb{R}^2} f(x,t)e^{-ix'\cdot\xi'}\,dx',\\
&\mathrm{Re}s > 0, \quad s = \gamma + i\xi_0,
\end{aligned}
\tag{3.2}
$$

where $\xi' = (\xi_1,\xi_2)$, $x' = (x_1,x_2)$, $x'\cdot\xi' = x_1\xi_1 + x_2\xi_2$, we obtain problem (3.1) in the form

$$
\mu\frac{d^2\widetilde{\omega}_k}{dx_3^2} + \nu i\xi_k\frac{d\widetilde{\omega}_3}{dx_3} - (s+\mu\xi^2)\widetilde{\omega}_k - \nu\xi_k\xi_j\widetilde{\omega}_j = 0,\ k = 1,2, \quad x_3 > 0,
\tag{3.3}
$$

$$
(\mu+\nu)\frac{d^2\widetilde{\omega}_3}{dx_3^2} + \nu i\xi_j\frac{d\widetilde{\omega}_j}{dx_3} - (s+\mu\xi^2)\widetilde{\omega}_3 = 0, \qquad x_3 > 0,
$$

$$
\mu\frac{d\widetilde{\omega}_k}{dx_3} + \mu i\xi_k\widetilde{\omega}_3 = \tilde{b}_k, \qquad k = 1,2, \quad x_3 = 0,
\tag{3.4}
$$

$$
\widetilde{\omega}_3 = \widetilde{b}_3, \qquad x_3 = 0,
$$

$\widetilde{\omega} \to 0$ as $x_3 \to \infty$, where $\xi = (\xi_1,\xi_2)$, $\xi^2 = \xi_1^2 + \xi_2^2$ and the summation convention over the repeated indices is assumed.

Every solution to (3.3) vanishing at infinity has the form

$$
\widetilde{\omega} = \Phi(\xi,s)e^{-\tau_1x_3} + \psi(\xi,s)(\xi_1,\xi_2,i\tau_2)e^{-\tau_2x_3},
\tag{3.5}
$$

where $\Phi(\xi,s) = (\phi_1,\phi_2,(i/\tau_1)\xi\cdot\phi)$, $\phi_j = \phi_j(\xi,s)$, $j = 1,2$, $\tau_1 = \sqrt{s/\mu+\xi^2}$, $\tau_2 = \sqrt{s/(\mu+\nu)+\xi^2}$, $\arg\tau_j \in (-\pi/4,\pi/4)$, $j = 1,2$, $\xi\cdot\phi = \xi_1\phi_1 + \xi_2\phi_2$, $\phi = (\phi_1,\phi_2)$.

Inserting (3.5) into (3.4) yields

$$\tag{3.6} \begin{aligned} &\tau_1\phi_k + \tau_2\xi_k\psi = i\widetilde{b}_3\xi_k - \widetilde{b}_k/\mu\,, \quad k=1,2,\\ &\xi\cdot\phi + \tau_1\tau_2\psi = -i\tau_1\widetilde{b}_3. \end{aligned}$$

From (3.6) we have

$$\tag{3.7} \tau_1\phi\cdot\xi + \tau_2\xi^2\psi = i\widetilde{b}_3\xi^2 - \widetilde{b}\cdot\xi/\mu, \quad \xi\cdot\phi + \tau_1\tau_2\psi = -i\tau_1\widetilde{b}_3.$$

Solving (3.7), we obtain

$$\tag{3.8} \phi\cdot\xi = \frac{\tau_1}{\tau_1^2-\xi^2}\left(2\xi^2 i\widetilde{b}_3 - \frac{1}{\mu}\widetilde{b}\cdot\xi\right), \quad \psi = \frac{1}{\tau_2(\tau_1^2-\xi^2)}\left((\tau_1^2-\xi^2)i\widetilde{b}_3 + \frac{1}{\mu}\widetilde{b}\cdot\xi\right),$$

where

$$\tag{3.9} \tau_1^2 - \xi^2 = s/\mu.$$

Using (3.8) and (3.9) in (3.6) implies

$$\tag{3.10} \phi_k = -\frac{1}{s\tau_1}\widetilde{b}\cdot\xi\xi_k - \frac{1}{\mu\tau_1}\widetilde{b}_k, \quad k=1,2, \qquad \psi = \frac{1}{\tau_2}\left(i\widetilde{b}_3 + \frac{1}{s}\widetilde{b}\cdot\xi\right).$$

Let

$$\tag{3.11} e_i = e^{-\tau_i x_3}, \quad i=1,2, \qquad e_0 = \frac{e_1-e_2}{\tau_1-\tau_2}.$$

Then we write (3.5) in the form

$$\tag{3.12} \widetilde{\omega} = \psi\begin{pmatrix}\xi_1\\ \xi_2\\ i\tau_2\end{pmatrix}(e_2-e_1) + \begin{pmatrix}\phi_1+\psi\xi_1\\ \phi_2+\psi\xi_2\\ i\phi\cdot\xi/\tau_1 + i\tau_2\psi\end{pmatrix}e_1 \equiv Ve_0 + We_1,$$

where

$$\tag{3.13} \begin{aligned} V_j &= \frac{1}{\tau_2}\left(i\widetilde{b}_3 + \frac{1}{s}\widetilde{b}\cdot\xi\right)(\tau_2-\tau_1)\xi_j\\ &= -\frac{c_0}{\tau_2(\tau_1+\tau_2)}(is\widetilde{b}_3 + \widetilde{b}\cdot\xi)\xi_j, \quad j=1,2,\\ V_3 &= i\left(i\widetilde{b}_3 + \frac{1}{s}\widetilde{b}\cdot\xi\right)(\tau_2-\tau_1) = -\frac{ic_0}{\tau_1+\tau_2}(is\widetilde{b}_3 + \widetilde{b}\cdot\xi), \end{aligned}$$

where we used

$$\tau_1^2 - \tau_2^2 = \frac{\nu}{\mu(\mu+\nu)}s \equiv c_0 s,$$

and

$$\tag{3.14} \begin{aligned} W_j &= \phi_j + \psi\xi_j = \frac{1}{s}\left(\frac{1}{\tau_2} - \frac{1}{\tau_1}\right)\widetilde{b}\cdot\xi\xi_j - \frac{1}{\mu\tau_1}\widetilde{b}_j + \frac{i}{\tau_2}\widetilde{b}_3\xi_j\\ &= \frac{c_0}{\tau_1\tau_2(\tau_1+\tau_2)}\widetilde{b}\cdot\xi\xi_j - \frac{1}{\mu\tau_1}\widetilde{b}_j + \frac{i}{\tau_2}\widetilde{b}_3\xi_j, \quad j=1,2,\\ W_3 &= \widetilde{b}_3. \end{aligned}$$

We need the following result (see [10], [5], [8]).

Lemma 3.1. *For $\xi \in \mathbb{R}^2$, $s = \gamma + i\xi_0$, $\gamma \in \mathbb{R}_+$, $\xi_0 \in \mathbb{R}$, $\gamma > 0$, and for any nonnegative integer j and $\kappa \in (0,1)$,*

$$\|\partial_{x_3}^j e_i\|_{0,\mathbb{R}_+}^2 \le c_1|\tau_i|^{2j-1}, \quad i = 1,2,$$

$$\int_0^\infty \int_0^\infty |\partial_{x_3}^j e_i(x_3+z) - \partial_{x_3}^j e_i(x_3)|^2 \frac{dx_3\,dz}{z^{1+2\kappa}} \le c_2|\tau_i|^{2(j+\kappa)-1}, \quad i = 1,2, \tag{3.15}$$

where c_1, c_2 do not depend on τ_i, $i = 1,2$ and $|\xi|$.

Moreover,

$$\|\partial_{x_3}^j e_0\|_{0,\mathbb{R}_+}^2 \le c_3 \frac{|\tau_1|^{2j-1} + |\tau_2|^{2j-1}}{|\tau_1|^2},$$

$$\begin{aligned}\int_0^\infty \int_0^\infty |\partial_{x_3}^j e_0(x_3+z) - \partial_{x_3}^j e_0(x_3)|^2 \frac{dx_3\,dz}{z^{1+2\kappa}} \\ \le c_4 \frac{|\tau_1|^{2(j+\kappa)-1} + |\tau_2|^{2(j+\kappa)-1}}{|\tau_1|^2},\end{aligned} \tag{3.16}$$

where c_3, c_4 do not depend on τ_i, $i = 1,2$ and $|\xi|$.

Using Lemma 3.1, we obtain

Lemma 3.2. *We assume that*

$$\begin{aligned} &b_i \in H_\gamma^{1/2+\alpha,1/4+\alpha/2}(\mathbb{R}_\infty^3), \quad i = 1,2,\\ &b_3 \in H_\gamma^{3/2+\alpha,3/4+\alpha/2}(\mathbb{R}_\infty^3), \quad \gamma > 0,\ \alpha > 0.\end{aligned}$$

Then there exists a solution to problem (3.1) in $H_\gamma^{2+\alpha,1+\alpha/2}(\mathbb{D}_\infty^4)$ and the following estimate holds

$$\sum_{i=1}^3 |||\omega_i|||_{2+\alpha,\gamma,\mathbb{D}_\infty^4} \le c(\gamma)\bigg(\sum_{i=1}^2 |||b_i|||_{1/2+\alpha,\gamma,\mathbb{R}_\infty^3} + |||b_3|||_{3/2+\alpha,\gamma,\mathbb{R}_\infty^3}\bigg), \tag{3.17}$$

where $c(\gamma)$ remains bounded for $\gamma \ge \gamma_0 > 0$.

Proof. The existence follows from constructions (3.12)–(3.14). To obtain (3.17) we consider

$$\begin{aligned}|||\omega|||_{2+\alpha,\gamma,\mathbb{D}_\infty^4}^2 &= \sum_{j\le 2} \int_{\mathbb{R}^2} d\xi \int_{-\infty}^\infty \|\partial_{x_3}^j \widetilde{\omega}(\xi,x_3,s)\|_{0,\mathbb{R}_+^1}^2 (|s| + |\xi|^2)^{2+\alpha-j}\,d\xi_0 \\ &\quad + \int_{\mathbb{R}^2} d\xi \int_{-\infty}^\infty \|\widetilde{\omega}(\xi,x_3,s)\|_{2+\alpha,\mathbb{R}_+^1}^2\,d\xi_0 \\ &\le \sum_{j\le 2} \int_{\mathbb{R}^2} d\xi \int_{-\infty}^\infty (|V|^2 \|\partial_{x_3}^j e_0\|_{0,\mathbb{R}_+^1}^2 + |W|^2 \|\partial_{x_3}^j e_1\|_{0,\mathbb{R}_+^1}^2) \\ &\quad \cdot (|s| + |\xi|^2)^{2+\alpha-j}\,d\xi_0 \\ &\quad + \int_{\mathbb{R}^2} d\xi \int_{-\infty}^\infty (|V|^2 \|e_0\|_{2+\alpha,\mathbb{R}_+^1}^2 + |W|^2 \|e_1\|_{2+\alpha,\mathbb{R}_+^1}^2)\,d\xi_0 \equiv I.\end{aligned}$$

Using the inequalities (see [10], [5])

$$c_1 \le \frac{|\tau_1|}{|\tau_2|} \le c_2, \quad |\tau_1+\tau_2| \ge \frac{1}{\sqrt{2}}(|\tau_1|+|\tau_2|),$$
$$c_3(|s|+\xi^2) \le |\tau_i|^2 \le c_4(|s|+\xi^2), \quad i=1,2,$$

we obtain

$$|V|^2 \le c(|\tau|^2|\widetilde{b}_3| + |\widetilde{b}'|^2), \quad |W|^2 \le c\bigg(\frac{1}{|\tau|^2}|\widetilde{b}'|^2 + |\widetilde{b}_3|\bigg), \tag{3.18}$$

where τ replaces either τ_1 or τ_2, and $\widetilde{b}' = (\widetilde{b}_1, \widetilde{b}_2)$.

In view of (3.18) and Lemma 3.1 we obtain

$$\begin{aligned} I &\le c\int_{\mathbb{R}^2} d\xi \int_{-\infty}^{\infty} (|\widetilde{b}'|^2(1+|\tau|^{1+2\alpha}) + |\widetilde{b}_3|^2(1+|\tau|^{3+2\alpha}))\, d\xi_0 \\ &\le c(\|b'\|^2_{1/2+\alpha,\gamma,\mathbb{R}^3_\infty} + \|b_3\|^2_{3/2+\alpha,\gamma,\mathbb{R}^3_\infty}). \end{aligned}$$

This concludes the proof. □

Now we consider the nonhomogeneous problem

$$\begin{aligned} &\omega_t - \operatorname{div}\mathbb{D}(\omega) = f && x_3>0, \\ &\mu\bigg(\frac{\partial\omega_i}{\partial x_3} + \frac{\partial\omega_3}{\partial x_i}\bigg) = b_i\,,\ i=1,2, && x_3=0, \\ &\omega_3 = b_3 && x_3=0, \\ &\omega|_{t=0} = 0 && x_3>0. \end{aligned} \tag{3.19}$$

Lemma 3.3. *Let $\alpha>0$, $\gamma>0$. If we assume that $f\in H^{\alpha,\alpha/2}_\gamma(\mathbb{D}^4_T)$, $b'=(b_1,b_2)\in H^{\alpha+1/2,\alpha/2+1/4}_\gamma(\mathbb{R}^3_T)$, $b_3\in H^{\alpha+3/2,\alpha/2+3/4}_\gamma(\mathbb{R}^3_T)$, $T>0$, then the problem* (3.19) *has a unique solution for $\nu>\mu/3$ such that $\omega\in H^{2+\alpha,1+\alpha/2}_\gamma(\mathbb{D}^4_T)$ and*

$$\|\omega\|_{2+\alpha,\gamma,\mathbb{D}^4_T} \le c(\|f\|_{\alpha,\gamma,\mathbb{D}^4_T} + \|b'\|_{\alpha+1/2,\gamma,\mathbb{R}^3_T} + \|b_3\|_{\alpha+3/2,\gamma,\mathbb{R}^3_T}). \tag{3.20}$$

Proof. We extend f to a function f' on $\mathbb{R}^4_\infty$ in such a way that $f'\in H^{\alpha,\alpha/2}_\gamma(\mathbb{R}^4_\infty)$ (see (2.2), (2.3)) and

$$\|f'\|_{\alpha,\gamma,\mathbb{R}^4_\infty} \le c\|f\|_{\alpha,\gamma,\mathbb{D}^4_T}. \tag{3.21}$$

Let ω' be a solution of the problem

$$L(\partial_x,\partial_t)\omega' \equiv \omega'_t - \operatorname{div}\mathbb{D}(\omega') = f' \quad \text{in } \mathbb{R}^4_\infty. \tag{3.22}$$

Applying the Fourier–Laplace transformation (2.1) to (3.22) yields

$$L(i\xi,s)\widetilde{\omega}' = \widetilde{f}', \tag{3.23}$$

so $\widetilde{\omega}' = L^{-1}(i\xi,s)\widetilde{f}'$. Since $\det L(i\xi,s) = (s+\mu\xi^2)^2(s+(\mu+\nu)\xi^2)$ and $s=\gamma+i\xi_0$ with $\gamma\ge\gamma_0>0$, one obtains easily $\|\omega'\|_{2+\alpha,\gamma,\mathbb{R}^4_\infty} \le c\|f'\|_{\alpha,\gamma,\mathbb{R}^4_\infty}$, and so

$$\|\omega\|_{2+\alpha,\gamma,\mathbb{D}^4_T} \le c\|f\|_{\alpha,\gamma,\mathbb{D}^4_\infty}. \tag{3.24}$$

Now $v=\omega-\omega'$ is a solution to the problem

$$\begin{aligned}
&v_t - \operatorname{div}\mathbb{D}(v) = 0,\\
&\mu\bigg(\frac{\partial v_i}{\partial x_3}+\frac{\partial v_3}{\partial x_i}\bigg) = h_i, \quad i=1,2,\\
&v_3 = h_3,\\
&v|_{t=0} = 0,
\end{aligned} \tag{3.25}$$

where

$$\begin{aligned}
h_i &= b_i - \mu\bigg(\frac{\partial \omega_i'}{\partial x_3}+\frac{\partial \omega_3'}{\partial x_i}\bigg), \quad i=1,2,\\
h_3 &= b_3 - \omega_3'.
\end{aligned} \tag{3.26}$$

Using (3.26) we have

$$\begin{aligned}
\|h_i\|_{1/2+\alpha/2,\gamma,\mathbb{R}^3_T} &\le c(\|b_i\|_{1/2+\alpha/2,\gamma,\mathbb{R}^3_T} + \|\omega'\|_{2+\alpha,\gamma,\mathbb{D}^4_T}), \quad i=1,2,\\
\|h_3\|_{3/2+\alpha/2,\gamma,\mathbb{R}^3_T} &\le c(\|b_3\|_{3/2+\alpha/2,\gamma,\mathbb{R}^3_T} + \|\omega'\|_{2+\alpha,\gamma,\mathbb{D}^4_T}),
\end{aligned} \tag{3.27}$$

so applying Lemma 3.2 to problem (3.25) we see that $\omega = v+\omega'$ is a solution to (3.19) and the estimate (3.20) holds. □

4. Existence of solutions to problem (1.12)

First we consider the problem (1.12) with vanishing initial data

$$\begin{aligned}
&L(\partial_x,\partial_t)u \equiv u_t - \operatorname{div}\mathbb{D}(u) = f && \text{in } \Omega\times(-\infty,T),\\
&B_{1i}(x,\partial_x)u \equiv \overline{\tau}_i\cdot\mathbb{D}(u)\cdot\overline{n} = b_i,\ i=1,2, && \text{on } S\times(-\infty,T),\\
&B_2(x)u \equiv u\cdot\overline{n} = b_3 && \text{on } S\times(-\infty,T).
\end{aligned} \tag{4.1}$$

We write shortly $B(x,\partial_x)u = (B_1(x,\partial_x)u, B_2(x)u)$.

Let $f^{(k)}(x,t) = \zeta^{(k)}(x,t)f(x,t)$. We denote by $R^{(k)}$, $k\in\mathcal{M}$, the operator

$$u^{(k)}(x,t) = R^{(k)}f^{(k)}(x,t), \tag{4.2}$$

where $u^{(k)}(x,t)$ is a solution of the Cauchy problem

$$L(\partial_x,\partial_t)u^{(k)}(x,t) = f^{(k)}(x,t). \tag{4.3}$$

For $k\in\mathcal{N}$ we define $R^{(k)}$ to be the operator

$$\widehat{u}^{(k)}(z,t) = R^{(k)}(\widehat{f}^{(k)}(z,t),\widehat{b}^{(k)}(z,t)), \tag{4.4}$$

where $\widehat{u}^{(k)}(z,t)$ is a solution to the boundary value problem

$$L(\partial_z,\partial_t)\widehat{u}^{(k)}(z,t) = \widehat{f}^{(k)}(z,t), \quad B(z,\partial_z)\widehat{u}^{(k)}(z,t) = \widehat{b}^{(k)}(z,t), \tag{4.5}$$

where $\widehat{u}^{(k)}(z,t) = Z_k^{-1}u^{(k)}(x,t)$ and Z_k describes the transformation between $\widehat{u}^{(k)}(z,t)$ and $u^{(k)}(x,t)$. Then we define an operator R (called a regularizer) by the formula (see [2], [6])

$$Rh = \sum_k \eta^{(k)}(x)u^{(k)}(x,t), \tag{4.6}$$

where

$$h^{(k)}(x,t) = \begin{cases} f^{(k)}(x,t) & k \in \mathcal{M}, \\ Z_k\{\widehat{f}^{(k)}(z,t), \widehat{b}^{(k)}(z,t)\} & k \in \mathcal{N}, \end{cases}$$

and

$$u^{(k)}(x,t) = \begin{cases} R^{(k)}f^{(k)}(x,t) & k \in \mathcal{M}, \\ Z_kR^{(k)}(Z_k^{-1}f^{(k)}(x,t), Z_k^{-1}b^{(k)}(x,t)) & k \in \mathcal{N}. \end{cases}$$

Lemma 3.3 implies the existence of solutions of problems (4.3), (4.5) and the estimates

$$\|u^{(k)}\|_{2+\alpha,\gamma,\mathbb{R}^4_\infty} \le c\|f^{(k)}\|_{\alpha,\gamma,\mathbb{R}^4_\infty}, \quad k \in \mathcal{M}, \tag{4.7}$$

and

$$\begin{aligned} \|\widehat{u}^{(k)}\|_{2+\alpha,\gamma,\mathbb{R}^4_\infty} \le{} & c(\|\widehat{f}^{(k)}\|_{\alpha,\gamma,\mathbb{D}^4_T} + \|\widehat{b}'^{(k)}\|_{1/2+\alpha/2,\gamma,\mathbb{R}^3_\infty} \\ & + \|\widehat{b}_3^{(k)}\|_{3/2+\alpha/2,\gamma,\mathbb{R}^3_\infty}), \quad k \in \mathcal{N}. \end{aligned} \tag{4.8}$$

Let

$$h = (f, b', b'') \in H_\gamma^{\alpha,\alpha/2}(\Omega^T) \times H_\gamma^{1/2+\alpha,1/4+\alpha/2}(S^T) \times H_\gamma^{3/2+\alpha,3/4+\alpha/2}(S^T) \equiv H_\gamma^\alpha$$

and let $V_\gamma^\alpha = H_\gamma^{2+\alpha,1+\alpha/2}(\Omega^T)$. Inequalities (4.7) and (4.8) imply

Lemma 4.1 (see [10], [11]). *Let $S \in C^{2+\alpha}$, $h \in H_\gamma^\alpha$, $\alpha > 1/2$ and let γ be sufficiently large. Then there exists a bounded linear operator $R : H_\gamma^\alpha \to V_\gamma^\alpha$ (defined by (4.6)) such that*

$$\|Rh\|_{V_\gamma^\alpha} \le c\|h\|_{H_\gamma^\alpha}, \tag{4.9}$$

where c does not depend on γ or h.

We write problem (4.1) in the following short form

$$Au = h\,, \quad A = (L, B). \tag{4.10}$$

Lemma 4.2. *Let $S \in H^{3+\alpha}$, $h \in H_\gamma^\alpha$ with γ sufficiently large and $\alpha > 1/2$. Then*

$$ARh = h + Th, \tag{4.11}$$

where T is a bounded operator in H^α_γ with a small norm for small λ and large γ.

Proof. We have

$$\begin{aligned} LRh = & \sum_{k\in\mathcal{M}\cup\mathcal{N}} (L(\partial_x,\partial_t)\eta^{(k)}u^{(k)} - \eta^{(k)}L(\partial_x,\partial_t)u^{(k)}) \\ & + \sum_{k\in\mathcal{N}} \eta^{(k)} Z_k(L(\partial_z - \nabla F\partial_{z_3},\partial_t) - L(\partial_z,\partial_t))Z_k^{-1}u^{(k)}(x,t) \\ & + \sum_{k\in\mathcal{M}} \eta^{(k)}L(\partial_x,\partial_t)u^{(k)}(x,t) + \sum_{k\in\mathcal{N}} \eta^{(k)}Z_kL(\partial_z,\partial_t)Z_k^{-1}u^{(k)}(x,t) \\ = & f + T_1h \end{aligned}$$

and

$$\begin{aligned} BRh = & \sum_{k\in\mathcal{N}} (B(x,\partial_x)\eta^{(k)}u^{(k)} - \eta^{(k)}B(x,\partial_x)u^{(k)}) \\ & + \sum_{k\in\mathcal{N}} \eta^{(k)}(B(x,\partial_x) - B(\xi^{(k)},\partial_x))u^{(k)} \\ & + \sum_{k\in\mathcal{N}} \eta^{(k)}Z_k(B(\xi^{(k)},\partial_z - \nabla F\partial_{z_3}) - B(\xi^{(k)},\partial_z))Z_k^{-1}u^{(k)}(x,t) \\ & + \sum_{k\in\mathcal{N}} \eta^{(k)}Z_kB(\xi^{(k)},\partial_z)Z_k^{-1}u^{(k)}(x,t) = b + T_2h. \end{aligned}$$

Now, we estimate operators T_1 and T_2. By Lemmas 2.4, 2.5 and Lemma 3.3 the first term in T_1h is estimated in the following way

$$\begin{aligned} \Big\| \sum_{k\in\mathcal{M}\cup\mathcal{N}} (L\eta^{(k)}u^{(k)} - \eta^{(k)}Lu^{(k)})\Big\|_{\alpha,\gamma,\Omega^T} & \le c \sum_{k\in\mathcal{M}\cup\mathcal{N}} \|u^{(k)}\|_{1+\alpha,\gamma,Q^{(k)}} \\ & \le c(\varepsilon^{\delta_1} + c_0(\varepsilon)\gamma^{-\delta_2}) \sum_{k\in\mathcal{M}\cup\mathcal{N}} \|u^{(k)}\|_{2+\alpha,\gamma,Q^{(k)}} \\ & \le c(\varepsilon^{\delta_1} + c_0(\varepsilon)\gamma^{-\delta_2})\|h\|_{H^\alpha_\gamma}, \end{aligned}$$

where $\delta_i > 0$, $i = 1,2$, $Q^{(k)} = \Omega^{(k)} \times (0,T)$ and $c_0(\varepsilon)$ is a decreasing function.

The second term in T_1h is bounded by

$$\begin{aligned} c \sum_{k\in\mathcal{N}} (& \|\nabla\widetilde{F}\nabla^2\widetilde{F}\nabla\widehat{u}^{(k)}|_{z=\Phi_k(y(x))}\|_{\alpha,\gamma,Q^{(k)}} \\ & + \|(\nabla\widetilde{F}(1+\nabla\widetilde{F})\nabla^2\widehat{u}^{(k)}|_{z=\Phi_k(y(x))}\|_{\alpha,\gamma,Q^{(k)}} \\ & + \|(\nabla^2\widetilde{F}\nabla\widehat{u}^{(k)}|_{z=\Phi_k(y(x))}\|_{\alpha,\gamma,Q^{(k)}}) \\ \le c & \sum_{k\in\mathcal{N}} (p(\|\nabla\widetilde{F}\|_{1+\alpha,Q^{(k)}})\|u^{(k)}\|_{1+\alpha,\gamma,Q^{(k)}} \\ & + |\nabla\widetilde{F}|_{\infty,Q^{(k)}}(1 + |\nabla\widetilde{F}|_{\infty,Q^{(k)}})\|u^{(k)}\|_{2+\alpha,\gamma,Q^{(k)}}) \equiv I, \end{aligned}$$

where p is an increasing function. Using $|\nabla\widetilde{F}|_{\infty,\Omega^{(k)}} \le c\lambda^a\|\nabla\widetilde{F}\|_{1+\alpha,\Omega^{(k)}}$, $a>0$, the interpolation inequalities from Lemma 2.4 and Lemma 3.3, we have

$$I \le c(\varepsilon^{\delta_1} + c_0(\varepsilon)(\lambda^{\delta_2} + \gamma^{-\delta_3}))\|h\|_{H^\alpha_\gamma}, \quad \delta_i > 0,\ i=1,2,3,$$

and $c_0(\varepsilon)$ is a decreasing function.

Now, we consider the second term in T_2h. Therefore, we have to estimate the expression by

$$\begin{aligned}\sum_{k\in\mathcal{N}}&(\|\eta^{(k)}(B_1(x,\partial_x) - B_1(\xi^{(k)},\partial_x))u^{(k)}\|_{1/2+\alpha,\gamma,Q^{(k)}\cap S}\\ &+ \|\eta^{(k)}(B_2(x,\partial_x) - B_2(\xi^{(k)},\partial_x))u^{(k)}\|_{3/2+\alpha,\gamma,Q^{(k)}\cap S})\\ &\le \sum_{k\in\mathcal{N}} p(\|\nabla\widetilde{F}\|_{1+\alpha,\Omega^{(k)}})\|\nabla\widetilde{F}\|_{2+\alpha,\Omega^{(k)}}\|u^{(k)}\|_{2+\alpha,\gamma,Q^{(k)}}\\ &\le c(\varepsilon^{\delta_1} + c_0(\varepsilon)\lambda^{\delta_2})\|h\|_{H^\alpha_\gamma}, \quad \delta_i>0,\ i=1,2,\end{aligned}$$

where $p=p(\,\cdot\,)$ is a polynomial and $c_0(\varepsilon)$ is a decreasing function.

Similar considerations can be applied to the other terms of T_1 and T_2. Summarizing we have

$$\|Th\|_{H^\alpha_\gamma} \le c[\varepsilon^{\delta_1} + c_0(\varepsilon)(\lambda^{\delta_2} + \gamma^{-\delta_3})]\|h\|_{H^\alpha_\gamma}. \tag{4.12}$$

This concludes the proof. □

Lemma 4.3. *Let $S\in H^{3+\alpha}$, $\alpha>1/2$. Then for every $v\in V^\alpha_\gamma$,*

$$RAv = v + Wv, \tag{4.13}$$

where W is a bounded operator in V^α_γ whose norm can be made small for small λ and large γ, because

$$\|Wv\|_{V^\alpha_\gamma} \le c(\varepsilon^{\delta_1} + c_0(\varepsilon)(\lambda^{\delta_2} + \gamma^{-\delta_3}))\|v\|_{V^\alpha_\gamma}, \tag{4.14}$$

$\varepsilon\in(0,1)$, δ_i, $i=1,2,3$, and $c_0(\varepsilon)$ has the same properties as before.

Proof. We have

$$\begin{aligned}RAv =& \sum_{k\in\mathcal{N}}\eta^{(k)}Z_kR^{(k)}[Z_k^{-1}\zeta^{(k)}L(\partial_x,\partial_t)v, Z_k^{-1}\zeta^{(k)}B(x,\partial_x)v|_S]\\ &+ \sum_{k\in\mathcal{M}}\eta^{(k)}R^{(k)}\zeta^{(k)}L(\partial_x,\partial_t)v\\ =& \sum_{k\in\mathcal{N}}\eta^{(k)}Z_kR^{(k)}[Z_k^{-1}(\zeta^{(k)}L(\partial_x,\partial_t)v - L(\partial_x,\partial_t)\zeta^{(k)}v),\\ &Z_k^{-1}(\zeta^{(k)}B(x,\partial_x)v - B(x,\partial_x)\zeta^{(k)}v)|_S]\\ &+ \sum_{k\in\mathcal{N}}\eta^{(k)}Z_kR^{(k)}[0, Z_k^{-1}(B(x,\partial_x) - B(\xi^{(k)},\partial_x))\zeta^{(k)}v|_S]\\ &+ \sum_{k\in\mathcal{N}}\eta^{(k)}Z_kR^{(k)}[Z_k^{-1}L(\partial_x,\partial_t)\zeta^{(k)}v, Z_k^{-1}B(\xi^{(k)},\partial_x)\zeta^{(k)}v|_S]\end{aligned} \tag{4.15}$$

$$+\sum_{k\in\mathcal{M}}\eta^{(k)}R^{(k)}(\zeta^{(k)}L(\partial_x,\partial_t)v-L(\partial_x,\partial_t)\zeta^{(k)}v)$$
$$+\sum_{k\in\mathcal{M}}\eta^{(k)}R^{(k)}L(\partial_x,\partial_t)\zeta^{(k)}v.$$

By uniqueness, for the Cauchy problem (4.3) we have

$$R^{(k)}L(\partial_x,\partial_t)\zeta^{(k)}v=\zeta^{(k)}v,\quad k\in\mathcal{M}, \tag{4.16}$$

so the last term on the r.h.s. of (4.15) is equal to v. The third term on the r.h.s. of (4.15) has the form

$$\begin{aligned}\sum_{k\in\mathcal{N}}&\eta^{(k)}Z_kR^{(k)}[L(\partial_z-\nabla\widetilde{F}(z)\partial_{z_3},\partial_t)\widetilde{v}^{(k)},\\ &B^{(k)}(\xi^{(k)},\partial_z-\nabla\widetilde{F}(z)\partial_{z_3})\widetilde{v}^{(k)}|_{z_3=0}]\\ =&\sum_{k\in\mathcal{N}}\eta^{(k)}(Z_kR^{(k)}[(L(\partial_z-\nabla\widetilde{F}(z)\partial_{z_3},\partial_t)-L(\partial_z,\partial_t))\widetilde{v}^{(k)},\\ &(B^{(k)}(\xi^{(k)},\partial_z-\nabla\widetilde{F}(z)\partial_{z_3})-B^{(k)}(\xi^{(k)},\partial_z))\widetilde{v}^{(k)}|_{z_3=0}]+Z_k\widetilde{v}^{(k)}),\end{aligned} \tag{4.17}$$

where $\widetilde{v}^{(k)}$ is a solution to problem (4.5), so

$$Z_kR^{(k)}[L(\partial_z,\partial_t)\widetilde{v}^{(k)},B^{(k)}(\xi^{(k)},\partial_z)\widetilde{v}^{(k)}|_{z_3=0}]=Z_k\widetilde{v}^{(k)}=\zeta^{(k)}(x)v(x,t),$$

for $k\in\mathcal{N}$ and $B^{(k)}$ is obtained from B by applying Z_k^{-1}. Therefore, the operator W is determined by the first, second and fourth expressions on the r.h.s. of (4.15) and the first term on the r.h.s. of (4.17).

The fourth term on the r.h.s. of (4.15) is estimated in the following way

$$\begin{aligned}\|\eta^{(k)}R^{(k)}(\zeta^{(k)}L(\partial_x,\partial_t)v-L(\partial_x,\partial_t)\zeta^{(k)}v)\|_{2+\alpha,\gamma,\Omega^T}&\\ \le c(\|\nabla\zeta^{(k)}\nabla v\|_{\alpha,\gamma,\Omega^T}+\|\nabla^2\zeta^{(k)}v\|_{\alpha,\gamma,\Omega^T})&\\ \le c\|v\|_{1+\alpha,\gamma,Q^{(k)}}&\\ \le c(\varepsilon^{\delta_1}+c_0(\varepsilon)\gamma^{-\delta_2})\|v\|_{2+\alpha,\gamma,Q^{(k)}},&\end{aligned} \tag{4.18}$$

where (2.6) has been used and $\delta_i>0$, $i=1,2$.

Next, we consider the first term on the r.h.s. of (4.17). The first part of this term is bounded by

$$\begin{aligned}c\sum_{k\in\mathcal{N}}(\|\nabla_z\widetilde{F}\nabla_z^2\widetilde{F}\nabla_z\widetilde{v}^{(k)}\|_{\alpha,\gamma,Q^{(k)}}+\|\nabla_z\widetilde{F}(1+\nabla_z\widetilde{F})\nabla_z^2\widetilde{v}^{(k)}\|_{\alpha,\gamma,Q^{(k)}}&\\ \le cp(\|\widetilde{F}\|_{2+\alpha,\Omega^T})(\varepsilon^{\delta_1}+c_0(\varepsilon)(\lambda^{\delta_2}+\gamma^{-\delta_3}))\|v\|_{V_\gamma^\alpha}.&\end{aligned} \tag{4.19}$$

Continuing we obtain (4.14). This concludes the proof. □

Summarizing we have

Theorem 4.4. *Let's assume that* $S \in H^{3+\alpha}$, $f \in H_\gamma^{\alpha,\alpha/2}(\Omega \times (-\infty, T))$, $b' \in H_\gamma^{1/2+\alpha,1/4+\alpha/2}(S \times (-\infty, T))$, $b'' \in H_\gamma^{3/2+\alpha,3/4+\alpha/2}(S \times (-\infty, T))$, $\alpha > 1/2$ *and* γ *is sufficiently large. Then there exists a unique solution to problem* (4.1) *such that* $u \in H_\gamma^{2+\alpha,1+\alpha/2}(\Omega \times (-\infty, T))$ *and*

$$\|u\|_{2+\alpha,\gamma,\Omega^T} \le c(\|f\|_{\alpha,\gamma,\Omega^T} + \|b'\|_{1/2+\alpha,\gamma,S^T} + \|b''\|_{3/2+\alpha,\gamma,S^T}), \tag{4.20}$$

where c *does not depend on* u *and* γ.

Now, we consider problem (4.1) with nonvanishing initial data. Therefore, we formulate it in the form

$$\begin{aligned} &u_t - \operatorname{div} \mathbb{D}(u) = f_1 && \text{in } \Omega^T,\\ &\overline{\tau} \cdot \mathbb{D}(u) \cdot \overline{n} = b_1' && \text{on } S^T,\\ &u \cdot \overline{n} = b_1'' && \text{on } S^T,\\ &u|_{t=0} = u_0 && \text{in } \Omega. \end{aligned} \tag{4.21}$$

We have

Lemma 4.5. *We assume that* $f_1 \in W_2^{\alpha,\alpha/2}(\Omega^T)$, $b_1' \in W_2^{1/2+\alpha/2,1/4+\alpha/4}(S^T)$, $b_1'' \in W_2^{3/2+\alpha/2,3/4+\alpha/4}(S^T)$, $u_0 \in W_2^{1+\alpha}(\Omega)$, $S \in W_2^{3+\alpha}$, $\alpha \in (1/2, 1)$, *and use the following compatibility conditions*

$$b_1'|_{t=0} - \overline{\tau} \cdot \mathbb{D}(u_0) \cdot \overline{n} = 0, \quad b_1''|_{t=0} - u_0 \cdot \overline{n} = 0. \tag{4.22}$$

Then there exists a solution to problem (4.21) *such that* $u \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ *and the estimate holds*

$$\|u\|_{2+\alpha,\Omega^T} \le c(\|f_1\|_{\alpha,\Omega^T} + \|b_1'\|_{1/2+\alpha/2,S^T} + \|b_1''\|_{3/2+\alpha/2,S^T} + \|u_0\|_{1+\alpha,\Omega}). \tag{4.23}$$

Proof. Since $u_0 \in W_2^{1+\alpha}(\Omega)$ there exists a function $\widetilde{u}_0 \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ such that $\widetilde{u}_0|_{t=0} = u_0$ and $\|\widetilde{u}_0\|_{2+\alpha,\Omega^T} \le c\|u_0\|_{1+\alpha,\Omega}$. Introducing the function $v = u - \widetilde{u}_0$ we see that it is a solution to the problem

$$\begin{aligned} &v_t - \operatorname{div} \mathbb{D}(v) = f_2 && \text{in } \Omega^T,\\ &\overline{\tau} \cdot \mathbb{D}(v) \cdot \overline{n} = b_2' && \text{on } S^T,\\ &v \cdot \overline{n} = b_2'' && \text{on } S^T,\\ &v|_{t=0} = 0 && \text{in } \Omega, \end{aligned} \tag{4.24}$$

where

$$\begin{aligned} f_2 &= f_1 - (\widetilde{u}_{0t} - \operatorname{div} \mathbb{D}(\widetilde{u}_0)) \in W_2^{\alpha,\alpha/2}(\Omega^T),\\ b_2' &= b_1' - \overline{\tau} \cdot \mathbb{D}(\widetilde{u}_0) \cdot \overline{n} \in W_2^{1/2+\alpha/2,1/4+\alpha/4}(S^T),\\ b_2'' &= b_1'' - \widetilde{u}_0 \cdot \overline{n} \in W_2^{3/2+\alpha/2,3/4+\alpha/4}(S^T), \end{aligned} \tag{4.25}$$

and we have the following estimates

$$
\begin{aligned}
\|f_2\|_{\alpha,\Omega^T} &\le c(\|f_1\|_{\alpha,\Omega^T} + \|u_0\|_{1+\alpha,\Omega}),\\
\|b_2'\|_{1/2+\alpha/2,S^T} &\le c(\|b_1'\|_{1/2+\alpha/2,S^T} + \|u_0\|_{1+\alpha,\Omega}),\\
\|b_2''\|_{3/2+\alpha/2,S^T} &\le c(\|b_1''\|_{3/2+\alpha/2,S^T} + \|u_0\|_{1+\alpha,\Omega}).
\end{aligned}
\tag{4.26}
$$

Since the compatibility conditions (4.22) are satisfied the functions f_2, b_2', b_2'' can be extended by zero for $t<0$, and the extended functions $\widetilde{f}_2$, $\widetilde{b}_2'$, $\widetilde{b}_2''$ are such that

$$
\begin{aligned}
&\widetilde{f}_2 \in H_0^{\alpha,\alpha/2}(\Omega^T),\\
&\widetilde{b}_2' \in H_0^{1/2+\alpha/2,1/4+\alpha/4}(S^T),\\
&\widetilde{b}_2'' \in H_0^{3/2+\alpha/2,3/4+\alpha/4}(S^T),
\end{aligned}
\tag{4.27}
$$

and

$$
\begin{aligned}
\|\widetilde{f}_2\|_{\alpha,0,\Omega^T} &\le c\|f_2\|_{\alpha,\Omega^T},\\
\|\widetilde{b}_2'\|_{1/2+\alpha/2,0,S^T} &\le c\|b_2'\|_{1/2+\alpha/2,S^T},\\
\|\widetilde{b}_2''\|_{3/2+\alpha/2,0,S^T} &\le c\|b_2''\|_{3/2+\alpha/2,S^T}.
\end{aligned}
\tag{4.28}
$$

Since $T<\infty$ the norms $H_\gamma^{\alpha,\alpha/2}(\Omega^T)$ and $H_0^{\alpha,\alpha/2}(\Omega^T)$ are equivalent (and similarly for boundary norms) we have that

$$
\begin{aligned}
&\widetilde{f}_2 \in H_\gamma^{\alpha,\alpha/2}(\Omega^T),\\
&\widetilde{b}_2' \in H_\gamma^{1/2+\alpha/2,1/4+\alpha/4}(S^T),\\
&\widetilde{b}_2'' \in H_\gamma^{3/2+\alpha/2,3/4+\alpha/4}(S^T),
\end{aligned}
\tag{4.29}
$$

and

$$
\begin{aligned}
\|\widetilde{f}_2\|_{\alpha,\gamma,S^T} &\le c(\gamma)\|\widetilde{f}_2\|_{\alpha,0,\Omega^T},\\
\|\widetilde{b}_2'\|_{1/2+\alpha/2,\gamma,S^T} &\le c(\gamma)\|\widetilde{b}_2'\|_{1/2+\alpha/2,0,S^T},\\
\|\widetilde{b}_2''\|_{3/2+\alpha/2,\gamma,S^T} &\le c(\gamma)\|\widetilde{b}_2''\|_{3/2+\alpha/2,0,S^T}.
\end{aligned}
\tag{4.30}
$$

By virtue of (4.23)–(4.30) and Theorem 4.4 we obtain the existence of solutions to problem (4.24) such that $v\in H_\gamma^{2+\alpha,1+\alpha/2}(\Omega^T)$ and the following estimate holds

$$
\begin{aligned}
\|v\|_{2+\alpha,\gamma,\Omega^T} \le{}& c(\|f_1\|_{\alpha,\Omega^T} + \|b_1'\|_{1/2+\alpha/2,S^T}\\
&+ \|b_1''\|_{3/2+\alpha/2,S^T} + \|u_0\|_{1+\alpha,\Omega}).
\end{aligned}
\tag{4.31}
$$

Now, in view of the definition of v, we obtain the existence of solutions to (4.21) such that $u\in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ and by (4.31) we have

$$
\|u\|_{2+\alpha,\Omega^T} \le c(\|v\|_{2+\alpha,\Omega^T} + \|u_0\|_{1+\alpha,\Omega} \le c(\|v\|_{2+\alpha,\gamma,\Omega^T} + \|u_0\|_{1+\alpha,\Omega}),
\tag{4.32}
$$

hence, (4.23) holds. This concludes the proof. □

5. Existence of solutions to (1.13)

Using a partition of unity in Ω^T we have (see [12]):

Lemma 5.1. *If we assume that*

$$\eta \in C^\beta(\Omega^T), \quad \beta > 0, \quad 1/\eta \in L_\infty(\Omega^T), \quad \eta > 0, \quad F_1 \in W_2^{\alpha,\alpha/2}(\Omega^T),$$
$$G_{1j} \in W_2^{1/2+\alpha,1/4+\alpha/2}(S^T), \quad j = 1,2, \quad G_{13} \in W^{3/2+\alpha,3/4+\alpha/2}(S^T),$$
$$w_0 \in W_2^{1+\alpha}(\Omega), \quad S \in H^{3+\alpha}.$$

Then there exists a solution to problem (1.13) *such that* $w \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ *and the estimate holds*

$$\|w\|_{2+\alpha,\Omega^T} \le c(\|\eta\|_{C^\beta(\Omega^T)}, |1/\eta|_{\infty,\Omega^T}) \cdot \bigg(\|F_1\|_{\alpha,\Omega^T} + \sum_{j=1}^2 \|G_{1j}\|_{1/2+\alpha,S^T} + \|G_{13}\|_{3/2+\alpha,S^T} + \|w_0\|_{1+\alpha,\Omega}\bigg). \tag{5.1}$$

6. Existence of solutions to problem (1.1)

To prove the existence of solutions to problem (1.1) we formulate it in Lagrangian coordinates. To simplify the considerations we introduce them once again using another notation. By Lagrangian coordinates we mean the initial data for the following Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x|_{t=0} = \xi \in \Omega, \quad \xi = (\xi_1, \xi_2, \xi_3). \tag{6.1}$$

Integrating (6.1) we obtain the following transformation between the Eulerian x and the Lagrangian ξ coordinates of the same fluid particle. Hence

$$x = \xi + \int_0^t u(\xi,s)\,ds \equiv x(\xi,t), \tag{6.2}$$

where $u(\xi,t) = v(x(\xi,t),t)$.

The condition $(1.1)_5$ implies that

$$\Omega \ni x = x(\xi,t) \quad \text{for } \xi \in \Omega, \qquad S \ni x = x(\xi,t) \quad \text{for } \xi \in S.$$

Using the Lagrangian coordinates, we can formulate the problem (1.1) in the form

$$\begin{aligned}
&\eta u_t - \operatorname{div}_u \mathbb{D}_u(u) + \nabla_u q = \eta g && \text{in } \Omega^T,\\
&\eta_t + \eta \nabla_u \cdot u = 0 && \text{in } \Omega^T,\\
&\overline{\tau}_{u\alpha} \cdot \mathbb{D}_u(u) \cdot \overline{n}_u + \gamma u \cdot \overline{\tau}_{u\alpha} = 0,\ \alpha = 1,2, && \text{on } S^T,\\
&\overline{n}_u \cdot u = 0 && \text{on } S^T,\\
&u|_{t=0} = v_0, \quad \eta|_{t=0} = \rho_0 && \text{in } \Omega,
\end{aligned} \tag{6.3}$$

where $\eta(\xi,t) = \rho(x(\xi,t),t)$, $q(\xi,t) = p(x(\xi,t),t)$, $g(\xi,t) = f(x(\xi,t),t)$, $\nabla_u = \partial_x \xi_i \nabla_{\xi_i}$, $\partial_{\xi_i} = \nabla_{\xi_i}$, $\mathbb{D}_u(u) = \{\mu(\nabla_{ui}u_j + \nabla_{uj}u_i) + (\nu-\mu)\delta_{ij}\nabla_u\cdot u\}$, $\nabla_u\cdot u = \partial_{x_i}\xi_j\nabla_{\xi_j}u_i$, $\nabla_{ui} = \partial_{x_i}\xi_k\partial_{\xi_k}$ and the summation convention over the repeated indices is assumed. Moreover, $\overline{n}_u(\xi,t) = \overline{n}(x(\xi,t),t)$, $\overline{\tau}_u(\xi,t) = \overline{\tau}(x(\xi,t),t)$.

Let $A_{x(\xi,t)}$ be the Jacobi matrix of the transformation $x = x(\xi,t)$ with elements $a_{ij}(\xi,t) = \delta_{ij} + \int_0^t u_{i\xi_j}\,d\tau$ and let $J_{x(\xi,t)} = \det\{a_{ij}\}_{i,j=1,2,3}$ be the Jacobian. Then

$$\frac{\partial J_{x(\xi,t)}}{\partial t} = \frac{\partial a_{ij}}{\partial t}A_{ij} = A_{ij}\frac{\partial u_i}{\partial \xi_j}, \quad J_{x(\xi,0)} = 1,$$

where A_{ij} are the algebraic complements of a_{ij} and $\mathcal{A} = \{A_{ij}\}_{i,j=1,2,3}$.

Hence

$$J_{x(\xi,t)} = 1 + \int_0^t A_{ij}\frac{\partial u_i}{\partial \xi_j}\,d\tau = 1 + \int_0^t \mathcal{A}\cdot\nabla\cdot u\,d\tau. \tag{6.4}$$

Moreover, since $A_{ij}\partial u_i/\partial\xi_j = A_{ij}a_{kj}\partial v_i/\partial x_k = \nabla\cdot v(x,t)|_{x=x(\xi,t)}J_{x(\xi,t)}$ it follows that

$$J_{x(\xi,t)} = \exp\bigg(\int_0^t \nabla\cdot v|_{x(\xi,t)}\,d\tau\bigg) = \exp\bigg(\int_0^t \nabla_u\cdot u\,d\tau\bigg),$$

where $\nabla_u = J^{-1}_{x(\xi,t)}\mathcal{A}\cdot\nabla$.

Therefore from $(6.3)_{2,5}$ we have

$$\eta(\xi,t) = \rho_0(\xi)\exp\bigg(-\int_0^t \nabla_u\cdot u\,d\tau\bigg) = \rho_0(\xi)J^{-1}_{x(\xi,t)}(\xi,t). \tag{6.5}$$

From (6.4) and (6.5) we obtain also

$$\eta(\xi,t) = \rho_0(\xi)\bigg(1 + \int_0^t \mathcal{A}\cdot\nabla\cdot u\,d\tau\bigg)^{-1}. \tag{6.6}$$

Lemma 6.1. *We assume that $\rho_0 \in W_2^{1+\alpha}(\Omega)$, $\alpha\in(1/2,1)$, $\rho_0(\xi)\ge\rho_*>0$, and*

$$T^{1/2}\|u\|_{2+\alpha,\Omega^T} \le \delta, \tag{6.7}$$

where δ is sufficiently small. Then solutions of problem $(6.3)_{2,5}$ satisfy

$$\begin{aligned} &\|\eta(\,\cdot\,,t)\|_{1+\alpha,\Omega} \le \phi(a)\|\rho_0\|_{1+\alpha,\Omega},\\ &[\eta_\xi]_{\alpha/2,\Omega^T,t} \le \|\rho_0\|_{1+\alpha,\Omega}\phi(a,T)a,\\ &\sup_t\int_\Omega\int_0^T \frac{|\eta_\xi(t)-\eta_\xi(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt'\\ &\quad\le \|\rho_0\|^2_{1+\alpha,\Omega}\phi(a)T^{1-\alpha}\int_0^T\|u\|^2_{2+\alpha,\Omega}\,dt\cdot\bigg(1+\int_0^T\|u\|^2_{2+\alpha,\Omega}\,dt\bigg), \end{aligned} \tag{6.8}$$

where $a = T^{\overline{a}}\|u\|_{2+\alpha,\Omega^T}$, $\overline{a} > 0$ *and* ϕ *is an increasing positive function.*

Proof. $(6.8)_1$ follows easily from (6.5). To show $(6.8)_2$ we get from (6.5)

$$\begin{aligned}
&\int_\Omega\int_0^T\int_0^T \frac{|\eta_\xi(t)-\eta_\xi(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt' \\
&\quad\le \|\rho_0\|^2_{1+\alpha,\Omega}\phi(a)\int_\Omega\int_0^T\int_0^T\biggl(\frac{|\int_t^{t'}u_\xi\,d\tau|^2}{|t-t'|^{1+\alpha}} \\
&\qquad+\frac{|\int_t^{t'}u_\xi\,d\tau|^2|\int_0^t u_{\xi\xi}\,d\tau|^2}{|t-t'|^{1+\alpha}}+\frac{|\int_t^{t'}u_{\xi\xi}\,d\tau|^2}{|t-t'|^{1+\alpha}}\biggr)\,d\xi\,dt\,dt' \\
&\quad\le \|\rho_0\|^2_{1+\alpha,\Omega}\phi(a)T^{2-\alpha}\int_\Omega d\xi\biggl(\int_0^T(u_\xi^2+u_{\xi\xi}^2)\,d\tau+\int_0^T u_\xi^2\,d\tau\int_0^T u_{\xi\xi}^2\,d\tau\biggr) \\
&\quad\le \|\rho_0\|^2_{1+\alpha,\Omega}\phi(a)T^{2-\alpha}\int_0^T\|u\|^2_{2+\alpha,\Omega}\,d\tau\biggl(1+\int_0^T\|u\|^2_{2+\alpha,\Omega}\,d\tau\biggr).
\end{aligned}$$

Therefore, $(6.8)_2$ has been proved.

A proof of $(6.8)_3$ is similar to the proof of $(6.8)_2$. □

To prove the existence of solutions to problem (1.1) we use the following method of successive approximations

$$(6.9)\qquad \begin{aligned}
&\eta_m u_{m+1t}-\mathrm{div}_{u_m}\mathbb{D}_{u_m}(u_{m+1})+\nabla_{u_m}q(\eta_m)=\eta_m g && \text{in } \Omega^T,\\
&\overline{\tau}_{u_m\alpha}\cdot\mathbb{D}_{u_m}(u_{m+1})\cdot\overline{n}_{u_m}=-\gamma u_m\cdot\overline{\tau}_{u_m\alpha},\ \alpha=1,2, && \text{on } S^T,\\
&\overline{n}_{u_m}\cdot u_{m+1}=0 && \text{on } S^T,\\
&u_{m+1}|_{t=0}=v_0 && \text{in } \Omega,
\end{aligned}$$

where η_m, u_m are given, and

$$(6.10)\qquad \begin{aligned}
&\eta_{mt}+\eta_m\mathrm{div}_{u_m}u_m=0 && \text{in } \Omega^T,\\
&\eta_m|_{t=0}=\rho_0 && \text{in } \Omega,
\end{aligned}$$

where u_m is given.

To apply Lemma 5.1 we write (6.9) in the form

$$(6.11)\qquad \begin{aligned}
&\eta_m u_{m+1t}-\mathrm{div}\mathbb{D}(u_{m+1})=-(\mathrm{div}\,\mathbb{D}(u_{m+1})\\
&\quad-\mathrm{div}_{u_m}\mathbb{D}_{u_m}(u_{m+1}))-\nabla_{u_m}q(\eta_m)+\eta_m g\equiv l_1+l_2+l_3 && \text{in } \Omega^T,\\
&\overline{\tau}_\alpha\cdot\mathbb{D}(u_{m+1})\cdot\overline{n}=(\overline{\tau}_\alpha\cdot\mathbb{D}(u_{m+1})\cdot\overline{n}\\
&\quad-\overline{\tau}_{u_m\alpha}\mathbb{D}_{u_m}(u_{m+1})\cdot\overline{n}_{u_m})-\gamma u_m\cdot\overline{\tau}_{u_m\alpha}\equiv l_4+l_5,\ \alpha=1,2, && \text{on } S^T,\\
&\overline{n}\cdot u_{m+1}=(\overline{n}-\overline{n}_{u_m})\cdot u_{m+1}\equiv l_6 && \text{on } S^T,\\
&u_{m+1}|_{t=0}=v_0 && \text{in } \Omega,
\end{aligned}$$

where $\overline{n}=\overline{n}(\xi,t)$, $\overline{\tau}_\alpha=\overline{\tau}_\alpha(\xi,t)$, $\alpha=1,2$. First we show

Lemma 6.2. *We assume that* $S \in W_2^{3+\alpha}$, $\rho_0 \in W_2^{1+\alpha}(\Omega)$, $v_0 \in W_2^{1+\alpha}(\Omega)$, $\alpha \in (1/2,1)$, $\rho_0(\xi) \ge \rho_* > 0$, *where* ρ_* *is a constant, and* $u_m \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ *and* (6.7) *holds with* δ *sufficiently small. Then for solutions of the problem* (6.11) *such that* $u_{m+1} \in W_2^{2+\alpha,1+\alpha/2}(\Omega^T)$ *the following inequality holds*

$$\|u_{m+1}\|_{2+\alpha,\Omega^t} \le G(\gamma, t^a\|u_m\|_{2+\alpha,\Omega^t}, t), \quad t \le T,\ a > 0, \tag{6.12}$$

where G *is an increasing positive function of its arguments and*

$$\gamma = \|\rho_0\|_{1+\alpha,\Omega} + \|g\|_{\alpha,\Omega^T} + \|v_0\|_{1+\alpha,\Omega},$$

and $G(\gamma,0,0) = G_0(\gamma) > 0$.

Proof. Applying Lemma (5.1) to (6.11) we have

$$\begin{aligned}\|u_{m+1}\|_{2+\alpha,\Omega^T} &\le \phi_1(\|\eta_m\|_{C^\beta(\Omega^T)}, |1/\eta_m|_{\infty,\Omega^T}) \\ &\quad\cdot\bigg(\sum_{i=1}^3 \|l_i\|_{\alpha,\Omega^T} + \sum_{i=4}^5 \|l_i\|_{1/2+\alpha,S^T} + \|l_6\|_{3/2+\alpha,S^T} + \|v_0\|_{1+\alpha,\Omega}\bigg),\end{aligned} \tag{6.13}$$

where ϕ_1 is an increasing positive function and $\beta < \alpha - 1/2$.

Now we have to estimate the norms from the r.h.s. of (6.13). To estimate l_1 we write it in the qualitative form

$$l_1 \approx \psi_1(b)u_{m+1\xi\xi} + \psi_2(b)b_\xi u_{m+1\xi},$$

where $b = \{b_{ij}\} = \{\int_0^t u_{i\xi_j}(\xi,\tau)\,d\tau\}$, and ψ_1, ψ_2 are matrix functions such that $\psi_1 = \xi_x\xi_x$, $\psi_2 = \xi_x\xi_{x,b}$, where the products are the matrix products.

We shall restrict our considerations to estimate the highest derivatives in the considered norms only.

$$\begin{aligned}[l_1]^2_{\alpha,\Omega^T,x} \le \phi_2(a_m)\int_0^T\int_\Omega\int_\Omega \bigg(&\frac{|\int_0^t (u_{m\xi} - u_{m\xi'})\,d\tau|^2}{|\xi-\xi'|^{3+2\alpha}}|u_{m+1\xi\xi}|^2 \\ &+ \frac{|u_{m+1\xi\xi} - u_{m+1\xi'\xi'}|^2|\int_0^t u_{m\xi}\,d\tau|^2}{|\xi-\xi'|^{3+2\alpha}} \\ &+ \frac{|\int_0^t (u_{m\xi} - u_{m\xi'})\,d\tau|^2|\int_0^t u_{m\xi\xi}\,d\tau|^2|u_{m+1\xi}|^2}{|\xi-\xi'|^{3+2\alpha}} \\ &+ \frac{|\int_0^t (u_{m\xi\xi} - u_{m\xi'\xi'})\,d\tau|^2|u_{m+1\xi}|^2}{|\xi-\xi'|^{3+2\alpha}} \\ &+ \bigg|\int_0^t u_{m\xi\xi}\,d\tau\bigg|^2 \frac{|u_{m+1\xi} - u_{m+1\xi'}|^2}{|\xi-\xi'|^{3+2\alpha}}\bigg)\,d\xi\,d\xi'\,dt \equiv \phi_2\sum_{i=1}^5 I_i,\end{aligned}$$

where

$$a_m = T^a\bigg(\int_0^T \|u_m\|^2_{2+\alpha\Omega}\,d\tau\bigg)^{1/2}, \quad a > 0.$$

Now, we estimate the expression I_i, $i = 1, \dots, 5$.

$$I_1 \le T^a \int_0^T \left(\int_\Omega \int_\Omega \frac{|u_{m\xi} - u_{m\xi'}|^{2p_1}}{|\xi - \xi'|^{3+2p_1(3\mu_1/2-3/2p_1+\alpha)}} \, d\xi \, d\xi' \right)^{1/p_1} dt$$
$$\cdot \int_0^T \left(\int_\Omega \int_\Omega \frac{|u_{m+1\xi\xi}(\xi)|^{2p_2}}{|\xi - \xi'|^{3\mu_2 p_2}} \, d\xi \, d\xi' \right)^{1/p_2} dt \equiv I_1,$$

where $a > 0$, $1/p_1 + 1/p_2 = 1$, $\mu_1 + \mu_2 = 1$ and $\mu_2 p_2 < 1$. Using the imbeddings $W_2^{2+\alpha}(\Omega) \subset W_{2p_1}^{1+\alpha+3\mu_1/2-3/2p_1}(\Omega)$ and $W_2^{2+\alpha}(\Omega) \subset L_{2p_2}(\Omega)$, which holds simultaneously because $3/2 - 3/2p_1 + 3\mu_1/2 - 3/2p_1 + \alpha \le 1 + \alpha$, $3/2 - 3/2p_2 \le \alpha$ are valid for $\alpha > 1/2$, we obtain $I_1' \le T^a \|u_m\|_{2+\alpha,\Omega^T} \|u_{m+1}\|_{2+\alpha,\Omega^T}$. Moreover, we have

$$I_2 \le T[u_{m+1\xi\xi}]^2_{\alpha,\Omega^T,x} \int_0^T |u_{m\xi}|^2_{\infty,\Omega} \, d\tau \le T \int_0^T \|u_m\|^2_{2+\alpha,\Omega} \, d\tau [u_{m+1\xi\xi}]^2_{\alpha,\Omega^T,x}.$$

Continuing we see that I_3, I_5 can be estimated in the similar way as I_1 and I_4 as I_2. Summarizing we have $[l_1]^2_{\alpha,\Omega^T,x} \le \phi_3(a_m) T^a \|u_m\|_{2+\alpha,\Omega^T} \|u_{m+1}\|_{2+\alpha,\Omega^T}$, where $a > 0$. Next, we estimate

$$\begin{aligned}
[l_1]^2_{\alpha/2,\Omega^T,t} \le{}& \phi_4(a_m) \int_\Omega \int_0^T \int_0^T \Bigg(\frac{|\int_t^{t'} u_{m\xi} \, d\tau|^2}{|t - t'|^{1+\alpha}} |u_{m+1\xi\xi}|^2 \\
&+ \frac{|u_{m+1\xi\xi}(t) - u_{m+1\xi\xi}(t')|^2}{|t - t'|^{1+\alpha}} \left| \int_0^t u_{m\xi} \, d\tau \right|^2 \\
&+ \frac{|\int_t^{t'} u_{m\xi} \, d\tau|^2 |\int_0^t u_{m\xi\xi} \, d\tau|^2 |u_{m+1\xi}|^2}{|t - t'|^{1+\alpha}} + \frac{|\int_t^{t'} u_{m\xi\xi} \, d\tau|^2 |u_{m+1\xi}|^2}{|t - t'|^{1+\alpha}} \\
&+ \left| \int_0^t u_{m\xi\xi} \, d\tau \right|^2 \frac{|u_{m+1\xi}(t) - u_{m+1\xi}(t')|^2}{|t - t'|^{1+\alpha}} \Bigg) d\xi \, d\xi' \, dt \\
\le{}& \phi_4(a_m) \Bigg(T^{1-\alpha} \int_\Omega d\xi \int_0^T |u_{m\xi}|^2 \, d\tau \int_0^T |u_{m+1\xi\xi}|^2 \, d\tau \\
&+ T[u_{m+1\xi\xi}]^2_{\alpha/2,\Omega^T,t} \int_0^T |u_{m\xi}|^2_{\infty,\Omega} \, d\tau \\
&+ T^{2-\alpha} \int_\Omega d\xi \int_0^T |u_{m\xi}|^2 \, d\tau \int_0^T u^2_{m\xi\xi} \, d\tau \int_0^T u^2_{m+1\xi} \, d\tau \\
&+ T^{1-\alpha} \int_\Omega d\xi \int_0^T u^2_{m\xi\xi} \, d\tau \int_0^T u^2_{m+1\xi} \, d\tau \\
&+ T \int_\Omega d\xi \int_0^T u^2_{m\xi\xi} \, d\tau \int_0^T \int_0^T \frac{|u_{m+1\xi}(t) - u_{m+1\xi}(t')|^2}{|t - t'|^{1+\alpha}} \, dt \, dt' \Bigg) \\
\equiv{}& \sum_{i=6}^{10} I_i^2,
\end{aligned}$$

where $I_6+I_8+I_9\le\phi_5(a_m)$, $a_m\|u_{m+1}\|_{2+\alpha,\Omega^T}$, and

$$\begin{aligned}I_{10}\le{}&\phi_6(a_m)T^{1/2}\bigg(\int_0^T|u_{m\xi\xi}|^2_{2p_1,\Omega}\,dt\bigg)^{1/2}\\&\cdot\bigg(\int_0^T\int_0^T\frac{|u_{m+1\xi}(t)-u_{m+1\xi}(t')|^2_{2p_2,\Omega}}{|t-t'|^{1+\alpha}}\bigg)^{1/2}\\\le{}&\phi_6(a_m)T^{1/2}\|u_m\|_{2+\alpha,\Omega^T}\|u_{m+1}\|_{2+\alpha,\Omega^T},\end{aligned}$$

where we used imbeddings $D^2W_2^{2+\alpha}(\Omega)\subset L_{2p_1}(\Omega)$, $D^2W_2^{2+\alpha}(\Omega)\subset L_{2p_2}(\Omega)$, which hold because $3/2-3/2p_1\le\alpha$, $3/2-3/2p_2\le1$, $1/p_1+1/p_2=1$ are satisfied together for $\alpha>1/2$.

Summarizing, we have shown

$$[l_1]_{\alpha/2,\Omega^T,t}\le\phi_7(a_m)T^a\|u_m\|_{2+\alpha,\Omega^T}\|u_{m+1}\|_{2+\alpha,\Omega^T}.$$

Now, we consider $l_2=\psi_3(b)\dot q(\eta_m)\eta_{m\xi}$, where the dot denotes the derivative with respect to the argument,

$$\begin{aligned}[l_2]_{\alpha,\Omega^T,x}\le{}&\phi_8(a_m,\sup_t\|\eta_m\|_{1+\alpha,\Omega})\bigg(\int_0^T\int_\Omega\int_\Omega\bigg[\frac{|\int_0^t(u_{m\xi}-u_{m\xi'})\,d\tau|^2|\eta_{m\xi}|^2}{|\xi-\xi'|^{3+2\alpha}}\\&+\frac{|\eta_m(\xi)-\eta_m(\xi')|^2|\eta_{m\xi}|^2}{|\xi-\xi'|^{3+2\alpha}}\bigg)\,dt\,d\xi\,d\xi'+[\eta_{m\xi}]^2_{\alpha,\Omega^T,x}\bigg)\\\le{}&\phi_9(a_m,\sup_t\|\eta_m\|_{1+\alpha,\Omega})T^a\sup_t\|\eta_m\|_{1+\alpha,\Omega},\end{aligned}$$

where to estimate the first two integrals on the r.h.s of the first inequality we used the same method as in the case of I_1. Next,

$$\begin{aligned}[l_2]_{\alpha/2,\Omega^T,t}\le{}&\phi_{10}(a_m,\sup_t\|\eta_m\|_{1+\alpha,\Omega})\int_\Omega\int_0^T\int_0^T\bigg(\frac{|\int_t^{t'}u_{m\xi}\,d\tau|^2|\eta_{m\xi}|^2}{|t-t'|^{1+\alpha}}\\&+\frac{|\eta_m(t)-\eta_m(t')|^2}{|t-t'|^{1+\alpha}}|\eta_{m\xi}|^2+\frac{|\eta_{m\xi}(t)-\eta_{m\xi}(t')|^2}{|t-t'|^{1+\alpha}}\bigg)\,d\xi\,dt\,dt'\\\equiv{}&\phi_{10}I_{11}.\end{aligned}$$

Continuing,

$$\begin{aligned}I_{11}\le{}&T^{2-\alpha}\int_0^T\|u_m\|^2_{2+\alpha,\Omega}\,d\tau\sup_t\|\eta_m\|^2_{1+\alpha,\Omega}\\&+\int_0^T\int_0^T\frac{|\eta_m(t)-\eta_m(t')|^2_{2p_1,\Omega}}{|t-t'|^{1+\alpha}}|\eta_{m\xi}|^2_{2p_2,\Omega}\,dt\,dt'+[\eta_{m\xi}]_{\frac{\alpha}{2},\Omega^T,t}\equiv I_{12},\end{aligned}$$

where $1/p_1+1/p_2=1$. To estimate the middle term in I_{12} we use the imbeddings $W_2^1(\Omega)\subset L_{2p_1}(\Omega)$, $DW_2^{1+\alpha}(\Omega)\subset L_{2p_2}(\Omega)$, which hold together because the relations $3/2-3/2p_1\le1$, $3/2-3/2p_2\le1+\alpha$ are satisfied for $\alpha>1/2$. Using (6.8) we obtain

$$[l_2]_{\alpha/2,\Omega^T,t}\le\phi_{11}(a_m,\sup_t\|\eta_m\|_{1+\alpha,\Omega})a_m.$$

In view of Lemma 2.6 we have

$$\|l_3\|_{\alpha,\Omega^T} \le \sup_t \|\eta_m\|_{1+\alpha,\Omega}\|g\|_{\alpha,\Omega^T}.$$

Next we have

$$\|l_4\|_{1/2+\alpha,S^T} \le c\|l_4\|_{1+\alpha,\Omega^T},$$

which can be estimated in the same way as l_1. Next

$$\begin{aligned}\|l_5\|_{1/2+\alpha,S^T} &\le c\|l_5\|_{1+\alpha,\Omega^T} \le c\|u_m\|_{1+\alpha,\Omega^T} \le \varepsilon\|u_m\|_{2+\alpha,\Omega^T} + c(\varepsilon)\|u_m\|_{0,\Omega^T}\\ &\le \varepsilon\|u_m\|_{2+\alpha,\Omega^T} + c(\varepsilon)T^{1/2}(\|u_m\|_{2+\alpha,\Omega^T} + \|v_0\|_{1+\alpha,\Omega}).\end{aligned}$$

Finally,

$$\|l_6\|_{3/2+\alpha,S^T} \le c\|l_6\|_{2+\alpha,\Omega^T} \le T^a\|u_m\|_{2+\alpha,\Omega^T}\|u_{m+1}\|_{2+\alpha,\Omega^T}.$$

Summarizing the above considerations we obtain (6.12). This concludes the proof. □

Let

$$\alpha_m(t) = \|u_m\|_{2+\alpha,\Omega^T}. \tag{6.14}$$

Let $A > 0$ be sufficiently large such that $G_0(\gamma) < A$, $\alpha_m(t) \le A$. Then there exists a time T_* such that for $t \le T_*$ we have

$$\alpha_{m+1}(t) \le G(\gamma, t^a A, t) \le A. \tag{6.15}$$

Hence we obtain

$$\alpha_m(t) \le A, \quad \text{for } m = 0, 1, \dots, \text{ and } t \le T_*. \tag{6.16}$$

Finally, we define the zero approximation function u_0 as a solution to the following problem

$$\begin{aligned}&u_{0t} - \operatorname{div}\mathbb{D}(u_0) = 0 && \text{in } \Omega^T,\\ &\overline{\tau}_\alpha \cdot \mathbb{D}(u_0)\cdot\overline{n} + \gamma u_0\cdot\overline{\tau}_\alpha = 0, \quad \alpha = 1,2, && \text{on } S^T,\\ &u_0\cdot\overline{n} = 0 && \text{on } S^T,\\ &u_0|_{t=0} = v_0 && \text{in } \Omega.\end{aligned}$$

Therefore, we have proved

Lemma 6.3. *If we assume that data are such that $\gamma < \infty$, and that $\alpha > 1/2$, then the sequence constructed by* (6.9) *and* (6.10) *is bounded for $T \le T_*$.*

Now we show that the sequence $\{\eta_m, u_m\}$ converges. To this end we consider the differences

$$H_m = \eta_m - \eta_{m-1}, \qquad U_m = u_m - u_{m-1}, \tag{6.17}$$

which are solutions of the following problems

$$
\begin{aligned}
\eta_m U_{m+1t} &- \operatorname{div}\mathbb{D}(U_{m+1}) \\
&= -(\operatorname{div}\mathbb{D}(U_{m+1}) - \operatorname{div}_{u_m}\mathbb{D}_{u_m}(U_{m+1})) \\
&\quad + (\operatorname{div}_{u_m}\mathbb{D}_{u_m}(u_m) - \operatorname{div}_{u_{m-1}}\mathbb{D}_{u_{m-1}}(u_m)) \\
&\quad - \dot q(\eta_m)\nabla_{u_m}H_m + \dot q(\eta_m)(\nabla_{u_m} - \nabla_{u_{m-1}})(\eta_{m-1}) \\
&\quad - (\dot q(\eta_m) - \dot q(\eta_{m-1}))\nabla_{u_{m-1}}\eta_{m-1} + H_m g - H_m u_{mt} \equiv \sum_{i=1}^{7} L_i,
\end{aligned}
$$

$$
\begin{aligned}
&\overline\tau_\alpha\cdot\mathbb{D}(U_{m+1})\cdot\overline n \\
&\quad = (\overline\tau_\alpha\cdot\mathbb{D}(U_{m+1})\cdot\overline n - \overline\tau_{u_m\alpha}\cdot\mathbb{D}_{u_m}(U_{m+1})\cdot\overline n_{u_m}) \\
&\qquad - (\overline\tau_{u_m\alpha}\cdot\mathbb{D}_{u_m}(u_m)\cdot\overline n_{u_m} - \overline\tau_{u_{m-1}\alpha}\cdot\mathbb{D}_{u_{m-1}}(u_m)\cdot\overline n_{u_{m-1}}) \\
&\qquad - \gamma(u_m\cdot\overline\tau_{u_m\alpha} - u_{m-1}\cdot\overline\tau_{u_{m-1}\alpha}) = \sum_{i=8}^{10} L_i, \quad \alpha = 1,2,
\end{aligned}
\tag{6.18}
$$

$$
\overline n\cdot U_{m+1} = (\overline n - \overline n_{u_m})\cdot U_{m+1} - (\overline n_{u_m} - \overline n_{u_{m-1}})\cdot u_m \equiv \sum_{i=11}^{12} L_i,
$$

$$
U_{m+1}|_{t=0} = 0,
$$

and

$$
\begin{aligned}
H_{mt} + H_m\operatorname{div}_{u_m}u_m &= -\eta_{m-1}(\operatorname{div}_{u_m}u_m - \operatorname{div}_{u_{m-1}}u_m), \\
H_m|_{t=0} &= 0.
\end{aligned}
\tag{6.19}
$$

To show the convergence of the sequence $\{u_m, \eta_m\}$ we need

Lemma 6.4. *Let the assumptions of Lemma* 6.3 *be satisfied. Then*

$$
\|U_{m+1}\|_{2+\alpha,\Omega^T} \le c(A)T^a\|U_m\|_{2+\alpha,\Omega^T},
\tag{6.20}
$$

where $a > 0$ and A is the bound from (6.16).

Proof. From Lemma 5.1 we have

$$
\begin{aligned}
\|U_{m+1}\|_{2+\alpha,\Omega^T} &\le \phi_1(\|\eta_m\|_{C^\beta(\Omega^T)}, \|1/\eta_m\|_{L_\infty(\Omega^T)}) \\
&\quad\cdot\bigg(\sum_{i=1}^{7}\|L_i\|_{\alpha,\Omega^T} + \sum_{i=8}^{10}\|L_i\|_{1/2+\alpha,S^T} + \sum_{i=11}^{12}\|L_i\|_{3/2+\alpha,S^T}\bigg).
\end{aligned}
\tag{6.21}
$$

Since $L_1 = \psi_1(b_m)b_m U_{m+1\xi\xi} + \psi_2(b_m)b_{m\xi}U_{m+1\xi}$ we have

$$
\|L_1\|_{\alpha,\Omega^T} \le \phi_2(A,T)T^a\|U_{m+1}\|_{2+\alpha,\Omega^T}.
$$

Since $L_2 = \psi_3(b_m, b_{m-1})\int_0^t U_{m\xi}\,d\tau u_{m\xi\xi} + \psi_4(b_m, b_{m-1})\int_0^t U_{m\xi\xi}\,d\tau u_{m\xi}$ we obtain

$$
\|L_2\|_{\alpha,\Omega^T} \le \phi_3(A,T)T^a\|U_m\|_{2+\alpha,\Omega^T}.
$$

Next, we have that $L_3 = \dot{q}(\eta_m)\psi_5(b_m)H_{m\xi}$. Therefore

$$
\begin{aligned}
[L_3]^2_{\alpha,\Omega^T,x} &\le \phi_4(A,T)\int_0^T\int_\Omega\int_\Omega\bigg(\frac{|\eta_m(\xi)-\eta_m(\xi')|^2|H_{m\xi}|^2}{|\xi-\xi'|^{3+2\alpha}} \\
&\quad + \frac{|\int_0^t(u_{m\xi}-u_{m\xi'})\,d\tau|^2|H_{m\xi}|^2}{|\xi-\xi'|^{3+2\alpha}} + \frac{|H_{m\xi}-H_{m\xi'}|^2}{|\xi-\xi'|^{3+2\alpha}}\bigg)\,dt\,d\xi\,d\xi' \\
&\le \phi_5(A,T)T\sup_t\|H_m\|_{1+\alpha,\Omega},
\end{aligned}
$$

and

$$
\begin{aligned}
[L_3]^2_{\alpha,\Omega^T,t} &\le \phi_6(A,T)\int_\Omega\int_0^T\int_0^T\bigg(\frac{|\eta_m(t)-\eta_m(t')|^2|H_{m\xi}|^2}{|t-t'|^{1+\alpha}} \\
&\quad + \frac{|\int_t^{t'}u_{m\xi}\,d\tau|^2|H_{m\xi}|^2}{|t-t'|^{1+\alpha}} + \frac{|H_{m\xi}(t)-H_{m\xi}(t')|^2}{|t-t'|^{1+\alpha}}\bigg)\,d\xi\,dt\,dt' \\
&\le \phi_7(A,T)T^a\sup_t\|H_m\|_{1+\alpha,\Omega} \\
&\quad + \phi_6(A,T)\int_\Omega\int_0^T\int_0^T\frac{|H_{m\xi}(t)-H_{m\xi}(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt\,dt'.
\end{aligned}
$$

To estimate the last expression we integrate (6.19) to get

$$
\begin{aligned}
(6.22)\quad H_m(\xi,t) &= -\exp\bigg(-\int_0^t \operatorname{div}_{u_m}u_m\,d\tau\bigg) \\
&\quad\cdot\int_0^t\exp\bigg(\int_0^{t'}\operatorname{div}_{u_m}u_m\,d\tau\bigg)\cdot\eta_{m-1}(\operatorname{div}_{u_m}u_m-\operatorname{div}_{u_{m-1}}u_{m-1})\,dt'.
\end{aligned}
$$

Using the fact that H_m has the following qualitative form

$$
H_m = \psi_6(b_m(t))\int_0^t[\psi_7(b_m(\tau))U_{m\xi}(\tau)+\psi_8(b_m(\tau))\int_0^t U_{m\tau}(\tau')\,d\tau u_{m-1}(\tau)]\,d\tau
$$

we obtain that

$$
(6.23)\qquad \int_\Omega\int_0^T\frac{|H_{m\xi}(t)-H_{m\xi}(t')|^2}{|t-t'|^{1+\alpha}}\,d\xi\,dt' \le \phi_7(A,T)T^a\int_0^T\|U_m\|^2_{2+\alpha,\Omega}\,dt.
$$

Similarly, we have

$$
(6.24)\qquad \sup_t\|H_m\|^2_{1+\alpha,\Omega} \le \phi_8(A,T)T^a\int_0^T\|U_m\|^2_{2+\alpha,\Omega}\,dt.
$$

Using (6.23) and (6.24) in the estimation of L_3 we obtain

$$
\|L_3\|_{\alpha,\Omega^T} \le \phi_9(A,T)T^a\|U_m\|_{2+\alpha,\Omega^T}.
$$

Similarly L_3 we estimate L_5. Moreover,

$$
\|L_4\|_{\alpha,\Omega^T} \le \phi_{10}(A,T)T^a\|U_m\|_{2+\alpha,\Omega^T}.
$$

Let us consider L_6. We have

$$[L_6]^2_{\alpha,\Omega^T,x} \le \int_0^T \|L_6\|^2_{\alpha,\Omega}\,dt \le \int_0^T \|H_m\|^2_{1+\alpha,\Omega}\|g\|^2_{\alpha,\Omega}\,dt \le \sup_t \|H_m\|^2_{1+\alpha,\Omega}\|g\|^2_{\alpha,\Omega^T},$$

and

$$\begin{aligned}[L_6]^2_{\alpha/2.\Omega^T,t} &\le \int_\Omega\int_0^T\int_0^T \bigg(\frac{|H_m(t)-H_m(t')|^2|g(t)|^2}{|t-t'|^{1+\alpha}} \\ &\quad + \frac{|H_m(t')|^2|g(t)-g(t')|^2}{|t-t'|^{1+\alpha}}\bigg)\,d\xi\,dt\,dt' \\ &\le \int_0^T\int_0^T \bigg(\frac{\|H_m(t)-H_m(t')\|^2_{1,\Omega}}{|t-t'|^{1+\alpha}}\|g\|^2_{1+\alpha,\Omega} \\ &\quad + \frac{\|H_m(t')\|^2_{1+\alpha,\Omega}|g(t)-g(t')|^2_{2,\Omega}}{|t-t'|^{1+\alpha}}\bigg)\,dt\,dt' \\ &\le \bigg(\sup_t\int_0^T dt'\,\frac{\|H_m(t)-H_m(t')\|^2_{1,\Omega}}{|t-t'|^{1+\alpha}} + \sup_t\|H_m\|^2_{1+\alpha,\Omega}\bigg)\|g\|^2_{\alpha,\Omega^T}.\end{aligned}$$

Using (6.23) and (6.24) we have the estimate

$$\|L_6\|_{\alpha,\Omega^T} \le \phi_{11}(A,T)T^a\|U_m\|_{2+\alpha,\Omega^T}\|g\|_{\alpha,\Omega^T}.$$

Similarly, we have

$$\|L_7\|_{\alpha,\Omega^T} \le \phi_{12}(A,T)T^a\|U_m\|_{2+\alpha,\Omega^T}\|u_{mt}\|_{\alpha,\Omega^T}.$$

Continuing the considerations we prove the lemma. □

Summarizing we have

Theorem 6.5. *Let the assumptions of Lemma* 6.3 *be satisfied. Then there exists a time $T_{**} \le T_*$ such that for $t \le T_{**}$ there exists a solution to problem* (1.1) *such that $v \in W_2^{2+\alpha,1+\alpha/2}(\Omega^t)$, $\rho \in W_2^{1+\alpha,1/2+\alpha/2}(\Omega^t)$.*

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