OBSTRUCTIONS TO THE EXTENSION PROBLEM
OF SOBOLEV MAPPINGS

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ABSTRACT. Let M and N be compact manifolds with $\partial M \neq \emptyset$. We show that when $1 < p < \dim M$, there are two different obstructions to extending a map in $W^{1-1/p,p}(\partial M, N)$ to a map in $W^{1,p}(M, N)$. We characterize one of these obstructions which is topological in nature. We also give properties of the other obstruction. For some cases, we give a characterization of $f \in W^{1-1/p,p}(\partial M, N)$ which has an extension $F \in W^{1,p}(M, N)$

1. Introduction

Let M and N be two compact connected Riemannian manifolds of $\dim M = m$ and $\dim N = n$. We assume M has a smooth boundary and $\partial N = \emptyset$. However, our argument also applies to the case $\partial N \neq \emptyset$ under a suitable assumption. To define the Sobolev spaces defined between manifolds, it is convenient to assume that N is isometrically imbedded in some Euclidean space $\mathbb{R}^k$. Indeed, by Nash ([15]), this is always satisfied for some large k. For $1 < p$ we define

$$W^{1,p}(M, N) := \{f \in W^{1,p}(M, \mathbb{R}^k) : f(x) \in N \text{ for a.e. } x \in M\}.$$

For the basic properties of this space, see [2], [12], [13], [18] and [19].

Recall that there is a well-defined continuous surjective linear map, called the *trace operator*, $\gamma \colon W^{1,p}(M, \mathbb{R}^k) \to W^{1-1/p,p}(\partial M, \mathbb{R}^k)$ such that for $g \in$

$C^1(M, \mathbb{R}^k)$, $\gamma g = g|_{\partial M}$ (see [1]). Here the *trace space* $W^{1-1/p,p}(\partial M, \mathbb{R}^k)$ is the subspace of $L^p(\partial M, \mathbb{R}^k)$ with a norm

$$\|f\|_{1-1/p,p} := \|f\|_{L^p(\partial M)} + \left\{ \int_{\partial M} \int_{\partial M} \frac{|f(x) - f(y)|^p}{d(x,y)^{p+m-2}} \, d\mu(x) \, d\mu(y) \right\}^{1/p},$$

where $d(\cdot, \cdot)$ is the distance function defined by the Riemannian structure of ∂M and $d\mu$ is the Riemannian density associated to the Riemannian metric on ∂M. We are concerned with the trace space defined between manifolds, $W^{1-1/p,p}(\partial M, N)$, defined as

$$W^{1-1/p,p}(\partial M, N) := \{f \in W^{1-1/p,p}(\partial M, \mathbb{R}^k) : f(x) \in N \text{ for a.e. } x \in \partial M\}.$$

One natural question, first treated systematically in [4], is the following:

QUESTION. For $f \in W^{1-1/p,p}(\partial M, N)$, is there a function F in $W^{1,p}(M, N)$ with $\gamma F = f$?

For $N = \mathbb{R}^k$, this question is, of course, solved positively for any $f \in W^{1-1/p,p}(\partial M, \mathbb{R}^k)$. For $p > \dim M$, by using the Sobolev imbedding theorem $W^{1-1/p,p}(\partial M, N) \hookrightarrow C(\partial M, N)$, one can easily prove that f has an $W^{1,p}(M, N)$-extension if and only if f can be extended to M as a continuous map $M \to N$ (see [4, Theorem 1]). Less obvious case is the limiting case $p = \dim M$. In this case, Bethuel and Demengel proved that any $f \in W^{1-1/p,p}(\partial M, N)$ has a $W^{1,p}(M, N)$-extension if and only if any continuous map from ∂M to N has a continuous extension $M \to N$ (see [4, Theorem 2]). Their argument in fact proves the following:

THEOREM 1.1 (Bethuel–Demengel). *Assume $p \geq \dim M$. Let f be a map in $W^{1-1/p,p}(\partial M, N)$. Recall that f has a well-defined homotopy class which is defined by the homotopy class of $g \in W^{1-1/p,p}(\partial M, N) \cap C(\partial M, N)$ with $\|g - f\|_{1-1/p,p}$ sufficiently small (see [3, Lemma 1]). We denote the homotopy class of f by $[f]$. Then f has a $W^{1,p}(M, N)$-extension if and only if any $h \in C(\partial M, N)$ with $h \in [f]$ has a continuous extension $M \to N$.*

The problem is much more complex when $1 < p < \dim M$. Also in this case, for some M and N, there exists $f \in W^{1-1/p,p}(\partial M, N)$ such that the above question is negative for f, that is, there is no $F \in W^{1,p}(M, N)$ with $\gamma F = f$. The first example is given by Hardt and Lin (see [14]). They considered the case $W^{1/2,2}(\mathbb{B}^3, \mathbb{S}^1)$ and showed that the map $f(x) = x'/|x'|$ ($\mathbb{B}^3$ is the unit ball in $\mathbb{R}^3$, $x = (x', x_3) \in \mathbb{B}^3$ and $|x'|$ is the Euclidean norm of $x' \in \mathbb{R}^2$) is in $W^{1/2.2}(\partial \mathbb{B}^3, \mathbb{S}^1)$, but it has no $W^{1,2}(\mathbb{B}^3, \mathbb{S}^1)$-extension (see Section 3 for more detailed discussion including this case). Later in [4], Bethuel and Demengel proved that the above question is negative for some $f \in W^{1-1/p,p}(\partial M, N)$ provided $\pi_{[p]-1}(N) \neq 0$ (and any M), or $\pi_j(N) \neq 0$ for some $j \leq [p] - 1$ (and for a suitable M), or $M = \mathbb{B}^m$, $N = \mathbb{S}^1$ and $3 \leq p < m$ ($\mathbb{B}^m$ is the unit ball in $\mathbb{R}^m$) (see also Section 3). The only

known positive answer for the case $1 < p < \dim M$ is due to Hardt and Lin (see [14, Theorem 6.2]). They proved that if $\pi_1(N) = \ldots = \pi_{[p]-1}(N) = 0$, then the question is positively solved for any $f \in W^{1-1/p,p}(\partial M, N)$ (M is arbitrary). Note that the case $p = 1$ is a special case. In this case, the trace space of $W^{1,1}(M, \mathbb{R}^k)$ is $L^1(\partial M, \mathbb{R}^k)$ and it is shown in [4] that $\gamma W^{1,1}(M, N) = L^1(\partial M, N)$. Note also that White ([18]) considered the case $f \in \mathrm{Lip}(\partial M, N)$, however, little is known for the general case about the above question.

Our main purpose of this paper is to clarify the following: For a given $f \in W^{1-1/p,p}(\partial M, N)$, what is the obstruction to extending f to M as a $W^{1,p}(M, N)$-map?

At least to the author's knowledge, it is not well understood. Our first observation is that, contrary to the case $p \geq \dim M$, there are two different kinds of obstructions to this problem. Roughly speaking, they are defined as follows: The first obstruction, we denote it by $\mathfrak{o}_A(f)$, is the obstruction to extending f to the collar neighbourhood of ∂M and the other, we denote it by $\mathfrak{o}_B(f)$, is the obstruction to extending the map defined in the collar to the whole M. (However, we note here that $\mathfrak{o}_B(f)$ will be defined whether f is extended to the collar or not, see below). We will see that these behave very differently. In this paper, we completely characterize the second type of obstruction $\mathfrak{o}_B(f)$. The other obstruction $\mathfrak{o}_A(f)$ is defined (see below) and some of its properties are studied. We also give, for some cases, a characterization of the space $T^p(\partial M, N) := \{f \in W^{1-1/p,p}(\partial M, N) : \exists F \in W^{1,p}(M, N), \; \gamma F = f\}$. (We use the notation introduced in [4]).

To state our results, we introduce some notations. We denote by $\mathcal{C}(\partial M)$ the collar neighbourhood of ∂M in M. It is a neighbourhood of ∂M in M diffeomorphic to $[0, 2) \times \partial M$. It is unique up to diffeomorphism. Let $\varphi \colon \mathcal{C}(\partial M) \to [0, 2) \times \partial M$ be a diffeomorphism. We denote by $\mathcal{C}_t(\partial M) := \varphi^{-1}([0, t) \times \partial M)$ for $t \in [0, 2]$. For $u \in W^{1,p}(\mathcal{C}_t(\partial M), \mathbb{R}^k)$, define

$$(1.1) \qquad E_{\varepsilon,t}(u) = \int_{\mathcal{C}_t(\partial M)} |\nabla u|^p + \frac{1}{\varepsilon} d_0(u(x), N)^p \, |dx|,$$

where d_0 is the Euclidean distance in $\mathbb{R}^k$ and $|dx|$ is the Riemannian density of M. $E_{\varepsilon,t}$ is an approximation of p-Dirichlet energy defined in $W^{1,p}(\mathcal{C}_t(\partial M), N)$.

For $f \in W^{1-1/p,p}(\partial M, N)$, define

$$(1.2) \qquad \mathfrak{o}_A(f) = \lim_{t \downarrow 0} \lim_{\varepsilon \downarrow 0} \mathcal{E}_{\varepsilon,t}(f),$$

where $\mathcal{E}_{\varepsilon,t}(f) := \inf\{E_{\varepsilon,t}(u) : u \in W^{1,p}(\mathcal{C}_t(\partial M), \mathbb{R}^k) \text{ with } \gamma u = f \text{ on } \partial M\}$.

Note that $\mathfrak{o}_A(f)$ is well defined for all $f \in W^{1-1/p,p}(\partial M, N)$ and $\mathfrak{o}_A(f) \in [0, \infty]$. In fact, in Section 2, we show that $\mathfrak{o}_A(f) = 0$ or $\mathfrak{o}_A(f) = \infty$ for any $f \in W^{1-1/p,p}(\partial M, N)$. Also note that it only depends on ∂M (not on M),

N and f. (In fact, we will show that $\mathfrak{o}_A(f)$ depends only on f and the topologies of ∂M and N.)

In this paper, we denote by M^k the k-skeleton of M with respect to some CW-structure of M. Throughout this paper, we consider $(M, \partial M)$ as a relative CW-complex, thus M is obtained by attaching cells to ∂M. Then $(\partial M)^k$ is defined by $(\partial M)^k := \partial M \cap \bigcup\{\partial e^l : \text{for all } l\text{-cell } e^l \text{ of } M \text{ with } l \leq k+1\}$. For our purpose, the "cubeulation" of M given in [18] (see also [2]) is sufficient.

We first give a result which characterizes maps in $T^p(\partial M, N)$ in terms of $\mathfrak{o}_A(f)$ and $\mathfrak{o}_B(f)$ ($\mathfrak{o}_B(f)$ is defined below):

THEOREM 1.2. *Assume $1 < p < \dim M$ and $p \notin \mathbb{Z}$. Let f be given map in $W^{1-1/p,p}(\partial M, N)$. There exists $F \in W^{1,p}(M, N)$ satisfying $\gamma F = f$ if and only if the following conditions hold:*

(a) $\mathfrak{o}_A(f) = 0$.

(b) $\mathfrak{o}_B(f)$: $f|_{(\partial M)^{[p]-1}} \in C((\partial M)^{[p]-1}, N)$ *has a continuous extension to $M^{[p]}$ for a generic relative pair $(M^{[p]}, (\partial M)^{[p]-1})$, that is, there exists $u_f \in C(M^{[p]}, N)$ such that $u_f|_{(\partial M)^{[p]-1}} = f|_{(\partial M)^{[p]-1}}$.*

The case $p \in \mathbb{Z}$ is treated in the following theorem.

THEOREM 1.3. *Assume $1 < p < \dim M$ and $p \in \mathbb{Z}$. Let f be given map in $W^{1-1/p,p}(\partial M, N)$. There exists $F \in W^{1,p}(M, N)$ satisfying $\gamma F = f$ if and only if the following conditions hold:*

(a) $\mathfrak{o}_A(f) = 0$.

(b) $\mathfrak{o}_{B'}(f)$: *For a generic relative pair $(M^p, (\partial M)^{p-1})$, there exists a sequence $\{f_i\} \subset \text{Lip}((\partial M)^{p-1}, N)$ such that $f_i \to f$ in $W^{1-1/p,p}((\partial M)^{p-1}, N)$ and f_i ($i \geq 1$) has a Lipschitz extension F_i to M^p, that is, $F_i \in \text{Lip}(M^p, N)$ and $F_i|_{(\partial M)^{p-1}} = f_i$.*

It is easy to show that $\mathfrak{o}_A(f) = 0$ if and only if f is extended in some neighbourhood of ∂M (see proof of Lemma 2.1). As noted before, the above definition shows that $\mathfrak{o}_B(f)$ is defined whether $\mathfrak{o}_A(f) = 0$ or not (i.e. for any $f \in W^{1-1/p,p}(\partial M, N)$). The obstructions $\mathfrak{o}_B(f)$ and $\mathfrak{o}_{B'}(f)$ are purely topological, that is, they only depend on the topologies of the pair $(M^{[p]}, (\partial M)^{[p]-1})$, N and the homotopy class of $f|_{(\partial M)^{[p]-1}} : (\partial M)^{[p]-1} \to N$ (in the case $p \notin \mathbb{Z}$. In the case $p \in \mathbb{Z}$, we take the homotopy class of $f \in W^{1-1/p,p}((\partial M)^{p-1}, N)$ as the generalized sense as in [3, Lemma 1]). Thus by the above theorems, once f is extended to a neighbourhood of ∂M, there is only a topological obstruction to extending f to M. Sufficient conditions for $\mathfrak{o}_B(f)$ and $\mathfrak{o}_{B'}(f)$ are given by the vanishing of cohomology groups $H^{k+1}(M^{[p]}, (\partial M)^{[p]-1}; \pi_k(N)) = 0$ ($0 \leq k \leq [p]-1$) or $H^{k+1}(M, (\partial M)^{[p]-1}; \pi_k(N)) = 0$ ($0 \leq k \leq [p]-2$), $\pi_{[p]-1}(N) = 0$, see Section 3.

On the other hand, it turns out that the structure of $\mathfrak{o}_A(f)$ is more complicated. The following result gives a simple condition for f implying $\mathfrak{o}_A(f) = 0$ (for the definition of $\mathrm{VMO}(\partial M)$, see [9] and [10]):

PROPOSITION 1.4. *Let* $1 < p < \dim M$. *Assume* $f \in W^{1,p}(\partial M, N)$ *or* $f \in W^{1-1/p,p}(\partial M, N) \cap \mathrm{VMO}(\partial M)$. *Then* $\mathfrak{o}_A(f) = 0$.

COROLLARY 1.5. *Let* $1 < p < \dim M$. *Assume*

$$H^{k+1}(M^{[p]}, (\partial M)^{[p]-1}; \pi_k(N)) = 0 \quad for\ 0 \le k \le [p] - 1,$$
$$or \quad H^{k+1}(M, (\partial M)^{[p]-1}; \pi_k(N)) = 0 \quad for\ 0 \le k \le [p] - 2,$$

and $\pi_{[p]-1}(N) = 0$. *For any* $f \in W^{1,p}(\partial M, N)$ *or* $f \in W^{1-1/p,p}(\partial M, N) \cap \mathrm{VMO}(\partial M)$, *there exists* $F \in W^{1,p}(M, N)$ *satisfying* $\gamma F = f$ *on* ∂M.

In general, for a given $f \in W^{1-1/p,p}(\partial M, N)$, deciding whether $\mathfrak{o}_A(f) = 0$ seems a difficult problem. In some cases, however, we can characterize a map $f \in W^{1-1/p,p}(\partial M, N)$ with $\mathfrak{o}_A(f) = 0$. Also, for some cases, we can give a concrete characterization of $T^p(\partial M, N)$. In the following, we give two results in this direction. By the theorem of Hardt–Lin (see [14, Theorem 6.2]), when $1 < p < 2$ (i.e. $[p] = 1$), we always have $T^p(\partial M, N) = W^{1-1/p,p}(\partial M, N)$ for any N. Thus by the same theorem, the first non-trivial case where $T^p(\partial M, N) \ne W^{1-1/p,p}(\partial M, N)$ occurs is the case $2 \le p < 3$ and $\pi_1(N) \ne 0$. The first result concerns the case where $\pi_1(N)$ is finite.

THEOREM 1.6. *Let* $2 \le p < 3$. *Let* M *be a compact Riemannian manifold with boundary with* $\dim M > p$. *Assume* N *is a compact Riemannian manifold with finite* $\pi_1(N)$. *Let* $\pi\colon \widetilde{N} \to N$ *be the universal covering of* N. *Then we have:*

(a) *Assume moreover that* $\pi_1(\partial M) = 0$, *then*

$$\{f \in W^{1-1/p,p}(\partial M; N) : \mathfrak{o}_A(f) = 0\} = \{\pi(\widetilde{f}) : \widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{N})\}.$$

(b) *Assume* $\pi_1(\partial M) = 0$ *or* $\pi_1(M) = 0$. *Then we have*

$$T^p(\partial M, N) = \{\pi(\widetilde{f}) : \widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{N})\}.$$

The next result treats the case where $\pi_1(N)$ is not necessarily finite, but N has some additional structure, that is, N is a compact Lie group. This class of N is also important since the Sobolev space with values into a Lie group naturally arises in gauge theory and harmonic maps into Lie groups (known as classical solutions of the Chiral Model), see [17]

THEOREM 1.7. *Let* $2 \le p < 4$. *Let* M *be a compact Riemannian manifold with boundary with* $\dim M > p$. *Assume that* $N = G$ *is a compact Lie group. Let*

$\pi \colon \widetilde{G} \to G$ be the universal covering of G. Then we have:

    (a) *Assume moreover that* $\pi_1(\partial M) = 0$, *then*

$$\{f \in W^{1-1/p,p}(\partial M, G) : \mathfrak{o}_A(f) = 0\} = \{\pi(\widetilde{f}) : \widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{G})\}.$$

    (b) *Assume* $\pi_1(\partial M) = 0$ *or* $\pi_1(M) = 0$, *then*

$$T^p(\partial M, G) = \{\pi(\widetilde{f}) : \widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{G})\}.$$

One may wonder whether the above theorems (especially, Theorem 1.6) hold for general compact N. For general compact N, $\widetilde{N}$ is non-compact and the definition of the Sobolev space $W^{1-1/p,p}(\partial M, \widetilde{N})$ is not so clear. (In the cases treated in the theorems, there is a natural definition of the Sobolev space $W^{1-1/p,p}(\partial M, \widetilde{N})$, see Section 3. This is the reason why we restrict our consideration to the above two cases). Thus to prove Theorem 1.6 for general N, we need a suitable definition of the Sobolev space $W^{1-1/p,p}(\partial M, \widetilde{N})$. For such a "Sobolev space" $W^{1-1/p,p}(\partial M, \widetilde{N})$, Theorem 1.6 will continue to hold. This observation indicates some aspects of difficulty involved in the extension problem.

Theorems 1.6 and 1.7 show that the extension problem is closely related to the lifting problem. Recall that $N = \mathbb{S}^1$ is a compact Lie group (this is the most simple compact Lie group) and our Theorem 1.7 shows that under appropriate conditions, the problem of finding an extension of $f \in W^{1-1/p,p}(\partial M, \mathbb{S}^1)$ is equivalent to finding a lifting of f to $\mathbb{R}$ (that is, to find $\phi \in W^{1,p}(\partial M, \mathbb{R})$ satisfying $f = e^{i\phi}$). In this special case $N = \mathbb{S}^1$, the lifting problem is extensively studied by Bourgain, Brezis and Mironescu. In their papers [6], [7], they showed that there are both topological and analytical obstructions to the lifting problem. Thus the above equivalence ($\mathfrak{o}_A(f) = 0$ if and only if f has a lifting) shows that, contrary to the obstruction $\mathfrak{o}_B(f)$, $\mathfrak{o}_A(f)$ contains both topological and analytical information.

Theorems 1.6 and 1.7 are also related to the conjecture of Bethuel–Demengel (see [4, Conjecture 2]). In [4], Bethuel–Demengel conjectured that if $\pi_{[p]-1}(N) = 0$ and $\pi_j(N)$ is finite for all $j \leq [p] - 2$, then $T^p(\partial M, N) = W^{1-1/p,p}(\partial M, N)$. In view of the theorem of Hardt–Lin (see [14, Theorem 6.2]), the first non-trivial case is the case $3 \leq p < 4$ (i.e. $[p] = 3$). In this case, their conjecture is: When $\pi_2(N) = 0$ and $\pi_1(N)$ is finite, $T^p(\partial M, N) = W^{1-1/p,p}(\partial M, N)$ for $3 \leq p < 4$. Thus the conjecture of Bethuel–Demengel is closely related to the problem treated in Theorems 1.6 and 1.7. We show that this conjecture is not true in general, see Example 3.4. However, one may conjecture that it is true under some additional assumptions on M, for example, $\pi_1(\partial M) = 0$ or $M = \mathbb{B}^m$ etc.

Please recall the examples stated after Theorem 1.1. We will show in Section 3 that the obstruction $\pi_{[p]-1}(N) \neq 0$ is contained in $\mathfrak{o}_A$ in the sense that

if $\mathfrak{o}_A(f) = 0$ for any f, it is necessary $\pi_{[p]-1}(N) = 0$. But the condition $\pi_{[p]-1}(N) = 0$ is not sufficient to imply $\mathfrak{o}_A(f) = 0$ for any f (as we have seen, it also depends on analytical nature of f). Other topological obstructions (namely $\pi_j(N) \neq 0$ for some $j < [p] - 1$) is contained in $\mathfrak{o}_B(f)$. This is also explained in Section 3.

In the next section, we give proofs of Theorems 1.2 and 1.3.

2. Proofs of Theorems 1.2 and 1.3

We first prove elementary properties of $\mathfrak{o}_A(f)$ for $f \in W^{1-1/p,p}(\partial M, N)$.

LEMMA 2.1. *For any* $f \in W^{1-1/p,p}(\partial M, N)$, $\mathfrak{o}_A(f) = 0$ *or* $\mathfrak{o}_A(f) = \infty$. *Moreover,* $\mathfrak{o}_A(f)$ *depends only on* f *and the topologies of* ∂M *and* N.

PROOF. We need to show $\mathfrak{o}_A(f) = 0$ if $\mathfrak{o}_A(f) < \infty$. So assume $\mathfrak{o}_A(f) < \infty$. By definition, there exists $t_0 \in (0, 2)$ such that $\lim_{\varepsilon \downarrow 0} \mathcal{E}_{\varepsilon,t_0}(f) < \infty$. Thus, we easily see that $\{u_{\varepsilon,t_0}\}$ is bounded in $W^{1,p}(\mathcal{C}_{t_0}(\partial M), \mathbb{R}^k)$, where $\gamma u_{\varepsilon,t_0} = f$ on ∂M and $E_{\varepsilon,t_0}(u_{\varepsilon,t_0}) = \mathcal{E}_{\varepsilon,t_0}$ for $u_{\varepsilon,t_0} \in W^{1,p}(\mathcal{C}_{t_0}(\partial M), N)$. From this, for some sequence $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$, we have $u_{\varepsilon_n,t_0} \rightharpoonup u_{t_0}$ weakly in $W^{1,p}(\mathcal{C}_{t_0}(\partial M), \mathbb{R}^k)$ for some $u_{t_0} \in W^{1,p}(\mathcal{C}_{t_0}(\partial M), \mathbb{R}^k)$ with $\gamma u_{t_0} = f$ on ∂M. Since, by passing to the limit $\varepsilon_n \to 0$,

$$\sup_{\varepsilon > 0} \int_{\mathcal{C}_{t_0}(\partial M)} \frac{1}{\varepsilon} d(u(x), N)^p \, |dx|$$

is finite, we easily see that $u_{t_0} \in W^{1,p}(\mathcal{C}_{t_0}(\partial M), N)$. Then we have

$$\mathfrak{o}_A(f) \leq \lim_{t \downarrow 0} \lim_{\varepsilon \downarrow 0} E_{\varepsilon,t}(u_{t_0}) = \lim_{t \downarrow 0} \int_{\mathcal{C}_t(\partial M)} |\nabla u_{t_0}|^p \, |dx| = 0.$$

It is obvious that $\mathfrak{o}_A(f)$ is independent of the metric of N. Since any collar of ∂M is diffeomorphic to $[0, 2) \times \partial M$, one can easily prove that $\mathfrak{o}_A(f)$ is independent of a collar of ∂M and the metric of ∂M. In particular, $\mathfrak{o}_A(f)$ depends only on f and the topologies of ∂M and N.   □

Now we turn to proofs of Theorems 1.2 and 1.3. We first give the proof of necessity of $\mathfrak{o}_A(f) = 0$ and $\mathfrak{o}_B(f)$ (or $\mathfrak{o}_{B'}(f)$). After that, we give the proof of sufficiency.

PROOF OF THEOREM 1.2 (Necessity). We assume $1 < p < \dim M$ and $p \notin \mathbb{Z}$. Assume that $f \in W^{1-1/p,p}(\partial M, N)$ has an extension $F \in W^{1,p}(M, N)$: $\gamma F = f$. Then we have

$$\mathfrak{o}_A(f) = \lim_{t \downarrow 0} \lim_{\varepsilon \downarrow 0} \mathcal{E}_{\varepsilon,t}(f) \leq \lim_{t \downarrow 0} \lim_{\varepsilon \downarrow 0} E_{\varepsilon,t}(F|_{\mathcal{C}_t(\partial M)}) = \lim_{t \downarrow 0} \int_{\mathcal{C}_t(\partial M)} |\nabla F|^p \, |dx| = 0.$$

Therefore, $\mathfrak{o}_A(f) = 0$.

Next we prove the necessity of $\mathfrak{o}_B(f)$. Since $p \notin \mathbb{Z}$, by the Sobolev imbedding $F \in \mathcal{C}(M^{[p]}, N)$ for generic $M^{[p]}$. Thus $F|_{M^{[p]}}$ is a continuous extension of

$f|_{(\partial M)^{[p]-1}}$ to $M^{[p]}$. Here $(\partial M)^{[p]-1}$ is taken as in Section 1, that is, $(M, \partial M)$ is considered as a relative CW-complex. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

PROOF OF THEOREM 1.3 (Necessity). The proof of $\mathfrak{o}_A(f) = 0$ is the same as in the proof of Theorem 1.2. To prove necessity of $\mathfrak{o}_{B'}(f)$, let F be a map in $W^{1,p}(M, N)$ with $\gamma F = f$. For generic M^p, $F|_{M^p} \in W^{1,p}(M^p, N)$. Since $\mathrm{Lip}(M^p, N)$ is dense in $W^{1,p}(M^p, N)$, there exists $\{u_j\} \subset \mathrm{Lip}(M^p, N)$ such that $u_j \to F|_{M^p}$ in $W^{1,p}(M^p, N)$. Trace theorem then implies that $u_j|_{(\partial M)^{p-1}} \to f|_{(\partial M)^{p-1}}$ in $W^{1-1/p,p}((\partial M)^{p-1}, N)$. $\qquad\qquad\qquad\qquad\square$

PROOF OF THEOREM 1.2 (Sufficiency). Let $p \notin \mathbb{Z}$. Since $\mathfrak{o}_A(f) = 0$, Lemma 2.1 and its proof show that there exists $t_0 \in [0, 2]$ and $u_{t_0} \in W^{1,p}(\mathcal{C}_{t_0}(\partial M), N)$ such that $\gamma u_{t_0} = f$ on ∂M. We may assume without loss of generality $t_0 = 2$ (since $\mathcal{C}_{t_0}(\partial M)$ is diffeomorphic to $\mathcal{C}(\partial M)$) and set $F := u_{t_0} \in W^{1,p}(\mathcal{C}(\partial M), N)$. We use the identification $\mathcal{C}(\partial M) \cong [0, 2) \times \partial M$ throughout the proof. Since for almost every $t \in [0, 1]$, $F|_{\{t\} \times \partial M} \in W^{1,p}(\partial M, N)$, we may assume without loss of generality $F|_{\{1\} \times \partial M} \in W^{1,p}(\partial M, N)$. Thus our problem is reduced to extending $F|_{\{1\} \times \partial M}$ to $M \setminus [0, 1] \times \partial M$ as a $W^{1,p}$-map (with values in N). To proceed, we need the following lemma:

LEMMA 2.2. *Assume* $f|_{(\partial M)^{[p]-1}} : (\partial M)^{[p]-1} \to N$ *has a continuous extension* $u_f : M^{[p]} \to N$ *for a generic pair* $(M^{[p]}, (\partial M)^{[p]-1})$. *Set* $\partial_1 M := \{1\} \times \partial M$. *Then for a generic pair* $((M \setminus [0, 1) \times \partial M)^{[p]}, (\partial_1 M)^{[p]-1})$, $F|_{(\partial_1 M)^{[p]-1}} : (\partial_1 M)^{[p]-1} \to N$ *has a continuous extension* $(M \setminus [0, 1) \times \partial M)^{[p]} \to N$. *Here* $(M \setminus [0, 1) \times \partial M, \partial_1 M)$ *is considered as a relative* CW-*complex as in Section* 1.

To prove the above lemma, we need

LEMMA 2.3. *For a generic* $[p] - 1$-*skeleton* $(\partial M)^{[p]-1}$ (*which comes from some relative* CW-*structure of* $(M, \partial M)$ *as before*), *we have* $F|_{[0,1] \times (\partial M)^{[p]-1}} \in W^{1,p}([0, 1] \times (\partial M)^{[p]-1}, N)$. *Here, as always, we use the identification* $\mathcal{C}(\partial M) \cong [0, 2) \times \partial M$.

PROOF. Give a $[p] - 1$-skeleton $(\partial M)^{[p]-1}$ (coming from some relative CW-structure of $(M, \partial M)$) arbitrary. We may assume without loss of generality ∂M is isometrically imbedded in $\mathbb{R}^r$ for some $r > 1$. Let $\mathcal{O}(\partial M) \subset \mathbb{R}^r$ be a tubular neighbourhood of ∂M and $\pi : \mathcal{O}(\partial M) \to \partial M$ a nearest point fibration, i.e. $d_0(x, \pi(x)) = d_0(x, \partial M)$ (d_0 is the Euclidean distance in $\mathbb{R}^r$). We take $\varepsilon > 0$ so that $\{x \in \mathbb{R}^r : d_0(x, \partial M) < \varepsilon\} \subset \mathcal{O}(\partial M)$. For $v \in \mathcal{O}(\partial M)$, define $\phi_v : \partial M \to \partial M$ by $\phi_v(x) = \pi(x + v)$. Then $\phi_v : \partial M \to \partial M$ is C^∞-isotopic to $\mathrm{Id}_{\partial M} : \partial M \to \partial M$. Define $\Phi_v : [0, 1] \times \partial M \to [0, 1] \times \partial M$ by $\Phi_v(t, x) = (t, \phi_v(x))$. Φ_v is C^∞-isotopic to $\mathrm{Id}_{[0,1] \times \partial M} : [0, 1] \times \partial M \to [0, 1] \times \partial M$.

By Fubini, we have

$$\int_{|v|<\varepsilon} \left(\int_{[0,1]\times(\partial M)^{[p]-1}} |\nabla(F\circ\Phi_v(t,x))|^p \, dt\, dx \right) dv$$

$$\leq C \int_{(\partial M)^{[p]-1}} \left(\int_{[0,1]} \int_{|v|<\varepsilon} |\nabla F(t,\phi_v(x))|^p \, dt\, dv \right) dx$$

$$\leq C \int_{\mathcal{C}(\partial M)} |\nabla F|^p \, dx < \infty.$$

Thus for almost every v with $|v| < \varepsilon$, $F\circ\Phi_v \in W^{1,p}([0,1]\times(\partial M)^{[p]-1}, N)$. Note that $\Phi_v^{-1}([0,1]\times(\partial M)^{[p]-1}) = [0,1]\times\phi_v^{-1}((\partial M)^{[p]-1})$.

To complete the proof, we need to prove that $\phi_v^{-1}((\partial M)^{[p]-1})$ is a $[p]-1$-skeleton of ∂M coming from some relative CW-structure of $(M,\partial M)$. For this, it is sufficient to prove that Φ_v has an extension $\widehat{\Phi}_v \colon M \to M$ which is also isotopic to $\mathrm{Id}_M \colon M \to M$. Then if $(M^{[p]}, (\partial M)^{[p]-1})$ is a relative pair, so is $(\widehat{\Phi}_v^{-1}(M^{[p]}), \phi_v^{-1}((\partial M)^{[p]-1}))$.

To extend Φ_v to M, we recall the identification $\mathcal{C}(\partial M) \cong [0,2)\times\partial M$. Let $\alpha \in C^\infty([1,2])$ be a function satisfying $\alpha(t) \equiv 1$ for t near 1, $\alpha(t) \equiv 0$ for t near 2 and $0 \leq \alpha(t) \leq 1$ for $t \in [1,2]$. Define $\widehat{\Phi}_v$ by $\widehat{\Phi}_v(t,x) = \Phi_v(t,x)$ for $(t,x) \in [0,1)\times\partial M$, $\widehat{\Phi}_v(t,x) = \Phi_{\alpha(t)v}(t,x)$ for $(t,x) \in [1,2)\times\partial M$ and $\widehat{\Phi} = \mathrm{Id}_{M\setminus\mathcal{C}(\partial M)}$ in $M\setminus\mathcal{C}(\partial M)$. Clearly, $\widehat{\Phi}_v$ is an extension of Φ_v to M and C^∞-isotopic to Id_M. $\square$

Proof of Lemma 2.2. By Lemma 2.3, for generic $[p]-1$-skeleton $(\partial M)^{[p]-1}$, $F \in W^{1,p}([0,1]\times(\partial M)^{[p]-1}, N)$. Since $p \notin \mathbb{Z}$, by the Sobolev imbedding theorem, $F|_{[0,1]\times(\partial M)^{[p]-1}} \in C([0,1]\times(\partial M)^{[p]-1}, N)$. Therefore $F|_{\{0\}\times(\partial M)^{[p]-1}} = f|_{(\partial M)^{[p]-1}}$ is homotopic to $F|_{(\partial_1 M)^{[p]-1}}$. By assumption, $f|_{(\partial M)^{[p]-1}}$ has a continuous extension $u_f \colon M^{[p]} \to N$. Thus by the homotopy extension property, see [8], [18], (since $(\partial M)^{[p]-1} \hookrightarrow M^{[p]}$ is a cofibration), $F|_{(\partial_1 M)^{[p]-1}}$ has a continuous extension $(M\setminus[0,1]\times\partial M))^{[p]} \to N$, too. $\square$

Our problem is reduced to the special case $f \in W^{1,p}(\partial M, N)$. Indeed, by Lemma 2.2, $F|_{\partial_1 M} \in W^{1,p}(\partial_1 M, N)$ has a continuous extension $(M\setminus[0,1)\times\partial M)^{[p]} \to N$ and to extend f to M as a $W^{1,p}(M,N)$-map, it is sufficient to extend $F|_{\partial_1 M}$ to $M\setminus[0,1)\times\partial M$ as a $W^{1,p}(M\setminus[0,1)\times\partial M, N)$-map. Thus in the following, we assume

$\mathfrak{o}_C(f)$: $f \in W^{1,p}(\partial M, N)$ and $f|_{(\partial M)^{[p]-1}}$ has a continuous extension $M^{[p]} \to N$ for a generic pair $(M^{[p]}, (\partial M)^{[p]-1})$.

We construct an extension of f by the following two steps.

Step 1. We extend f to $M^{[p]}\cup\partial M$ as a $W^{1,p}(M^{[p]}\cup\partial M, N)$-map.

Step 2. We extend the map obtained in Step 1 to M as a $W^{1,p}(M,N)$-map.

For the moment, we assume Step 1, that is, we assume f has an extension $F_1 \in W^{1,p}(M^{[p]} \cup \partial M, N)$ and give a construction of an extension of F_1 to M.

Construction of an extension of F_1 to M. By using the filtration $M^{[p]} \subset M^{[p]+1} \subset \ldots \subset M$ of M, we successively construct an extension. By Step 1, we have $F_1 \in W^{1,p}(M^{[p]} \cup \partial M, N)$ with $F_1|_{\partial M} = f$.

Assume that F_1 is extended to $M^{[p]+k-1}$ as a $W^{1,p}(M^{[p]+k-1} \cup \partial M, N)$-map F_k $(1 \leq k \leq m + 1 - [p])$. Let $\sigma^{[p]+k}$ be a $[p] + k$-cell of M. We may assume it is bi-Lipschitz equivalent to $[-1, 1]^{[p]+k}$, that is, there exists a bi-Lipschitz map

$$\varphi_\sigma \colon [-1, 1]^{[p]+k} \xrightarrow{\sim} \sigma^{[p]+k}.$$

Define F_{k+1} on $\sigma^{[p]+k}$ by homogeneous degree 0 extension of $F_k|_{\partial \sigma^{[p]+k}}$:

$$F_{k+1} := F_k \circ f_{\partial \sigma} \circ \varphi_\sigma \left(\frac{\varphi_\sigma^{-1}}{\|\varphi_\sigma^{-1}\|} \right),$$

where $f_{\partial \sigma}$ is the attaching map $\partial e^{[p]+k} \to M^{[p]+k-1}$ for the cell $\sigma^{[p]+k}$, $e^{[p]+k} := \{x \in \mathbb{R}^{[p]+k} : |x| \leq 1\}$ and $\|y\| := \max_{1 \leq i \leq [p]+k} |y_i|$ for $y = (y_1, \ldots, y_{[p]+k}) \in [-1, 1]^{[p]+k}$. Note that $F_{k+1} \in W^{1,p}(\sigma^{[p]+k}, N)$.

We carry out this construction for all $[p] + k$-cell and obtain an extension F_{k+1} of F_k to $M^{[p]+k}$ as a map in $W^{1,p}(M^{[p]+k}, N)$. By the induction on k, we finally obtain an extension $F_{m-[p]+1}$ of f to M as a map in $W^{1,p}(M, N)$. $\square$

We now give a construction of an extension of f to $\partial M \cup M^{[p]}$.

Construction of an extension of f to $\partial M \cup M^{[p]}$. Here the condition $\mathfrak{o}_B(f)$ is used. In other words, the topological obstruction $\mathfrak{o}_B(f)$ exists here. By the condition $\mathfrak{o}_B(f)$ (and our reduction of the problem), $f|_{(\partial M)^{[p]-1}} \colon (\partial M)^{[p]-1} \to N$ has a continuous extension $u_f \colon M^{[p]} \to N$. In the following, we successively construct an extension of f using the filtration $M^0 \subset \ldots \subset M^{[p]}$. In this process, we construct extensions $v_f^k \colon \partial M \cup M^k \to N$ $(k = 0, \ldots, [p])$ not only $v_f^k = f$ on ∂M but also satisfying:

(a) $v_f^k \subset v_f^{k+1}$ (this means v_f^{k+1} is an extension of v_f^k),
(b) $v_f^k \sim u_f$ on M^k ($v_f^k \sim u_f$ on M^k means that v_f^k is homotopic to u_f on M^k).

The existence of v_f^0 satisfying (a) and (b) is obvious. Assume we have constructed $v_f^0, \ldots, v_f^k$ satisfying (a) and (b) $(k + 1 \leq [p])$. Let σ^{k+1} be a $k + 1$-cell of M. We may assume σ^{k+1} is bi-Lipschitz equivalent to $[-1, 1]^{k+1}$. We define v_f^{k+1} by v_f^k on $\partial \sigma^{k+1} \cup \partial M$. By the induction assumption, $v_f^k \sim u_f$ on M^k. Since $u_f|_{M^k}$ has a continuous extension to $M^{[p]}$ (i.e. u_f), by the homotopy extension property (since $M^k \hookrightarrow M^{[p]}$ is a cofibration), v_f^k has a continuous extension $V_f^k \colon M^{[p]} \to N$ which is homotopic to u_f on $M^{[p]}$. We use V_f^k to extend v_f^k to σ^{k+1}. We first extend v_f^k to a neighbourhood of $\partial \sigma^{k+1}$ in σ^{k+1}. For this,

we identify the collar of $\partial\sigma^{k+1}$ by $[0,1]\times\partial\sigma^{k+1}$ and define for $x\in\partial\sigma^{k+1}$ and $0<h<1$:

$$v(h,x) := \frac{1}{\mathcal{H}^k(\mathbb{B}^k_h(x))}\int_{\mathbb{B}^k_h(x)} v^k_f\,d\mathcal{H}^k,$$

where $\mathbb{B}^k_h(x)\subset\partial\sigma^{k+1}$ is the metric k-ball of radius h with center at x and $\mathcal{H}^k$ is the k-dimensional measure induced from the Riemannian metric on M.

It is well known that $v\in\mathrm{Lip}((0,1)\times\partial\sigma^{k+1})\cap W^{1,p}((0,1)\times\partial\sigma^{k+1},\mathbb{R}^k)$ and $v(h,\cdot)\to v^k_f$ in $W^{1-1/p,p}(\partial\sigma^{k+1},\mathbb{R}^k)$ as $h\downarrow 0$. Moreover, for small $h>0$, $v(h,\cdot)\in\mathcal{O}(N)$, where $\mathcal{O}(N)$ is a tubular neighbourhood of N in $\mathbb{R}^k$. The last claim easily follows from the Sobolev imbedding theorem and $k+1\le[p]$. We may assume without loss of generality $v(h,\cdot)\in\mathcal{O}(N)$ for $h\in(0,1)$.

Let $\pi_N:\mathcal{O}(N)\to N$ be the nearest point projection. Then $\pi_N\circ v$ is well-defined and in $\mathrm{Lip}((0,1)\times\partial\sigma^{k+1})\cap W^{1,p}((0,1)\times\partial\sigma^{k+1},N)$. Since $v^k_f|_{\partial\sigma^{k+1}}$ has a continuous extension V^k_f in σ^{k+1}, $\pi_N\circ v$ also has an continuous extension to σ^{k+1} (since $\pi_N\circ v|_{\partial\sigma^{k+1}}=v^k_f|_{\partial\sigma^{k+1}}$). Since $\pi_N\circ v$ is Lipschitz in $(0,1)\times\partial\sigma^{k+1}$, we can take this extension also Lipschitz in σ^{k+1} and homotopic to V^k_f in σ^{k+1}. In σ^{k+1}, we define v^{k+1}_f by this extension.

By construction $v^{k+1}_f\in W^{1,p}(M^k\cup_{f_{\partial\sigma^{k+1}}}\sigma^{k+1},N)$, $v^{k+1}_f|_{M^k}=v_f$, $v^{k+1}_f\sim V^k_f\sim u_f$ on $M^k\cup_{f_{\partial\sigma^{k+1}}}\sigma^{k+1}$. We continue this construction for all $k+1$-cell of M and obtain $v^{k+1}_f:M^{k+1}\cup\partial M\to N$ with $v^{k+1}_f\in W^{1,p}(\partial M\cup M^{k+1},N)$, $v^k_f\subset v^{k+1}_f$ and $v^{k+1}_f\sim u_f$ in M^{k+1}. By induction, we obtain $v^0_f,v^1_f,\ldots,v^{[p]}_f$ satisfying (a) and (b). Clearly, $v^{[p]}_f\in W^{1,p}(\partial M\cup M^{[p]},N)$ is an extension of f to $\partial M\cup M^{[p]}$. $\qquad\square$

Proof of Theorem 1.3 (Sufficiency). We assume $p\in\mathbb{Z}$ and $1<p<\dim M$. We prove Theorem 1.3 by the same steps in the proof of Theorem 1.2. If f has an extension to $\partial M\cup M^p$ as a map in $W^{1,p}(\partial M\cup M^p,N)$ (Step 1), then by the same argument given in the proof of Theorem 1.2, f can be extended to M as a $W^{1,p}(M,N)$-map. Thus we only need to prove the assertion: Under the assumption of Theorem 1.3, f has an extension to $\partial M\cup M^p$ as a map in $W^{1,p}(\partial M\cup M^p,N)$.

By Lemma 2.1 and its proof (see also the argument in the proof of Theorem 1.2), there exists $F\in W^{1,p}(\mathcal{C}(\partial M),N)$ with $\gamma F|_{\partial M}=f$. The following lemma is the analogue of Lemma 2.2 when $p\in\mathbb{Z}$.

Lemma 2.4. *Under the assumption $\mathfrak{o}_{B'}(f)$, there exists $\{v_i\}\subset\mathrm{Lip}((M\setminus\mathcal{C}(\partial M))^p,N)$ such that $v_i|_{(\partial_1 M)^{p-1}}\to F|_{(\partial_1 M)^{p-1}}$ in $W^{1-1/p,p}((\partial_1 M)^{p-1},N)$.*

Proof. As usual, we identify $\mathcal{C}(\partial M)\cong[0,2)\times\partial M$. By Lemma 2.3, for generic $p-1$-skeleton $(\partial M)^{p-1}$, $F\in W^{1,p}((0,1)\times(\partial M)^{p-1},N)$.

Since $\mathrm{Lip}([0,1]\times(\partial M)^{p-1},N)$ is dense in $W^{1,p}((0,1)\times(\partial M)^{p-1},N)$ (see [16], [12]), there exists $F_i \in \mathrm{Lip}([0,1]\times(\partial M)^{p-1},N)$ such that $F_i|_{(0,1)\times(\partial M)^{p-1}} \to F|_{(0,1)\times(\partial M)^{p-1}}$ in $W^{1,p}((0,1)\times(\partial M)^{p-1},N)$.

Set $f_i^0 = F_i|_{\{0\}\times(\partial M)^{p-1}}$ and $f_i^1 = F_i|_{\{1\}\times(\partial M)^{p-1}}$. Note that $f_i^0 \sim f_i^1$ on $(\partial M)^{p-1}$ (by the homotopy F_i) and by the trace theorem, $f_i^0 \to f$ in $W^{1-1/p,p}((\partial M)^{p-1},N)$ and $f_i^1 \to F|_{\{1\}\times(\partial M)^{p-1}}$ in $W^{1-1/p,p}((\partial M)^{p-1},N)$.

Since $f_i \to f$ and $f_i^0 \to f$ in $W^{1-1/p,p}((\partial M)^{p-1},N)$, by [3, Lemma 1], $f_i^0 \sim f_i$ on $(\partial M)^{p-1}$ if i is large. Thus $f_i \sim f_i^0 \sim f_i^1$ on $(\partial M)^{p-1}$.

By assumption, f_i has a continuous extension $M^p \to N$, so by the homotopy extension property (since $(\partial M)^{p-1} \hookrightarrow M^p$ is a cofibration), f_i^1 also has a continuous extension $v_i \colon M^p \to N$.

Since f_i^1 is Lipschitz, one can take v_i as a Lipschitz map. Here note that by our identification, M^p is identified with $(M \setminus \mathcal{C}(\partial M))^p$. Clearly, $\{v_i\}$ satisfies the required property. $\square$

As in the case of Theorem 1.2, we may assume $F|_{\partial_1 M} \in W^{1,p}(\partial_1 M, N)$. Thus as in the proof of Theorem 1.2, by Lemma 2.4, we have reduced the problem to the case $f \in W^{1,p}(\partial M, N)$. Therefore to complete the proof of Theorem 1.3, we need to extend f to M under the assumption

$\mathfrak{o}_{C'}(f)\colon\ f \in W^{1,p}(\partial M, N)$ and there exists $\{f_i\} \subset \mathrm{Lip}((\partial M)^{p-1},N)$ such that $f_i \to f|_{(\partial M)^{p-1}}$ in $W^{1-1/p,p}((\partial M)^{p-1},N)$ and f_i has a Lipschitz extension $F_i \colon M^p \to N$.

We also need the following lemma:

LEMMA 2.5. *Let $\{f_i\}$ and $\{F_i\}$ be sequences satisfying $\mathfrak{o}_{C'}(f)$. Then we can assume that $\{F_i\}$ satisfies $F_i \sim F_j$ on M^p for any $i \neq j$.*

PROOF. Since $f_i \to f$ in $W^{1-1/p,p}((\partial M)^{p-1},N)$, by Bethuel ([3, Lemma 1]), $f_i \sim f_j$ on $(\partial M)^{p-1}$ if i, j are large. Discarding finitely many f_i if necessary, we may assume $f_i \sim f_j$ on $(\partial M)^{p-1}$ for all $i \neq j$. Then $f_1 \sim f_i$ on $(\partial M)^{p-1}$ for all i and since f_1 has a continuous extension $F_1\colon M^p \to N$, by the homotopy extension property f_i has a continuous extension $F_i'\colon M^p \to N$ satisfying $F_1 \sim F_i'$ on M^p. Since f_i is Lipschitz, we can take F_i' also Lipschitz. Replacing $\{F_i\}$ by $\{F_1, F_2', \ldots, F_i', \ldots\}$, we complete the proof. $\square$

By Lemma 2.5, we may assume that the sequence $\{F_i\}$ in $\mathfrak{o}_{C'}(f)$ satisfies $F_i \in [\alpha]$, where $[\alpha]$ is a fixed homotopy class of continuous maps from M^p to N.

We now complete the proof of Theorem 1.3. The idea is the same as in the case $p \notin \mathbb{Z}$: We use the filtration $M^0 \subset \ldots \subset M^p$ and construct an extension successively.

COMPLETION OF THE PROOF OF THEOREM 1.3. As in the proof of Theorem 1.2, we are going to construct $v_f^k\colon M^k \to N$ ($k = 0, \ldots, p$) such that

$v_f^0 \subset \ldots \subset v_f^p$ and $v_f^k \in [\alpha]$ on M^k (the meaning of the last condition is that if $G: M^p \to N$ is a representative of $[\alpha]$, then $v_f^k \sim G$ on M^k). The existence of v_f^0 satisfying the above condition is obvious.

Assume that we have constructed $v_f^0 \subset \ldots \subset v_f^k$ satisfying $v_f^j \in [\alpha]$ on M^j for $0 \le j \le k$ $(k+1 \le p)$. Let σ^{k+1} be a (generic) $k+1$-cell of M. Note that $v_f^k|_{\partial \sigma^{k+1}} \in W^{1,p}(\partial \sigma^{k+1}, N) \hookrightarrow C(\partial \sigma^{k+1}, N)$ by the Sobolev imbedding. Since $v_f^k \in [\alpha]$ on M^k and $[\alpha] = [G]$ for some continuous $G: M^p \to N$, by the homotopy extension property (since $M^k \hookrightarrow M^p$ is a cofibration), v_f^k has a continuous extension $V_f^k: M^p \to N$ with $V_f^k \in [\alpha]$ in M^p. Then we proceed as in the proof of Theorem 1.2 and obtain an extension of v_f^k to $M^k \cup_{f_{\partial \sigma^{k+1}}} \sigma^{k+1}$ as a $W^{1,p}(M^k \cup_{f_{\partial \sigma^{k+1}}} \sigma^{k+1}, N)$-map. We continue this construction for all $k+1$-cell of M and obtain $v_f^{k+1}: M^{k+1} \cup \partial M \to N$ with $v_f^{k+1} \in W^{1,p}(\partial M \cup M^{k+1}, N)$, $v_f^k \subset v_f^{k+1}$ and $v_f^{k+1} \in [\alpha]$ in M^{k+1}. By induction, we construct $v_f^0, \ldots, v_f^p$ satisfying the required property. Clearly, $v_f^p \in W^{1,p}(\partial M \cup M^p, N)$ is an extension of f to $\partial M \cup M^p$. $\qquad\square$

3. Obstructions $\mathfrak{o}_A(f)$ and $\mathfrak{o}_B(f)$ (and $\mathfrak{o}_{B'}(f)$)

Since the structures of the obstructions $\mathfrak{o}_B(f)$ and $\mathfrak{o}_{B'}(f)$ are simpler than that of the obstruction $\mathfrak{o}_A(f)$, we first study $\mathfrak{o}_B(f)$ and $o_{B'}(f)$.

3.1. The obstructions $\mathfrak{o}_B(f)$ and $\mathfrak{o}_{B'}(f)$. We first point out that the obstructions $\mathfrak{o}_B(f)$, $\mathfrak{o}_{B'}(f)$ are topological one, that is, it depends only on the topology of $(M^{[p]}, (\partial M)^{[p]-1})$, N and the homotopy class of $f|_{(\partial M)^{[p]-1}}: (\partial M)^{[p]-1} \to N$. A sufficient condition to hold $o_B(f)$ (or $o_{B'}(f)$) for any $f \in W^{1-1/p,p}(\partial M, N)$ is given by the following:

PROPOSITION 3.1. *Let $1 < p < \dim M$. We assume that N is simple and $H^{k+1}(M^{[p]}, (\partial M)^{[p]-1}; \pi_k(N)) = 0$ for $1 \le k \le [p]-1$. Then the conditions $\mathfrak{o}_B(f)$ and $\mathfrak{o}_{B'}(f)$ hold for any $f \in W^{1-1/p,p}(\partial M, N)$.*

Before we give the proof, we explain the terminology "simple" in the above proposition. Recall that (see [8], [20]) there is a natural action of $\pi_1(N)$ on $\pi_k(N)$ for any $k \ge 1$ (it is defined by the condition that $\gamma \cdot \alpha$, the action of $\gamma \in \pi_1(N)$ on $\alpha \in \pi_k(N)$, is freely homotopic to α along γ). N is *simple* if $\pi_1(N)$ acts trivially on $\pi_k(N)$ for any $k \ge 1$. For example if N is simply connected, N is simple. If $N = \mathbb{S}^1$, N is simple since $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ is abelian and $\pi_k(\mathbb{S}^1) = 0$ for $k \ge 2$.

PROOF OF PROPOSITION 3.1. This is a direct consequence of the obstruction theory in topology. The obstruction theory concerns, for example, the problem of extending a continuous function $g: A \to Y$ to X, where (X, A) is a relative complex and Y is a topological space. By the obstruction theory, obstructions to extending g to X are (essentially) cohomology classes in $H^{k+1}(X, A; \pi_k(Y))$

$(k \geq 0)$ (for more precise statement and details of this theory, see [8] and [20]). In particular if $H^{k+1}(X, A; \pi_k(Y)) = 0$ for all $k \geq 0$, g has an extension to X. Returning to our problem, under the assumption of the proposition, we have $H^{k+1}(M^{[p]}, (\partial M)^{[p]-1}; \pi_k(N)) = 0$ for all k. Thus for the case $p \notin \mathbb{Z}$, the condition $\mathfrak{o}_B(f)$ follows directly from the obstruction theory. When $p \in \mathbb{Z}$, since $\mathrm{Lip}((\partial M)^{[p]-1}, N)$ is dense in $W^{1-1/p,p}((\partial M)^{[p]-1}, N)$ (see [3]), there exists $\{f_i\} \in \mathrm{Lip}((\partial M)^{[p]-1}, N)$ such that $f_i \to f$ in $W^{1-1/p,p}((\partial M)^{[p]-1}, N)$. Under the assumption, f_i has an continuous (and hence Lipschitz) extension to $M^{[p]}$ and the condition $\mathfrak{o}_{B'}(f)$ follows. $\qquad\square$

We will show in Section 3.2 that if $\mathfrak{o}_A(f) = 0$ for any $f \in W^{1-1/p,p}(\partial M, N)$, it is necessary $\pi_{[p]-1}(N) = 0$. (The importance of the condition $\pi_{[p]-1}(N) = 0$ for the extension problem was first pointed out by Bethuel [3], and Bethuel–Demengel [4]). Thus it seems useful to restate the above proposition under the condition $\pi_{[p]-1}(N) = 0$.

PROPOSITION 3.2. *Let $1 < p < \dim M$. Let $f \in W^{1-1/p,p}(\partial M, N)$. Assume N is simple, $\pi_{[p]-1}(N) = 0$ and $H^{k+1}(M, (\partial M)^{[p]-1}; \pi_k(N)) = 0$ for $1 \leq k \leq [p] - 2$, then $\mathfrak{o}_B(f)$ and $\mathfrak{o}_{B'}(f)$ hold.*

PROOF. Under the assumption, we need to show that

$$H^{k+1}(M^{[p]}, (\partial M)^{[p]-1}; \pi_k(N)) = 0 \quad \text{for } 1 \leq k \leq [p] - 1.$$

Since $\pi_{[p]-1}(N) = 0$, $H^{[p]}(M^p, (\partial M)^{[p]-1}; \pi_{[p]-1}(N)) = 0$. The remaining cases follow from the cohomology exact sequence of $(M, M^{[p]}, (\partial M)^{[p]-1})$:

$$\cdots \to H^k(M, M^{[p]}) \to H^k(M, (\partial M)^{[p]-1})$$
$$\to H^k(M^{[p]}, (\partial M)^{[p]-1}) \to H^{k+1}(M, M^{[p]}),$$

where the coefficient of the cohomology is an arbitrary group G. (Of course, we are interested in the case $G = \pi_{k-1}(N)$.) From the above sequence, if $k \leq [p] - 1$, $0 = H^k(M, (\partial M)^{[p]-1}) \simeq H^k(M^{[p]}, (\partial M)^{[p]-1})$. Thus the assertion of Proposition 3.2 follows from Proposition 3.1. $\qquad\square$

EXAMPLE 3.3 (Hardt–Lin [14] and Bethuel–Demengel [4]). Bethuel–Demengel (and Hardt–Lin for a special case) produced an example under the condition $\pi_j(N) \neq 0$ for some $1 \leq j \leq [p] - 1$. Under this condition, they proved that for some M there exists $f \in W^{1-1/p,p}(\partial M, N)$ which is not in $T^p(\partial M, N)$. They choose $M = \mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}$ (with $m > p$).

We will see in the next section that if $\pi_{[p]-1}(N) \neq 0$, there always exists $f \in W^{1-1/p,p}(\partial M, N)$ which does not admit an extension to M as a $W^{1,p}(M, N)$-map, thus we may assume $1 \leq j < [p] - 1$. f is constructed as follows (cf. [14] and [4]): By $\pi_j(N) \neq 0$, there exists $u \in C^\infty(\mathbb{S}^j, N)$ such that u is not homotopic to 0. For $(x, y) \in \partial M = \mathbb{S}^j \times \mathbb{S}^{m-j-1}$, define $f(x, y) := u(x)$. Obviously

$f \in W^{1-1/p,p}(\mathbb{S}^j \times \mathbb{S}^{m-j-1}, N)$. Note that in this case f satisfies $\mathfrak{o}_A(f) = 0$, since f obviously has an smooth (and hence $W^{1,p}$) extension to a collar neighbourhood of ∂M.

We claim that f does not admit any $W^{1,p}(M, N)$-extension. There are two proofs.

The first proof goes as follows (see [14], [4]). Assume contrary, there exists $F \in W^{1,p}(M, N)$ such that $\gamma F = f$. By Fubini's theorem, for almost every $y_0 \in \mathbb{S}^{m-j-1}$, $F(\cdot, y_0) \in W^{1,p}(\mathbb{B}^{j+1}, N)$. Clearly, $\gamma F(\cdot, y_0) = u$ on $\partial \mathbb{B}^{j+1} = \mathbb{S}^j$. Then there exists $V \in W^{1,p}(\mathbb{B}^{j+1}, N)$ which minimizes the functional $W^{1,p}(\mathbb{B}^{j+1}, N) \ni v \mapsto \int_{\mathbb{B}^{j+1}} |\nabla v|^p \, dx$ under the Dirichlet condition $\gamma v = u$ on $\partial \mathbb{B}^{j+1}$. Since $j + 1 \leq [p]$, we know $V \in C^1(\overline{\mathbb{B}^{j+1}}, N)$ (see [14]). This is a contradiction since $u \not\sim 0$.

The second proof uses Theorems 1.2 and 1.3. For the case $p \notin \mathbb{Z}$, one can easily show the existence of a relative CW-structure of $(M, \partial M)$ such that $(\partial M)^{[p]-1}$ contains $\mathbb{S}^j \times \{y_0\}$ and $M^{[p]}$ contains $\mathbb{B}^j \times \{y_0\}$ for a given $y_0 \in \mathbb{S}^{m-j-1}$ (cf. the construction below). Then the assertion readily follows from Theorem 1.2.

The case $p \in \mathbb{Z}$ is similar by using Theorem 1.3. In fact, if f has an extension, by Theorem 1.3 there exists $\{f_i\}$ such that $\mathfrak{o}_{B'}(f)$ holds. By [4], $f|_{(\partial M)^{p-1}} \sim f_i|_{(\partial M)^{p-1}}$ for large i and by the homotopy extension property, $f|_{(\partial M)^{p-1}}$ has an continuous extension to M^p since $f_i|_{(\partial M)^{p-1}}$ has an continuous extension to M^p. Then we have a contradiction as in the case $p \notin \mathbb{Z}$.

As noted above, $\mathfrak{o}_A(f) = 0$ in this case. Thus the obstruction to the extension is in $\mathfrak{o}_B(f)$. In view of Proposition 3.2, it follows that

$$H^{k+1}(M, (\partial M)^{[p]-1}, \pi_k(N)) \neq 0 \quad \text{for some } 1 \leq k \leq [p] - 2.$$

In fact, we have the following claim:

CLAIM 1. $H^{j+1}(M, (\partial M)^{[p]-1}; \pi_j(N)) \neq 0$ *for some relative* CW-*structure of* $(M, \partial M)$.

PROOF. As a CW-structure of ∂M, take $\partial M = \mathbb{S}^j \times \mathbb{S}^{m-j-1} = e^0 \cup e^j \cup e^{m-j-1} \cup e^{m-1}$, where e^k is a k-dimensional cell. Attaching a $j+1$-cell e^{j+1} to $\mathbb{S}^j$ by the identity $\mathrm{Id} \colon \partial e^{j+1} \to \mathbb{S}^j$, we obtain a relative CW-structure of $(M, \partial M)$. We have

$$(\partial M)^{[p]-1} = (\mathbb{S}^j \times \mathbb{S}^{m-j-1})^{[p]-1}$$
$$= \begin{cases} S^j \times \{*\} & (m - j - 1 \geq [p]), \\ S^j \times \{*\} \cup \{*\} \times S^{m-j-1} & (m - j - 1 < [p]). \end{cases}$$

Case $m - j - 1 \geq [p]$. In this case,

$$H^{j+1}(M, (\partial M)^{[p]-1}; \pi_j(N)) = H^{j+1}(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}, (\mathbb{S}^j \times \{*\}; \pi_j(N)).$$

By the cohomology exact sequence of the pair $(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}, \mathbb{S}^j \times \{*\})$, we have

$$(3.1) \quad H^j(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}; \pi_j(N)) \to H^j(\mathbb{S}^j \times \{*\}; \pi_j(N))$$
$$\to H^{j+1}(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}, \mathbb{S}^j \times \{*\}; \pi_j(N))$$
$$\to H^{j+1}(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}; \pi_j(N)) \to \cdots$$

By assumption $m - j - 1 \geq [p]$, we have $1 \leq j < m - j - 1$ and $H^j(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}; \pi_j(N)) = 0$. Thus the assertion $H^{j+1}(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}, \mathbb{S}^j \times \{*\}; \pi_j(N)) \neq 0$ follows easily from (3.1).

Case $m - j - 1 < [p]$. In this case, $H^{j+1}(M, (\partial M)^{[p]-1}; \pi_j(N)) = H^{j+1}(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}, \mathbb{S}^j \times \{*\} \cup \{*\} \times \mathbb{S}^{m-j-1}; \pi_j(N))$. By the cohomology exact sequence of the pair $(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}, \mathbb{S}^j \times \{*\} \cup \{*\} \times \mathbb{S}^{m-j-1})$, we have

$$(3.2) \quad H^j(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}; \pi_j(N)) \xrightarrow{i} H^j(\mathbb{S}^j \times \{*\} \cup \{*\} \times \mathbb{S}^{m-j-1}; \pi_j(N))$$
$$\xrightarrow{j} H^{j+1}(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}, \mathbb{S}^j \times \{*\} \cup \{*\} \times \mathbb{S}^{m-j-1}; \pi_j(N)) \to \cdots$$

Since $p_1^* \omega \in H^j(\mathbb{S}^j \times \{*\} \cup \{*\} \times \mathbb{S}^{m-j-1}; \pi_j(N))$ is not in the image of i and

$$\text{image}(j) \simeq H^j(\mathbb{S}^j \times \{*\} \cup \{*\} \times \mathbb{S}^{m-j-1}; \pi_j(N))/\text{image}(i),$$

where $p_1 : \mathbb{S}^j \times \mathbb{S}^{m-j-1} \to \mathbb{S}^j$ is the projection to the first factor and $\omega \in H^j(\mathbb{S}^j; \pi_j(N)) \cong \pi_j(N)$ is a generator, we have $\text{image}(j) \neq 0$. In particular, $H^{j+1}(\mathbb{B}^{j+1} \times \mathbb{S}^{m-j-1}, \mathbb{S}^j \times \{*\} \cup \{*\} \times \mathbb{S}^{m-j-1}; \pi_j(N)) \neq 0$.

Combining the two cases, we complete the proof of the claim. $\qquad\square$

When $H^{k+1}(M^{[p]}, (\partial M)^{[p]-1}; \pi_k(N)) \neq 0$ for some $1 \leq k \leq [p] - 1$, we do not know whether there exists $f \in W^{1-1/p,p}(\partial M, N)$ such that f does not admit any extension.

3.2. The obstruction $\mathfrak{o}_A(f)$. Our proof of Lemma 2.1 shows that, for $f \in W^{1-1/p,p}(\partial M, N)$, $\mathfrak{o}_A(f) = 0$ is equivalent to the condition that f has an extension $F \in W^{1,p}(\mathcal{C}(\partial M), N)$. Since $\mathfrak{o}_A(f)$ does not depend on specific extension of f to $\mathcal{C}(\partial M)$, the condition $\mathfrak{o}_A(f) = 0$ may be considered as a "universal" form of the expression "f has an $W^{1,p}(\mathcal{C}(\partial M), N)$-extension to some collar $\mathcal{C}(\partial M)$".

Assume $\mathfrak{o}_A(f) = 0$, and so there exists an extension $F \in W^{1,p}(\mathcal{C}(\partial M), N)$ of f. Since $\mathcal{C}(\partial M) \cong [0, 2) \times \partial M$, F is parameterized by $(t, x) \in [0, 2) \times \partial M$. Under the identification $\mathcal{C}(\partial M) \cong [0, 2) \times \partial M$, set $\varphi(t)(x) := F(t, x)$. Then by the trace theorem, $\varphi|_{[0,1] \times \partial M} \in C([0, 1], W^{1-1/p,p}(\partial M, N))$ and

$$\int_{[0,1] \times \partial M} |\nabla \varphi(t, x)|^p \, dt \, d\mu(x) < \infty.$$

Conversely, the existence of such a family of maps $\{\varphi(t)\}_{t\in[0,1]}$ implies $\mathfrak{o}_A(f) = 0$. The family $\{\varphi(t)\}_{t\in[0,1]}$ may be considered as a "regularization" of f, since $\varphi(t) \in W^{1,p}(\partial M, N)$ for a.e. $t \in [0,1]$ and $\varphi(t) \to f$ in $W^{1-1/p,p}(\partial M, N)$ as $t \to 0$.

From this observation, one may think that we can extend f to a neighbourhood of ∂M by regularizing f (for example, by using the method of Friedrichs mollifier, etc.). In fact, for the cases $f \in W^{1,p}(\partial M, N)$ or $f \in W^{1-1/p,p}(\partial M, N) \cap$ VMO, such method works well. (However, it does not work for general f.)

PROOF OF PROPOSITION 1.4. We first consider the case $f \in W^{1,p}(\partial M, N)$. In this case, $\mathfrak{o}_A(f) = 0$. Indeed, as always under the identification $\mathcal{C}(\partial M) \cong [0,2) \times \partial M$, we set $\varphi(t, x) = f(x)$. Clearly, this satisfies the above condition and therefore $\mathfrak{o}_A(f) = 0$. Thus the only obstruction of extending $f \in W^{1,p}(\partial M, N)$ is $\mathfrak{o}_B(f)$ (or $o_{B'}(f) = 0$).

Next, we consider the case $f \in W^{1-1/p,p}(\partial M, N) \cap \mathrm{VMO}(\partial M)$. Our procedure is based on the work of Brezis–Nirenberg (see [10]).

We define $\varphi: \mathcal{C}(\partial M) \to \mathbb{R}^k$ by (recall N is isometrically imbedded in $\mathbb{R}^k$)

$$(3.4) \qquad \phi(x) = \frac{1}{\mathcal{H}^{m-1}(B_{d(x)}(P(x)))} \int_{B_{d(x)}(P(x))} f \, d\mathcal{H}^{m-1},$$

where $d(x) := \mathrm{dist}(x, \partial M)$, $P: \mathcal{C}(\partial M) \to \partial M$ is the nearest point retraction and $B_{d(x)}(P(x))$ is the geodesic ball in ∂M with center $P(x)$ and radius $d(x)$. By the result of Brezis and Nirenberg (see [10, Example 3 and Lemma 7]) it follows that $\phi \in \mathrm{VMO}(\mathcal{C}(\partial M))$. It is also a fundamental result that $\phi \in W^{1,p}(\mathcal{C}(\partial M), \mathbb{R}^k)$ and $\gamma\phi = f$ on ∂M. Since $\phi \in \mathrm{VMO}(\mathcal{C}(\partial M), \mathbb{R}^k)$ and f takes values in N on ∂M, it follows from [9] and [10] that $\phi(x)$ takes values in $\mathcal{O}(N)$ if $d(x) < \varepsilon$ for some small $\varepsilon > 0$. Then $\varphi := \pi_N \circ \phi$ is defined in some collar neighbourhood of ∂M and we have $\mathfrak{o}_A(f) = 0$. Note that by our identification $\mathcal{C}(\partial M) \cong [0,2) \times \partial M$,

$$\phi(t, x) = \frac{1}{\mathcal{H}^{m-1}(B_t(x))} \int_{B_t(x)} f \, d\mathcal{H}^{m-1}. \qquad \square$$

As for the case $f \in W^{1-1/p,p}(\partial M, N) \cap \mathrm{VMO}(\partial M)$ in the above proof, we may replace ϕ in (3.4) by h, the harmonic extension of f in $\mathcal{C}(\partial M)$. In fact, Brezis and Nirenberg ([9]) proved that if $f \in \mathrm{VMO}(\partial M, \mathbb{R}^k)$, its harmonic extension also belongs to VMO. Thus, by the same reason as above, $\varphi := \pi_N \circ h$ is defined in some collar neighbourhood of ∂M and belongs to $W^{1,p}$. In this case, denoting by $P_t(x, y) = P((t, x), y)$ the Poisson kernel, where $(t, x) \in [0, 2) \times \partial M$ and $y \in \partial M$, $\varphi_t(x) = \pi_N \circ \phi_t(x)$, where $\phi_t(x) = \int_{\partial M} P_t(x, y) f(y) \, dy$. Note that the case $f \in \mathrm{Lip}(\partial M, N)$ (which is considered in [19]) is included in both of the above cases.

Under more restrictive hypothesis on N, other method is possible. Hardt–Lin (see [14]) showed that when $\pi_1(N) = \ldots = \pi_{[p]-1}(N) = 0$, any extension

$F' \in W^{1,p}(M, \mathbb{R}^k)$ of $f \in W^{1-1/p,p}(\partial M, N)$ can be suitably projected to N to produce an extension $F \in W^{1,p}(M, N)$ of f. For general N, of course, this method does not work well.

Next we give proofs of Theorems 1.6 and 1.7.

PROOF OF THEOREM 1.6. Since the fiber of $\pi \colon \widetilde{N} \to N$ is naturally identified with $\pi_1(N)$ and $\pi_1(N)$ is finite, $\widetilde{N}$ is a compact Riemannian manifold (its Riemannian structure is defined by requiring $\pi \colon \widetilde{N} \to N$ is a local isometry). Thus the Sobolev space $W^{1-1/p,p}(\partial M, \widetilde{N})$ is defined by imbedding $\widetilde{N}$ into some Euclidean $\mathbb{R}^l$.

We first prove (a) and assume $\pi_1(\partial M) = 0$. Suppose $\mathfrak{o}_A(f) = 0$. Then there exists $F \in W^{1,p}(\mathcal{C}(\partial M), N)$ such that $\gamma F = f$ on ∂M. By the approximation theorem of Bethuel ([2, Theorem 2]), F can be strongly approximated by maps $\{F_n\} \in R_p^\infty(\mathcal{C}(\partial M), N)$, where $R_p^\infty(\mathcal{C}(\partial M), N)$ is defined as follows: $u \in R_p^\infty(\mathcal{C}(\partial M), N)$ if and only if u is smooth except on a singular set $\Sigma(u)$, where $\Sigma(u) = \sum_{j=1}^r \Sigma_j$ ($r \in \mathbb{N}$), where for $j = 1, \dots, r$, Σ_j is a subset of a submanifold of $\mathcal{C}(\partial M)$ of dimension $n - [p] - 1$, and the boundary of Σ_j is smooth.

By assumption $2 \leq p < 3$, we have $\operatorname{codim} \Sigma(F_n) \geq 3$ and the inclusion $\iota_n \colon \mathcal{C}(\partial M) \setminus \Sigma(F_n) \hookrightarrow \mathcal{C}(\partial M)$ induces an isomorphism

$$(\iota_n)_* \colon \pi_1(\mathcal{C}(\partial M) \setminus \Sigma(F_n)) \xrightarrow{\sim} \pi_1(\mathcal{C}(\partial M)) \simeq \pi_1(\partial M) = 0$$

for all n. Thus there exists a lift $\widetilde{F}_n \colon \mathcal{C}(\partial M) \setminus \Sigma(F_n) \to \widetilde{N}$ of F_n for all n. Note that $\widetilde{F}_n \in R_p^\infty(\mathcal{C}(\partial M), \widetilde{N})$ and since $F_n \to F$ in $W^{1,p}(\mathcal{C}(\partial M), N)$, $\{\widetilde{F}_n\}$ is Cauchy in $W^{1,p}(\mathcal{C}(\partial M), \widetilde{N})$ (recall that $\pi \colon \widetilde{N} \to N$ is a local isometry).

Let $\widetilde{F} \in W^{1,p}(\mathcal{C}(\partial M), \widetilde{N})$ be the limit of $\{\widetilde{F}_n\}$. Define $\widetilde{f} := \gamma \widetilde{F}$. $\widetilde{f}$ satisfies $\widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{N})$ and $f = \pi(\widetilde{f})$. On the other hand, for any $\widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{N})$, we have $\pi(\widetilde{f}) \in W^{1-1/p,p}(\partial M, N)$. This completes the proof of (a).

We next prove (b). Assume first $\pi_1(M) = 0$. Suppose $f \in T^p(\partial M, N)$. Then there exists $F \in W^{1,p}(M, N)$ such that $\gamma F = f$. By arguing as in the proof of (a), we can find a lifting $\widetilde{F} \in W^{1,p}(M, \widetilde{N})$ of F. Set $\widetilde{f} := \gamma \widetilde{F} \in W^{1-1/p,p}(\partial M, \widetilde{N})$. We then have $f = \pi(\widetilde{f})$.

On the other hand, assume $f = \pi(\widetilde{f})$ for some $\widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{N})$. Since $\pi_1(\widetilde{N}) = 0$, by the result of Hart–Lin in [14] ($[p] - 1 = 1$ under our assumption), $\widetilde{f}$ has an extension $\widetilde{F} \in W^{1,p}(M, \widetilde{N})$. Set $F := \pi(\widetilde{F})$. Then $F \in W^{1,p}(M, N)$ and $\gamma F = f$, that is, $f \in T^p(\partial M, N)$.

The proof for the case $\pi_1(\partial M) = 0$ is similar. Assume $f \in T^p(\partial M, N)$. Let $F \in W^{1,p}(M, N)$ be such that $\gamma F = f$. Since $F|_{\mathcal{C}(\partial M)} \in W^{1,p}(\mathcal{C}(\partial M), N)$, arguing as in (a), we find $\widetilde{F} \in W^{1,p}(\mathcal{C}(\partial M), \widetilde{N})$ such that $\pi(\widetilde{F}) = F|_{\mathcal{C}(\partial M)}$. Define $\widetilde{f} = \gamma \widetilde{F}$ on ∂M. We clearly have $f = \pi(\widetilde{f})$ and $\widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{N})$.

The proof of the converse is the same as in the case $\pi_1(M) = 0$. This completes the proof. $\qquad\square$

Before to give the proof of Theorem 1.7, we need to give the definition of the Sobolev space $W^{1-1/p,p}(\partial M, \widetilde{G})$ since $\widetilde{G}$ is in general a non-compact manifold. However, as we will see, when G is a compact Lie group, there is a natural definition of it.

We recall briefly the structural theory of compact Lie groups. Let $\mathfrak{g}$ be the Lie algebra of G. By the structural theory of compact Lie algebras (see [11]), $\mathfrak{g}$ splits into the direct sum of its center $\mathfrak{c}$ and its simple ideals $\mathfrak{g}_1, \dots, \mathfrak{g}_l$, namely

$$(3.5) \qquad \mathfrak{g} = \mathfrak{c} \oplus \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_l.$$

Let $k = \dim_{\mathbb{R}} \mathfrak{c}$ (where $\mathfrak{c}$ is considered as a vector space over $\mathbb{R}$) and G_j for $j = 1, \dots, l$ the connected, simply connected Lie group with the Lie algebra $\mathfrak{g}_j$. By (3.5), the universal covering $\widetilde{G}$ of G is $\widetilde{G} = \mathbb{R}^k \times G_1 \times \dots \times G_l$. Since G_j $(j = 1, \dots, l)$ is a compact manifold, the Sobolev spaces $W^{1,p}(M, G_j)$ and $W^{1-1/p,p}(\partial M, G_j)$ are defined as in Section 1 and we define

$$(3.6) \qquad W^{1,p}(M, \widetilde{G}) = W^{1,p}(M, \mathbb{R}^k) \times W^{1,p}(M, G_1) \times \dots \times W^{1,p}(M, G_l),$$

$$(3.7) \quad W^{1-1/p,p}(\partial M, \widetilde{G}) = W^{1-1/p,p}(\partial M, \mathbb{R}^k) \times W^{1-1/p,p}(\partial M, G_1)$$
$$\times \dots \times W^{1-1/p,p}(\partial M, G_l).$$

Under these preparations, we now prove Theorem 1.7.

PROOF OF THEOREM 1.7. We first prove (a). Assume $f \in W^{1-1/p,p}(\partial M, G)$ is written $f = \pi(\widetilde{f})$ for $\widetilde{f} = (\widetilde{f}_0, \dots, \widetilde{f}_l) \in W^{1-1/p,p}(\partial M, \widetilde{G})$. Since G_j in (3.6) and (3.7) is a connected, simply connected compact Lie group and $\pi_2(G) = 0$ for any Lie group G, it is in fact 2-connected, that is, $\pi_k(G_j) = 0$ for $k = 0, 1, 2$. Thus applying the usual trace theorem to $\widetilde{f}_0$ and the result of Hardt–Lin (see [14]) to each component $\widetilde{f}_j$ $(j = 1, \dots, l)$ (recall $2 \le p < 4$), there exists $\widetilde{F} \in W^{1,p}(M, \widetilde{G})$ such that $\gamma \widetilde{F} = \widetilde{f}$. Define $F := \pi(\widetilde{F})$. We have $\gamma F = f$ and $\mathfrak{o}_A(f) = 0$.

Conversely, suppose $\mathfrak{o}_A(f) = 0$. There exists $F \in W^{1,p}(\mathcal{C}(\partial M), G)$ such that $\gamma F = f$ on ∂M. Let Z^0 be the connected center of G, that is, the connected component of the center $Z = \{x \in G : xg = gx \text{ for all } g \in G\}$ of G containing the identity. Then it follows from the above direct sum decomposition of $\mathfrak{g}$ (3.5) that $Z^0 = \mathbb{T}^k$ (k-dimensional torus) and

$$\pi' \colon G' := Z^0 \times G_1 \times \dots \times G_l \to G; \quad (g_0, \dots, g_l) \mapsto g_0 \dots g_l$$

is a covering homomorphism with a finite kernel. By arguing as in the proof of Theorem 1.6, there exists $F' \in W^{1,p}(\mathcal{C}(\partial M), G')$ such that $\pi'(F') = F$.

Let $p \colon \mathbb{R}^k \to \mathbb{T}^k$ be the universal covering of $\mathbb{T}^k$ given by the exponential

$$p \colon (t_1, \dots, t_k) \mapsto (e^{it_1}, \dots, e^{it_k}).$$

Here we have assumed without loss of generality that $Z^0 = \mathbb{T}^k$ is the standard torus $\mathbb{T}^k = \{(e^{it_1}, \dots, e^{it_k}) : (t_1, \dots, t_k) \in \mathbb{R}^k\}$. Let

$$\pi'' = p \times \mathrm{id} \times \dots \times \mathrm{id} \colon \widetilde{G} \to Z^0 \times G_1 \times \dots \times G_l.$$

Here we recall the results of Bethuel–Zheng ([5, Lemma 1], see also [6] for a simpler proof). Their result asserts that for simply connected manifold Ω, $u \in W^{1,p}(\Omega, \mathbb{S}^1)$ can be written as $u = e^{i\varphi}$ for some $\varphi \in W^{1,p}(\Omega, \mathbb{R})$. Applying this to the Z^0 factor of F', we have $\widetilde{F} \in W^{1,p}(\mathcal{C}(\partial M), \widetilde{G})$ such that $F' = \pi''(\widetilde{F})$. Set $\widetilde{f} := \gamma \widetilde{F} \in W^{1-1/p,p}(\partial M, \widetilde{G})$. Since $\pi = \pi' \circ \pi''$, we have $\pi(\widetilde{f}) = f$. This completes the proof of (a).

One can prove the assertion of (b) by combining the arguments in the proof of Theorems 1.6(b) and 1.7(a), so we complete the proof of Theorem 1.7. $\qquad \square$

Under the assumptions of Theorems 1.6 or 1.7, one may wonder whether any $f \in W^{1-1/p,p}(\partial M, N)$ can be written as $f = \pi(\widetilde{f})$ for some $\widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{N})$. This is not true in general. We give the following example.

EXAMPLE 3.4. Let $3 \le p < 4$, $M = \mathbb{B}^2 \times \mathbb{S}^2$ and $N = SO(3)$. Recall that $SO(3) = \{A \in M_3(\mathbb{R}) : A^t A = I,\ \det A = 1\}$ is a compact connected Lie group and its universal covering is the 2-fold covering $\pi \colon \mathbb{S}^3 \to SO(3)$ (there are many ways describing it, see [11]). From this $\pi_1(SO(3)) = \mathbb{Z}_2$ is finite and $\pi_2(SO(3)) \simeq \pi_2(\mathbb{S}^3) = 0$. Note also that $\pi_1(M) = 0$. We show that there exists $f \in W^{1-1/p,p}(\partial M, N)$ which can not be written as $f = \pi(\widetilde{f})$ for $\widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{N})$.

Let $\gamma \colon \partial B^2 = \mathbb{S}^1 \to SO(3)$ be an essential smooth loop. Define $f(x,y) = \gamma(x)$ for $(x,y) \in \partial \mathbb{B}^2 \times \mathbb{S}^2$. Clearly $f \in W^{1-1/p,p}(\partial M, N)$. Assume that there exists $\widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{N})$ such that $f = \pi(\widetilde{f})$. Let $h_{\widetilde{f}} \in W^{1,p}(M, \mathbb{R}^4)$ be the harmonic extension of $\widetilde{f}$.

By Fubini $h_{\widetilde{f}}(\cdot, y) \in W^{1,p}(\mathbb{B}^2, \mathbb{R}^4)$ for a.e. $y \in \mathbb{S}^2$. Thus for a.e. $y \in \mathbb{S}^2$, $h_{\widetilde{f}}(\cdot, y)|_{\partial \mathbb{B}^2} = \widetilde{f}(\cdot, y) \in W^{1-1/p,p}(\partial \mathbb{B}^2, \mathbb{S}^3)$. Since $3 \le p < 4$, by the Sobolev imbedding, $\widetilde{f}(\cdot, y) \in C^0(\partial \mathbb{B}^2, \mathbb{S}^3)$ for a.e. $y \in \mathbb{S}^2$ and $\pi(\widetilde{f}(\cdot, y)) = f(\cdot, y) = \gamma$ for a.e. $y \in \mathbb{S}^2$. Since $\pi_1(\mathbb{S}^3) = 0$, it follows that γ is homotopic to 0. This is a contradiction.

As noted in the introduction, in view of Theorem 1.7, this example shows that the conjecture 2 in [4] is not true in general. However, under the assumption that $3 \le p < 4$, $\pi_1(N)$ is finite and some additional assumption on M, for example the case $\pi_1(\partial M) = 0$ or $M = \mathbb{B}^m$ etc. it is reasonable to conjecture that any $f \in W^{1-1/p,p}(M, N)$ can be written as $f = \pi(\widetilde{f})$ for some $\widetilde{f} \in W^{1-1/p,p}(\partial M, \widetilde{N})$ and Conjecture 2 in [4] holds at least for such a case.

We recall here the important role played by $\pi_{[p]-1}(N)$ in the extension problem (see [3], [4]). The following result holds (which is also due to Bethuel and

Demengel [4]). For readers convenience, we give a proof using our Theorems 1.2 and 1.3. (Note that the proof given in [4] relies on the approximation theorem of Bethuel [2]. Our proof does not rely on such an approximation theorem).

PROPOSITION 3.5. *Let* $1 < p < \dim M$. *Assume* $\pi_{[p]-1}(N) \neq 0$, *then there exists* $f \in W^{1-1/p,p}(\partial M, N)$ *such that* f *has no* $W^{1,p}(M, N)$-*extension.*

PROOF. By assumption $\pi_{[p]-1}(N) \neq 0$, there exists $\varphi_0 \colon \mathbb{S}^{[p]-1} \to N$ such that φ_0 is not homotopic to a constant map. We inductively define $\varphi_k \colon \mathbb{S}^{[p]+k-1} \to N$ for $0 \leq k \leq m - [p]$ by

$$\varphi_k(x) := \varphi_{k-1}\left(\frac{x'}{|x'|}\right), \quad \text{where } x \in \mathbb{S}^{[p]+k-1}, \ x = (x', x_{[p]+k}).$$

One can easily verify $\varphi_k \in W^{1-1/p,p}(\mathbb{S}^{[p]+k-1}, N)$ for $k = 0, \dots, m - [p]$.

Define $\Phi := \varphi_{m-[p]}$. Φ is in $W^{1-1/p,p}(\mathbb{S}^{m-1}, N)$ and smooth away from a $m - [p] - 1$-dimensional closed set. Let $x_0 \in \mathbb{S}^{m-1}$ be a smooth point of Φ and consider the geodesic ball $\mathbb{B}_{2r}(x_0) \subset \mathbb{S}^{m-1}$ with center x_0 and radius $2r$. Choosing $r > 0$ small enough, we may assume Φ is smooth in $\mathbb{B}_{2r}(x_0)$. Modifying Φ in $\mathbb{B}_{2r}(x_0)$, one can easily construct $\widetilde{\Phi}$ satisfying $\widetilde{\Phi} = \Phi$ in $\mathbb{S}^{m-1} \setminus \mathbb{B}_{2r}(x_0)$ and $\widetilde{\Phi} \equiv a$ in $\mathbb{B}_r(x_0)$, where $a \in N$. Let $y_0 \in \partial M$ and consider the geodesic ball $\mathbb{B}_\rho(y_0)$ in ∂M. Since $\mathbb{S}^{m-1} \setminus \mathbb{B}_r(x_0) \cong \overline{\mathbb{B}_\rho(y_0)}$, there exists a diffeomorphism $G \colon \mathbb{B}_\rho(x_0) \xrightarrow{\sim} \mathbb{S}^{m-1} \setminus \mathbb{B}_r(x_0)$. Define f by

$$f := \begin{cases} a & \text{on } \partial M \setminus \mathbb{B}_\rho(y_0), \\ \widetilde{\Phi} \circ G & \text{on } \mathbb{B}_\rho(y_0). \end{cases}$$

Then $f \in W^{1-1/p,p}(\partial M, N)$. We claim that f does not have $W^{1,p}(M, N)$-extension. We prove this by contradiction. Suppose there exists $F \in W^{1,p}(M, N)$ such that $\gamma F = f$. In the case $p \notin \mathbb{Z}$, by Theorem 1.2 $f|_{(\partial M)^{[p]-1}}$ has a continuous extension $M^{[p]} \to N$ for any generic pair $(M^{[p]}, (\partial M)^{[p]-1})$. By the construction of f, there exists a $[p]$-cell $\sigma^{[p]}$ of M such that $f|_{\partial\sigma^{[p]}} \colon \partial\sigma^{[p]} \cong \mathbb{S}^{[p]-1} \to N$ is continuous, $F|_{\sigma^{[p]}} \in W^{1,p}(\sigma^{[p]}, N)$ and $f|_{\partial\sigma^{[p]}}$ is not homotopic to a constant map. This is a contradiction, since $F_{\sigma^{[p]}}$ is a continuous extension of $f_{\partial\sigma^{[p]}}$. In the case $p \in \mathbb{Z}$, by the construction of f, there exists a p-cell σ^p such that $f|_{\partial\sigma^p} \colon \partial\sigma^p \to N$ is continuous, $F|_{\sigma^p} \in W^{1,p}(\sigma^p, N)$ and $f|_{\partial\sigma^p}$ is not homotopic to a constant map. By Theorem 1.3, there exists $\{f_i\} \subset \operatorname{Lip}(\sigma^p, N)$ such that $f_i \to f$ in $W^{1-1/p,p}(\partial\sigma^p, N)$. Then by [3, Lemma 1], $f_i \sim f$ on $\partial\sigma^p$ for large i. This is a contradiction since $f_i \sim 0$ on $\partial\sigma^p$. $\qquad\square$

In fact, the above proof also works if f has only a $W^{1,p}(\mathcal{C}(\partial M), N)$-extension. Thus we have

COROLLARY 3.6. *If* $\mathfrak{o}_A(f) = 0$ *for any* $f \in W^{1-1/p,p}(\partial M, N)$, *then it is necessary* $\pi_{[p]-1}(N) = 0$.

One may wonder whether the condition $\pi_{[p]-1}(N) = 0$ is also sufficient to conclude $\mathfrak{o}_A(f) = 0$ for any $f \in W^{1-1/p,p}(\partial M, N)$, that is, $\pi_{[p]-1}(N) = 0$ is equivalent to $\mathfrak{o}_A(f) = 0$ for any $f \in W^{1-1/p,p}(\partial M, N)$? It turns out that this is not true in general. We explain this by using Theorem 1.7 for the special case $G = \mathbb{S}^1$ ($\mathbb{S}^1$ is the simplest compact Lie group). Note that in this special case, Theorem 1.7 holds for any $2 \le p < \dim M$ since $\widetilde{\mathbb{S}^1} = \mathbb{R}$ is contractible.

By Theorem 1.7, $\mathfrak{o}_A(f) = 0$ for $f \in W^{1-1/p,p}(\partial M, \mathbb{S}^1)$ if and only if $f = e^{i\varphi}$ for some $\varphi \in W^{1-1/p,p}(\partial M, \mathbb{R})$ (recall that $e^{i\cdot} \colon \mathbb{R} \to \mathbb{S}^1$ is the universal covering). That is, the extension problem is equivalent to the lifting problem. The latter problem is extensively studied in [6] and [7]. In their papers [6] and [7], Bourgain et al. studied the problem of lifting $W^{s,p}(\Omega, \mathbb{S}^1)$ to $W^{s,p}(\Omega, \mathbb{R})$ for simply connected Ω and $0 < s < \infty$, $1 < p < \infty$. One of their results asserts that when $1 \le sp < \dim M$, there exists $f \in W^{1,p}(M, \mathbb{S}^1)$ which does not admit a $W^{s,p}(M, \mathbb{R})$-lifting, that is, there is no $\varphi \in W^{s,p}(M, \mathbb{R})$ satisfying $f = e^{i\varphi}$. Applying their result to our case (we replace Ω by ∂M and take $s = 1 - 1/p$), there exists $f \in W^{1-1/p,p}(\partial M, \mathbb{S}^1)$ which does not admit a $W^{1-1/p,p}(\partial M, \mathbb{R})$-lifting when $2 \le p < \dim M$.

By Theorem 1.7 and the above remark, we in particular derive the existence of $f \in W^{1-1/p,p}(\partial M, \mathbb{S}^1)$ ($2 \le p < \dim M$) which does not satisfy $\mathfrak{o}_A(f) = 0$. Note that when $p \ge 3$, $\pi_{[p]-1}(\mathbb{S}^1) = 0$. We thus conclude that in general $\mathfrak{o}_A(f) = 0$ for any $f \in W^{1-1/p,p}(\partial M, N)$ is not equivalent to the condition $\pi_{[p]-1}(N) = 0$.

As noticed in the beginning of this subsection, the condition $\mathfrak{o}_A(f) = 0$ may be seen as a regularization condition, that is, $\varphi(t, \cdot)$ may be seen as a regularization of f (see the beginning of this subsection). From this observation, one may ask the following question: Assume that there exists $\{f_n\}$ in $W^{1,p}(\partial M, N)$ such that $f_n \to f$ in $W^{1-1/p,p}(\partial M, N)$. Under this assumption, $\mathfrak{o}_A(f) = 0$ holds?

Unfortunately, the answer is no even if we assume $f_n \in C^\infty(\partial M, N)$. In fact, consider the case $N = \mathbb{S}^1$ and $p = 2$. It is shown in [7] that there exists $f \in \overline{C^\infty(\partial M, \mathbb{S}^1)}^{H^{1/2}}$ (the strong closure of $C^\infty(\partial M, \mathbb{S}^1)$ in $H^{1/2}(\partial M, \mathbb{S}^1)$) which does not admit a $H^{1/2}(\partial M, \mathbb{R})$-lifting, that is, there is no $\varphi \in H^{1/2}(\partial M, \mathbb{R})$ such that $f = e^{i\varphi}$. For such f the above assumption is satisfied (for $f_n \in C^\infty(\partial M, \mathbb{S}^1)$), however, by Theorem 1.7, f does not satisfy $\mathfrak{o}_A(f) = 0$. In [7], such f is constructed independent of the topology (and geometry) of ∂M, and the existence of such f seems to rely on the global structure of $\mathbb{S}^1$ and the analytical structure of $\mathbb{S}^1$-valued maps. From this, in general, one may conclude that the condition $\mathfrak{o}_A(f) = 0$ contains both topological and analytical information.

In Example 3.3 we have seen that, for some M, N and $1 < p < \dim M$, there exists $f \in W^{1-1/p,p}(\partial M, N)$ such that $\mathfrak{o}_A(f) = 0$ but $\mathfrak{o}_B(f)$ is not satisfied.

On the other hand, Theorems 1.6 and 1.7 show that for some cases there exists $f \in W^{1-1/p,p}(\partial M, N)$ such that $\mathfrak{o}_A(f) \neq 0$ but $\mathfrak{o}_B(f)$ is satisfied. Combining these two examples, one can produce an example of M, N, $1 < p < \dim M$ and $f \in W^{1-1/p,p}(\partial M, N)$ such that $\mathfrak{o}_A(f) \neq 0$ and $\mathfrak{o}_B(f)$ is not satisfied. Therefore, in general, $\mathfrak{o}_A$ and $\mathfrak{o}_B$ is essentially independent to each other.

In this paper, we have defined two obstructions $\mathfrak{o}_A(f)$ and $\mathfrak{o}_B(f)$. We have shown that $\mathfrak{o}_B(f)$ is completely characterized by the topology of the pair $(M^{[p]}, (\partial M)^{[p]-1})$, the topology of N and the topology of the map $f \colon (\partial M)^{[p]-1} \to N$. However, as for $\mathfrak{o}_A(f)$, we have characterized f satisfying $\mathfrak{o}_A(f) = 0$ only for some special cases. In general, giving a reasonable criterion of f satisfying $\mathfrak{o}_A(f) = 0$ still remains as a problem.

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