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FORCED SINGULAR OSCILLATORS AND THE METHOD OF LOWER AND UPPER SOLUTIONS

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Abstract. In this note, we study the existence of positive periodic solutions of the second order differential equation

$$u'' + g(u)u' + f(t,u) = h(t)$$

where $f(t,\,\cdot\,)$ has a singularity of repulsive type at the origin. We use the method of lower and upper solutions.

1. Introduction

In this note, we are interested in proving existence of solutions of the problem

$$\begin{gathered} u'' + g(u)u' + f(t,u) = h(t), \\ u(a) = u(b), \quad u'(a) = u'(b), \end{gathered} \tag{1.1}$$

where $f(t,\,\cdot\,)$ defined in $\mathbb{R}^+$ has a singularity at the origin.

The study of such singular second order boundary value problems goes back at least to A. C. Lazer and S. Solimini. Keeping in mind the model equations

$$u'' + \frac{1}{u^\nu} = h(t), \tag{1.2}$$

$$u'' - \frac{1}{u^\nu} = h(t), \tag{1.3}$$

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with $\nu > 0$, they proved in [11] the following result: Let f be a positive continuous function defined on $]0,\infty[$ such that $f(u) \to 0$ as $u \to \infty$, $f(u) \to \infty$ as $u \to 0^+$ and h be continuous on $[a,b]$. Then the problem

$$u'' + f(u) = h(t), \quad u(a) = u(b), \quad u'(a) = u'(b)$$

has a solution if and only if $\int_a^b h(s)\,ds > 0$. If moreover f satisfies

$$\int_0^1 f(s)\,ds = \infty, \tag{1.4}$$

then the problem

$$u''(t) - f(u) = h(t), \quad u(a) = u(b), \quad u'(a) = u'(b) \tag{1.5}$$

has a solution if and only if $\int_a^b h(s)\,ds < 0$.

This result has been extended in order to deal with nonlinearities f which are possibly unbounded at ∞. The result for the case of an attractive force at the origin was extended by P. Habets and L. Sanchez in [10]. In the repulsive case, the extension to equation (1.1) with an unbounded nonlinearity $f(t,\cdot)$ at ∞ is not complete. M. A. del Pino and R. Manásevich in [5] study the case of a nonlinearity superlinear at infinity while the case of an asymptotically linear nonlinearity at infinity is considered among others by M. A. del Pino, R. Manásevich and A. Montero ([6]), A. Fonda ([7]), P. Omari and W. Ye ([13]) and M. Zhang ([15], [16]). In all these papers, it is always assumed a certain kind of nonresonance condition. For example, in [6], it is assumed that

$$\frac{k^2\pi^2}{(b-a)^2} < \liminf_{u\to+\infty} \frac{f(t,u)}{u} \le \limsup_{u\to+\infty} \frac{f(t,u)}{u} < \frac{(k+1)^2\pi^2}{(b-a)^2},$$

uniformly in t for some $k \in \mathbb{N}$. In [7], considering only the case of a nonlinearity independent of the t-variable, the author assumes

$$\liminf_{u\to+\infty} \frac{2F(u)}{u^2} < \frac{\pi^2}{(b-a)^2},$$

where $F(u) = \int_1^u f(\xi)\,d\xi$. Hence the case $f(u) = \pi^2 u/(b-a)^2$ is always excluded. A first result in case f is asymptotically like $\pi^2 u/(b-a)^2$ is given by I. Rachunková, M. Tvrdý and I. Vrkoč in [14] but a major step is made by C. Fabry, D. Smets and D. Bonheure (see [1]). They prove for the model equation

$$\begin{gathered} u'' - \frac{1}{u^\nu} + \frac{k^2\pi^2}{(b-a)^2}u = h(t), \\ u(a) = u(b), \quad u'(a) = u'(b) \end{gathered} \tag{1.6}$$

with $\nu \geq 1$, that the existence of a solution of (1.6) is closely related to the properties of the function

$$\Phi(\theta) = \int_a^b h(t)\psi(t+\theta)\,dt,$$

where $\psi(t) = |\cos(k\pi((t-a)/(b-a)))|$. In another direction, P. Habets and L. Sanchez ([10]) replace the condition on f by a real damping $|g(u)| \geq A > 0$.

In Section 3, we extend the result of [11] to (1.1) in the repulsive case for a Carathéodory function f which can be asymptotically linear at ∞. We impose a limitation on $f(t,u)/u$ for large u (see condition (d) of Theorem 3.1) and a type of *strong force condition* in a form which is very similar to the one introduced by R. Martins in [12]. This section has to be compared with the results of [13].

In Section 4, we consider the undamped equation. We study the problem (1.1) with $g \equiv 0$ and f such that $f(t,u) \leq (\pi/(b-a))^2 u$. We show that (1.1) has at least one periodic solution if

$$\min_{t\in[a,b]} \int_t^{t+b-a} h(s)\sin\pi\frac{s-t}{b-a}\,ds \geq \delta > 0. \tag{1.7}$$

This result improves Corollary 3.7 of [14]. Compared with the results of [1] mentioned above for the problem (1.6), it corresponds to the case where Φ is positive. However, it should be pointed out that our result applies for a larger class of functions than in [1]. In particular, it also applies in case we have no singularity or a "weak repulsive force" at the origin as for example $f(t,u) = -1/\sqrt{u} + (\pi/(b-a))^2 u$.

To conclude this introduction, let us recall the following definition.

A function $f\colon D \subset [a,b] \times \mathbb{R} \to \mathbb{R}$ is said to be a *Carathéodory function* if

(a) for a.e. $t \in [a,b]$, the function $f(t,\cdot\,)$ with domain $\{u \in \mathbb{R} \mid (t,u) \in D\}$ is continuous,
(b) for all $u \in \mathbb{R}$, the function $f(\,\cdot\,,u)$ with domain $\{t \in [a,b] \mid (t,u) \in D\}$ is measurable.

If further, the Carathéodory function f satisfies

(c) for all $r > 0$, there exists $h \in L^1(a,b)$ such that for all $(t,u) \in D$ with $|u| \leq r$, $|f(t,u)| \leq h(t)$,

we say that f is a L^1-*Carathéodory function.*

2. Lower and upper solutions

Let us first prove the needed result concerning non well ordered lower and upper solutions. Further results concerning the method of lower and upper solutions can be found in [2] and [4].

Consider the following Liénard equation

$$u'' + g(u)u' + f(t,u) = 0, \qquad u(a) = u(b), \quad u'(a) = u'(b). \tag{2.1}$$

In this work we will use the following notion of lower and upper solutions.

Definitions 2.1. A function $\alpha \in \mathcal{C}([a,b])$ such that $\alpha(a) = \alpha(b)$ is a *lower solution* of (2.1) if its periodic extension on $\mathbb{R}$, defined by $\alpha(t) = \alpha(t+b-a)$ is such that for any $t_0 \in \mathbb{R}$ either $D_-\alpha(t_0) < D^+\alpha(t_0)$, or there exists an open interval I_0 such that $t_0 \in I_0$, $\alpha \in W^{2,1}(I_0)$ and, for a.e. $t \in I_0$,

$$\alpha''(t) + g(\alpha(t))\alpha'(t) + f(t,\alpha(t)) \geq 0.$$

A function $\beta \in \mathcal{C}([a,b])$ such that $\beta(a) = \beta(b)$ is an *upper solution* of (2.1) if its periodic extension on $\mathbb{R}$, defined by $\beta(t) = \beta(t+b-a)$ is such that for any $t_0 \in \mathbb{R}$ either $D^-\beta(t_0) > D_+\beta(t_0)$, or there exists an open interval I_0 such that $t_0 \in I_0$, $\beta \in W^{2,1}(I_0)$ and, for a.e. $t \in I_0$,

$$\beta''(t) + g(\beta(t))\beta'(t) + f(t,\beta(t)) \leq 0.$$

Definition 2.2. A lower solution α of (2.1) (resp. an upper solution β of (2.1)) is said *strict* if every solution u of (2.1) with $u \geq \alpha$ (resp. $u \leq \beta$) is such that $u(t) > \alpha(t)$ (resp. $u(t) < \beta(t)$) on $[a,b]$.

The idea of the following result is known. We give it here for completeness.

Theorem 2.3. *Let α and $\beta \in \mathcal{C}([a,b])$ be strict lower and upper solutions of* (2.1) *such that $\alpha < \beta$ on $[a,b]$. Define*

$$E = \{(t,u) \in [a,b] \times \mathbb{R} \mid \alpha(t) \leq u \leq \beta(t)\}.$$

Assume $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and $f\colon E \to \mathbb{R}$ is a L^1-Carathéodory function. Then, for every $C \in L^1(a,b)$ such that, for a.e. $t \in [a,b]$, $|g(\alpha(t))| < C(t)$, $|g(\beta(t))| < C(t)$ and for every $(t,u) \in E$, $|f(t,u)| < C(t)$,

$$\deg(I - T, \Omega) = 1,$$

where $T\colon \mathcal{C}^1([a,b]) \to \mathcal{C}^1([a,b])$ is the fixed point operator defined by

$$(Tu)(t) := -\int_a^b G(t,s)[g(u(s))u'(s) + f(s,u(s)) + C(s)u(s)]\,ds,$$

with $G(t,s)$ the Green function corresponding to

$$u'' - C(t)u = f(t), \quad u(a) = u(b),\ u'(a) = u'(b), \tag{2.2}$$

and

$$\Omega = \{u \in \mathcal{C}^1([a,b]) \mid \textit{for all } t \in [a,b],\ \alpha(t) < u(t) < \beta(t),\ |u'(t)| < R\}$$

(*with* $R > 0$ *large enough*). *In particular, the problem* (2.1) *has at least one solution* $u \in W^{2,1}(a,b)$ *such that*

$$\alpha(t) < u(t) < \beta(t) \quad \textit{for all } t \in [a,b].$$

PROOF. *Step 1.* The modified problem. We consider the modified problem

$$\begin{gathered} u'' - C(t)u + g(\gamma(t,u))u' + f(t,\gamma(t,u)) + C(t)\gamma(t,u) = 0, \\ u(a) = u(b), \quad u'(a) = u'(b), \end{gathered} \tag{2.3}$$

where $\gamma(t,u) := \max\{\alpha(t), \min\{u, \beta(t)\}\}$.

Step 2. The solution u of (2.3) is such that $\alpha < u < \beta$ on $[a,b]$. This step follows from the definition of $C(t)$ and the arguments used in the proof of Theorem 1.12 in [3].

Step 3. Degree estimations. Consider the homotopy

$$\begin{gathered} u'' - C(t)u + \lambda g(\gamma(t,u))u' + f(t,\gamma(t,u)) + C(t)\gamma(t,u) = 0, \\ u(a) = u(b), \quad u'(a) = u'(b). \end{gathered} \tag{2.4}$$

Observe first that arguing again as in Theorem 1.12 of [3], it is easily seen that every solution u of (2.4) satisfies $\overline{\alpha} = \min_t \alpha(t) - 2 \le u \le \overline{\beta} = \max_t \beta(t) + 2$.

Multiplying equation (2.4) by u and integrating we obtain

$$\begin{aligned} \|u'\|_{L^2}^2 \le & \int_a^b {u'}^2(t) + C(t)u^2(t)\,dt \\ = & \lambda \int_a^b g(\gamma(t,u(t)))u(t)u'(t)\,dt + \int_a^b f(t,\gamma(t,u(t)))u(t)\,dt \\ & + \int_a^b C(t)\gamma(t,u(t))u(t)\,dt \\ \le & C_1\|u'\|_{L^2} + C_2, \end{aligned}$$

for some constants C_1, $C_2 > 0$. It is now easy to obtain a bound for $\|u''\|_{L^1}$ and another for $\|u'\|_\infty$. It follows that there exists $\overline{R} > 0$ such that, for every $\lambda \in [0,1]$, every solution u of (2.4) satisfies $\|u\|_{\mathcal{C}^1} < \overline{R}$.

The problem (2.4) is equivalent to the fixed point problem

$$u = \overline{T}_\lambda u,$$

where $\overline{T}_\lambda \colon \mathcal{C}^1([a,b]) \to \mathcal{C}^1([a,b])$ is defined by

$$(\overline{T}_\lambda u)(t) = -\int_a^b G(t,s)[\lambda g(\gamma(t,u))u' + f(t,\gamma(t,u)) + C(t)\gamma(t,u)]\,ds,$$

with $G(t,s)$ the Green function corresponding to (2.2). Observe that $\overline{T}_\lambda$ is completely continuous and, increasing $\overline{R}$ if necessary, we can assume that $\Omega \subset$

$B(0,\overline{R})$ and $\overline{T}_0(\mathcal{C}^1([a,b])) \subset B(0,\overline{R})$. Hence we have, by the properties of the degree,

$$\deg(I-\overline{T}_0, B(0,\overline{R})) = 1.$$

By Step 2, every fixed point of $\overline{T}_1$ is in Ω and by the properties of the degree we obtain $\deg(I-T,\Omega) = 1$. The existence of a solution u such that for all $t \in [a,b]$, $\alpha(t) < u(t) < \beta(t)$ follows now from the properties of the degree. □

The main ingredient of this paper is the following result.

Theorem 2.4. *Let α and $\beta \in \mathcal{C}([a,b])$ be lower and upper solutions of* (2.1) *such that $\alpha \not\leq \beta$. Assume that $g\colon \mathbb{R} \to \mathbb{R}$ is continuous and $f\colon [a,b]\times\mathbb{R} \to \mathbb{R}$ is a L^1-Carathéodory function such that, for some a_-, $b_+ \in L^1(a,b)$ such that $b_+ \geq 0$,*

$$\limsup_{u\to-\infty} f(t,u) \leq a_-(t) \quad \textit{and} \quad \limsup_{u\to\infty} \frac{f(t,u)}{u} \leq b_+(t),$$

uniformly in t. Assume further that for any $p \in L^1(a,b)$ with $p \leq b_+$ a.e. on $[a,b]$ and any $\overline{t} \in [a,b[$, the problem

$$\begin{aligned} &u'' + p(t)u = 0,\\ &u(\overline{t}) = 0, \quad u(\overline{t}+b-a) = 0, \end{aligned} \tag{2.5}$$

has only the trivial solution (where $p(t) = p(t-b+a)$ for $t \in]b, \overline{t}+b-a]$). Then, there exists a solution $u \in \mathcal{S}$ of (2.1)*, where*

$$\mathcal{S} := \{u \in \mathcal{C}([a,b]) \mid \exists t_1, t_2 \in [a,b], u(t_1) \geq \beta(t_1),\ u(t_2) \leq \alpha(t_2)\}. \tag{2.6}$$

To prove this result, we need the following lemma

Lemma 2.5. *Let $\gamma \in L^1(a,b)$ be such that, for any $p \in L^1(a,b)$ with $p \leq \gamma$ a.e. on $[a,b]$ and any $\overline{t} \in [a,b]$, the problem*

$$\begin{aligned} &u'' + p(t)u = 0,\\ &u(\overline{t}) = 0, \quad u(\overline{t}+b-a) = 0, \end{aligned}$$

has only the trivial solution (where p is extended by periodicity on $]b, \overline{t}+b-a]$). Then there exists $\varepsilon > 0$ such that, for all $\overline{t} \in [a,b]$, all $u \in H_0^1(\overline{t}, \overline{t}+b-a)$, we have

$$\int_{\overline{t}}^{\overline{t}+b-a} (u'^2(t) - \gamma(t)u^2(t))\,dt \geq \varepsilon \|u\|_{H^1}^2.$$

Proof. First observe that arguing as in [8], we can associate to any $\overline{t} \in [a,b]$, a positive constant $\varepsilon(\overline{t})$ such that, for every $u \in H_0^1(\overline{t}, \overline{t}+b-a)$, we have

$$\int_{\overline{t}}^{\overline{t}+b-a} (u'^2(t) - \gamma(t)u^2(t))\,dt \geq \varepsilon(\overline{t}) \|u\|_{H^1}^2.$$

Now, let us prove the claim. By contradiction, assume there exist sequences $(t_n)_n \subset [a,b]$ and $(u_n)_n \subset H^1(t_n, t_n+b-a)$ with $\|u_n\|_{H^1} = 1$ and

$$\int_{t_n}^{t_n+b-a} ({u'}_n^2(t) - \gamma(t)u_n^2(t))\,dt \le \frac{1}{n}. \tag{2.7}$$

Extending u_n by 0, we have, for all n, $u_n \in H_0^1(a, 2b-a)$, $\|u_n\|_{H^1(a,2b-a)} = 1$ and hence, passing to a subsequence, $t_n \to \bar{t} \in [a,b]$, the sequence $(u_n)_n$ converges weakly in $H_0^1(a, 2b-a)$ and strongly in $\mathcal{C}([a, 2b-a])$ to some function u with $u = 0$ on $[a, 2b-a] \setminus [\bar{t}, \bar{t}+b-a]$ and

$$\int_{\bar{t}}^{\bar{t}+b-a} ({u'}^2(t) - \gamma(t)u^2(t))\,dt \le 0.$$

By the first part of the proof, $\|u\|_{H^1} = 0$ i.e. $u \equiv 0$. We deduce now from (2.7) that $\int_a^{2b-a} {u'}_n^2(t)\,dt \to 0$ which contradicts $\|u_n\|_{H^1} = 1$. □

PROOF OF THEOREM 2.4. For each $r > 0$, we define

$$f_r(t,u) = \begin{cases} f(t,u) & \text{if } |u| < r, \\ (1 + r - |u|)f(t,u) - (|u| - r)u/r & \text{if } r \le |u| < r+1, \\ -u/r & \text{if } r+1 \le |u|, \end{cases}$$

and consider the modified problem

$$\begin{gathered} u'' + g(u)u' + f_r(t,u) = 0, \\ u(a) = u(b), \quad u'(a) = u'(b). \end{gathered} \tag{2.8}$$

Claim 1. There exists $L^+ > 0$ such that for any $r > 0$, the solutions u of (2.8), which are in $\mathcal{S}$, are such that $\max u \le L^+$.

Let $u \in \mathcal{S}$ be a solution of (2.8) such that

$$\max_t u(t) > R := \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}.$$

From the definition of $\mathcal{S}$, there exists $t_2 \in [a,b]$ so that $u(t_2) \le \alpha(t_2) \le R$. Hence, extending u by periodicity if necessary, we can find $a' < b'$ so that $b' - a' < b - a$, $u(a') = u(b') = R$, $u(t) > R$ on $]a', b'[$ and $\max_{a' \le t \le b'} u(t) = \max_{a \le t \le b} u(t)$.

Let now $\varepsilon > 0$ be given by Lemma 2.5 and choose $M > R$ such that for all $u \ge M$ and a.e. $t \in [a,b]$

$$\frac{f(t,u)}{u} \le b_+(t) + \frac{\varepsilon}{2}.$$

Define then

$$p_r(t,u) = \begin{cases} f_r(t,u)/u & \text{for } u \ge M, \\ f_r(t,M)/M & \text{for } u < M, \end{cases}$$

and

$$q_r(t,u) = f_r(t,u) - p_r(t,u)u.$$

For $r > 0$, these functions are such that for a.e. $t \in [a,b]$ and all $u \geq R$,

$$p_r(t,u) \leq b_+(t) + \varepsilon/2 \quad \text{and} \quad q_r(t,u) \leq k(t)$$

for some $k \in L^1(a,b)$.

The function $v = u - R$ is nonnegative on $[a',b']$ and solves the problem

$$\begin{gathered} v''(t) + g(v(t)+R)v'(t) + p_r(t,u(t))(v(t)+R) + q_r(t,u(t)) = 0, \\ v(a') = 0, \quad v(b') = 0. \end{gathered}$$

Hence, we compute

$$\begin{aligned} \int_{a'}^{b'} v'^2(t)\,dt &- \int_{a'}^{b'} (b_+(t)+\varepsilon/2)(v(t)+R)v(t)\,dt \\ &\leq -\int_{a'}^{b'} v(t)(v''(t) + g(v(t)+R)v'(t) + p_r(t,u(t))(v(t)+R))\,dt \\ &= \int_{a'}^{b'} q_r(t,u(t))v(t)\,dt \end{aligned}$$

and using Lemma 2.5 we obtain

$$\begin{aligned} \frac{\varepsilon}{2}\|v\|^2_{H^1(a',b')} &\leq \varepsilon\|v\|^2_{H^1(a',b')} - \frac{\varepsilon}{2}\|v\|^2_{L^2(a',b')} \\ &\leq \int_{a'}^{b'} \left(v'^2(t) - \left(b_+(t) + \frac{\varepsilon}{2}\right)v^2(t)\right)dt \\ &\leq \int_{a'}^{b'} \left(q_r(t,u(t)) + \left(b_+(t) + \frac{\varepsilon}{2}\right)R\right)v(t)\,dt \\ &\leq \int_{a'}^{b'} \left(k(t) + \left(b_+(t) + \frac{\varepsilon}{2}\right)R\right)v(t)\,dt, \end{aligned}$$

i.e. $\|v\|^2_{H^1(a',b')} \leq K\|v\|_\infty$, for some $K > 0$ independent of r. It follows that, for all $t \in [a',b']$,

$$v(t) = \int_{a'}^{t} v'(s)\,ds \leq K^{1/2}\|v\|_\infty^{1/2}(b-a)^{1/2}$$

so that $\|v\|_\infty \leq K(b-a)$ and $\max_t u(t) \leq R + K(b-a) =: L^+$.

Claim 2. There exists L^- so that any solution $u \in \mathcal{S}$ of (2.8) with $r \geq 8(b-a)^2$ satisfies $\min_t u(t) \geq L^-$.

By assumption, there exists $k \in L^1(a,b)$ such that for a.e. $t \in [a,b]$ and every $u \leq L^+$, $f(t,u) \leq k(t)$.

Using Claim 1, we can write $f(t,u(t)) \leq k(t)$ for a.e. $t \in [a,b]$, if $u \in \mathcal{S}$ is a solution of (2.8). Next, integrating (2.8) or multiplying this equation by u and integrating, we obtain

$$\int_a^b f_r(t,u(t))\,dt = 0 \quad \text{and} \quad \int_a^b u'^2(t)\,dt = \int_a^b f_r(t,u(t))u(t)\,dt.$$

It follows then that

$$\|u'\|_{L^2}^2 = \int_a^b f_r(t,u(t))(\|u\|_\infty + u(t))\,dt \le 2(\|k\|_{L^1} + \frac{\|u\|_\infty}{r}(b-a))\|u\|_\infty$$

i.e.

$$\|u'\|_{L^2} \le (2\|k\|_{L^1}\|u\|_\infty)^{1/2} + \frac{\|u\|_\infty}{2(b-a)^{1/2}}.$$

Define now $t_1 \in [a,b]$ such that $u(t_1) \ge \beta(t_1) \ge -R$. Extending u by periodicity, we can write for $t \in [t_1, t_1+b-a]$

$$u(t) = u(t_1) + \int_{t_1}^t u'(s)\,ds \ge -R - (2(b-a)\|k\|_{L^1}\|u\|_\infty)^{1/2} - \frac{\|u\|_\infty}{2}.$$

It is now easy to obtain L^- so that $u(t) \ge L^-$ on $[a,b]$.

Conclusion. Consider the problem (2.8), with $r > \max\{L^+, -L^-, 8(b-a)^2\}$. It is easy to see that $\alpha_1(t) = -r-2$ and $\beta_2(t) = r+2$ are strict lower and upper solutions of (2.8).

Assume that β is not a strict upper solution. Then, there exists a solution u of (2.8) such that $u \le \beta$ and for some $t_1 \in [a,b]$, $u(t_1) = \beta(t_1)$. As further $\alpha \not\le \beta$, there exists $t_2 \in [a,b]$ such that $\alpha(t_2) > \beta(t_2)$. It follows that $\alpha(t_2) > u(t_2)$, $u \in \mathcal{S}$, and we deduce from the claim that $\|u\|_\infty \le \max\{L^+, -L^-\}$. Hence, u is a solution of (2.1) in $\mathcal{S}$. We come to the same conclusion if α is not a strict lower solution.

Suppose now that $\beta_1 = \beta$ and $\alpha_2 = \alpha$ are strict upper and lower solutions. In that case, we apply Theorem 2.3 successively with

$$\begin{aligned}
\Omega_{1,1} &= \{u \in \mathcal{C}^1([a,b]) \mid \text{for all } t \in [a,b],\ \alpha_1(t) < u(t) < \beta_1(t),\ \|u'\|_\infty < R\},\\
\Omega_{2,2} &= \{u \in \mathcal{C}^1([a,b]) \mid \text{for all } t \in [a,b],\ \alpha_2(t) < u(t) < \beta_2(t),\ \|u'\|_\infty < R\},\\
\Omega_{1,2} &= \{u \in \mathcal{C}^1([a,b]) \mid \text{for all } t \in [a,b],\ \alpha_1(t) < u(t) < \beta_2(t),\ \|u'\|_\infty < R\}.
\end{aligned}$$

We have $\deg(I-T, \Omega_{1,1}) = 1$ and $\deg(I-T, \Omega_{2,2}) = 1$ and moreover,

$$\begin{aligned}
1 &= \deg(I-T, \Omega_{1,2})\\
&= \deg(I-T, \Omega_{1,1}) + \deg(I-T, \Omega_{2,2}) + \deg(I-T, \Omega_{1,2} \setminus (\overline{\Omega}_{1,1} \cup \overline{\Omega}_{2,2}))
\end{aligned}$$

which implies $\deg(I-T, \Omega_{1,2} \setminus (\overline{\Omega}_{1,1} \cup \overline{\Omega}_{2,2})) = -1$. The existence of a solution u of (2.8) in $\Omega_{1,2} \setminus (\overline{\Omega}_{1,1} \cup \overline{\Omega}_{2,2}) \subset \mathcal{S}$ follows and hence, by the claims, $\|u\|_\infty < r$ and $u \in \mathcal{S}$ is a solution of (2.1). □

To conclude this section, we give concrete conditions on b_+ in order to satisfy the assumption of Theorem 2.4.

Proposition 2.6. *Let $b_+ \in L^1(a,b)$. Assume that $b_+(t) \le \pi^2/(b-a)^2$ a.e. on $[a,b]$ with strict inequality on a subset of positive measure. Then the problem*

$$\begin{gathered} u'' + pu = 0, \\ u(\bar t) = 0, \quad u(\bar t + b - a) = 0, \end{gathered}$$

has only the trivial solution for any $\bar t \in [a,b]$.

Proof. This result can be deduced easily from the fact that $\pi^2/(b-a)^2$ is the first eigenvalue of the Dirichlet problem

$$u'' + \lambda u = 0, \quad u(a) = 0, \quad u(b) = 0. \qquad \square$$

3. Main result

In this section, we prove an existence result for the case of a repulsive force. A model example is

$$\begin{gathered} u'' + g(u)u' - \frac{1}{u^3} + \gamma(t)u = h(t), \\ u(a) = u(b), \quad u'(a) = u'(b). \end{gathered} \tag{3.1}$$

Theorem 3.1. *Let $g \in \mathcal{C}(\mathbb{R}^+)$, $h \in L^1(a,b)$ and assume $f\colon [a,b] \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfies a Carathéodory condition together with*

(a) *for any $0 < r < s$, there exists a function $k \in L^1(a,b)$ such that, for a.e. $t \in [a,b]$ and all $u \in [r,s]$, we have $|f(t,u)| \le k(t)$.*

Assume moreover that

(b) *there exists lower and upper solutions α and $\beta \in \mathcal{C}([a,b])$ of (1.1) such that $0 < \alpha$, $0 < \beta$ on $[a,b]$ and $\alpha \not\geq \beta$,*

(c) (strong force) *there exist $\rho > 0$, $\ell \in L^1(a,b)$ and $\widehat f \in \mathcal{C}(]0,\rho])$ with*

$$\int_0^\rho \widehat f^-(u)\,du = +\infty \quad \text{and} \quad \widehat f^-(u) = \max\{-\widehat f(u), 0\},$$

such that for all $u \in\,]0,\rho]$ and a.e. $t \in [a,b]$

$$f(t,u) \le \widehat f(u) \quad \text{and} \quad f(t,u) \le \ell(t),$$

(d) *there exists a function $\gamma \in L^1(a,b)$ such that $\gamma(t) \le (\pi/(b-a))^2$ a.e. on $[a,b]$ with strict inequality on a subset of positive measure and*

$$\limsup_{u\to+\infty} \frac{f(t,u)}{u} \le \gamma(t),$$

uniformly in $t \in [a,b]$.

Then the problem (1.1) *has at least one positive solution.*

REMARK. We can of course generalize the condition (d) in the same spirit as in Theorem 2.4.

The main role of the strong force condition (c) is to ensure the existence of a lower a-priori bound on the solutions. It is important to notice that here we do not impose that $f(t,u)$ goes to $-\infty$ as u goes to zero as it is the case in [6], [10], [13] and [16], but that the bound $\widehat{f}$ has a negative part whose primitive is unbounded as u goes to zero. As oscillating singularities that enter into the framework of our result, we can consider for example the function

$$f(t,u) = \min\left(\frac{-1}{u^2\sqrt{t-a}}\sin\frac{1}{u}, \frac{1}{\sqrt{t-a}}\right) + \frac{\pi^2}{(b-a)^2}u\sin\frac{\pi(t-a)}{b-a}.$$

Notice that some control on the singularity such as assumption (c) is necessary. This is clear from [11] where it is proved in particular that the problem

$$u'' - \frac{1}{\sqrt{u}} = h(t),$$
$$u(a) = u(b), \quad u'(a) = u'(b),$$

has no solution for some negative $h \in \mathcal{C}([a,b])$.

At last, assumption (d) forces the nonlinearity to be, for large values of u, "under" the asymptote of the first nontrivial Fučík curve for the periodic problem, which is also the first eigenvalue of the Dirichlet problem

$$u'' + \lambda u = 0, \quad u(a) = 0, \quad u(b) = 0.$$

Such a condition is somewhat natural if we realize that large solutions of the periodic problem (3.1) look like solutions of a Dirichlet problem. We also refer to [16] for a discussion about the relationship between the periodic and the Dirichlet problem. Also, such a condition cannot be avoided. Indeed, it is proved in [1] that for some $h \in \mathcal{C}([0,2\pi])$ the problem

$$u'' - \frac{1}{u^3} + \frac{1}{4}u = h(t),$$
$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

has no solution.

PROOF OF THEOREM 3.1. Without loss of generality, we can assume that $\rho \in \,]0, \min\{\min_t \beta(t), \min_t \alpha(t)\}[$.

Let $\delta \in \,]0,\rho]$, define the truncated function

$$f_\delta(t,u) = \begin{cases} f(t,u) & \text{if } u \ge \delta, \\ f(t,\delta) & \text{if } u < \delta, \end{cases}$$

and consider the modified problem

$$
\begin{gathered}
u'' + g(|u|)u' + f_\delta(t,u) = h(t),\\
u(a) = u(b),\ u'(a) = u'(b).
\end{gathered}
\tag{3.2}
$$

We deduce from Theorem 2.4 that there exists a solution u of (3.2) in

$$\mathcal{S} = \{u \in \mathcal{C}([a,b]) \mid \exists t_1, t_2 \in [a,b],\ u(t_1) \geq \beta(t_1),\ u(t_2) \leq \alpha(t_2)\}.$$

Claim 1. There exists $R > 0$ so that any solution $u \in \mathcal{S}$ of (3.2) with $0 < \delta \leq \rho$ satisfies $\|u\|_{\mathcal{C}^1} \leq R$.

Repeating the arguments used to prove Claims 1 and 2 of Theorem 2.4, we obtain $\overline{R} > 0$ so that, any solution $u \in \mathcal{S}$ of (3.2) with $0 < \delta \leq \rho$ satisfies $\|u\|_\infty \leq \overline{R}$. Hence, there exists $C \in L^1(a,b)$ such that,

$$f_\delta(t, u(t)) \leq C(t) \qquad \text{a.e. on } [a,b].$$

Multiplying (3.2) by u and integrating, we obtain

$$
\begin{aligned}
\|u'\|_{L^2}^2 &= \int_a^b (f_\delta(t,u(t)) - h(t))u(t)\,dt\\
&= \int_a^b (f_\delta(t,u(t)) - h(t))(\|u\|_\infty + u(t))\,dt\\
&\leq \int_a^b (C(t) - h(t))(\|u\|_\infty + u(t))\,dt \leq 2\overline{R}(\|C\|_{L^1} + \|h\|_{L^1}).
\end{aligned}
$$

It is now easy to deduce a bound for $\|u''\|_{L^1}$ and another for $\|u'\|_\infty$.

Claim 2. There exists $\xi \in]0,\rho]$ so that any solution $u \in \mathcal{S}$ of (3.2) with $0 < \delta \leq \xi$ satisfies $u(t) \geq \xi$ on $[a,b]$.

Define $\xi > 0$ such that

$$\int_\xi^\rho \widehat{f}^-(u)\,du > (\|h\|_{L^1} + \|\ell\|_{L^1})R + \bigg(\frac{1}{2} + (b-a) \max_{-R \leq u \leq R} |g(|u|)|\bigg) R^2$$

with R given by Claim 1. Let $u \in \mathcal{S}$ be a solution of (3.2). It follows from the definition of $\mathcal{S}$ that $\max_t u > \rho$. Define the set $A = \{t \in [a,b] \mid u'(t) \geq 0\}$ and suppose by contradiction that there exist $t_1,\ t_2 \in A$ so that $u(t_1) = \xi$, $u(t_2) = \rho$ and $\xi \leq u(t) \leq \rho$ on $[t_1, t_2]$. Multiplying (3.2) by u' and integrating on $B = [t_1, t_2] \cap A$, we get

$$\frac{u'^2(t_2)}{2} - \frac{u'^2(t_1)}{2} + \int_B g(|u(s)|)u'^2(s)\,ds + \int_B f_\delta(s,u(s))u'(s)\,ds = \int_B h(s)u'(s)\,ds.$$

From the a priori bounds on $\|u\|_\infty$ and $\|u'\|_\infty$, we obtain then

$$\int_B g(|u(s)|)u'^2(s)\,ds \leq (b-a)R^2 \max_{-R \leq u \leq R} |g(|u|)|$$

and it follows that

$$-\int_B f_\delta(s,u(s))u'(s)\,ds \le R\|h\|_{L^1} + \frac{R^2}{2} + (b-a)R^2 \max_{-R\le u\le R} |g(|u|)|.$$

On the other hand, let $B^- = \{t \in B \mid \widehat{f}(u(t)) \le 0\}$ and $B^+ = B \setminus B^-$. We then have the following inequalities

$$\int_{B^+} f_\delta(s,u(s))u'(s)\,ds \le R\|\ell\|_{L^1}$$

and

$$\begin{aligned} -\int_{B^-} f_\delta(s,u(s))u'(s)\,ds &\ge -\int_{B^-} \widehat{f}(u(s))u'(s)\,ds = \int_B \widehat{f}^-(u(s))u'(s)\,ds \\ &\ge \int_{t_1}^{t_2} \widehat{f}^-(u(s))u'(s)\,ds = \int_\xi^\rho \widehat{f}^-(u)\,du. \end{aligned}$$

We now deduce that

$$\int_\xi^\rho \widehat{f}^-(u)\,du \le R\|\ell\|_{L^1} + R\bigg(\|h\|_{L^1} + \frac{R}{2}\bigg) + (b-a)R^2 \max_{-R\le u\le R} |g(|u|)|,$$

which contradicts the definition of ξ.

Conclusion. From the first part of the proof, we know that problem (3.2) with $\delta = \xi$ has a solution. According to Claim 2, this solution solves (1.1). □

Next we give conditions that ensure the existence of the required lower and upper solutions.

COROLLARY 3.2. *Let $g \in \mathcal{C}(\mathbb{R}^+)$, $h \in L^1(a,b)$ and assume $f\colon [a,b] \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfies a Carathéodory condition together with*

(a) *any $0 < r < s$, there exists a function $k \in L^1(a,b)$ such that, for a.e. $t \in [a,b]$ and all $u \in [r,s]$, we have $|f(t,u)| \le k(t)$.*

Assume moreover that

(b) *for some $\beta > 0$ and a.e. $t \in [a,b]$, $f(t,\beta) - h(t) \le 0$,*
(c) (Strong force) *there exist $\rho \in \,]0,\beta[$, $\ell \in L^1(a,b)$ and $\widehat{f} \in \mathcal{C}(]0,\rho])$ with*

$$\int_0^\rho \widehat{f}^-(u)\,du = +\infty \quad \textit{and} \quad \widehat{f}^-(u) = \max\{-\widehat{f}(u), 0\},$$

such that $f(t,u) \le \widehat{f}(u)$ and $f(t,u) \le \ell(t)$ for all $u \in \,]0,\rho]$ and a.e. $t \in [a,b]$,
(d) *there exist $R > \beta$ and $f_0 \in L^1(a,b)$ such that, for a.e. $t \in [a,b]$, if $u \ge R$,*

$$f(t,u) \ge f_0(t) \quad \textit{and} \quad \int_a^b f_0(t)\,dt \ge \int_a^b h(t)\,dt,$$

(e) *there exists a function* $\gamma \in L^1(a,b)$ *such that* $\gamma(t) \le (\pi/(b-a))^2$ *a.e. on* $[a,b]$ *with strict inequality on a subset of positive measure and*

$$\limsup_{u\to+\infty} \frac{f(t,u)}{u} \le \gamma(t), \quad \textit{uniformly in } t \in [a,b].$$

Then the problem (1.1) *has at least one positive solution.*

Proof. We apply Theorem 3.1. We only have to prove the existence of the lower and upper solutions. We first deduce from (b) that the constant function $\beta(t) = \beta$ is an upper solution for (1.1).

Next, we deduce a lower solution from condition (d) using the following argument. Let us introduce the function

$$\phi(t) = f_0(t) - h(t) - \frac{1}{b-a}\int_a^b (f_0(t) - h(t))\,dt \in L^1(a,b)$$

and prove that, for all $c \in \mathbb{R}$, the problem

$$\begin{gathered} u''(t) + g(|u(t)+c|)u'(t) + \phi(t) = 0, \\ u(a) = u(b), \quad u'(a) = u'(b) \end{gathered} \tag{3.3}$$

has a solution $\alpha_0 \in W^{2,1}(a,b)$ such that $\overline{\alpha}_0 = (1/(b-a))\int_a^b \alpha_0(t)\,dt = 0$. To this aim consider the homotopy

$$\begin{gathered} u''(t) + \lambda[g(|u(t)+c|)u'(t) + \phi(t)] = 0, \\ u(a) = u(b), \quad u'(a) = u'(b). \end{gathered} \tag{3.4}$$

Solutions of this problem with mean value zero are fixed points of the operator

$$T_\lambda : \widetilde{\mathcal{C}}^1([a,b]) \to \widetilde{\mathcal{C}}^1([a,b]), \quad u \mapsto T_\lambda u,$$

where $\widetilde{\mathcal{C}}^1([a,b]) = \{u \in \mathcal{C}^1([a,b]) \mid \int_a^b u(s)\,ds = 0\}$,

$$T_\lambda u = \lambda \int_a^b G(t,s)[g(|u(s)+c|)u'(s) + \phi(s)]\,ds$$

and $G(t,s)$ is the corresponding Green function.

Let us prove that the fixed points of T_λ are a priori bounded in $\mathcal{C}([a,b])$. Multiply (3.4) by u and integrate, we obtain, using [9],

$$\|u'\|_{L^2}^2 = \lambda \int_a^b \phi(t)u(t)\,dt \le \|\phi\|_{L^1}\|u\|_\infty \le \sqrt{\frac{b-a}{12}}\,\|\phi\|_{L^1}\|u'\|_{L^2}.$$

This implies $\|u'\|_{L^2} \le \sqrt{(b-a)/12}\,\|\phi\|_{L^1}$ and $\|u\|_\infty \le ((b-a)/12)\|\phi\|_{L^1}$. It is now easy to bound $\|u''\|_{L^1}$ and next $\|u'\|_\infty$. Hence, by the invariance of the degree along the homotopy, for every $c \in \mathbb{R}$, the problem (3.3) has a solution α_0 such that $\overline{\alpha}_0 = 0$ and $\|\alpha_0\|_\infty \le ((b-a)/12)\|\phi\|_{L^1}$.

It remains to observe that, if we choose $c = \|\phi\|_{L^1}(b-a)/12 + R$ and α_0 the corresponding solution of (3.3),

$$\alpha(t) = \alpha_0(t) + c \geq R$$

is the desired lower solution. □

EXAMPLE 3.3. It is easy to see from Corollary 3.2 that the model example (3.1) has a solution if $\gamma \in L^1(a,b)$ is such that

$$0 \leq \gamma(t) \lneqq \left(\frac{\pi}{b-a}\right)^2, \qquad \int_a^b \gamma(t)\,dt > 0,$$

and $h \in L^1(a,b)$ is lower bounded.

In the following situation, we do not impose a lower bound on f near $+\infty$. We use in the sequel the notations $\overline{h} = (1/(b-a))\int_a^b h(t)\,dt$ and $\widetilde{h}(t) = h(t) - \overline{h}$.

COROLLARY 3.4. *Let $g \in \mathcal{C}(\mathbb{R}^+)$, $h \in L^1(a,b)$ and assume $f\colon [a,b] \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfies a Carathéodory condition together with*

(a) *for any $0 < r < s$, there exists a function $k \in L^1(a,b)$ such that, for a.e. $t \in [a,b]$ and all $u \in [r,s]$, we have $|f(t,u)| \leq k(t)$.*

Assume moreover that

(b) *for some $\beta > 0$ and a.e. $t \in [a,b]$, $f(t,\beta) - h(t) \leq 0$,*

(c) (Strong force) *there exist $\rho \in \,]0,\beta[$, $\ell \in L^1(a,b)$ and $\widehat{f} \in \mathcal{C}(]0,\rho])$ with*

$$\int_0^\rho \widehat{f}^-(u)\,du = +\infty \quad \text{and} \quad \widehat{f}^-(u) = \max\{-\widehat{f}(u), 0\},$$

such that for all $u \in \,]0,\rho]$ and a.e. $t \in [a,b]$

$$f(t,u) \leq \widehat{f}(u) \quad \text{and} \quad f(t,u) \leq \ell(t),$$

(d) *there exist $R > \beta$ and $f_0 \in L^1(a,b)$ such that, for a.e. $t \in [a,b]$ and all $u \in [R, R + ((b-a)/6)\|\widetilde{h}\|_{L^1}]$,*

$$f(t,u) \geq \overline{h},$$

(e) *there exists a function $\gamma \in L^1(a,b)$ such that $\gamma(t) \leq (\pi/(b-a))^2$ a.e. on $[a,b]$ with strict inequality on a subset of positive measure and*

$$\limsup_{u\to+\infty} \frac{f(t,u)}{u} \leq \gamma(t),$$

uniformly in $t \in [a,b]$.

Then the problem (1.1) *has at least one positive solution.*

Proof. We apply again Theorem 3.1. We only have to prove the existence of the lower and upper solutions. We first deduce from (b) that the constant function $\beta(t) = \beta$ is an upper solution for (1.1).

Next, we deduce a lower solution from condition (d) using the following argument. As in the proof of Corollary 3.2, we know that, for all $c \in \mathbb{R}$, the problem

$$\begin{aligned} u''(t) + g(|u(t) + c|)u'(t) &= \widetilde{h}(t), \\ u(a) = u(b), \quad u'(a) &= u'(b), \end{aligned} \tag{3.5}$$

has a solution $\alpha_0 \in W^{2,1}(a,b)$ such that $\overline{\alpha}_0 = 0$ and $\|\alpha_0\|_\infty \leq ((b-a)/12)\|\widetilde{h}\|_{L^1}$. Hence $\alpha(t) = R + ((b-a)/12)\|\widetilde{h}\|_{L^1} + \alpha_0(t)$ is the required lower solution. □

The following model equation

$$u'' - \frac{1}{u} = \frac{-1}{\sqrt{t-a}}, \quad u(a) = u(b), \quad u'(a) = u'(b)$$

is neither covered by Corollary 3.2 nor by Corollary 3.5 while Theorem 3.12 in [11] ensures the existence of at least one solution. The problem is that, as $h(t) = -1/\sqrt{t-a}$ is not bounded below, condition (b) of Corollaries 3.2 and 3.4 is not satisfied and we do not have, as above, a constant upper solution. The following result is the counterpart of Corollary 3.2 in case h is not bounded below.

Theorem 3.5. *Let $g \in \mathcal{C}(\mathbb{R}^+)$, $h \in L^1(a,b)$ and assume $f\colon [a,b] \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfies a Carathéodory condition together with*

(a) *for any $0 < r < s$, there exists a function $k \in L^1(a,b)$ such that, for a.e. $t \in [a,b]$ and all $u \in [r,s]$, we have $|f(t,u)| \leq k(t)$.*

Assume moreover that

(b) (Strong force) *there exist $\rho > 0$, $\ell \in L^1(a,b)$ and $\widehat{f} \in \mathcal{C}(]0,\rho])$ with*

$$\int_0^\rho \widehat{f}^-(u)\,du = +\infty \quad \text{and} \quad \widehat{f}^-(u) = \max(-\widehat{f}(u), 0),$$

such that for all $u \in\,]0,\rho]$ and a.e. $t \in [a,b]$

$$f(t,u) \leq \widehat{f}(u) \quad \text{and} \quad f(t,u) \leq \ell(t),$$

(c) $\int_a^b \ell(t)\,dt < \int_a^b h(t)\,dt$,

(d) *there exist $R > \rho$ and $f_0 \in L^1(a,b)$ such that, for a.e. $t \in [a,b]$, if $u \geq R$*

$$f(t,u) \geq f_0(t) \quad \text{and} \quad \int_a^b f_0(t)\,dt \geq \int_a^b h(t)\,dt,$$

(e) *there exists a function* $\gamma \in L^1(a,b)$ *such that* $\gamma(t) \le (\pi/(b-a))^2$ *a.e. on* $[a,b]$ *with strict inequality on a subset of positive measure and*

$$\limsup_{u\to+\infty} \frac{f(t,u)}{u} \le \gamma(t),$$

uniformly in $t \in [a,b]$.

Then the problem (1.1) *has at least one positive solution.*

PROOF. For any $0 < \delta \le \rho$, consider the modified problem

$$\begin{gathered} u'' + g(|u|)u' + f_\delta(t,u) = h(t), \\ u(a) = u(b), \quad u'(a) = u'(b), \end{gathered} \tag{3.6}$$

where

$$f_\delta(t,u) = \begin{cases} f(t,\delta) + u - \delta & \text{if } u \le \delta, \\ f(t,u) & \text{if } u > \delta. \end{cases}$$

Step 1. Existence of a solution of (3.6). As in the proof of Corollary 3.2, we deduce from (d) the existence of a lower solution $\alpha \ge R$ and we know that, for all $c \in \mathbb{R}$, the problem

$$\begin{gathered} w'' + g(|w+c|)w' = \widetilde{h}(t) - \widetilde{\ell}(t), \\ w(a) = w(b), \quad w'(a) = w'(b), \end{gathered}$$

has a solution w with $\overline{w} = 0$ and $\|w\|_\infty \le ((b-a)/12)\|\widetilde{h} - \widetilde{\ell}\|_{L^1}$. Hence, for every $0 < \delta \le \rho$, $\beta(t) = w(t) - ((b-a)/12)\|\widetilde{h} - \widetilde{\ell}\|_{L^1}$ is an upper solution of (3.6) and we can apply Theorem 2.4 to prove the existence of a solution of (3.6) in $\mathcal{S}$.

Step 2. A priori bounds. Repeating the arguments used to prove Claims 1 and 2 of Theorem 2.4 and Claim 1 of Theorem 3.1, we obtain $R > 0$ such that every solution $u \in \mathcal{S}$ of (3.6) with $0 < \delta \le \rho$ satisfies $\|u\|_{\mathcal{C}^1} \le R$. Now, integrating (3.6) on $[a,b]$, we deduce from (c) that $\max u \ge \rho$. Hence, as in Claim 1 of Theorem 3.1, we have $\xi \in \,]0,\rho]$ such that, any solution $u \in \mathcal{S}$ of (3.6) with $0 < \delta \le \rho$ satisfies $u(t) \ge \xi$ on $[a,b]$.

Conclusion. By Step 1, we know that problem (3.6) with $\delta = \xi$ has a solution in $\mathcal{S}$ and by Step 2, this solution solves (1.1). □

4. A problem without damping

In this section, we treat the case of a nonlinearity $f(t,u)$ which can be of the form $f(t,u) = g(t,u) + (\pi/(b-a))^2 u$ with g bounded from above in L^1. Hence the first resonant case is included. Moreover, the strong force condition becomes unnecessary. Here we consider the problem

$$\begin{gathered} u'' + f(t,u) = h(t), \\ u(a) = u(b), \quad u'(a) = u'(b). \end{gathered} \tag{4.1}$$

The following result which present an explicit existence condition on h, is an alternative to Corollary 3.2.

Theorem 4.1. *Let $h \in L^1(a,b)$ and assume $f\colon [a,b] \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfies a Carathéodory condition together with*

(a) *for any $0 < r < s$, there exists a function $k \in L^1(a,b)$ such that, for a.e. $t \in [a,b]$, and all $u \in [r,s]$, we have $|f(t,u)| \le k(t)$.*

Assume moreover that

(b) *for all $u > 0$ and a.e. $t \in [a,b]$,*
$$f(t,u) \le \left(\frac{\pi}{b-a}\right)^2 u,$$

(c) *there exist $R > 0$ and $f_0 \in L^1(a,b)$ such that, for a.e. $t \in [a,b]$, if $u \ge R$*
$$f(t,u) \ge f_0(t) \quad \text{and} \quad \int_a^b f_0(t)\,dt \ge \int_a^b h(t)\,dt,$$

(d) *there exists $\delta > 0$ so that*
$$\min_{t\in[a,b]} \int_t^{t+b-a} h(s) \sin \pi \frac{s-t}{b-a}\,ds \ge \delta.$$

Then the problem (4.1) *has at least one positive solution.*

Proof. The modified problem: For $r > \beta_0 := \delta(b-a)/2\pi$, we define the truncated function
$$f_r(t,u) = \begin{cases} f(t,\beta_0) + \left(\dfrac{\pi}{b-a}\right)^2 (u-\beta_0) & \text{if } u \le \beta_0, \\ f(t,u) & \text{if } \beta_0 < u \le r, \\ f(t,r) & \text{if } r < u, \end{cases}$$
and consider the modified problem
$$\begin{gathered} u'' + f_r(t,u) = h(t), \\ u(a) = u(b), \quad u'(a) = u'(b). \end{gathered} \tag{4.2}$$
The function $\beta(t) = w(t) - B$ is an upper solution of (4.2) if $B > 0$ is large enough and w solves the problem
$$\begin{gathered} w'' = h(t) - f(t,\beta_0) - \frac{1}{b-a}\int_a^b (h(s) - f(s,\beta_0))\,ds, \\ w(a) = w(b), \quad w'(a) = w'(b). \end{gathered}$$

Next, we deduce a lower solution as in the proof of Corollary 3.2. At last, we notice that for some $k \in L^1(a,b)$, all $u \in \mathbb{R}$ and a.e. $t \in [a,b]$,
$$f_r(t,u) \le \sup_{\beta_0 \le u \le r} f(t,u) \le k(t).$$

We deduce now from Theorem 2.4 the existence of a solution u of (4.2).

Claim 1. Solutions u of the modified problem (4.2) are such that $u(t) \geq \delta(b-a)/2\pi = \beta_0$ on $[a,b]$.

Let t_0 be such that $u(t_0) = \min_{t\in[a,b]} u(t)$. Multiplying equations (4.2) by $\sin\pi((t-t_0)/(b-a))$ and integrating on $[t_0, t_0+b-a]$, we obtain

$$\begin{aligned}\delta &\leq \int_{t_0}^{t_0+b-a} h(t)\sin\pi\frac{t-t_0}{b-a}\,dt = \int_{t_0}^{t_0+b-a}[u''(t)+f_r(t,u(t))]\sin\pi\frac{t-t_0}{b-a}\,dt \\ &\leq \int_{t_0}^{t_0+b-a}\left[u''(t)+\left(\frac{\pi}{b-a}\right)^2 u(t)\right]\sin\pi\frac{t-t_0}{b-a}\,dt = \frac{2\pi}{b-a}u(t_0).\end{aligned}$$

Claim 2. There exists $M>0$ so that for any $r>R$, solutions $u\in\mathcal{S}$ of (4.2) are such that $u\leq M$.

Let $k\in L^1(a,b)$ be so that for all $r>R$, $u\geq\beta_0$ and a.e. $t\in[a,b]$

$$h(t)-f_r(t,u)\leq h(t)-\min\{\min_{\beta_0\leq u\leq R} f(t,u), f_0(t)\}\leq k(t).$$

By periodicity, there exists $t_0\in[a,b]$ such that $u'(t_0)=0$ and, extending u by periodicity,

$$\begin{aligned}u'(t) &= \int_{t_0}^{t} u''(s)\,ds \leq \int_{t_0}^{t} k(s)\,ds \leq \|k\|_{L^1} \quad \text{for all } t\in[t_0,t_0+b-a],\\ u'(t) &= -\int_{t}^{t_0} u''(s)\,ds \geq -\|k\|_{L^1} \quad \text{for all } t\in[t_0-b+a,t_0],\end{aligned}$$

i.e. $\|u'\|_\infty\leq\|k\|_{L^1}$.

Next, as $u\in\mathcal{S}$, there exists some $\bar{t}\in[a,b]$ so that $u(\bar{t})\leq\|\alpha\|_\infty$. Hence, we can write

$$u(t)=u(\bar{t})+\int_{\bar{t}}^{t}u'(s)\,ds\leq\|\alpha\|_\infty+\|k\|_{L^1}(b-a)=:M.$$

Conclusion. It follows now from Claims 1 and 2 that problem (4.2) with $r\geq\max(M,R)$ has a solution $u\in[\beta_0,M]$. This solution solves the problem (4.1). □

Remark 4.2. We only need condition (b) satisfied for all $u\geq\delta(b-a)/2\pi$.

Example 4.3. Let $h\in L^1(a,b)$ and $\nu>0$. It is then easy to see from Theorem 4.1 that the problem

$$\begin{gathered}u''-\frac{1}{u^\nu}+\left(\frac{\pi}{b-a}\right)^2u=h(t),\\ u(a)=u(b),\quad u'(a)=u'(b)\end{gathered}$$

has at least one positive solution if h is bounded below by a positive constant as well as for

$$h(t) = -1 \quad \text{if } t \in [a, a+\varepsilon], \qquad h(t) = 1 \quad \text{if } t \in [a+\varepsilon, b],$$

if ε is small enough.

Notice also that, for $k > 0$, the existence of at least one solution of

$$\begin{gathered} u'' - \frac{k}{\sqrt{u}} = -h(t), \\ u(a) = u(b), \quad u'(a) = u'(b), \end{gathered}$$

can be deduced from Theorem 4.1 for h with $\|h^+\|_{L^1}$ small enough.

References

[1] D. Bonheure, C. Fabry and D. Smets, *Periodic solutions of forced isochronous oscillators at resonance*, Discrete Contin. Dynam. Systems **8** (2002), 907–930.

[2] C. De Coster and P. Habets, *Upper and lower solutions in the theory of ODE boundary value problems: classical and recent results*, Nonlinear Analysis and Boundary Value Problems for Ordinary Differential Equations (F. Zanolin, ed.), vol. 371, Springer, New York, C.I.S.M. Courses and Lectures, 1996, pp. 1–79.

[3] C. De Coster and P. Habets, *The lower and upper solutions method for boundary value problems*, Handbook of Differential Equations, Section: Ordinary Differential Equations (A. Cañada, P. Drabek and A. Fonda, eds.), Elsevier Science BV, North Holland (to appear).

[4] C. De Coster and M. Tarallo, *Foliations, associated reductions and lower-upper solutions*, Calc. Var. Partial Differential Equations **15** (2002), 25–44.

[5] M. A. del Pino and R. Manásevich, *Infinitely many T-periodic solutions for a problem arising in nonlinear elasticity*, J. Differential Equations **103** (1993), 260–277.

[6] M. A. del Pino, R. Manásevich and A. Montero, *T-periodic solutions for some second order differential equations with singularities*, Proc. Roy. Soc. Edinburgh Sect. A **120** (1992), 231–243.

[7] A. Fonda, *Periodic solutions of scalar second order differential equations with a singularity*, Mémoire de la Classe des Sciences Académie Royale de Belgique **8-IV** (1993).

[8] A. Fonda and J. Mawhin, *Quadratic forms, weighted eigenfunctions and boundary value problems for nonlinear second order ordinary differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **112** (1989), 145–153.

[9] J. P. Gossez, *Some nonlinear differential equations with resonance at the first eigenvalue*, Confer. Sem. Mat. Univ. Bari **167** (1979), 355–389.

[10] P. Habets and L. Sanchez, *Periodic solutions of some Lienard equations with singularities*, Proc. Amer. Math. Soc. **109** (1990), 1035–1044.

[11] A. C. Lazer and S. Solimini, *On periodic solutions of nonlinear differential equations with singularities*, Proc. Amer. Math. Soc. **99** (1987), 109–114.

[12] R. Martins, *Existence of periodic solutions for second-order differential equations and the strong force condition*, preprint (2000).

[13] P. Omari and W. Ye, *Necessary and sufficient conditions for the existence of periodic solutions of second-order ordinary differential equations with singular nonlinearities*, Differential Integral Equations **8** (1995), 1843–1858.

[14] I. RACHUNKOVÁ, M. TVRDÝ AND I. VRKOČ, *Existence of nonnegative and nonpositive solutions for second order periodic boundary value problems*, J. Differential Equations **176** (2001), 445–469.

[15] M. ZHANG, *Periodic solutions of Lienard equations with singular forces of repulsive type*, J. Math. Anal. Appl. **203** (1996), 254–269.

[16] M. ZHANG, *A relationship between the periodic and the Dirichlet BVPs of singular differential equations*, Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 1099–1114.

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