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A THREE CRITICAL POINTS THEOREM AND ITS APPLICATIONS TO THE ORDINARY DIRICHLET PROBLEM

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Abstract. The aim of this paper is twofold. On one hand we establish a three critical points theorem for functionals depending on a real parameter $\lambda \in \Lambda$, which is different from the one proved by B. Ricceri in [15] and gives an estimate of where Λ can be located. On the other hand, as an application of the previous result, we prove an existence theorem of three classical solutions for a two-point boundary value problem which is independent from the one by J. Henderson and H. B. Thompson ([10]). Specifically, an example is given where the key assumption of [10] fails. Nevertheless, the existence of three solutions can still be deduced using our theorem.

1. Introduction

Recently, B. Ricceri established a very interesting three critical points result ([15, Theorem 1]), that we recall in an equivalent formulation (see [3, Theorem 2.3 and Remark 2.2]):

Theorem A. *Let X be a separable and reflexive real Banach space, $\Phi\colon X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*,*

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$\Psi: X \to \mathbb{R}$ *a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that:*

(i) $\lim_{\|x\| \to +\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty$ *for all* $\lambda \in [0, +\infty[$,

(ii) *there are* $r \in \mathbb{R}$, $x_0, x_1 \in X$ *such that:*

$$\Phi(x_0) < r < \Phi(x_1),$$
$$\inf_{x \in \Phi^{-1}(]-\infty, r])} \Psi(x) > \frac{(\Phi(x_1) - r)\Psi(x_0) + (r - \Phi(x_0))\Psi(x_1)}{\Phi(x_1) - \Phi(x_0)}.$$

Then, there exists an open interval $\Lambda \subseteq]0, +\infty[$ *and a positive real number* q *such that, for each* $\lambda \in \Lambda$, *the equation*

$$\Phi'(x) + \lambda\Psi'(x) = 0 \tag{1.1}$$

has at least three solutions in X *whose norms are less than* q.

Applications of Theorem A to nonlinear boundary value problems have been given in [2]–[4], [6]–[8], [12], [15], (see also [13] for the non smooth case), establishing multiplicity results for equations depending on a parameter λ.

We note that Theorem A gives no estimate of where Λ can be located in $]0, +\infty[$. Very recently, another three critical point theorem was established (Theorem 2.1 of [5]), which provides an upper bound for Λ.

The aim of this paper is to establish some theorems ensuring the existence of at least three solutions for the equation (1.1) for each λ in an explicitly determined interval.

The main result of Section 2 is Theorem 2.1. Its proof is based on the variational principle of B. Ricceri ([16], see also [17]) and on the mountain pass theorem as expressed by P. Pucci and J. Serrin in [14].

The following is a particular case of Theorem 2.1.

Theorem B. *Let* X *be a reflexive real Banach space,* Φ, Ψ *be as in Theorem* A, *and assume that* (i) *of Theorem* A *holds. Further put*

$$\varphi_1(r) := \inf_{x \in \Phi^{-1}(]-\infty, r[)} \frac{\Psi(x) - \inf_{\overline{\Phi^{-1}(]-\infty, r[)}^w} \Psi}{r - \Phi(x)},$$
$$\varphi_2(r) := \inf_{x \in \Phi^{-1}(]-\infty, r[)} \sup_{y \in \Phi^{-1}([r, +\infty[)} \frac{\Psi(x) - \Psi(y)}{\Phi(y) - \Phi(x)},$$

for each $r > \inf_X \Phi$, *where* $\overline{\Phi^{-1}(]-\infty, r[)}^w$ *is the closure of* $\Phi^{-1}(]-\infty, r[)$ *in the weak topology, and assume that*

(ii$'$) *there is* $r \in \mathbb{R}$ *such that* $\inf_X \Phi < r$ *and* $\varphi_1(r) < \varphi_2(r)$.

Then, for each $\lambda \in]1/\varphi_2(r), 1/\varphi_1(r)[$, *the equation* (1.1) *has at least three solutions in* X.

However, $\varphi_1(r)$ in Theorem B could be 0 (see Theorem 2.2). In this and similar cases, here and in the sequel, we agree to read $1/0$ as $+\infty$.

In Theorem B, the separability of X is not required. Moreover, hypotheses (ii) and (ii′) in Theorems A and B respectively seem to be different. Theorem B gives a lower bound for Λ, whereas Theorem A assures the stability of the three solutions with respect to λ, namely the uniform boundedness of norms of solutions.

In Section 3, as an application of Theorem B and its consequences, we study the following ordinary autonomuous Dirichlet problems

$$\text{(ADE)} \qquad \begin{cases} -u'' = \lambda f(u), \\ u(0) = u(1) = 0, \end{cases}$$

and

$$\text{(AD)} \qquad \begin{cases} -u'' = f(u), \\ u(0) = u(1) = 0, \end{cases}$$

establishing the existence of three classical solutions under a suitable set of assumptions (see Theorems 3.1, 3.5 and 3.9).

Multiple solutions to the above mentioned problems have been obtained by several authors using different techniques. We refer to [2] and the references therein for problem (ADE) and to [1], [10], [11] for problem (AD).

In [2] (see also [3], [5]), using critical points theorems and set-valued analysis arguments, a λ-uniform norm-boundedness of the three solutions to problem (ADE) was established under assumptions which are very similar to ours (see Remark 3.2).

In the very interesting work [10], J. Henderson and H. B. Thompson ensured the existence of at least three solutions by using a method of lower and upper solutions. It is worth to note that their key assumption, which we recall in Remark 3.4, fails in examples where, on the contrary, we can apply our Theorem 3.1 (see Example 3.8).

The main result of Section 3 is Theorem 3.1. Here are two particular cases of it.

Theorem C. *Let $f:\mathbb{R}\to\mathbb{R}$ be a nonnegative and bounded continuous function such that*

$$2\int_0^{1/2} f(\xi)\,d\xi < 1 < \frac{1}{12}\int_0^1 f(\xi)\,d\xi.$$

Then, the problem (AD) *has at least three classical solutions.*

Theorem D. *Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function with $f(0)=0$ and $f(x)\le 0$ in a right neighbourhood of* 0*, and such that*

$$\lim_{x\to+\infty}\frac{f(x)}{x^q}\in\,]0,+\infty[\quad \textit{for some } q\in\,]0,1[.$$

Then, there exists a positive real number λ^* such that, for each $\lambda>\lambda^*$, the problem (ADE) *has at least two nontrivial and nonnegative classical solutions.*

2. Critical points theorems

In this section we establish some three critical points theorems for a suitable class of functionals depending on a real parameter λ. The main result is Theorem 2.1. As its consequences we obtain Theorem B given in Introduction, and Theorem 2.2.

Theorem 2.1. *Let X be a reflexive real Banach space, and let $\Phi,\Psi: X\to\mathbb{R}$ be two sequentially weakly lower semicontinuous functionals. Assume also that Φ is* (*strongly*) *continuous, satisfies $\lim_{\|x\|\to+\infty}\Phi(x)=+\infty$ and, for each $\lambda>0$, the functional $\Phi+\lambda\Psi$ is continuously Gâteaux differentiable, bounded below, and satisfies the Palais–Smale condition. Further, assume that there exists $r>\inf_X\Phi$ such that, given φ_1 and φ_2 as in Theorem* B, *$\varphi_1(r)<\varphi_2(r)$. Then, for each $\lambda\in\,]1/\varphi_2(r),1/\varphi_1(r)[$, the functional $\Phi+\lambda\Psi$ has at least three critical points.*

Proof. Fix $\lambda\in\,]1/\varphi_2(r),1/\varphi_1(r)[$ and consider the functional $\Psi+(1/\lambda)\Phi$. Since $1/\lambda>\varphi_1(r)$, thanks to Theorem 5 of [17], the functional $\Psi+(1/\lambda)\Phi$ has a local minimum, say x_0, which lies in $\Phi^{-1}(]-\infty,r[)$.

Moreover, from $1/\lambda<\varphi_2(r)$ we have that for every $x\in\Phi^{-1}(]-\infty,r[)$ there exists $y\in\Phi^{-1}([r,+\infty[)$ such that

$$\Psi(y)+\frac{1}{\lambda}\Phi(y)<\Psi(x)+\frac{1}{\lambda}\Phi(x),$$

hence x_0 is not a global minimum for $\Psi+(1/\lambda)\Phi$ in X.

On the other hand, by Theorem 38.F of [18], $\Psi+(1/\lambda)\Phi$ admits a global minimum, say x_1, in X.

Then, by Corollary 1 of [14], the functional $\Psi+(1/\lambda)\Phi$ admits a third critical point distinct from x_0 and x_1.

Of course, also the functional $\Phi+\lambda\Psi$ has the same three distinct critical points. □

Now, we give the proof of Theorem B stated in the Introduction.

Proof of Theorem B. The compactness of Ψ' implies that Ψ is sequentially weakly continuous ([18, Corollary 41.9]). Moreover, $\Phi+\lambda\Psi$ satisfies the Palais–Smale condition (see, for instance, Example 38.25 of [18]) and is bounded below. □

THEOREM 2.2. *Let X, Φ, Ψ be as in Theorem* B *and assume that* (i) *of Theorem* A *holds. Further, assume that there are $r \in \mathbb{R}$, x_0, $x_1 \in X$ such that*

(j) $\Phi(x_0) < r < \Phi(x_1)$,

(jj) $\inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi = \Psi(x_0) > \Psi(x_1)$.

Then, for each $\lambda \in](\Phi(x_1) - \inf_{\Phi^{-1}(]-\infty,r[)} \Phi)/(\Psi(x_0) - \Psi(x_1)), +\infty[$, the functional $\Phi + \lambda\Psi$ has at least three critical points.

PROOF. Thanks to our assumptions, we have

$$\varphi_1(r) = 0 \quad \text{and} \quad \varphi_2(r) \geq \frac{\Psi(x_0) - \Psi(x_1)}{\Phi(x_1) - \inf_{\Phi^{-1}(]-\infty,r[)} \Phi} > 0.$$

Thus, the conclusion follows by Theorem B. $\square$

3. Applications to the ordinary Dirichlet problem

In this section, we apply Theorem 2.1 and its consequences to the Dirichlet problems (ADE) and (AD).

Let us assume $f\colon \mathbb{R} \to \mathbb{R}$ continuous and put

$$g(\xi) := \int_0^\xi f(t)\,dt, \quad \text{for } \xi \in \mathbb{R}.$$

The main result of this section is the following

THEOREM 3.1. *Assume that there exist four positive constants c, d, a, s, with $c < d$ and $s < 2$, such that:*

(k) $\dfrac{\max_{|\xi|\leq c} g(\xi)}{c^2} < \dfrac{1}{4}\dfrac{g(d) + (1/d)\int_0^d g(t)dt - 2\max_{|\xi|\leq c} g(\xi)}{d^2}$,

(kk) $g(\xi) \leq a(1 + |\xi|^s)$ *for all* $\xi \in \mathbb{R}$.

Then, for each

$$\lambda \in \left] \frac{8d^2}{g(d) + (1/d)\int_0^d g(t)\,dt - 2\max_{|\xi|\leq c} g(\xi)}, \frac{2c^2}{\max_{|\xi|\leq c} g(\xi)} \right[,$$

the problem (ADE) *admits at least three classical solutions.*

PROOF. Let X be the Sobolev space $W_0^{1,2}([0,1])$ endowed with the norm

$$\|u\| := \left(\int_0^1 |u'(t)|^2\,dt \right)^{1/2}.$$

For each $u \in X$ put

$$\Phi(u) := \frac{1}{2}\|u\|^2, \quad \Psi(u) := -\int_0^1 g(u(t))\,dt.$$

It is well known that the critical points in X of the functional $\Phi+\lambda\Psi$ are precisely the classical solutions of problem (ADE). So, our end is to apply Theorem B to Φ and Ψ.

Clearly, Φ and Ψ are as in Theorem A. Furthermore, thanks to (kk) and to Hölder inequality, we have

$$\lim_{\|u\|\to+\infty}(\Phi(u)+\lambda\Psi(u))=+\infty \quad \text{for all } \lambda\in[0,+\infty[.$$

In order to prove (ii′) of Theorem B, we claim that:

$$\varphi_1(r)\le\frac{\max_{|\xi|\le\sqrt{r/2}}g(\xi)}{r} \tag{C1}$$

for each $r>0$, and

$$\varphi_2(r)\ge 2\frac{\int_0^1 g(y(t))\,dt-\max_{|\xi|\le\sqrt{r/2}}g(\xi)}{\|y\|^2} \tag{C2}$$

for each $r>0$ and every $y\in X$ such that

$$\frac{1}{2}\|y\|^2\ge r \quad \text{and} \quad \int_0^1 g(y(t))\,dt\ge\max_{|\xi|\le\sqrt{r/2}}g(\xi).$$

In fact, for $r>0$, taking into account that $\overline{\Phi^{-1}(]-\infty,r[)}^{w}=\Phi^{-1}(]-\infty,r])$, we have

$$\varphi_1(r)\le\frac{\sup_{\|x\|^2/2\le r}\int_0^1 g(x(t))\,dt}{r}.$$

Thus, since $\max_{t\in[0,1]}|x(t)|\le\|x\|/2$ for every $x\in X$, we obtain

$$\frac{\sup_{\|x\|^2/2\le r}\int_0^1 g(x(t))\,dt}{r}\le\frac{\max_{|\xi|\le\sqrt{r/2}}g(\xi)}{r}.$$

So, (C1) is proved.

Moreover, for each $r>0$ and each $y\in X$ such that $\|y\|^2/2\ge r$, we have

$$\varphi_2(r)\ge\inf_{\|x\|^2/2<r}\frac{\int_0^1 g(y(t))\,dt-\int_0^1 g(x(t))\,dt}{\|y\|^2/2-\|x\|^2/2},$$

thus, since $\max_{t\in[0,1]}|x(t)|\le\|x\|/2$ for every $x\in X$, we obtain

$$\begin{aligned}\inf_{\|x\|^2/2<r}&\frac{\int_0^1 g(y(t))\,dt-\int_0^1 g(x(t))\,dt}{\|y\|^2/2-\|x\|^2/2}\\ &\ge 2\inf_{\|x\|^2/2<r}\frac{\int_0^1 g(y(t))\,dt-\max_{|\xi|\le\sqrt{r/2}}g(\xi)}{\|y\|^2-\|x\|^2},\end{aligned}$$

from which, being $0 < \|y\|^2 - \|x\|^2 \leq \|y\|^2$ for every $x \in X$ such that $\|x\|^2/2 < r$, and under further condition

$$\int_0^1 g(y(t))\,dt \geq \max_{|\xi| \leq \sqrt{r/2}} g(\xi),$$

we can write

$$2 \inf_{\|x\|^2/2 < r} \frac{\int_0^1 g(y(t))\,dt - \max_{|\xi| \leq \sqrt{r/2}} g(\xi)}{\|y\|^2 - \|x\|^2} \geq 2 \frac{\int_0^1 g(y(t))\,dt - \max_{|\xi| \leq \sqrt{r/2}} g(\xi)}{\|y\|^2}.$$

So, (C2) is also proved.

Now, in order to prove (ii′) of Theorem B, taking into account (C1) and (C2), it suffices to find $r > 0$ and $y \in X$ such that

$$\frac{1}{2}\|y\|^2 \geq r, \qquad \int_0^1 g(y(t))\,dt \geq \max_{|\xi| \leq \sqrt{r/2}} g(\xi),$$

and

$$\frac{\max_{|\xi| \leq \sqrt{r/2}} g(\xi)}{r} < 2 \frac{\int_0^1 g(y(t))\,dt - \max_{|\xi| \leq \sqrt{r/2}} g(\xi)}{\|y\|^2}. \tag{3.1}$$

To this end, we define

$$y(t) := \begin{cases} 4\,dt & \text{if } t \in [0, 1/4[, \\ d & \text{if } t \in [1/4, 3/4], \\ 4\,d(1-t) & \text{if } t \in]3/4, 1], \end{cases}$$

and $r := 2c^2$. Clearly, $y \in X$ and $\|y\|^2 = 8d^2$. Hence, since $c < d$, we have $\|y\|^2/2 > r$. Moreover, we have

$$\int_0^1 g(y(t))\,dt = \frac{1}{2} g(d) + \frac{1}{2d} \int_0^d g(t)\,dt,$$

so that

$$2 \frac{\int_0^1 g(y(t))\,dt - \max_{|\xi| \leq \sqrt{r/2}} g(\xi)}{\|y\|^2} = \frac{g(d) + (1/d) \int_0^d g(t)\,dt - 2 \max_{|\xi| \leq c} g(\xi)}{8d^2},$$

hence hypothesis (k) gives (3.1) and $\int_0^1 g(y(t))\,dt > \max_{|\xi| \leq \sqrt{r/2}} g(\xi)$.

Thus, the conclusion follows by Theorem B, taking into account that, writing (C1) and (C2) with the $y(t)$ and r defined above,

$$\frac{1}{\varphi_2(r)} \leq \frac{4d^2}{(1/2) g(d) + (1/(2d)) \int_0^d g(t)\,dt - \max_{|\xi| \leq c} g(\xi)}$$

and

$$\frac{1}{\varphi_1(r)} \geq \frac{2c^2}{\max_{|\xi| \leq c} g(\xi)}. \qquad \square$$

Remark 3.2. In Theorem 3.1 instead of hypothesis (k) we can also use the following less general, but simpler:

(k_1) $\int_0^d g(\xi)\,d\xi \geq 0$,

(k_2) $g(\xi)/c^2 < g(d)/(6d^2)$, for every $\xi \in [-c,c]$.

In fact, taking into account that $0 < c < d$, using (k_1) and (k_2) we obtain

$$\frac{1}{6}\frac{g(d)}{d^2} < \frac{1}{4}\frac{g(d)}{d^2} - \frac{1}{2}\frac{\max_{|\xi|\leq c} g(\xi)}{d^2} \leq \frac{1}{4}\frac{g(d) + (1/d)\int_0^d g(t)\,dt - 2\max_{|\xi|\leq c} g(\xi)}{d^2}$$

thus, using again (k_2), hypothesis (k) of Theorem 3.1 is fulfilled.

We observe that the assumptions (k_1) and (k_2) are very similar to those of Theorem 2 of [2] (see Remark 2 of [2]) and Theorem 3.1 of [5]. Here we have a precise estimate of the interval of parameters for which the problem has at least three solutions, while in those theorems the uniform boundedness of the norms of the solutions with respect to λ is obtained.

Remark 3.3. In Theorem 3.1 the assumption (kk), together with (k), ensures the third solution and cannot be dropped as the function $f(u) = e^u$ shows (see [9]).

Also the assumption (k) cannot be dropped as the function $f(u) = 1$ shows (see also Remark 3.7).

We now give a simple example of application of Theorem 3.1.

Example 3.4. It is simple to verify that the function

$$g(u) = e^{-u}u^{11} + \frac{3}{5}(u+1)^{5/3} - \frac{3}{5},$$

besides (kk) of Theorem 3.1, satisfies (k_1) and (k_2) of Remark 3.2 by choosing, for instance, $c = 1$ and $d = 2$; moreover, we have

$$\left]\frac{1}{8}, \frac{11}{10}\right[\subseteq \left]\frac{8d^2}{g(d) + (1/d)\int_0^d g(t)\,dt - 2\max_{|\xi|\leq c} g(\xi)}, \frac{2c^2}{\max_{|\xi|\leq c} g(\xi)}\right[.$$

Therefore, thanks to Theorem 3.1, for each $\lambda \in \,]1/8, 11/10[$, the problem

$$\begin{cases} -u'' = \lambda(e^{-u}u^{10}(11-u) + \sqrt[3]{(u+1)^2}), \\ u(0) = u(1) = 0, \end{cases}$$

admits at least three non trivial classical solutions.

An immediate consequence of Theorem 3.1 is the following

Theorem 3.5. *Assume that there exist four positive constants c, d, a, s, with $c < d$ and $s < 2$, such that:*

(k') $\displaystyle\frac{\max_{|\xi|\leq c} g(\xi)}{c^2} < 2 < \frac{1}{4}\frac{g(d) + (1/d)\int_0^d g(t)\,dt - 2\max_{|\xi|\leq c} g(\xi)}{d^2}$,

(kk) $g(\xi) \leq a(1+|\xi|^s)$ *for all* $\xi \in \mathbb{R}$.

Then, the problem (AD) *admits at least three classical solutions.*

Proof. It is clear that Theorem 3.1 can be used. So, it is enough to observe that, owing to (k′), we have

$$1 \in \left] \frac{8d^2}{g(d) + (1/d)\int_0^d g(t)\,dt - 2\max_{|\xi|\le c} g(\xi)}, \frac{2c^2}{\max_{|\xi|\le c} g(\xi)} \right[. \qquad \square$$

Remark 3.6. On the basis of Remark 3.2, in Theorem 3.5 instead of hypothesis (k′) we can use the following simpler:

(k_1) $\int_0^d g(\xi)\,d\xi \ge 0$,
(k_2') $g(\xi)/c^2 < 2 < g(d)/(6d^2)$, for every $\xi \in [-c, c]$.

Proof of Theorem C. Taking into account Remark 3.6, we can choose $c = 1/2$, $d = 1$ and apply Theorem 3.5. $\square$

Remark 3.7. Problem (AD) has been studied, for instance, in [1], [10] and [11]. The key assumption in [10] is (see (iii) in Theorem 2 of [11])

(HT) there exist $b > 0$ and $0 < e < 1/2$ such that $f(y) \ge 2b/(e(1-2e))$ for every $y \in [b, b(2e+1)/(4e)]$,

and the authors give an example (see Remark 7 of [10]) where (HT) fails and the problem has only the trivial solution. The following example shows a problem that admits at least two positive classical solutions even if the assumption (HT) is not verified.

Example 3.8. Let $h\colon \mathbb{R} \to \mathbb{R}$ be the function defined as follows

$$h(\xi) := \begin{cases} 0 & \text{if } \xi \in\,]-\infty, 1/5], \\ 20\xi - 4 & \text{if } \xi \in\,]1/5, 1], \\ -20\xi + 36 & \text{if } \xi \in\,]1, 9/5], \\ 0 & \text{if } \xi \in\,]9/5, +\infty[. \end{cases}$$

By choosing, for instance, $c = 1/5$ and $d = 1$, it is simple to verify all the assumptions of Theorem 3.5. So, taking into account that h is nonnegative and vanishes at 0, from the maximum principle the problem

$$\begin{cases} -u'' = h(u), \\ u(0) = u(1) = 0, \end{cases}$$

admits at least two positive classical solutions. On the other hand, the assumption (HT) fails, as it is simple to see.

As application of Theorem 2.2 we give the following

Theorem 3.9. *Assume that there exist four positive constants c, d, a, s, with $c < d$ and $s < 2$, such that:*

(k_1') $\int_0^d g(\xi)\,d\xi > 0$,
(k_2') $\max_{|\xi|\le c} g(\xi) = 0$,
(kk) $g(\xi) \le a(1+|\xi|^s)$ *for all* $\xi \in \mathbb{R}$.

Then, for each $\lambda \in \,]2d^3/\int_0^d g(t)\,dt, +\infty[$, *the problem* (ADE) *admits at least two nontrivial and nonnegative classical solutions.*

Proof. Since assumption (k_2') implies that $f(0) = 0$, it is not restrictive to suppose that $f(0) = 0$ for $x < 0$. Clearly, the solutions of the problem (ADE) with such an f are nonnegative and they are also solutions of the problem (ADE) with the original one.

Now, let X, Φ, Ψ be as in proof of Theorem 3.1, and define

$$x_1(t) := \begin{cases} 2dt & \text{if } t \in [0,1/2], \\ 2d(1-t) & \text{if } t \in \,]1/2,1], \end{cases}$$

$x_0(t) := 0$ for every $t \in [0,1]$, and $r := 2c^2$.

Clearly, we have $\Phi(x_0) = 0$, $\Psi(x_0) = 0$, $\Phi(x_1) = 2d^2$ and $-\Psi(x_1) = (1/d)\int_0^d g(t)\,dt$.

Since $c < d$, one has that $\Phi(x_0) < r < \Phi(x_1)$. Moreover, taking into account that $\max_{t\in[0,1]} |x(t)| \le \|x\|/2$ for every $x \in X$, we have $-\Psi(x) \le \max_{|\xi|\le c} g(\xi) = 0$ for every $x \in X$ such that $\Phi(x) \le r$. Then,

$$\inf_{\overline{\Phi^{-1}(]-\infty,r[)}^w} \Psi = \inf_{\Phi^{-1}(]-\infty,r])} \Psi = \Psi(x_0),$$

and, thanks to (k_1'), $\Psi(x_1) < \Psi(x_0)$. Hence, using Theorem 2.2, since

$$\frac{\Phi(x_1) - \inf_{\Phi^{-1}(]-\infty,r[)} \Phi}{\Psi(x_0) - \Psi(x_1)} = \frac{\Phi(x_1)}{-\Psi(x_1)} \frac{2d^3}{\int_0^d g(t)\,dt},$$

we have the conclusion. □

Proof of Theorem D. As in the proof of Theorem 3.9, we can suppose $f(x) = 0$ for $x < 0$. Clearly, there exists $c > 0$ such that $\max_{|t|\le c} g(t) = 0$. Moreover, since $\lim_{x\to+\infty} f(x)/x^q \in \,]0,+\infty[$, there exists $d > c$ such that $\int_0^d g(\xi)\,d\xi > 0$, and there exists $a > 0$ such that $g(\xi) \le a(1+|\xi|^{1+q})$ for all $\xi \in \mathbb{R}$. Therefore, we can use Theorem 3.9 to reach the conclusion. □

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