Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 22, 2003, 345–353

NONLINEAR SUBMEANS ON SEMIGROUPS

Anthony To-Ming Lau — Wataru Takahashi

To Professor Andrzej Granas with admiration and respect

Abstract. The purpose of this paper is to study some algebraic structure of submeans on certain spaces X of bounded real valued functions on a semigroup and to find local conditions on X in terms of submean for the existence of a left invariant mean.

1. Introduction

Let S be a semigroup and X be a subspace of $\ell^\infty(S)$ containing constants, where $\ell^\infty(S)$ denotes the Banach space of bounded real-valued functions on S with the supremum norm. A continuous linear functional μ on X is called a *mean* if $\|\mu\| = \mu(1) = 1$. As well known, μ is a mean on X if and only if

$$\inf_{s\in S} f(s) \leq \mu(f) \leq \sup_{s\in S} f(s) \quad \text{for each } f \in X.$$

By a (nonlinear) *submean* on X, we shall mean a real-valued function μ on X with the following properties:

(1) $\mu(f+g) \leq \mu(f) + \mu(g)$ for every $f, g \in X$,
(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geq 0$,
(3) for $f, g \in X$, $f \leq g$ implies $\mu(f) \leq \mu(g)$,
(4) $\mu(c) = c$ for every constant function c.

2000 *Mathematics Subject Classification.* 43A07, 47H20, 47H09.

Key words and phrases. Invariant means, submean, left reversible semigroup, amenable semigroup, nonlinear ergodic and fixed point theorems.

This research is supported by NSERC grant A7679 and by Grant-in-Aid for General Scientific Research No. 12640157, the Ministry of Education, Science, Sports and Culture, Japan.

Clearly every mean is a submean. The notion of submean was first introduced by Mizoguchi and Takahashi in [8]. A semigroup S is called *left reversible* if $aS \cap bS \neq \phi$ for any $a, b \in S$. As well known, if $\ell^\infty(S)$ has a left invariant mean (i.e. S is *left amenable*), then S is left reversible but the converse is not true (e.g. free group or semigroup on two generators). On the other hand, a semigroup S is left reversible if and only if $\ell^\infty(S)$ has a left invariant submean (see [6]).

The purpose of this paper is to study some algebraic structures of submeans on certain subspaces X of $\ell^\infty(S)$, and find local conditions on X in terms of submean for the existence of a left invariant mean on X.

The notion of submean turns out to be an effective notion in non-linear fixed point and ergodic theorems (see [6], [7], [8] and [12]). It has a strong relation with normal structure in Banach spaces (see Theorem 2.3 and Theorem 2.4 in [7]). When G is a locally compact group, the invariant submean $\mu(f) = \sup\{f(s) : s \in G\}$, $f \in \ell^\infty(G)$, plays an important role (in place of an invariant mean when G is amenable) in the proof that the group algebra of G is weakly amenable (see [1] and [5]).

2. Preliminaries and some notations

All topologies in this paper are assumed to be Hausdorff. If E is a Banach space and E^* is its continuous dual, then the value of $f \in E^*$ at $x \in E$ will be denoted by $f(x)$ or $\langle f, x\rangle$.

Let S be a semigroup. Then a subspace X of $\ell^\infty(S)$ is called *left* (resp. *right*) *translation invariant* if $\ell_a(X) \subseteq X$ (resp. $r_a(X) \subseteq X$) for all $a \in S$, where $(\ell_a f)(s) = f(as)$ and $(r_a f)(s) = f(sa)$ for all $s \in S$.

Let S be a semitopological semigroup (i.e. a semigroup with a topology such that the multiplication is separately continuous) and let $LUC(S)$ denote all bounded continuous real-valued functions $f \in \ell^\infty(S)$ such that the map $a \mapsto \ell_a f$ from S into $\ell^\infty(S)$ is continuous when $\ell^\infty(S)$ has the norm topology. Then as known, $LUC(S)$ is a left and right translation invariant closed subspace of $\ell^\infty(S)$. It is precisely the space of bounded right uniformly continuous functions on S when S is a topological group.

3. Algebraic properties of submeans

Let S be a semitopological semigroup and X be a closed left translation invariant subspace of $\ell^\infty(S)$ containing constants.

Remark 3.1. (a) Let SM_X denote the set of submeans on X. Then SM_X is a compact convex subset of the product topological space $\prod_{f \in X} R_f$, where each $R_f = R$.

(b) If S is left reversible, then $\mu_0(f) = \inf_s \sup_t f(st)$ is a left invariant submean on $\ell^\infty(S)$ (see [6, Proposition 3.6]). Also, if μ is any other left invariant submean of $\ell^\infty(S)$, then for each $f \in \ell^\infty(S)$,

$$\mu_1(f) \leq \mu(f) \leq \mu_0(f)$$

where $\mu_1(f) = \sup_s \inf_t f(st)$. In particular, μ_0 is the maximal left invariant submean on $\ell^\infty(S)$.

(c) Let SM be the set of submeans on $\ell^\infty(S)$. For $\mu \in SM$ and $f \in \ell^\infty(S)$, define

$$\mu_\ell(f)(s) = \mu(\ell_s f)$$

for each $s \in S$. Then

$$\|f\| \leq \inf(\ell_s f)(t) \leq \mu(\ell_s f)(t) \leq \sup(\ell_s f) \leq \|f\|.$$

for each $s \in S$. So $\mu_\ell f \in \ell^\infty(S)$. Hence if $\psi, \mu \in SM$, we may define

$$\langle \psi \bigodot \mu, f \rangle = \langle \psi, \mu_\ell(f) \rangle.$$

LEMMA 3.2. *If $\psi, \mu \in SM$, then $\psi \bigodot \mu \in SM$.*

PROOF. (1) If $f, g \in \ell^\infty(S)$, then

$$\begin{aligned} \mu_\ell(f+g)(s) &= \mu(\ell_s f + \ell_s g) \leq \mu(\ell_s f) + \mu(\ell_s g) \\ &= \mu_\ell(f)(s) + \mu_\ell(g)(s) = (\mu_\ell(f) + \mu_\ell(g))(s) \end{aligned}$$

for each $s \in S$. Hence

$$\begin{aligned} (\psi \bigodot \mu)(f+g) &= \langle \psi, \mu_\ell(f+g) \rangle \leq \langle \psi, \mu_\ell(f) + \mu_\ell(g) \rangle \\ &\leq \langle \psi, \mu_\ell(f) \rangle + \langle \psi, \mu_\ell(g) \rangle = \psi \bigodot \mu(f) + \psi \bigodot \mu(g). \end{aligned}$$

(2) If $f \in \ell^\infty(S)$ and $\alpha \geq 0$, then

$$\mu_\ell(\alpha f)(s) = \mu(\ell_s(\alpha f)) = \mu(\alpha(\ell_s f)) = \alpha(\mu(\ell_s f)) = \alpha \mu_\ell(f)(s)$$

for each $s \in S$, i.e. $\mu_\ell(\alpha f) = \alpha \mu_\ell(f)$. Hence

$$(\psi \bigodot \mu)(\alpha f) = \langle \psi, \mu_\ell(\alpha f) \rangle = \langle \psi, \alpha \mu_\ell(f) \rangle = \alpha \langle \psi, \mu_\ell(f) \rangle = \alpha(\psi \bigodot \mu)(f).$$

(3) If $f \leq g$, $\mu_\ell(f)(s) = \langle \mu,\ \ell_s f \rangle \leq \langle \mu,\ \ell_s g \rangle = \mu_\ell(g)(s)$ for all $s \in S$ (since $\ell_s f \leq \ell_s g$). So $\mu_\ell(f) \leq \mu_\ell(g)$. Hence

$$(\psi \bigodot \mu)(f) = \langle \psi, \mu_\ell(f) \rangle \leq \langle \psi, \mu_\ell(g) \rangle = (\psi \bigodot \mu)(g).$$

(4) If c is a constant, then $\mu_\ell(c) = c$. So $(\psi \bigodot \mu)(c) = c$. Hence $\psi \bigodot \mu \in SM$. $\square$

A semigroup S is called a *left zero semigroup* if all of its elements are left zeros which means that $xy = x$ for all $x, y \in S$. Similarly S is called a *right zero semigroup* if $xy = y$ for all $x, y \in S$. The (possibly empty) set of idempotents of a semigroup S is denoted by $E(S)$.

Let X, Y be nonempty sets and G be a group. Let $K = X \times G \times Y$. Given a map $\delta\colon X \times Y \to G$, we define a sandwich product on K by

$$(x, g, y) \circ (x', g', y') = (x, g\delta(y, x')g', y').$$

Then $(K, \circ)$ is a simple group (i.e. no proper two-sided ideals) and any semigroup isomorphic to a simple group of this kind is called a *paragroup*.

Let S be a compact semigroup. It is called a *right topological semigroup* if the translations $x \mapsto xs$ $(s \in S)$ are continuous.

Theorem 3.3. $\Pi = (SM, \odot)$ *is a compact right topological semigroup. Further, the following conditions hold:*

(a) Π *has a minimal ideal* K *and*

$$K \simeq E(p\Pi) \times p\Pi p \times E(\Pi p)$$

where p *is any idempotent of* K *and* $p\Pi = \{p \circ s : s \in \Pi\}$ *with similar definition for* $p\Pi p$ *and* Πp. *Also,* $E(p\Pi)$ *is a right zero semigroup,* $E(\Pi p)$ *is a left zero semigroup and* $p\Pi p = p\Pi \cap \Pi p$ *is a group.*

(b) *The minimal ideal* K *need not be a direct product, but is a paragroup with respect to the natural map*

$$\delta\colon E(p\Pi) \times E(\Pi p) \to p\Pi p\colon (x, y) \mapsto x \circ y.$$

(c) *For any idempotent* $p \in K$, $p\Pi$ *is a minimal right ideal and* Πp *is a minimal left ideal.*

(d) *The minimal left ideals in* Π *are closed and homeomorphic to each other.*

Proof. We first show that the multiplication $\odot$ on Π is associative. Indeed, if $\psi, \mu, \theta \in \Pi$ and $f \in \ell^\infty(S)$, then

$$\langle \psi \odot (\mu \odot \theta), f\rangle = \langle \psi, (\mu \odot \theta)_\ell(f)\rangle = \langle \psi, \mu_\ell(\theta_\ell(f))\rangle \tag{3.1}$$

since

$$(\mu \odot \theta)_\ell(f)(s) = \langle \mu \odot \theta, \ell_s f\rangle = \langle \mu, \theta_\ell(\ell_s f)\rangle = \langle \mu, \ell_s \theta_\ell(f)\rangle = \mu_\ell(\theta_\ell(f))(s).$$

Also,

$$\langle (\psi \odot \mu) \odot \theta, f\rangle = \langle \psi \odot \mu, \theta_\ell(f)\rangle = \langle \psi, \mu_\ell(\theta_\ell(f))\rangle. \tag{3.2}$$

So $(\psi \odot \mu) \odot \theta = \psi \odot (\mu \odot \theta)$ by (3.1) and (3.2). Also, if $\mu_\alpha \to \mu$, then for $f \in \ell^\infty(S)$ and $\psi \in \Pi$,

$$\langle \mu_\alpha \odot \psi, f\rangle = \langle \mu_\alpha, \psi_\ell(f)\rangle \to \langle \mu, \psi_\ell(f)\rangle = \langle \mu \odot \psi, f\rangle.$$

Hence $\mu_\alpha \odot \psi \to \mu \odot \psi$ in Π, i.e. Π is a compact right semitopological semigroup (see [4]) by Lemma 3.2. Hence Π must have an idempotent (see [2]). The structure of Π stated in (a)–(d) now follows from [9]. $\square$

REMARK 3.4. (a) Lemma 3.2 and Theorem 3.3 remain valid if SM is replaced by SM_X when X is a left translation invariant and left introverted subspace of $\ell^\infty(S)$ containing constants, i.e. for each $\mu \in SM_X$ and $f \in X$, the function $\mu_\ell(f) \in X$.

(b) If $X \subseteq \ell^\infty(S)$ is left translation invariant and left introverted and contains constants, then

(i) X is right translation invariant,
(ii) for each $f \in X$, $K_f =$ the w^*-closed convex hull of $\{r_a f : a \in S\} \subseteq X$.

PROOF. (i) Let $a \in S$ and $\mu = \delta_a$. Then $\mu_\ell(f) = r_a f$ for each $f \in X$. Hence $r_a f \in X$.

(ii) Since K_f is compact in the weak*-topology on $\ell^\infty(S)$, and the topology p of pointwise convergence is Hausdorff and weaker than the weak*-topology, it follows that p agrees with the weak*-topology on K_f. Let $g \in K_f$. There exists a net $h_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha r_{a_i^\alpha} f$ of convex combinations of $r_a f$, $0 \le \lambda_i^\alpha \le 1$, $\sum \lambda_i^\alpha = 1$, such that $h_\alpha \to g$ in the p-topology. Let $\mu_\alpha = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha \delta_{a_i^\alpha}$, and μ be a weak*-cluster point of $\{\mu_\alpha\}$. By passing to a subnet, we may assume that $\mu_\alpha \to \mu$ in the weak*-topology of $\ell_\infty(S)^*$. Now $\mu_{\alpha_\ell} f = \sum_{i=1}^{n_\alpha} \lambda_i^\alpha r_{a_i^\alpha} f$, and so $\mu_{\alpha_\ell}(f) \in X$ for each α. Now

$$(\mu_{\alpha_\ell} f)(s) = \left(\sum \lambda_i^\alpha r_{a_i^\alpha} f\right)(s) = \langle \mu_\alpha, \ell_s f\rangle \to \langle \mu, \ell_s f\rangle = (\mu_\ell f)(s)$$

for each $s \in S$ i.e. $\mu_{\alpha_\ell}(f) \to \mu_\ell(f)$ in p-topology. Hence $\mu_\ell(f) = g$. Since X is left introverted, we have $g \in X$. □

PROPOSITION 3.5. *Let $X \subseteq \ell^\infty(S)$ be left invariant and left introverted containing constants, and $\mu \in SM_X$, φ is a left invariant submean on X. Then*

(a) $\mu \odot \varphi = \varphi$,
(b) $\varphi \odot \mu$ *is also a left invariant submean on X.*

PROOF. (a) $\langle \mu \odot \varphi, f\rangle = \langle \mu, \varphi_\ell(f)\rangle = \langle \mu, \varphi(f)\rangle = \varphi(f)$.

(b) $\langle \varphi \odot \mu, \ell_a f\rangle = \langle \varphi, \mu_\ell(\ell_a f)\rangle = \langle \varphi, \ell_a(\mu_\ell(f))\rangle = \langle \varphi, \mu_\ell(f)\rangle = \langle \varphi \odot \mu, f\rangle$.□

PROPOSITION 3.6. *Let X be as above. If φ is a left invariant submean on X and μ is a right invariant submean, then $\varphi \odot \mu$ is an invariant submean on X.*

PROOF. We know that $\varphi \odot \mu \in LISM_X$ by above. Now if $a \in S$ and $f \in X$, then

$$\langle \varphi \odot \mu, r_a f\rangle = \langle \varphi, \mu_\ell(r_a f)\rangle = \langle \varphi, \mu_\ell(f)\rangle = \langle \varphi \odot \mu, f\rangle$$

since

$$\mu_\ell(r_a f)(s) = \langle \mu, \ell_s(r_a f)\rangle = \langle \mu, r_a(\ell_s f)\rangle = \langle \mu, \ell_s f\rangle = (\mu_\ell f)(s). \qquad \square$$

Corollary 3.7. *If X has a left invariant submean and a right invariant submean, then X has an invariant submean.*

The following is an analogue of Lemma 1 in [4] and the Localization Theorem (Theorem 5.2) in [13]:

Theorem 3.8. *Let X be a left invariant and left introverted subspace of $\ell^\infty(S)$ containing constants. Then X has a left invariant submean if and only if for each $f \in X$, there exists a submean μ (depending on f) such that $\mu(f) = \mu(\ell_s f)$ for all $s \in S$.*

Proof. Let $f \in X$, and $K_f = \{\mu \in SM_X : \mu(\ell_s f) = \mu(f)$ for all $s \in S\}$. Then by assumption, K_f is a non-empty closed subset of SM_X. It sufficies to show that for $n = 1, 2, \ldots$ and $f_1, \ldots, f_n \in X$, $\bigcap_{i=1}^n K_{f_i} \neq \phi$. We do this by induction. Indeed, assume that $\bigcap_{i=1}^{n-1} K_{f_i} \neq \phi$ and let $\mu \in \bigcap_{i=1}^{n-1} K_{f_i}$. Consider the function $\mu_\ell(f_n) = g$, and choose $\mu_0 \in K_g$. Then

$$\mu_0 \bigodot \mu \in \bigcap_{i=1}^{n-1} K_{f_i}.$$

(Note $\mu_0 \bigodot \mu \in LSM_X$ by Lemma 3.5). Indeed, if $1 \leq i \leq n-1$, we have

$$\mu_\ell(f_i)(s) = \mu(\ell_s f_i) = \mu(f_i)$$

for all $s \in S$ and $i = 1, \ldots, n-1$. So

$$\begin{aligned}(\mu_0 \bigodot \mu)(f_i) &= \langle \mu_0, \mu_\ell(f_i)\rangle = \langle \mu_0, \mu(f_i)\rangle = \mu(f_i),\\ (\mu_0 \bigodot \mu)(\ell_s f_i) &= \langle \mu_0, \mu_\ell(\ell_s f_i)\rangle = \langle \mu_0, \mu_\ell(f_i)\rangle = \mu(f_i)\end{aligned}$$

for all $i = 1, \ldots, n-1$. Hence $(\mu_0 \bigodot \mu)(f_i) = \mu(f_i)$ for all $i = 1, \ldots, n-1$. Also $\mu_0 \bigodot \mu \in K_{f_n}$ (since $\mu_0 \in K_g$). Hence

$$\mu_0 \bigodot \mu \in \bigcap_{i=1}^{n} K_{f_i}. \qquad \square$$

4. Submeans and invariant means

We now proceed to find necessary and sufficient conditions for the existence of a left invariant mean on an invariant subspace X in terms on submeans.

Theorem 4.1. *Let S be a semitopological semigroup. The following statements are equivalent:*

(a) *For each $f \in LUC(S)$, there exists a submean μ (depending on f) such that*
$$\mu(f - \ell_s f) = \mu(\ell_s f - f) = 0 \quad \text{for all } s \in S.$$
(b) *$LUC(S)$ has a left invariant mean.*

Proof. (b)⇒(a) is trivial.

(a)⇒(b) If (a) holds, then for each $\alpha \in R$ and $s \in S$, $\mu(\alpha(f - \ell_s f)) = 0$. Hence $\sup(\alpha(f - \ell_s f)) \geq 0$ and $\inf(\alpha(f - \ell_s f)) \leq 0$ for each $s \in S$ and $\alpha \in R$. Now if $\alpha, \beta \in R$ and $s_1, s_2 \in S$, then

$$\mu(\alpha(f - \ell_{s_1} f) + \beta(f - \ell_{s_2} f)) \leq \mu(\alpha(f - \ell_{s_1} f)) + \mu(\beta(f - \ell_{s_2} f)) = 0.$$

Hence we have

$$\mu(h) \leq 0 \quad \text{for each } h \in S_f,$$

where $S_f = \text{linear span}\{f - l_s f : s \in S\}$. Consequently, $\inf h \leq 0$ for each $h \in S_f$. So $\sup h \geq 0$ for each $h \in S_f$. Now define $p(h) = \sup h$. Then $p(h) \geq 0$ for $h \in S_f$. So by the Hahn–Banach theorem, there exists a linear functional m on $LUC(S)$ such that

$$m(h) = 0 \quad \text{for each } h \in S_f \quad \text{and} \quad m(h) \leq p(h) = \sup h \quad \text{for all } h \in LUC(S).$$

Hence we have

$$m(-h) \leq \sup(-h) \quad \text{for all } h \in LUC(S).$$

So $\inf h \leq m(h) \leq \sup h$ for all $h \in LUC(S)$, i.e. m is a mean, and $m(f - \ell_s f) = 0$ for all $s \in S$. Now by [4], $LUC(S)$ has a left invariant mean. ☐

Example 4.2. If μ is a left invariant mean on $\ell^\infty(S)$, then $\mu(h) = 0$ for any $h = (f_1 - \ell_{a_1} f_1) + \ldots + (f_n - \ell_{a_n} f_n)$, $f_1, \ldots, f_n \in \ell^\infty(S)$, $a_1, \ldots, a_n \in S$. But this is no long true for left invariant submean.

Let S = free group on two generators a, b. Define $\mu(f) = \sup f(s)$. Then $\mu(\ell_a f) = \mu(f)$ for all $a \in S$ (this is the case when $aS = S$ for all $a \in S$, i.e. μ is a left invariant submean on $\ell^\infty(S)$. But if A = all elements in S that begin with a or a^{-1} (reduced word), and $f = 1_A$, and

$$h = (\ell_{ba^{-1}} f - \ell_{ab^{-1}a}(\ell_{ba^{-1}} f) + ((-f) - \ell_{b^{-1}a^{-1}}(-f)),$$

then $\mu(h) < 0$ (see Theorem 4.4).

Remark 4.3. Theorem 4.1 remains valid if $LUC(S)$ is replaced by a left translation invariant subspace X containing constants and for each $\mu \in X^*$ and $f \in X$, $\mu_\ell(f) \in X$.

Theorem 4.4. *Let X be a left translation subspace of $\ell^\infty(S)$ containing constans. The following are equivalent:*

(a) *X has a left invariant mean.*
(b) *For $s_1, \ldots, s_n \in S$ and $f_1, \ldots, f_n \in X^+ = \{f \in X : f \geq 0\}$, there exists a submean μ on X such that*

$$\mu\Big(\sum_{i=1}^n f_i\Big) \leq \mu\Big(\sum_{i=1}^n \ell_{s_i} f_i\Big).$$

Proof. (a)⇒(b) is trivial. (b)⇒(a) If (b) holds, then

$$\mu(h) = \mu\biggl(\sum_{i=1}^{n} \ell_{s_i} f_i - \sum_{i=1}^{n} f_i\biggr) \geq 0.$$

So $\sup(h) \geq 0$. For any $\varepsilon > 0$, choose $t_\varepsilon \in S$ such that $h(t_\varepsilon) \geq -\varepsilon$. Let m be a weak*-cluster point of $\{\delta_{t_\varepsilon}\}$ in X^*. Then μ is a submean (in fact a mean) on X and $m(h) \geq 0$. So, by linearity of m,

$$m\biggl(\sum_{i=1}^{n} \ell_{s_i} f - \sum_{i=1}^{n} f_i\biggr) \geq 0, \quad \text{i.e. } m\biggl(\sum_{i=1}^{n} \ell_{s_i} f_i\biggr) \geq m\biggl(\sum_{i=1}^{n} f_i\biggr).$$

Now for $f \in X^*$, and $s \in S$, define the lower semicontinuous convex function $F_{f,s}$ on $K = \{m \in X^* : \|m\| = m(1) = 1\}$ (set of means on X) by

$$F_{f,s}(m) = m(f - \ell_s f) \quad \text{for all } m \in K.$$

Now K is a weak*-compact convex subset of X^*. It follows from above that for any $f_1, \ldots, f_n \in X^+$ and $s_1, \ldots, s_n \in S$, there exists $m \in K$ such that

$$m\biggl(\sum_{i=1}^{n} f_i - \sum_{i=1}^{n} \ell_{s_i} f_i\biggr) = \sum_{i=1}^{n} F_{f_i,s_i}(m) \leq 0.$$

So Fan's existence theorem for systems of convex inequalities (see [5]), there exists $m \in K$ such that $F_{f,s}(m) \leq 0$ for all $f \in X^+$ and $s \in S$, i.e. $m(f) \leq m(\ell_s f)$ for all $f \in X^+$ and $s \in S$. For $f \in X$, let $g = \|f\| \cdot 1$. Then $g - f \in X^+$, and we have

$$m(g - f) \leq m(\ell_s(g - f)) \quad \text{for all } s \in S.$$

But $\ell_s(g - f) - (g - f) = f - \ell_s f$. So $m(\ell_s f - f) \leq 0$ for all $f \in X$ and $s \in S$. Consequently $m(\ell_s f) = m(f)$ for all $s \in S$ and $f \in X$ by linearity. □

Remark 4.5. Theorem 4.4 is an improvement of Theorem 6 of [10].

References

[1] M. Despic and F. Ghahramani, *Weak amenability of group algebras of locally comact groups*, Canad. Math. Bull. **37** (1994), 165–167.

[2] R. Ellis, *Locally compact transformation groups*, Duke Math. J. **24** (1957), 119–125.

[3] Ky Fan, *Existence theorems and extreme solutions for inequalities concerning convex functions or linear transformations*, Math. Z. **68** (1957), 205–217.

[4] E. E. Granirer and A. T. Lau, *Invariant means on locally compact groups*, Illinois J. Math. 15 (1971), 249–257.

[5] B. Johnson, *Weak amenability of group algebras*, Bull. London Math. Soc. 23 (1991), 281–284.

[6] A. T. Lau and W. Takahashi, *Invariant means and fixed point properties for non-expansive representations of topological semigroups*, Topol. Methods Nonlinear Anal. **5** (1995), 39–57.

[7] ______, *Invariant submeans and semigroups of nonexpansive mappings in Banach spaces with normal structures*, J. Funct. Anal. **25** (1996), 79–88.

[8] N. Mizoguchi and W.Takahashi, *On the existence of fixed points and ergodic retractions for Lipshitzian semigroups in Hilbert spaces*, Nonlinear Anal. **14** (1990), 69–80.

[9] W. Ruppert, *Rechstopologische Halbgruppen*, J. Angew. Math. **261** (1973), 123–133.

[10] K. Sakamaki and W. Takahashi, *Systems of convex inequalities and their applications*, J. Math. Anal. Appl. 70 (1979), 445–459.

[11] W. Takahashi, *Fixed point theorem and nonlinear ergodic theorem for nonexpansive semigroups without convexity*, Canad. J. Math. **44** (1992), 880–887.

[12] W. Takahashi and D. H. Jeong, *Fixed Point theorem for nonexpansive semigroups on Banach space*, Proc. Amer. Math. Soc. **122** (1994), 1175–1179.

[13] J. C. S. Wong, *Topologically stationary locally compact groups and amenability*, Trans. Amer. Math. Soc. **144** (1969), 351–363.

Manuscript received September 13, 2001

Anthony To-Ming Lau
Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta T6G-2G1, CANADA

E-mail address: tlau@math.ualberta.ca

Wataru Takahashi
Department of Mathematical and Computing Sciences
Tokyo Institute of Technology
Ohokayama, Meguro-ku, Tokyo 152-8552, JAPAN

E-mail address: wataru@is.titech.ac.jp