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ON EXACT TOPOLOGICAL FLOWS

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Abstract. It is shown that group endomorphisms are exact flows if and only if they are exact in the measure-theoretic sense and that all flows which are exact with respect to an invariant measure with full support are exact. It is also proved that all locally eventually dense (led) flows have uniformly positive entropy (u.p.e.).

1. Introduction

The concept of an exact topological flow, an analogue of the well known measure-theoretic concept defined by Rokhlin ([5]), was introduced in [3]. It has been also observed that all locally eventually onto (leo) flows are exact. An important class of such flows is formed by rational mappings R with $\deg R \geq 2$ of the Riemann sphere restricted to Julia set (cf. [2]).

In this paper we show that some important classes of topological flows are exact. Namely, we prove that the group endomorphisms are exact if and only if they are exact with respect to the Haar measure. Next we show that any flow exact with respect to an invariant measure with full support is exact. The existence of exact flows with zero topological entropy has already been observed in [3]. In Theorem 3 we show that led, and so leo flows have u.p.e.

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2. Results

Let (X,d) be a compact metric space. We denote by $C(X)$ the algebra of all real continuous functions on X. Its subalgebra of all constant functions will be denoted by $\mathcal{C}$.

By $CER(X)$ we denote the set of all closed equivalence relations in $X \times X$. Let $(R_n)_0^\infty$ be a sequence of relations from $CER(X)$. We denote by $\mathcal{A}_n$ the subalgebra of $C(X)$ consisiting of functions constant on equivalence classes of R_n, $n \geq 0$. Let R_∞ denote the smallest closed equivalence relation containing all R_n, $n \geq 0$. By $\mathcal{B}_{R_n}$ we denote the σ-algebra of Borel sets which are unions of equivalence classes of R_n.

Let $T\colon X \to X$ be continuous and onto. For the notation and basic properties concerning topological entropy we refer the reader to [6].

Let us recall (cf. [1]) that an open cover $\alpha = (U,V)$ of X is said to be *standard* if U and V are non-dense.

A flow (X,T) has *uniformly positive entropy* (u.p.e.) if for any standard cover α of X the entropy $h(\alpha,T)$ is positive.

Now we recall the concept of an exact measure-theoretic flow (cf. [5]).

A measure preserving transformation T of a Lebesgue probability space $(X,\mathcal{B},\mu)$ is said to be *exact* with respect to μ if the σ-algebra $\bigcap_{n=0}^\infty T^{-n}\mathcal{B}$ is trivial mod μ.

Recall (cf. [3]) that a flow (X,T) is said to be *exact* if

$$\overline{\bigcup_{n=0}^{\infty} (T\times T)^{-n}(\Delta)} = X \times X.$$

It follows from Example 4 (cf. [3]) that the exactness does not imply the exactness in the measure-theoretic sense.

We say that a flow (X,T) is *led* (*leo*) if for any open set $U \neq \emptyset$ there exists a positive integer n with $\overline{T^nU} = X$ ($T^nU = X$). Clearly any led flow is exact.

Lemma. *If $R_n \in CER(X)$ and $\bigcup_{n=0}^\infty R_n$ is dense in $X\times X$ then*

$$\bigcap_{n=0}^{\infty} \mathcal{A}_n = \mathcal{C}.$$

Proof. Let $f \in \bigcap_{n=0}^\infty \mathcal{A}_n$, $x,y \in X$ and $\varepsilon > 0$ be arbitrary. Let $\delta > 0$ be such that $|f(u) - f(v)| < \varepsilon/2$ for all $u,v \in X$ with $d(u,v) < \delta$.

By the assumption there exist $n \geq 0$ and $(x_\delta, y_\delta) \in R_n$ such that $d(x,x_\delta) < \delta$ and $d(y,y_\delta) < \delta$. Since $f \in \mathcal{A}_n$ we have $f(x_\delta) = f(y_\delta)$ and so

$$|f(x) - f(y)| \leq |f(x) - f(x_\delta)| + |f(y) - f(y_\delta)| < \varepsilon,$$

i.e. $f \in \mathcal{C}$. $\square$

We are not able to decide whether the converse implication is true. Theorem 1 will show that this is the case for group endomorphisms and $R_n = (T \times T)^{-n}(\Delta)$, $n \geq 0$.

Notice that the following is true.

Remark. If $\bigcap_{n=0}^{\infty} \mathcal{A}_n = \mathcal{C}$ then $R_\infty = X \times X$.

Proof. Suppose there exists $(x_0, y_0) \in X \times X$ with $(x_0, y_0) \notin R_\infty$, i.e. the equivalence classes $[x_0]$, $[y_0]$ of R_∞ are different. By the Urysohn lemma there exists a function $f \in C(X)$ constant on the equivalence classes of R_∞ such that $f(x_0) \neq f(y_0)$.

Let $n \geq 0$ be arbitrary. It is clear that f is constant on the equivalence classes of R_n, i.e. $f \in \mathcal{A}_n$. Hence $f \in \bigcap_{n=0}^{\infty} \mathcal{A}_n$ which gives the result. $\square$

Let G be a compact topological abelian group with dual group $\widehat{G}$. Let T be a continuous algebraic endomorphism of G. We denote by $\widehat{T}$ the dual transformation of $\widehat{G}$: $\widehat{T}f = f \circ T$ for $f \in \widehat{G}$.

Theorem 1. *The following conditions are equivalent:*

(a) (G, T) *is exact,*

(b) (G, T) *is exact with respect to the Haar measure,*

(c) $\bigcap_{n=0}^{\infty} \mathcal{A}_n = \mathcal{C}$, *where* $\mathcal{A}_n$ *is associated with* $R_n = (T \times T)^{-n}(\Delta)$, $n \geq 0$.

Proof. (a)⇔(b) Assume that (G, T) is exact with respect to the Haar measure. Using Lemma 2 of [4] we know that it is equivalent to the fact that the set $\bigcup_{n=1}^{\infty} T^{-n}\{g\}$ is dense in G for all $g \in G$. In particular for the identity element $e \in G$ it implies that the set $\bigcup_{n=0}^{\infty} (T \times T)^{-n}\{(e, e)\}$ is dense in $G \times G$ which completes the first part of the proof.

Let us now suppose (G, T) is exact. We put $R_n = (T \times T)^{-n}(\Delta)$, $n \geq 0$.

It follows from the assumption that the set $\bigcup_{n=0}^{\infty} R_n$ is dense in $G \times G$. Hence Lemma gives $\bigcap_{n=0}^{\infty} \mathcal{A}_n = \mathcal{C}$ where $\mathcal{A}_n \subset C(G)$ consists of all functions constant on equivalence classes of R_n, $n \geq 0$.

Now we want to show that $\bigcap_{n=0}^{\infty} \widehat{T}^n \widehat{G} = \{1\}$.

We have

$$\mathcal{A}_n = \widehat{T}^n \mathcal{A}_0 = \widehat{T}^n C(G), \quad n \geq 0.$$

Since

$$\bigcap_{n=0}^{\infty} \widehat{T}^n \widehat{G} \subset \bigcap_{n=0}^{\infty} \widehat{T}^n C(G) = \bigcap_{n=0}^{\infty} \mathcal{A}_n = \mathcal{C},$$

we have $\bigcap_{n=0}^{\infty} \widehat{T}^n \widehat{G} = \{1\}$. Therefore (G, T) is exact with respect to the Haar measure by Lemma 2 of [4].

(a)⇔(c) The implication (a)⇒(c) follows at once from Lemma. The converse implication is an easy consequence of Remark and the observation that

$\overline{\bigcup_{n=0}^{\infty}(T\times T)^{-n}(\Delta)}$ is an equivalence relation since it is reflexive, symmetric and a subgroup of $G\times G$. □

In the next theorem we investigate the exactness of flows which are exact with respect to invariant measures with full support.

Theorem 2. *If (X,T) is exact with respect to an invariant measure with full support then (X,T) is exact.*

Proof. Suppose (X,T) is not exact, i.e. there exist $(x_0,y_0)\in X\times X$, open sets U_i,V_i, $i=0,1$ with $(x_0,y_0)\in U_1\times V_1$, $\overline{U_1}\subset U_0$, $\overline{V_1}\subset V_0$, $\overline{U_1}\cap\overline{V_1}=\emptyset$ and

$$U_0\times V_0\cap\bigcup_{n=0}^{\infty}R_n=\emptyset$$

where $R_n=(T\times T)^{-n}(\Delta)$, $n\geq 0$.

For a given $x\in X$ we denote by $[x]_n$ its equivalence class of R_n.

Let $\pi_n\colon X\to X/R_n$ be the quotient map. We put

$$\begin{aligned}
g_n(x)&=\inf\{d(u,v):u\in\pi_n^{-1}[x]_n,\ v\in\overline{U_1}\},\\
h_n(x)&=\inf\{d(u,v):u\in\pi_n^{-1}[x]_n,\ v\in\overline{V_1}\},\\
f_n(x)&=\frac{g_n(x)}{g_n(x)+h_n(x)}\ n\geq 0,\quad x\in X.
\end{aligned}$$

It is clear that for any $n\geq 0$ we have $0\leq f_n\leq 1$ and the function f_n is constant on equivalence classes of R_n. Let

$$f(x)=\overline{\lim_{n\to\infty}}f_n(x),\quad x\in X$$

and let μ be an invariant probability measure with full support.

Since the measurability of f with respect to $\bigcap_{n=0}^{\infty}\mathcal{B}_{R_n}$ is obvious, in order to get our result it is enough to check that f is not constant μ-a.e. We put

$$D=\{(x,y)\in X\times X:f(x)\neq f(y)\}.$$

Because $f_{|U_1}=0$, $f_{|V_1}=1$, we have $U_1\times V_1\subset D$. Therefore

$$(\mu\times\mu)(D)\geq(\mu\times\mu)(U_1\times V_1)=\mu(U_1)\cdot\mu(V_1)>0$$

and so f is not constant μ-a.e. □

As we mentioned above exact flows may have zero entropy. Our next goal is to show that if they are led flows then they have u.p.e.

The proof of the following theorem in the case of leo flows is a result of a discussion with B. Weiss. E. Glasner observed that its slight modification allows to extend it to the case of led flows.

Theorem 3. *If (X,T) is led then it has uniformly positive entropy.*

Proof. Let $\alpha = (U_0, U_1)$ be a standard cover of X, i.e. $\overline{U_0} \neq X \neq \overline{U_1}$, and let $V_0 = X \setminus \overline{U_1}$ and $V_1 = X \setminus \overline{U_0}$.

By assumption there exists a positive integer N with $T^N V_0$ and $T^N V_1$ dense in X. It is enough to show that $h(T^N, \alpha) > 0$.

Let $n \geq 1$ be arbitrary. First we show that for any $\eta = (\eta_0, \dots, \eta_{n-1}) \in \{0,1\}^n$ it holds

$$(*) \qquad V_{\eta_0} \cap T^{-N} V_{\eta_1} \cap \ldots \cap T^{-(n-1)N} V_{\eta_{n-1}} \neq \emptyset.$$

Set $W_0 = V_{\eta_0}$ and notice that, as $T^N W_0$ is dense in X, $W_1 = T^N W_0 \cap V_{\eta_1}$ is dense in V_{η_1}. Inductively define

$$\begin{aligned} W_1 &= T^N W_0 \cap V_{\eta_1}, \quad W_2 = T^N W_1 \cap V_{\eta_2}, \ \ldots, \\ W_{k+1} &= T^N W_k \cap V_{\eta_{k+1}}, \ldots, \\ W_{n-1} &= T^N W_{n-2} \cap V_{\eta_{n-1}}. \end{aligned}$$

Choose $x_{n-1} \in W_{n-1}$ and then inductively a sequence $\{x_k\}$ so that $x_k \in W_k$ and $x_k = T^N x_{k-1}$, $k = n-1, \ldots, 0$. Then

$$x_0 \in V_{\eta_0} \cap T^{-N} V_{\eta_1} \cap \ldots \cap T^{-(n-1)N} V_{\eta_{n-1}} \neq \emptyset.$$

Since $V_0 \cap V_1 = \emptyset$ we have

$$\bigcap_{k=0}^{n-1} T^{-kN} V_{\eta_k} \cap \bigcap_{k=0}^{n-1} T^{-kN} V_{\eta'_k} = \emptyset$$

for $\eta = (\eta_0, \dots, \eta_{n-1}) \neq \eta' = (\eta'_0, \dots, \eta'_{n-1})$. Now we consider the cover $\beta = (P_0, P_1)$ defined by $P_0 = X \setminus \overline{V_0}$, $P_1 = X \setminus \overline{V_1}$. Since $\beta < \alpha$ it is enough to show that $h(T^N, \beta) > 0$.

To show this we consider the cover $\bigvee_{j=0}^{n-1} T^{-jN} \beta$. This cover is minimal in the following sense. If we remove any set from it, it stops being a cover.

Indeed, let $\eta = (\eta_0, \dots, \eta_{n-1}) \in \{0,1\}^n$. We show that the family

$$\gamma = \bigvee_{j=0}^{n-1} T^{-jN} \beta \setminus \{P_{\eta_0} \cap T^{-N} P_{\eta_1} \cap \ldots \cap T^{-(n-1)N} P_{\eta_{n-1}}\}$$

is not a cover of X.

Let $\widetilde{\eta} = (\widetilde{\eta}_0, \dots, \widetilde{\eta}_{n-1})$ where $\widetilde{\eta}_i = 1 - \eta_i$, $0 \leq i \leq n-1$. It follows from $(*)$ that $V_{\widetilde{\eta}_0} \cap T^{-N} V_{\widetilde{\eta}_1} \cap \ldots \cap T^{-(n-1)N} V_{\widetilde{\eta}_{n-1}} \neq \emptyset$.

Let $x \in V_{\widetilde{\eta}_0} \cap T^{-N} V_{\widetilde{\eta}_1} \cap \ldots \cap T^{-(n-1)N} V_{\widetilde{\eta}_{n-1}}$. Since $V_0 \subset P_1$, $V_1 \subset P_0$ we have

$$V_{\widetilde{\eta}_0} \cap T^{-N} V_{\widetilde{\eta}_1} \cap \ldots \cap T^{-(n-1)N} V_{\widetilde{\eta}_{n-1}} \subset P_{\eta_0} \cap T^{-N} P_{\eta_1} \cap \ldots \cap T^{-(n-1)N} P_{\eta_{n-1}}.$$

We claim that

$$x \notin \bigcup_{(\eta'_0,\dots,\eta'_{n-1})\neq(\eta_0,\dots,\eta_{n-1})} P_{\eta'_0} \cap T^{-N} P_{\eta'_1} \cap \dots \cap T^{-(n-1)N} P_{\eta'_{n-1}}.$$

Suppose that $x \in P_{\eta'_0} \cap T^{-N} P_{\eta'_1} \cap \dots \cap T^{-(n-1)N} P_{\eta'_{n-1}}$ for some $(\eta'_0,\dots,\eta'_{n-1}) \neq (\eta_0,\dots,\eta_{n-1})$. Hence $\eta'_i \neq \eta_i$ for some $0 \le i \le n-1$ and so $T^{iN}x \in P_{\eta'_i}$ and $T^{iN}x \in V_{\eta'_i}$. This is impossible because $P_{\eta'_i} \cap V_{\eta'_i} = \emptyset$ for any $0 \le i \le n-1$. Therefore γ is not a cover.

By the use of minimality of $\bigvee_{j=0}^{n-1} T^{-jN}\beta$ we get

$$N\bigg(\bigvee_{j=0}^{n-1} T^{-jN}\beta\bigg) = \#\bigvee_{j=0}^{n-1} T^{-jN}\beta = 2^n.$$

Hence

$$h(T^N,\beta) = \lim_{n\to\infty} \frac{1}{n} H\bigg(\bigvee_{j=0}^{n-1} T^{-jN}\beta\bigg) = \log 2 > 0$$

which gives us the desired result. □

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