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PERIODIC SOLUTIONS OF A CLASS OF INTEGRAL EQUATIONS

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Abstract. Based on the fixed point index theory for a Banach space, nontrivial periodic solutions are found for a class of integral equation of the form

$$\phi(x) = \int_{[x,x+\omega]\cap\Omega} K(x,y)f(y,\phi(y-\tau(y)))\,dy, \quad x \in \Omega,$$

where Ω is a closed subset of $\mathbb{R}^N$ with perioidc structure.

Nonlinear Hammerstein integral equations of the form

$$\phi(x) = \int_{\Omega} K(x,y)f(y,\phi(y))\,dy$$

have been extensively studied under the assumptions that Ω is a bounded and closed subset of $\mathbb{R}^N$ with positive Lebesgue measure $\mu(\Omega)$, see e.g. [4], [5].

There are situations, however, where Ω is not fixed but depends on x. For instance, suppose we are concerned with the periodic solutions of the differential equation

$$\phi'(x) = -a(x)\phi(x) + f(\phi(x)), \quad x \in \mathbb{R}. \tag{1}$$

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Under the conditions that $a = a(x)$ is a positive continuous 2π-periodic function defined on $\mathbb{R}$, we may check that a 2π-periodic solution of

$$\phi(x) = \int_x^{x+2\pi} K(x,y) f(\phi(y))\, dy, \quad x \in \mathbb{R}, \tag{2}$$

where

$$K(x,y) = \frac{\exp \int_x^y a(t)\, dt}{\exp \int_0^\omega a(t)\, dt - 1}, \quad x, y \in \mathbb{R},$$

is also a 2π-periodic solution of (1), see e.g. [2], [3]. Therefore, it is desirable to study the equation (2).

More generally, let $\mathbb{R}^N$ be the N-dimensional Euclidean space endowed with componentwise ordering $\leq$. For any $u, v \in \mathbb{R}^N$, the "interval" $[u,v]$ is the set $\{x \in \mathbb{R}^N : u \leq x \leq v\}$. Let $\omega = (\omega_1, \ldots, \omega_N) \in \mathbb{R}^N$ with positive components and let $e^{(1)} = (1,0,\ldots,0), \ldots, e^{(N)} = (0,\ldots,0,1)$ be the standard orthonormal vectors in $\mathbb{R}^N$. Let Ω be a closed subset of $\mathbb{R}^N$ which has the following "periodic" structure:

$$x + \omega_i e^{(i)} \in \Omega \quad \text{for each } x \in \Omega$$

and, for each pair $y, z \in \Omega$,

$$\mu([y, y+\omega] \cap \Omega) = \mu([z, z+\omega] \cap \Omega) > 0.$$

A trivial example is $\Omega = \mathbb{R}$ with accompanying $\omega = 2\pi$. As a nontrivial example, Ω may be taken as

$$\{(x,y) \in \mathbb{R}^2 : 4n\pi \leq x, y \leq 4n\pi + 2\pi,\ n = 0, \pm 1, \pm 2, \ldots\} \tag{3}$$

with accompanying $\omega = (4\pi, 4\pi)$.

We will be concerned with integral equations of the form

$$\phi(x) = \int_{[x,x+\omega]\cap\Omega} K(x,y) f(y, \phi(y - \tau(y)))\, dy, \quad x \in \Omega, \tag{4}$$

where the functions K, f and τ satisfy the following conditions:

- $K \in C(\Omega \times \Omega, \mathbb{R}^+)$ and $K(x + \omega_i e^{(i)}, y + \omega_i e^{(i)}) = K(x,y)$ for any $(x,y) \in \Omega \times \Omega$ and $i \in \{1, \ldots, N\}$,
- $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ and $f(x + \omega_i e^{(i)}, u) = f(x,u)$ for $i \in \{1, \ldots, N\}$ and $x \in \Omega$,
- $\tau : \Omega \to \Omega$ is continuous and $\tau(x + \omega_i e^{(i)}) = \tau(x)$ for any $x \in \Omega$ and $i \in \{1, \ldots, N\}$.

As an example, let Ω be defined by (3) and let $a_1(t_1) = |\cos t_1|$, $a_2(t_2) = |\cos t_2|$, $\tau(t) = 0$, $f(x) = \sin x_1 \sin x_2$,

$$G_1(t_1, s_1) = \frac{\exp \int_{t_1}^{s_1} |\cos x_1|\, dx_1}{\exp \int_0^{2\pi} |\cos x_1|\, dx_1 - 1}$$

and

$$G_2(t_2, s_2) = \frac{\exp \int_{t_2}^{s_2} |\cos x_2| \, dx_2}{\exp \int_0^{2\pi} |\cos x_2| \, dx_2 - 1}.$$

Then the following equation

$$\begin{aligned} &\phi_1(t_1)\phi_2(t_2) \\ &= \iint_{[(t_1,t_2),(t_1+4\pi,t_2+4\pi)]\cap\Omega} G_1(t_1, s_1)G_2(t_2, s_2) \sin(\phi_1(s_1)) \sin(\phi_2(s_2)) \, ds_1 \, ds_2 \end{aligned}$$

is a special case of (4).

Our main concern will be the existence of periodic solutions of our equation (4). More precisely, we will look for solutions in the set of all real continuous functions of the form $\phi: \Omega \to \mathbb{R}$ such that $\phi(x + \omega_i e^{(i)}) = \phi(x)$ for $x \in \Omega$. This set will be denoted by $C(\Omega)$ in the sequel. Note that when endowed with the usual linear and ordering structure as well as the norm

$$\|\phi\| = \max_{z\in[x,x+\omega]\cap\Omega, x\in\Omega} |\phi(z)|,$$

$C(\Omega)$ is a normed ordered linear space with normal cone $P_0 = \{\phi \in C(\Omega) : \phi(x) \geq 0,\ x \in \Omega\}$. For the sake of convenience, we will use the norm $\|(\phi, \psi)\| = \max\{\|\phi\|, \|\psi\|\}$ for the naturally ordered product space $C(\Omega) \times C(\Omega)$. For the same reason, we will also set

$$\Omega(x) = [x, x + \omega] \cap \Omega.$$

Our proofs will involve the fixed point index, the basic properties of which are listed in the following lemma. A proof of this lemma based on the Leray–Schauder degree theory can be found in [1] and [4].

Lemma 1. *Let Q be a retract of a Banach space E. For every open subset U of Q and every completely continuous map $A: \overline{U} \to Q$ which has no fixed points on the boundary ∂U of U, there exists an integer $i(A, U, Q)$ satisfying:*

(a) *if $A: \overline{U} \to U$ is a constant map, then $i(A, U, Q) = 1$,*

(b) *if U_1 and U_2 are disjoint open subsets of U such that A has no fixed points on $U \setminus (U_1 \cup U_2)$, then $i(A, U, Q) = i(A, U_1, Q) + i(A, U_2, Q)$, where $i(A, U_k, Q) = i(A \setminus \overline{U}_k, U_k, Q)$ for $k = 1, 2$,*

(c) *if I is a compact interval in $\mathbb{R}$ and $h: I \times \overline{U} \to Q$ is a continuous map with relatively compact range such that $h(\lambda, x) \neq x$ for $(\lambda, x) \in I \times \partial U$, then $i(h(\lambda, \cdot), U, Q)$ is well-defined and independent of λ,*

(d) *if $i(A, U, Q) \neq 0$, then A has at least one fixed point in U,*

(e) *if Q_1 is a retract of Q and $A(\overline{U}) \subset Q_1$, then $i(A, U, Q) = i(A, U \cap Q_1, Q_1)$, where $i(A, U \cap Q_1, Q_1) = i(A \setminus \overline{U \cap Q_1}, U \cap Q_1, Q_1)$,*

(f) *if V is open in U and A has no fixed points in $U \setminus V$, then $i(A, U, Q) = i(A, V, Q)$.*

Theorem 2. *Suppose*

(H1) $K(x,y) \geq m > 0$ *for* $x, y \in \Omega(t)$ *and* $t \in \Omega$,

(H2) $f(x,u) = f_1(x,u) - f_2(x,u)$, *where* $f_i(x,u)$ *is nonnegative and continuous on* $\Omega \times \mathbb{R}$ *and* $f_i(x,0) = 0$ *for* $i = 1, 2$.

Suppose further that

$$\lim_{|u|\to 0} \frac{f_1(x,u)}{|u|} = \infty, \tag{5}$$

$$\limsup_{|u|\to 0} \frac{f_2(x,u)}{|u|} < \infty, \tag{6}$$

$$\lim_{u\to \infty} \frac{f_1(x,u)}{u} = 0, \tag{7}$$

$$\lim_{|u|\to \infty} \frac{f_2(x,u)}{|u|} = 0, \tag{8}$$

uniformly with respect to all $x \in \Omega$. *Then the integral equation* (4) *has at least one nontrivial periodic solution in* $C(\Omega)$.

Proof. Note that $M = \sup_{x,y\in\Omega(t),t\in\Omega} K(x,y) < \infty$. Thus, in view of (H1), $\widehat{c} = m/M > 0$. Furthermore, for any $x, y, z \in \Omega(t)$, we have

$$K(x,y) \geq \widehat{c}K(z,y). \tag{9}$$

Let $P = \{\phi \in C(\Omega) : \phi(x) \geq 0, \phi(x) \geq \widehat{c}\phi(z),\ \text{for all } x, z \in \Omega(t),\ t \in \Omega\}$. Then it is not difficult to check that P is a cone in $C(\Omega)$ and $P \times P$ is also a cone in $C(\Omega) \times C(\Omega)$. Let

$$\begin{aligned} A_1(\phi,\psi)(x) &= \int_{\Omega(x)} K(x,y) f_1(y, \phi(y-\tau(y)) - \psi(y-\tau(y)))\,dy, \\ A_2(\phi,\psi)(x) &= \int_{\Omega(x)} K(x,y) f_2(y, \phi(y-\tau(y)) - \psi(y-\tau(y)))\,dy, \end{aligned}$$

and $A(\phi,\psi)(x) = (A_1(\phi,\psi)(x), A_2(\phi,\psi)(x))$. Then it is easily seen that $A: P \times P \to C(\Omega) \times C(\Omega)$ is completely continuous. Furthermore, for any $x, z \in \Omega(t)$ where $t \in \Omega$,

$$\begin{aligned} A_i(\phi,\psi)(z) &= \int_{\Omega(z)} K(z,y) f_i(y, \phi(y-\tau(y)) - \psi(y-\tau(y)))\,dy \\ &\leq M \int_{\Omega(z)} f_i(y, \phi(y-\tau(y)) - \psi(y-\tau(y)))\,dy, \\ A_i(\phi,\psi)(x) &= \int_{\Omega(x)} K(x,y) f_i(y, \phi(y-\tau(y)) - \psi(y-\tau(y)))\,dy \\ &\geq m \int_{\Omega(x)} f_i(y, \phi(y-\tau(y)) - \psi(y-\tau(y)))\,dy \end{aligned}$$

$$= m \int_{\Omega(z)} f_i(y, \phi(y - \tau(y)) - \psi(y - \tau(y)))\, dy$$
$$\geq \widehat{c} A_i(\phi, \psi)(z)$$

for $i = 1, 2$. Thus A maps $P \times P$ into $P \times P$. From (6), there exist $\beta > 0$ and $r_1 > 0$ such that when $|u| \leq r_1$, we have

$$f_2(x, u) \leq \beta |u|, \quad x \in \Omega. \tag{10}$$

Let $0 < \varepsilon < \min\{1, \widehat{c}/(2 + 2M\beta\mu(\Omega(x)))\}$. Then when $(\phi, \psi) \in P \times P$, $\|(\phi, \psi)\| = r \leq r_1$ and $A_2(\phi, \psi) = \psi$, we have

$$\mu(\Omega_0) \geq \min\left\{\mu(\Omega(x)), \frac{\widehat{c}}{2M\beta}\right\}, \tag{11}$$

where

$$\Omega_0 = \{y \in \Omega(x) : |\phi(y - \tau(y)) - \psi(y - \tau(y))| \geq \varepsilon r\}.$$

Indeed, if $|\phi(y - \tau(y)) - \psi(y - \tau(y))| \geq \varepsilon r$ for any $y \in \Omega(x)$, then (11) is obvious. If there exists $x_1 \in \Omega(x)$ such that $|\phi(x_1 - \tau(x_1)) - \psi(x_1 - \tau(x_1))| < \varepsilon r$, then

$$\|\psi\| \geq \psi(x_1 - \tau(x_1)) > \phi(x_1 - \tau(x_1)) - \varepsilon r \geq \widehat{c}\|\phi\| - \varepsilon r,$$

hence, $\|\psi\| > (\widehat{c} - \varepsilon)r$. Suppose $\psi(x_2) = \|\psi\|$. Then in view of the fact that $A_2(\phi, \psi) = \psi$ and (10), we have

$$(\widehat{c} - \varepsilon)r \leq \psi(x_2) = \int_{\Omega(x_2)} K(x_2, y) f_2(y, \phi(y - \tau(y)) - \psi(y - \tau(y)))\, dy$$
$$\leq M\beta\left(\int_{\Omega_0} + \int_{\Omega(x_2)\setminus\Omega_0}\right)|\phi(y - \tau(y)) - \psi(y - \tau(y))|\, dy$$
$$\leq M\beta r(\mu(\Omega_0) + \varepsilon\mu(\Omega(x_2) \setminus \Omega_0)).$$

Hence, in view of the definition of ε and by a simple computation, $\mu(\Omega_0) \geq \widehat{c}/(2M\beta)$. Our assertion (11) thus holds.

Let $a = \min\{\mu(\Omega(x)), \widehat{c}/(2M\beta)\}$. Choose an α such that $\alpha \geq 1/ma\varepsilon$. In view of (5), there exists $r \leq r_1$ such that when $|u| \leq r$, we have

$$f_1(x, u) \geq \alpha |u|, \quad x \in \Omega. \tag{12}$$

Let $h(x) = \int_{\Omega(x)} K(x, y)\, dy$. Then $h \in P$. Furthermore, for any (ϕ, ψ) in

$$\partial(P \times P)_r = \{(\phi, \psi) \in P \times P : \|(\phi, \psi)\| = r\},$$

we have

$$(\phi, \psi) - A(\phi, \psi) \neq t(h, \theta), \quad t \geq 0. \tag{13}$$

Indeed, if there is $(\phi_0, \psi_0) \in \partial(P \times P)_r$ and $t_0 \geq 0$ such that $(\phi_0, \psi_0) - A(\phi_0, \psi_0) = t_0(h, \theta)$, then

$$\phi_0 - A_1(\phi_0, \psi_0) = t_0 h, \tag{14}$$

$$\psi_0 - A_2(\phi_0,\psi_0) = \theta. \tag{15}$$

If $t_0 = 0$, then (ϕ_0,ψ_0) is a fixed point of A. Thus, we suppose $t_0 > 0$. In view of (15), for above ε, (11) holds. From (14), we have $\phi_0 \geq t_0 h$.

Note that $t^* = \sup\{t : \phi_0 \geq th\} \geq t_0 > 0$. From (11), (12) and (14), we have

$$\begin{aligned}
\phi_0(x) &= t_0 h(x) + A_1(\phi_0,\psi_0)(x) \\
&= t_0 h(x) + \int_{\Omega(x)} K(x,y) f_1(y, \phi_0(y-\tau(y)) - \psi_0(y-\tau(y)))\,dy \\
&\geq t_0 h(x) + \int_{\Omega_0} K(x,y) f_1(y, \phi_0(y-\tau(y)) - \psi_0(y-\tau(y))\,dy \\
&\geq t_0 h(x) + \alpha \int_{\Omega_0} K(x,y) |\phi_0(y-\tau(y)) - \psi_0(y-\tau(y))|\,dy \\
&\geq t_0 h(x) + m\alpha\varepsilon r\mu(\Omega_0) \geq t_0 h(x) + ma\alpha\varepsilon t^* h(x) \geq (t_0 + t^*)h(x),
\end{aligned}$$

which is a contradiction. Thus (13) holds. Therefore (see e.g. [1], [4])

$$i(A, (P\times P)_r, P\times P) = 0. \tag{16}$$

Next, we will prove that there is $R > 0$ such that when $(\phi,\psi) \in \partial(P\times P)_R$,

$$A(\phi,\psi) \ngeq (\phi,\psi). \tag{17}$$

Indeed, choose c satisfying $0 < c < \widehat{c}/(M\mu(\Omega(x)))$. In view of (7) and (8), we see that there exists R_0 such that when $u \geq R_0$ and $|v| \geq R_0$, we have

$$f_1(x,u) \leq cu, \quad f_2(x,v) \leq c|v| \quad \text{for all } x \in \Omega.$$

Let

$$T_0 = \max\Big\{ \sup_{0\leq u\leq R_0, x\in\Omega} f_1(x,u), \sup_{0\leq |v|\leq R_0, x\in\Omega} f_2(x,v) \Big\}.$$

Then for any $u \geq 0$, $v \in \mathbb{R}$ and $x \in \Omega$,

$$f_1(x,u) \leq cu + T_0, \tag{18}$$

$$f_2(x,v) \leq c|v| + T_0. \tag{19}$$

Choose $R > \max\{r, R_0, MT_0\mu(\Omega(x))/(\widehat{c} - cM\mu(\Omega(x)))\}$. Then (17) will be satisfied for $(\phi,\psi) \in \partial(P\times P)_R$. Indeed, when $\|(\phi,\psi)\| = R$, if $\phi(x) \geq \psi(x)$ for any $x \in \Omega$, then from (18), we have

$$\begin{aligned}
A_1(\phi,\psi)(x) &= \int_{\Omega(x)} K(x,y) f_1(y, \phi(y-\tau(y)) - \psi(y-\tau(y)))\,dy \\
&\leq \int_{\Omega(x)} K(x,y)[c(\phi(y-\tau(y)) - \psi(y-\tau(y))) + T_0]\,dy \\
&\leq MRc\mu(\Omega(x)) + MT_0\mu(\Omega(x)) < R = \|\phi\|.
\end{aligned}$$

Thus, $A_1(\phi,\psi) \ngeq \phi$ and consequently $A(\phi,\psi) \ngeq (\phi,\psi)$. If there exists $x_0 \in \Omega$ such that $\phi(x_0) < \psi(x_0)$, then $\|\psi\| \geq \widehat{c}R$, and consequently from (19), we have

$$\begin{aligned} A_2(\phi,\psi)(x) &= \int_{\Omega(x)} K(x,y) f_2(y, \phi(y-\tau(y)) - \psi(y-\tau(y)))\,dy \\ &\leq \int_{\Omega(x)} K(x,y)[c|\phi(y-\tau(y)) - \psi(y-\tau(y))| + T_0]\,dy \\ &\leq MRc\mu(\Omega(x)) + mT_0\mu(\Omega(x)) \leq \widehat{c}R \leq \|\psi\|. \end{aligned}$$

Thus $A_2(\phi,\psi) \ngeq \psi$ and consequently $A(\phi,\psi) \ngeq (\phi,\psi)$. From (17) we have

$$i(A,(P\times P)_R, P\times P) = 1. \tag{20}$$

From (16) and (20), we have

$$i(A,(P\times P)_R \setminus (P\times P)_r, P\times P) = 1.$$

Thus by Lemma 1(d), there exists $(\phi^*,\psi^*) \in (P\times P)_R \setminus (P\times P)_r$ such that $A(\phi^*,\psi^*) = (\phi^*,\psi^*)$, i.e.

$$\begin{aligned} \phi^*(x) &= \int_{\Omega(x)} K(x,y) f_1(y, \phi^*(y-\tau(y)) - \psi^*(y-\tau(y)))\,dy, \\ \psi^*(x) &= \int_{\Omega(x)} K(x,y) f_2(y, \phi^*(y-\tau(y)) - \psi^*(y-\tau(y)))\,dy. \end{aligned}$$

Finally, from the assumption that $f_1(x,0) = f_2(x,0) = 0$ for all $x \in \Omega$, we know that $\phi^* \neq \psi^*$. (Indeed, if $\phi^* = \psi^*$, then $\phi^* = \psi^* = 0$, which is contrary to the fact that $(\phi^*,\psi^*) \in (P\times P)_R \setminus (P\times P)_r$). This shows that $\phi^* - \psi^*$ is a nontrivial periodic solution of (4) in $C(\Omega)$. The proof is complete. □

As a nontrivial example, consider the first-order functional differential equation

$$y'(t) = -a(t)y(t) + h(t)f(y(t-\tau(t))) \tag{21}$$

where $a = a(t)$, $h = h(t)$ and $\tau = \tau(t)$ are continuous T-periodic functions. We assume that $T > 0$, that $h = h(t)$ are nonnegative, that $\int_0^T a(t)\,dt > 0$ and $f = f(t)$ is continuous function satisfying $f(0) = 0$. Then it is easily checked that any T-periodic function $y(t)$ that satisfies the following integral equation

$$y(t) = \int_t^{t+T} G(t,s)h(s)f(y(s-\tau(s)))\,ds \tag{22}$$

where

$$G(t,s) = \frac{\exp\int_t^s a(u)du}{\exp\int_0^T a(u)\,du - 1}, \quad s,t \in \mathbb{R}.$$

is also a T-periodic solution of (21). Note that

$$G(t,s) \geq \min_{0\leq s,t\leq T} \frac{\exp \int_t^s a(u)\,du}{\exp \int_0^T a(u)\,du - 1} = m > 0, \quad |s-t| \leq T,$$

and $f(u) = f_1(u) - f_2(u)$ where f_1 and f_2 are nonnegative and continuous functions satisfying $f_1(0) = f_2(0) = 0$. Thus by Theorem 2, we may assert that if

$$\lim_{|u|\to 0} \frac{f_1(u)}{|u|} = \infty, \qquad \limsup_{|u|\to 0} \frac{f_2(u)}{|u|} < \infty,$$
$$\lim_{u\to\infty} \frac{f_1(u)}{u} = 0, \qquad \lim_{|u|\to\infty} \frac{f_2(u)}{u} = 0,$$

then equation (21) has at least one nontrivial T-periodic solution.

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