Topological Methods in Nonlinear Analysis
Journal of the Juliusz Schauder Center
Volume 21, 2003, 53–80

INDEX OF SOLUTION SET FOR PERTURBED FREDHOLM EQUATIONS AND EXISTENCE OF PERIODIC SOLUTIONS FOR DELAY DIFFERENTIAL EQUATIONS

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Abstract. We consider the index of the solution set of Fredholm equations with f-condensing type perturbations. This characteristic is applied to the existence of periodic solutions for delay differential equations.

1. Introduction

Let E, F be Banach spaces, $f, k: E \to F$ continuous maps. The authors of paper [2] consider the situation when the operator equation

$$f(x) + k(x) = 0 \tag{1.1}$$

possesses some regular properties only on a certain neighbourhood of the solution set. In particular, the main hypothesis of [2] includes the assumptions that the solution set of (1.1) is compact, restriction of f onto a certain neighbourhood of the solution set of (1.1) is a C^r-smooth ($r \geq 1$) Fredholm map of index $n \geq 0$ and k is a completely continuous map. Under these assumptions it was proposed to construct the topological index of the solution set of (1.1)

2000 *Mathematics Subject Classification*. 34K13, 47H11, 47H09.
Key words and phrases. Index of solution set, f-condensing map, measure of noncompactness, delay differential equation, periodic solution, existence theorem.
The research was supported by RFBR, grant No. 01-01-00425, and by CRDF, award No. VZ-010-0.

possessing the usual properties of the characteristic of this sort. By now this program has been realized only for the case of a non-oriented index (see [16]). In the mentioned paper some applications of the index to the solvability of the Dirichlet problem for the Monge–Ampére equation were also presented. In work [17] the index of the solution set for Fredholm equations with compactly restricted perturbations was introduced and investigated. As an application the solvability of the boundary-value problem for a system of ordinary differential equations unsolved with respect to the highest-order derivative was considered. Furthermore in [6] this approach was used also to solve the periodic problem for ordinary differential equations.

It is well known that the periodic problem, being one of classical objects of mathematics and mechanics, has attracted the attention of many researchers. We mention here only the works closely related to the subject of the present paper.

M. A. Krasnosel'skiĭ in [10] has reduced the periodic problem for the differential equation

$$\dot{x} = F(t, x) \tag{1.2}$$

to the study of a fixed point of the translation operator along solutions of equation (1.2) (the Poincare–Krasnosel'skiĭ map). Applying the theory of the rotation of vector fields and appropriate homotopies along the trajectories of solutions of (1.2) he proved the existence of periodic solutions under the condition that the Brouwer's degree of the map $F(0,\,\cdot\,)\colon B_r(0) \subset \mathbb{R}^n \to \mathbb{R}^n$ differs from zero on a ball $B_r(0)$.

Another approach, based on the coincidence topological degree, was proposed by J. Mawhin in [7] and [11], where the autonomous averaged vector field

$$f_0\colon z \mapsto \frac{1}{\omega} \int_0^\omega f(s, z)\, ds$$

was chosen as the final point of homotopies. A detailed survey of related results and further developments can be found in [4].

In paper [12] the existence of solutions for the periodic boundary-value problem for the equation

$$(p(t)x'(t))' = f(t, x(t), x'(t), x''(t)) + y(t)$$

is proved. Further, in papers [13] and [14], these results were generalized to the case of equations of a higher order. In these researches the degree theory for A-proper maps was used as the main tool.

Note that the use of the degree theory requires, as a rule, some strong conditions on the operators, generated by an equation, and therefore on the functions determining this equation. Usually these assumptions are imposed on the behavior of the functions on the entire domain. The application of the index of solution set needs structural conditions only on a certain neighbourhood of the solution set. This neighbourhood is chosen by some a priori arguments.

In the present paper the topological index of solution set, introduced in [17], is applied to the investigation of periodic solutions of the differential equation of the form

$$a(t, x(t), x(t-\tau), x'(t), x'(t-\tau)) = b(t, x(t), x(t-\tau), x'(t), x'(t-\tau)). \tag{1.3}$$

We suppose that the functions a, b are ω-periodic in the first variable and that the delay τ is commensurable with ω.

The paper consists of the introduction and 5 sections. In Section 2 we present basic notions and properties of the index of the solution set for equations with local Fredholm and local f-condensing maps. Section 3 contains auxiliary results concerning the properties of the superposition operator. In Section 4 we study conditions under which the index of solution set is well-posed for each differential equation from the studied family. The evaluation of the index for one of these equations is contained in Section 5. We reduce the evaluation of the index to that of the nonoriented degree for a finite dimensional map. Section 6 contains the main theorem on the existence of a periodic solution for differential equation (1.3). To prove it we evaluate the index of solution set using its main properties. As an application, the existence of a periodic solution for a nonlinear differential equation is considered.

2. The index of the solution set for Fredholm equations with f-condensing perturbations

Let E, F be real Banach spaces, X be an open subset of the space E, $f\colon X \to F$ be a C^r-smooth map, $r \geq 1$.

We recall some known notions [1], [3], [6].

Definition 2.1. The map $f\colon X \to F$ is called a *Fredholm map of the index* 0 (briefly $\Phi_0 C^r$*-map*) on the set $M \subset X$ if at any point $u \in M$ the Frechet derivative $f'(u)$ is a linear Fredholm operator of the index 0, i.e. $\dim \operatorname{Ker} f'(u) = \dim \operatorname{Coker} f'(u) < \infty$.

Definition 2.2. Let V be closed in E, $V \subset X$. We say that the restriction $\overline{f} = f|_V\colon V \to F$ is a *proper map on* V if $f^{-1}(K) \cap V$ is a compact set for any compact $K \subseteq F$.

Definition 2.3. The measure of noncompactness in a Banach space F is a function ψ assigning a nonnegative number $\psi(M)$ to any bounded set $M \subseteq F$ such that the following conditions are fulfilled:

(a) $\psi(\overline{\text{co}}(M)) = \psi(M)$, where $\overline{\text{co}}(M)$ is the closure of a convex hull of the set M,
(b) from inclusion $M_1 \subseteq M_2$ it follows that $\psi(M_1) \leq \psi(M_2)$.

We suppose below that all measures of noncompactness satisfy the following conditions:

(c) $\psi(M) = 0$ if and only if M is relatively compact,
(d) $\psi(M_1 \cup M_2) \leq \max\{\psi(M_1), \psi(M_2)\}$,
(e) $\psi(M_1 + M_2) \leq \psi(M_1) + \psi(M_2)$ for any bounded subset M_1, M_2 in F.

As the example of a measure of noncompactness satisfying above conditions we can present Kuratowski measure of noncompactness $\alpha(M)$:

$$\alpha(M) = \inf\{d > 0, \text{ for which } M \text{ may be covered by a finite number of sets of diameter } d\}.$$

Let U be an arbitrary subset of a Banach space E, $f, g: U \to F$ be maps acting from the set U to a Banach space F.

Definition 2.4. The map $g: U \to F$ is called *f-condensing* on the set U with respect to the measure of noncompactness ψ if $\psi(g(M)) < \psi(f(M))$ for any M in U such that $\psi(g(M)) \neq 0$.

Consider the equation

$$f(u) - g(u) = 0, \quad u \in X. \tag{2.1}$$

Denote by $Q \subseteq X$ the set of solutions of the equation (2.1), i.e. $Q = (f-g)^{-1}(0)$.

We suppose that the following conditions are fulfilled:

(C_1) The set Q is compact.
(C_2) There exists an open neighbourhood of the compact Q such that the map f is a $\Phi_0 C^1$-map on the set Q.
(C_3) There exists an open neighbourhood U of the compact Q such that $g|_{\overline{U}}$ is a f-condensing map on the set U with respect to the measure of noncompactness ψ.

Under the above conditions the index of the solution set of equation (2.1), $\text{ind}_2(f - g, X, 0)$, with values in the additive group Z_2, is defined in [6].

We point out some properties of the index that we will need in the sequel.

PROPERTY 2.5. *If an index* $\mathrm{ind}_2(f-g,X,0)$ *differs from zero then equation* (2.1) *has a solution in* X.

In order to formulate the second property we consider the family of operator equations

$$f(u,\lambda)-g(u,\lambda)=0,\quad u\in X,\ \lambda\in[0,1], \tag{2.2}$$

where $f,g\colon X\times[0,1]\to F$ are continuous maps. Denote by $Q_\Gamma\subseteq X\times[0,1]$ the set of all solutions (u,λ) of family (2.2).

We suppose that the following conditions are fulfilled:

(Γ_1) the set Q_Γ is compact in $X\times[0,1]$,

(Γ_2) there exists an open neighbourhood U of the compact Q_Γ with the property: on the closure of U the map f is continuous in λ uniformly with respect to u and $f(\,\cdot\,,\lambda)\colon U\cap(E\times\{\lambda\})\to F$ is a Φ_0C^1-map for any fixed λ, whose derivative $f'_u(u,\lambda)$ is continuous in u and λ,

(Γ_3) the map g is a f-condensing map with respect to the measure of noncompactness ψ on some neighbourhood U of Q_Γ.

Under the above conditions the following property holds.

PROPERTY 2.6 (The homotopy invariance property of the index). *Let conditions* (Γ_1)–(Γ_3) *be fulfilled. Then*

$$\mathrm{ind}_2(f_0-g_0,X,0)=\mathrm{ind}_2(f_1-g_1,X,0),$$

where $f_i(u)=f(u,i)$, $g_i(u)=g(u,i)$, $i=0,1$, $u\in X$.

By virtue of the compactness of the solution set it is sufficient to investigate the properties of f and g in some neighbourhood of each point $q\in Q$.

DEFINITION 2.7. A map $g\colon U\to F$ is called *kf-bounded with respect to the measure of noncompactness* ψ at a point $q\in U$ if for any $\varepsilon>0$ there exists $\delta>0$ such that $\psi[g(M)]\le(k+\varepsilon)\psi[f(M)]$ for any M from $B(q,\delta)$.

THEOREM 2.8. *Let a map f be continuously differentiable on some neighbourhood of the compact Q and $f'(q)$ be a Fredholm operator for any $q\in Q$. Let a map g be locally $kf'(q)$-bounded with respect to the measure of noncompactness α at any point $q\in Q$ and $k<1$. Then the map g is f-condensing with respect to the measure of noncompactness α on some open neighbourhood U of the compact Q.*

The proof can be found in [17].

3. Auxiliary results concerning the properties of a superposition operator

Here we present some necessary facts and statements concerning the properties of a superposition operator in the space of functions continuous on $[0,\omega]$.

Let x be a vector in the space $\mathbb{R}^n$ and $|x|$ be the norm of this vector.

Theorem 3.1 (Implicit Function Theorem). *Let the functions $\varphi,\mu\colon U\subset\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^n$ be continuous in a neighbourhood U of the point (x_0,y_0) and the following conditions be satisfied:*

(a) $\varphi(x_0,y_0)=\mu(x_0,y_0)$,

(b) *the function $\varphi(x,y)$ has a continuous derivative $\partial\varphi/\partial y$ on U and*

$$\det\frac{\partial\varphi}{\partial y}(x_0,y_0)\neq 0,$$

(c) $|\mu(x,y)-\mu(x,\overline{y})|\leq c|\partial\varphi/\partial y(x_0,y_0)(y-\overline{y})|$ *for all $(x,y),(x,\overline{y})\in U$ and any constant $c<1$.*

Then, in a certain neighbourhood of the point x_0, there exists a unique function $y=y(x)$ such that it is continuous, $y(x_0)=y_0$ and $\varphi(x,y(x))=\mu(x,y(x))$.

The statement follows from Banach fixed point theorem.

The following inverse function theorem is a simple corollary of Theorem 3.1.

Theorem 3.2 (Inverse Function Theorem). *Let the functions $\varphi,\mu\colon U\subset\mathbb{R}^n\to\mathbb{R}^n$ be continuous in a neighbourhood U of the point x_0 and let the following conditions be satisfied:*

(a) $y_0=\varphi(x_0)-\mu(x_0)$,

(b) *the function $\varphi(x)$ has a continuous derivative $d\varphi/dx$ on U and*

$$\det\frac{d\varphi}{dx}(x_0)\neq 0,$$

(c) $|\mu(x)-\mu(\overline{x})|\leq c|(d\varphi/dx)(x_0)(x-\overline{x})|$ *for all $x,\overline{x}\in U$ and any constant $c<1$.*

Then, in a certain neighbourhood V_ε of the point y_0, there exists a continuous function, inverse to $\varphi-\mu$.

The statement follows from Theorem 3.1 applied to the equation

$$\varphi(x)-\mu(x)-y=0\quad\text{for } x\in U,\ y\in\mathbb{R}^n.$$

Let $V_0\subset[0,\omega]\times\mathbb{R}^n$ be a bounded set, closed in $[0,\omega]\times\mathbb{R}^n$, and V be the closure of a certain bounded open neighbourhood of the set V_0.

Denote by $C([0,\omega],\mathbb{R}^n)$ the space of continuous functions with the norm $\|x\|_0=\max_{t\in[0,\omega]}|x(t)|$ (where $|\cdot|$ is the norm in $\mathbb{R}^n$, defined by the formula $|u|=\max_{1\leq i\leq n}|u_i|$, $u=(u_1,\dots,u_n)\in\mathbb{R}^n$). As usual, $C^k([0,\omega],\mathbb{R}^n)$ denotes the space of functions having continuous derivatives up to order k with the norm $\|x\|_k = \sum_{j=0}^{k} \|x^{(j)}\|_0$.

Define the subsets W_0, W in the space $C([0,\omega],\mathbb{R}^n)$ in the following way:

$$\begin{aligned} W_0 &= \{u \in C([0,\omega],\mathbb{R}^n) : (t,u(t)) \in V_0, \quad \text{for all } t \in [0,\omega]\}, \\ W &= \{u \in C([0,\omega],\mathbb{R}^n) : (t,u(t)) \in V, \ \text{for all } t \in [0,\omega]\}. \end{aligned}$$

Denote by D_0 the set

$$D_0 = \{(t,u) \in V : \varphi(t,u) - \mu(t,u) = 0\}.$$

Let $\{u_k(t)\}$ be an arbitrary sequence of functions from W. We say that the sequence $\{u_k(t)\}$ converges to the set V_0, if for any $\varepsilon > 0$ there exists a number k_0 such that for every $k \ge k_0$ the graphs of the functions $\{(t,u_k(t))\}$ are contained in the ε-neighbourhood V_ε of V_0.

Theorem 3.3. *Let the functions $\varphi,\mu\colon V \subset [0,\omega] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and for any point $(t_0,u_0) \in D_0 \cap V_0$ there exists a neighbourhood such that:*

(a) *the function $\varphi(t,u)$ has a continuous derivative $\partial\varphi/\partial u$ on V and*

$$\det \frac{\partial\varphi}{\partial u}(t_0,u_0) \ne 0,$$

(b) *there exists a constant $c < 1$ such that*

$$|\mu(t,u) - \mu(t,\overline{u})| \le c\left|\frac{\partial\varphi}{\partial u}(t_0,u_0)(u-\overline{u})\right|$$

for any $(t,u),(t,\overline{u})$ from this neighbourhood.

Then if the sequence $\{u_k(t)\}$ of functions from W converges to V_0 and the sequence

$$y_k(t) = \varphi(t,u_k(t)) - \mu(t,u_k(t))$$

uniformly converges to zero-function on $[0,\omega]$, there exists a subsequence, uniformly converging on $[0,\omega]$.

Proof. The set V is bounded and compact, $(0,u_k(0)) \in V$, hence, without loss of generality, we can suppose the sequence $u_k(0)$ to be converging to u^0. By the conditions of the theorem $(0,u^0) \in V_0$ and $(0,u^0) \in D_0$. By Implicit Function Theorem 3.1 there exists a unique solution $u_0(t)$ of the equation $\varphi(t,u) = \mu(t,u)$, whose graph starts from the point $(0,u^0)$. From the same theorem it follows that the function $u_0(t)$ can be extended either on the entire interval $[0,\omega]$ or up to the moment when the graph leaves the set V_0.

Let the function $u_0(t)$ be extended on $[0,t_0]$. By the Implicit Function Theorem there exists $\varepsilon > 0$ such that the ε-neighbourhood of the graph of $u_0(t)$ does not contain the points $(t,u) \in D_0$, $t \in [0,t_0]$ except the points of the graph.

Denote by $U_{\varepsilon/3}D_0$ the $\varepsilon/3$-neighbourhood of D_0 and by $V(t_0)$ the set $V(t_0) = \{(t,u) \in V : t \in [0,t_0]\}$. Then on the set $\overline{V(t_0) \setminus U_{\varepsilon/3}D_0}$ the function $|\varphi(t,u) - \mu(t,u)|$ is positive and for some $\delta > 0$ we have $|\varphi(t,u) - \mu(t,u)| > \delta$ for all $(t,u) \in \overline{V(t_0) \setminus U_{\varepsilon/3}D_0}$.

Since the sequence $y_k(t)$ converges to zero uniformly on $[0,t_0]$, $|y_k(t)| < \delta$ for all $t \in [0,t_0]$ starting from a certain member of the sequence. Thus the graphs of the functions $u_k(t)$ are contained in the set $U_{\varepsilon/3}D_0$.

By the hypothesis, the initial points of the graphs $(t,u_k(t))$ are contained in the $\varepsilon/3$-neighbourhood of $(0,u^0)$ for k sufficiently large. Hence the graphs of these functions belong to the $\varepsilon/3$-neighbourhood of the graph of $u_0(t)$ for all $t \in [0,t_0]$. Thus the sequence $u_k(t)$ converges to $u_0(t)$ uniformly on $[0,t_0]$ since ε can be chosen arbitrarily small and δ is getting smaller.

If $[0,t_0] = [0,\omega]$, the theorem follows. Suppose that $(t_0,u_0(t_0)) \notin V_0$ and $t_0 < \omega$. Denote by ε the distance from $(t_0,u_0(t_0))$ to V_0. Since $(t_0,u_k(t_0))$ converges to $(t_0,u_0(t_0))$, $(t_0,u_k(t_0))$ does not belong to $V_{\varepsilon/2}$ for k sufficiently large. This contradicts to the hypothesis that the sequence $\{u_k(t)\}$ converges to the set V_0. Hence the graph of $u_0(t)$ is contained in V_0 and $[0,t_0] = [0,\omega]$. □

4. Conditions for the well-posedness of the index of the solution set of equation (1.3)

Consider the equation

$$a_0(t,x(t),x(t-\tau),x'(t),x'(t-\tau)) = b_0(t,x(t),x(t-\tau),x'(t),x'(t-\tau)). \tag{4.1}$$

We suppose that the maps

$$a_0, b_0 \colon \mathbb{R} \times \underbrace{\mathbb{R}^n \times \ldots \times \mathbb{R}^n}_{4} \to \mathbb{R}^n$$

are continuous and ω-periodic with respect to the first variable.

Suppose that the delay τ is commensurable with ω, that is there exists $\tau_0 > 0$ such that $p\tau_0 = \omega$ and $k\tau_0 = \tau$ for some integer p and k, $p > k$.

Consider the existence problem for ω-periodic solutions of equation (4.1) of a class $C^1(\mathbb{R},\mathbb{R}^n)$. Let $C^1_\omega(\mathbb{R}^n)$ be its subspace consisting of ω-periodic functions endowed with the norm from $C^1([0,\omega],\mathbb{R}^n)$. (In what follows we assume that the norm of a vector $u = (u_1,\ldots,u_n) \in \mathbb{R}^n$ is defined as $|u| = \max_{1\le i\le n}|u_i|$.)

Introduce the maps

$$\begin{aligned} f_0 &\colon C^1_\omega(\mathbb{R}^n) \to C_\omega(\mathbb{R}^n), & f_0(x)(t) &= a_0(t,x(t),x(t-\tau),x'(t),x'(t-\tau)),\\ g_0 &\colon C^1_\omega(\mathbb{R}^n) \to C_\omega(\mathbb{R}^n), & g_0(x)(t) &= b_0(t,x(t),x(t-\tau),x'(t),x'(t-\tau)). \end{aligned}$$

Then equation (4.1) is equivalent to the operator equation

$$f_0(x) = g_0(x). \tag{4.2}$$

We use the index of solution set to prove the solvability of this equation. We consider the family of differential equations

$$(4.1_\lambda)\quad a(t,x(t),x(t-\tau),x'(t),x'(t-\tau),\lambda)=b(t,x(t),x(t-\tau),x'(t),x'(t-\tau),\lambda)$$

and the corresponding family of operator equations

$$(4.2_\lambda)\qquad f_\omega(x,\lambda)=g_\omega(x,\lambda),\quad \lambda\in[0,1].$$

We suppose that the functions a and b are continuous, ω-periodic with respect to the first variable and that

$$a(\,\cdot\,,0)=a_0,\quad a(t,x(t),x(t-\tau),x'(t),x'(t-\tau),1)=a_1(t,x(t),x'(t)),$$
$$b(\,\cdot\,,0)=b_0,\quad b(\,\cdot\,,1)=0.$$

In the sequel we will see that under some conditions on the functions a and b the solvability of the equation

$$(4.1_1)\qquad a_1(t,x(t),x'(t))=0,$$

implies the solvability of (4.1) and follows from the solvability in $\mathbb{R}^n$ of the equation

$$a_1(0,u,0)=0.$$

Let $V_0\subset[0,\omega]\times\mathbb{R}^n\times\mathbb{R}^n$ be a bounded set, closed in $[0,\omega]\times\mathbb{R}^n\times\mathbb{R}^n$, and let the multivalued function $\Gamma(t)=\{(u,v)\in\mathbb{R}^n\times\mathbb{R}^n:(t,u,v)\in V_0\}$ be continuous on $[0,\omega]$ and $\Gamma(0)=\Gamma(\omega)$. Denote by V the closure of a certain bounded neighbourhood of V_0. For simplicity we use the same notation V_0 for the subset in $\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n$, which coincides with V_0 for all $t\in[0,\omega]$ and has the property

$$V_0\cap(\{t+\omega\}\times\mathbb{R}^n\times\mathbb{R}^n)=V_0\cap(\{t\}\times\mathbb{R}^n\times\mathbb{R}^n)$$

for all $t\in\mathbb{R}$. We do the same for V.

In $C^1([0,\omega],\mathbb{R}^n)$ we define subsets W_0,W in the following way:

$$W_0=\{x\in C^1([0,\omega],\mathbb{R}^n):(t,x(t),x'(t))\in V_0,\ \text{for all } t\in[0,\omega]\},$$
$$W=\{x\in C^1([0,\omega],\mathbb{R}^n):(t,x(t),x'(t))\in V,\ \text{for all } t\in[0,\omega]\}.$$

Similarly, in $C^1_\omega(\mathbb{R}^n)$, we define the subsets

$$W_{0,\omega}=\{x\in C^1_\omega(\mathbb{R}^n):x|_{[0,\omega]}\in W_0\},\qquad W_\omega=\{x\in C^1_\omega(\mathbb{R}^n):x|_{[0,\omega]}\in W\}.$$

We assume that $W_{0,\omega},W_\omega$ are nonempty sets.

Below we get conditions for a family of operator equations to be admissible on W_ω, i.e. the assumptions (Γ_1)–(Γ_3) to be fulfilled. For this purpose we introduce

$$A,B:\mathbb{R}\times\underbrace{\mathbb{R}^n\times\ldots\times\mathbb{R}^n}_{2p}\times[0,1]\to\underbrace{\mathbb{R}^n\times\ldots\times\mathbb{R}^n}_{p},$$

$$A(t,u_1,\dots,u_p,v_1,\dots,v_p,\lambda) = (a(t,u_1,u_{k+1},v_1,v_{k+1},\lambda),\\ a(t-\tau_0,u_2,u_{k+2},v_2,v_{k+2},\lambda),\dots,a(t-(p-1)\tau_0,u_p,u_k,v_p,v_k,\lambda)),$$

$$B(t,u_1,\dots,u_p,v_1,\dots,v_p,\lambda) = (b(t,u_1,u_{k+1},v_1,v_{k+1},\lambda),\\ b(t-\tau_0,u_2,u_{k+2},v_2,v_{k+2},\lambda),\dots,b(t-(p-1)\tau_0,u_p,u_k,v_p,v_k,\lambda)).$$

It is clear that the maps A, B are ω-periodic with respect to the first variable.

For sets V_0 and V we define corresponding sets in $[0,\omega]\times\mathbb{R}^{pn}\times\mathbb{R}^{pn}$:

$$\widetilde{V}_0 = \{(t,u_1,\dots,u_p,v_1,\dots,v_p)\in[0,\omega]\times\mathbb{R}^{pn}\times\mathbb{R}^{pn}:\\ (t-(i-1)\tau_0,u_i,v_i)\in V_0,\ i=1,\dots,p\},$$
$$\widetilde{V} = \{(t,u_1,\dots,u_p,v_1,\dots,v_p)\in[0,\omega]\times\mathbb{R}^{pn}\times\mathbb{R}^{pn}:\\ (t-(i-1)\tau_0,u_i,v_i)\in V,\ i=1,\cdots,p\}.$$

We define also corresponding sets in the function space $C^1([0,\omega],\mathbb{R}^{pn})$:

$$\widetilde{W_0} = \{x\in C^1([0,\omega],\mathbb{R}^{pn}) : (t,x(t),x'(t))\in\widetilde{V}_0,\ \text{for all } t\in[0,\omega]\},$$
$$\widetilde{W} = \{x\in C^1([0,\omega],\mathbb{R}^{pn}) : (t,x(t),x'(t))\in\widetilde{V},\ \text{for all } t\in[0,\omega]\}.$$

In the same way we define the subsets $\widetilde{W}_{0,\omega},\widetilde{W}_\omega$ in the space $C^1_\omega(\mathbb{R}^{pn})$:

$$\widetilde{W}_{0,\omega} = \{x\in C^1_\omega(\mathbb{R}^{pn}) : x|_{[0,\omega]}\in\widetilde{W_0}\},\quad \widetilde{W}_\omega = \{x\in C^1_\omega(\mathbb{R}^{pn}) : x|_{[0,\omega]}\in\widetilde{W}\}.$$

We consider a family of differential equations

$$A(t,y(t),y'(t),\lambda) = B(t,y(t),y'(t),\lambda),\quad \lambda\in[0,1]. \tag{4.3$_\lambda$}$$

It is not difficult to see that every ω-periodic solution $x(t)$ of equation (4.1_λ) yields the solution

$$y(t) = (x(t),x(t-\tau_0),x(t-2\tau_0),\dots,x(t-(p-1)\tau_0))$$

of equation (4.3_λ).

Let us denote by D_λ the set $D_\lambda = \{(t,u,v)\in\widetilde{V} : A(t,u,v,\lambda) = B(t,u,v,\lambda)\}$ and let $D = \bigcup_{\lambda\in[0,1]} D_\lambda$.

Theorem 4.1. *Let the continuous functions $A(t,u,v,\lambda)$ and $B(t,u,v,\lambda)$ satisfy the conditions:*

- (A1) *for any $\lambda_0\in[0,1]$ and $(t_0,u_0,v_0)\in D$ there exists a neighbourhood of (t_0,u_0,v_0,λ_0) where the function $A(t,u,v,\lambda)$ has continuous derivatives A'_u, A'_v, A'_λ and $\det A'_v(t_0,u_0,v_0,\lambda_0)\neq 0$,*
- (B1) *for any $\lambda_0,\lambda_0\in[0,1]$, and $(t_0,u_0,v_0)\in D_{\lambda_0}$ there exists an $\varepsilon>0$ such that*

$$|B(t_0,u,v,\lambda) - B(t_0,u,\overline{v},\lambda)| \le c|A'_v(t_0,u_0,v_0,\lambda_0)(v-\overline{v})|$$

for all λ such that $|\lambda - \lambda_0| < \varepsilon$, and $(t_0, u, v), (t_0, u, \overline{v})$ from an ε-neighbourhood of (t_0, u_0, v_0), where $c < 1$ is a constant,

(C1) *the equations of family (4.1_λ), $\lambda \in [0,1]$, do not have solutions on the boundary of $W_{0,\omega}$.*

Then the family of operator equations (4.2_λ), $\lambda \in [0,1]$, is admissible on the domain $W_{0,\omega}$ and

$$\mathrm{ind}_2(f_0 - g_0, W_{0,\omega}, 0) = \mathrm{ind}_2(f(\,\cdot\,,\lambda) - g(\,\cdot\,,\lambda), W_{0,\omega}, 0).$$

Proof. Introduce the maps

$$F\colon \widetilde{W}_\omega \times [0,1] \to C_\omega(\mathbb{R}^{pn}), \quad (y,\lambda) \mapsto A(t, y(t), y'(t), \lambda),$$
$$G\colon \widetilde{W}\omega \times [0,1] \to C_\omega(\mathbb{R}^{pn}), \quad (y,\lambda) \mapsto B(t, y(t), y'(t), \lambda),$$

and the family of operator equations

$$F(y,\lambda) = G(y,\lambda). \tag{4.4_λ}$$

On the set $\widetilde{W}_\omega$, this family is equivalent to the family of differential equations (4.3_λ).

We will check the conditions (Γ_1)–(Γ_3) for family (4.2_λ) via the investigation of family (4.4_λ). Denote by $\widetilde{C}_\omega(\mathbb{R}^{pn})$ the space of functions

$$y(t) = (y_1(t), \ldots, y_p(t)) = (x(t), x(t-\tau_0), x(t-2\tau_0), \ldots, x(t-(p-1)\tau_0)),$$

where $x \in C^1_\omega(\mathbb{R}^n)$.

The map $\pi_0\colon C_\omega(\mathbb{R}^n) \to \widetilde{C}_\omega(\mathbb{R}^{pn})$, $\pi_0\colon x \mapsto y$, is an isomorphism. Let $\widetilde{C}^1_\omega(\mathbb{R}^{pn}) = C^1_\omega(\mathbb{R}^{pn}) \cap \widetilde{C}_\omega(\mathbb{R}^{pn})$. The map $\pi_0\colon C^1_\omega(\mathbb{R}^n) \to \widetilde{C}^1_\omega(\mathbb{R}^{pn})$ is also an isomorphism.

Equation (4.2_λ) can be rewritten in the form

$$\pi_0^{-1} \circ F(\,\cdot\,,\lambda) \circ \pi_0(x) = \pi_0^{-1} \circ G(\,\cdot\,,\lambda) \circ \pi_0(x). \tag{4.5_λ}$$

We have $F(\,\cdot\,,\lambda), G(\,\cdot\,,\lambda)\colon \widetilde{C}^1_\omega(\mathbb{R}^{pn}) \cap \widetilde{W}_\omega \to \widetilde{C}_\omega(\mathbb{R}^{pn})$, and the map π_0^{-1} determines the first component $y_1(t)$ of $y(t)$.

Let us verify the condition (Γ_1) for family (4.2_λ). Since the solution set Q_Γ is closed, and family (4.2_λ) is presented in form (4.5_λ), this will follow from the compactness of the solution set of family (4.4_λ).

Lemma 4.2. *Let the functions A, B satisfy conditions* (A1), (B1) *of Theorem* 4.1. *Then the set of solutions $(y,\lambda) \in \widetilde{W}_{0,\omega} \times [0,1]$ of (3.4_λ) is compact in $C^1_\omega(\mathbb{R}^{pn}) \times [0,1]$.*

Proof. Take an arbitrary sequence (y_k, λ_k) such that $y_k \in \widetilde{W}_{0,\omega}$ is a solution of equation (4.4_{λ_k}) and $\lambda_k \in [0,1]$. Since the sequence λ_k is bounded, then (passing to a subsequence if necessary) we may assume that the sequence λ_k

converges and $\lim_{k\to\infty}\lambda_k=\lambda_0$. Let us demonstrate that the sequence y_k has a converging subsequence. This will complete the proof.

Since the embedding $\widetilde{W}_{0,\omega}\subset C_\omega(\mathbb{R}^{pn})$ is completely continuous, without loss of generality we may assume that $y_k\to y_0$ uniformly on $[0,\omega]$. Rewrite the equations

$$A(t,y_k(t),y_k'(t),\lambda_k)=B(t,y_k(t),y_k'(t),\lambda_k)$$

in the form

$$\begin{aligned}A(t,y_0(t),y_k'(t),\lambda_0)-B(t,y_0(t),y_k'(t),\lambda_0)&=A(t,y_0(t),y_k'(t),\lambda_0)\\ -A(t,y_k(t),y_k'(t),\lambda_k)+B(t,y_k(t),y_k'(t),\lambda_k)&-B(t,y_0(t),y_k'(t),\lambda_0).\end{aligned}$$

Denote by $z_k(t)$ the right-hand side of the last equation. Then it takes the form

$$A(t,y_0(t),y_k'(t),\lambda_0)-B(t,y_0(t),y_k'(t),\lambda_0)=z_k(t). \tag{4.6}$$

Notice that for k large enough the points $(t,y_0(t),y_k'(t))$ get into a small neighbourhood of $(t,y_k(t),y_k'(t))$ and so into $\widetilde{V}$. Thus equation (4.6) is well-posed.

Show that the sequence z_k converges to zero uniformly on $[0,\omega]$. Since A and B are uniformly continuous on $\widetilde{V}\times[0,1]$ and since $y_k\to y_0$, $\lambda_k\to\lambda_0$ then the functions

$$\begin{aligned}&A(t,y_0(t),y_k'(t),\lambda_0)-A(t,y_k(t),y_k'(t),\lambda_k)\\ \text{and}\quad &B(t,y_k(t),y_k'(t),\lambda_k)-B(t,y_0(t),y_k'(t),\lambda_0)\end{aligned}$$

converge to zero uniformly on $[0,\omega]$. Hence z_k converges to zero also uniformly on $[0,\omega]$.

Define the maps

$$\varphi(t,v)=A(t,y_0(t),v,\lambda_0)\quad\text{and}\quad\mu(t,v)=B(t,y_0(t),v,\lambda_0).$$

They are continuous on the set $U=\{(t,v)\in[0,\omega]\times\mathbb{R}^{pn}:(t,y_0(t),v)\in\widetilde{V}\}$. Then equations (4.6) take the form

$$\varphi(t,y_k'(t))-\mu(t,y_k'(t))=z_k(t).$$

Let $U_0=\{(t,v)\in[0,\omega]\times\mathbb{R}^{pn}):(t,y_0(t),v)\in\widetilde{V}_0\}$, $Z=\{z\in C_\omega(\mathbb{R}^{pn}):(t,z(t))\in U,\ \text{for all } t\in[0,\omega]\}$. Then $y_k'\in Z$ for a sufficiently large k and the sequence $\{y_k'\}$ converges to the set U_0. It is easy to verify that the maps φ and μ satisfy all conditions of Theorem 3.3 on U and so the sequence $\{y_k'\}$ has a subsequence uniformly converging on $[0,\omega]$. Without loss of generality we may assume that $y_k'\to\breve{y}_0$ uniformly on $[0,\omega]$. But by our assumption y_k tens to y_0. It follows that $y_0'(t)=\breve{y}_0(t)$. Thus, $y_k\to y_0$ in the norm of $C^1([0,\omega],\mathbb{R}^{pn})$. This proves the lemma. □

Let us verify the condition (Γ_2) for family (4.2_λ).

Lemma 4.3. *Let the function A satisfy condition* (A1) *of Theorem* 4.1. *Let for the function $x \in W_{0,\omega}$ the corresponding function $y = \pi_0(x)$ be such that $(t, y(t), y'(t)) \in D$ for all $t \in [0, \omega]$. Then, for every $\lambda \in [0, 1]$, the Frechét derivative $f'_{\omega,x}(x, \lambda)$ of the map f_ω with respect to x is a Fredholm operator of index zero and $f'_{\omega,x}(\widetilde{x}, \widetilde{\lambda})$ is continuous with respect to $(\widetilde{x}, \widetilde{\lambda})$ on some neighbourhood of (x, λ).*

Proof. Let $(x, \lambda) \in Q_\Gamma$ be an arbitrary solution of equation (4.2_λ) and $y = \pi(x)$. Since

$$f'_{\omega,x}(x, \lambda) = \pi_0^{-1} \circ F'_y(y, \lambda) \circ \pi_0, \tag{4.7}$$

we have, for arbitrary $h \in C^1_\omega(\mathbb{R}^n)$,

$$\begin{aligned}(f'_{\omega,x}&(x, \lambda)h)(t) \\ &= \pi_0^{-1} \circ (A'_u(t, y(t), y'(t), \lambda) \circ (\pi_0 h)(t) + A'_v(t, y(t), y'(t), \lambda) \circ (\pi_0 h')(t)).\end{aligned}$$

Introduce the auxiliary operators

$$\begin{aligned}\Phi_u(y, \lambda)&: C^1_\omega(\mathbb{R}^n) \to C_\omega(\mathbb{R}^n), \quad h \mapsto \pi_0^{-1} \circ A'_u(t, y(t), y'(t), \lambda) \circ (\pi_0 h)(t), \\ \Phi_v(y, \lambda)&: C^1_\omega(\mathbb{R}^n) \to C_\omega(\mathbb{R}^n), \quad h \mapsto \pi_0^{-1} \circ A'_v(t, y(t), y'(t), \lambda) \circ (\pi_0 h')(t).\end{aligned}$$

Then $(f'_{\omega,x}(x, \lambda)h)(t) = (\Phi_u(y, \lambda)h)(t) + (\Phi_v(y, \lambda)h)(t)$.

As the embedding $C^1_\omega(\mathbb{R}^n) \subseteq C_\omega(\mathbb{R}^n)$ is completely continuous, the map $\Phi_u(y, \lambda)$ is also completely continuous. Therefore it is sufficient to show that the map $\Phi_v(y, \lambda)$ is a Fredholm map of zero index. Let us present this map as the superposition of maps:

$$\begin{aligned}h(t) &\xrightarrow{\frac{d}{dt}} h'(t) \xrightarrow{\pi_0} H'(t) \\ &\xrightarrow{A'_v} A'_v(t, y(t), y'(t), \lambda)H'(t) \xrightarrow{\pi_0^{-1}} \pi_0^{-1} \circ A'_v(t, y(t), y'(t), \lambda)H'(t).\end{aligned}$$

It is well known that the operator

$$\frac{d}{dt}: C^1_\omega(\mathbb{R}^n) \to C_\omega(\mathbb{R}^n)$$

is a Fredholm map of zero index and π_0 is an isomorphism of the space $C_\omega(\mathbb{R}^n)$ and $\widetilde{C}_\omega(\mathbb{R}^{pn})$. As $(t, y(t), y'(t)) \in D$, the matrices $A'_v(t, y(t), y'(t), \lambda)$ are invertible for all $t \in [0, \omega]$, therefore the operator

$$A'_v: C(\mathbb{R}, \mathbb{R}^{pn}) \to C(\mathbb{R}, \mathbb{R}^{pn}), \quad H' \mapsto A'_v(t, y(t), y'(t), \lambda)H'(t),$$

is also invertible.

It is easy to verify that its restriction

$$A'_v: C_\omega(\mathbb{R}^{pn}) \to C_\omega(\mathbb{R}^{pn}) \quad \text{and} \quad \pi_0^{-1} \circ A'_v: C_\omega(\mathbb{R}^{pn}) \to C_\omega(\mathbb{R}^n)$$

is also invertible. Hence, operators $\Phi_v(y,\lambda)$ and $f'_{\omega,x}(x,\lambda)$ are Fredholm of zero index. From the definition of $\Phi_u(y,\lambda)$, $\Phi_v(y,\lambda)$ and condition (A1) of Theorem 4.1 it is easily seen, that $f'_{\omega,x}(\widetilde{y},\widetilde{\lambda})$ is continuous with respect to $\widetilde{y}$ and $\widetilde{\lambda}$.□

Since the set of linear Fredholm operators is open in the set of continuous linear operators [8], there exists a neighbourhood of the point (x,λ), in which the derivative $f'_{\omega,x}(\widetilde{x},\widetilde{\lambda})$ is the Fredholm operator of zero index and it is continuous with respect to $(\widetilde{x},\widetilde{\lambda})$.

Let us verify the condition (Γ_3). Let $(x,\lambda)\in Q_\Gamma$ be an arbitrary solution of (4.2_λ). By Theorem 2.1 it is sufficient to show, that the map g is locally $cf'_\omega(x,\lambda)$-bounded with respect to the measure of noncompactness α at the point (x,λ), where c is the constant from condition (B1). As

$$f'_\omega(x,\lambda)(h,\mu)=f'_{\omega,x}(x,\lambda)h+f'_{\omega,\lambda}(x,\lambda)\mu$$

and $f'_{\omega,\lambda}(x,\lambda)$ is a finite-dimensional map, it is sufficient to analyze the $cf'_{\omega,x}(x,\lambda)$-boundedness of the map g with respect to the measure of noncompactness α at the point (x,λ). Note that the map π_0 does not change the measure of noncompactness of a set, therefore representation (4.7) allows us to prove the equivalent inequality

$$\alpha(F'_y(y,\lambda)(\widetilde{M}))\le(c+\varepsilon)\alpha(G(\widetilde{M})),$$

instead of the required inequality

$$\alpha(\pi_0^{-1}\circ F'_y(y,\lambda)\circ\pi_0(M))\le(c+\varepsilon)\alpha(\pi_0^{-1}\circ G\circ\pi_0(M)),$$

for $M\subset U$ and $\widetilde{M}=\pi_0(M)$.

Thus, it is sufficient to establish the $cF'_y(y,\lambda)$-boundedness of the map G with respect to the measure of noncompactness α at the point (y,λ).

Lemma 4.4. *Let the functions A, B satisfy conditions* (A1), (B1) *of Theorem* 4.1. *Then for any function $y\in\widetilde{W}_{0,\omega}$ and the arbitrary number $\lambda_0\in[0,1]$, such that $(t,y(t),y'(t))\in D_{\lambda_0}$ for all $t\in[0,\omega]$, the map G is $cF'(x,\lambda_0)$-bounded with respect to the measure of noncompactness α at the point (y,λ_0) with a constant $c<1$.*

Proof. Let (y,λ_0) satisfy the above assumptions. The lemma will be proved if, given $\varepsilon>0$, there exists $\delta>0$ such that

$$\alpha(G(\widetilde{M}))\le(c+\varepsilon)\alpha(F'_y(y,\lambda_0)\widetilde{M})$$

for $\widetilde{M}\subset B_\delta(y)\times(\lambda_0-\delta,\lambda_0+\delta)$. Introduce the operators

$$F_u(y,\lambda)\colon C^1_\omega(\mathbb{R}^n)\to C_\omega(\mathbb{R}^n),\quad H\mapsto A'_u(t,y(t),y'(t),\lambda)H(t),$$
$$F_v(y,\lambda)\colon C^1_\omega(\mathbb{R}^n)\to C_\omega(\mathbb{R}^n),\quad H\mapsto A'_v(t,y(t),y'(t),\lambda)H'(t).$$

As $F'_y(y,\lambda_0)H = F'_u(y,\lambda_0)H + F'_v(y,\lambda_0)H$, using the properties of the measure of noncompactness and the fact that the map $F'_v(y,\lambda_0)$ is completely continuous, we obtain

$$\begin{aligned}\alpha(F'_v(y,\lambda_0)\widetilde{M}) &\le \alpha(F'(y,\lambda_0)\widetilde{M} - F'_u(y,\lambda_0)\widetilde{M})\\ &\le \alpha(F'(y,\lambda_0)\widetilde{M}) + \alpha(F'_u(y,\lambda_0)\widetilde{M}) = \alpha(F'(y,\lambda_0)\widetilde{M}).\end{aligned}$$

The similar arguments give the inequality $\alpha(F'(y,\lambda_0)\widetilde{M}) \le \alpha(F'_v(y,\lambda_0)\widetilde{M})$. Hence, $\alpha(F'(y,\lambda_0)\widetilde{M}) = \alpha(F'_v(y,\lambda_0)\widetilde{M})$ and the required inequality takes the form

$$\alpha(G(\widetilde{M})) \le (c+\varepsilon)\alpha(F'_v(y,\lambda_0)\widetilde{M}).$$

To prove the last inequality, assume the contrary. Then there exists $\varepsilon > 0$ such that for every $\delta > 0$ there exists a set $\widetilde{M} \subset B_\delta(y) \times (\lambda_0 - \delta, \lambda_0 + \delta)$ such that the inequality

$$\alpha(G(\widetilde{M})) \ge (c+\varepsilon)\alpha(F'_v(y,\lambda_0)\widetilde{M})$$

is valid.

Let $\alpha(F'_v(y,\lambda_0)\widetilde{M}) = d_0$. By the definition for any small $\delta_1 > 0$ the set $F'_v(y,\lambda_0)\widetilde{M}$ can be covered by a finite number of subsets $F'_v(y,\lambda_0)\widetilde{M}_i$, $i = 1,\dots,k$ with $\operatorname{diam}(F'_v(y,\lambda_0)\widetilde{M}_i) \le d_0 + \delta_1$. Moreover, as $\widetilde{M}$ is relatively compact in $C_\omega(\mathbb{R}^n) \times (\lambda_0 - \delta, \lambda_0 + \delta)$ and the function B is uniformly continuous on $\widetilde{V} \times (\lambda_0 - \delta, \lambda_0 + \delta)$, the cover can be chosen such that

$$|B(t,u(t),u'(t),\lambda_1) - B(t,v(t),u'(t),\lambda_2)| < \delta_2$$

for any small $\delta_2 > 0$ and for all $(u,\lambda_1),(v,\lambda_2) \in \widetilde{M}_i$, $t \in [0,\omega]$. Hence

$$\begin{aligned}\|G(u,\lambda_1) - G(v,\lambda_2)\|_0 &\le \delta_2 + c\|F'_v(y,\lambda_0)(u-v)\|_0\\ &\le \delta_2 + c\cdot\operatorname{diam}(F'_v(y,\lambda_0)\widetilde{M}_i) \le \delta_2 + c(d_0+\delta_1).\end{aligned}$$

Since (u,λ_1), (v,λ_2) are arbitrary points in $\widetilde{M}_i$, we have

$$\operatorname{diam}(G(\widetilde{M}_i)) \le \delta_2 + cd_0 + c\delta_1.$$

The inequality is valid for every subset $\widetilde{M}_i$, $i = 1,\dots,k$, therefore

$$\alpha(G(\widetilde{M}) \le \delta_2 + cd_0 + c\delta_1.$$

We choose constants δ_1 and δ_2 such that $\delta_2 + c\delta_1 < \varepsilon d_0$. Then

$$\alpha(G(\widetilde{M}) \le (c+\varepsilon)d_0.$$

This contradicts the assumption $\alpha(G(\widetilde{M})) \ge (c+\varepsilon)\alpha(F'_v(y,\lambda_0)\widetilde{M}) = (c+\varepsilon)d_0$.$\square$

5. Evaluation of the index of the solution set of equation (4.2_1)**.**

In this section we evaluate the index of the solution set on $W_{0,\omega}$ of equation $f_\omega(x,1)=0$.

Let $a(t,u_1,u_2,v_1,v_2,1) = a_1(t,u_1,v_1)$. We rewrite equation (4.1_1) in the form

$$a_1(t,x(t),x'(t))=0, \quad t\in[0,\omega].$$

We call the problem

$$a_1(t,x(t),x'(t)) = 0, \qquad t\in[0,\omega], \tag{5.1}$$

$$x(0) = x(\omega), \tag{5.2}$$

a periodic boundary value problem. If the solution x satisfies the condition

$$x'(0)=x'(\omega), \tag{5.3}$$

it is said to be periodic. Such a solution $x(t)$ can be extended to an ω-periodic function in $C^1(\mathbb{R},\mathbb{R}^n)$. Notice that not any solution of periodic boundary value problem (5.1), (5.2) is a periodic solution.

We consider the family of boundary value problems

$$\begin{aligned} a_1(t,x(t),x'(t)) &= 0, \quad t\in[0,\lambda\omega],\\ \frac{x(\lambda\omega)-x(0)}{\lambda\omega} &= 0, \quad \lambda\in(0,1]. \end{aligned} \tag{5.4_λ}$$

Define the set

$$W(\lambda)=\{x\in C^1([0,\lambda\omega],\mathbb{R}^n) : (t,x(t),x'(t))\in V_0 \text{ for all } t\in[0,\lambda\omega]\}$$

and maps

$$\begin{aligned} &f_1(\lambda)\colon W(\lambda)\to C([0,\lambda\omega],\mathbb{R}^n), && x\mapsto a_1(t,x(t),x'(t)),\\ &l_1(\lambda)\colon W(\lambda)\to \mathbb{R}^n, && x\mapsto \frac{x(\lambda\omega)-x(0)}{\lambda\omega}. \end{aligned}$$

Then the family of operator equations

$$(f_1(\lambda),l_1(\lambda))(x)=(0,0). \tag{5.5_λ}$$

is equivalent to the family of periodic boundary value problems (5.4_λ).

We state the main result of this section.

Theorem 5.1. *Let the continuous function $a_1(t,u,v)$ satisfy the following conditions:*

(A1') *for any point $(t_0,u_0,v_0)\in D_1$ there exists a neighbourhood where the function a_1 has continuous derivatives $a'_{1,u}$, $a'_{1,v}$ and*

$$\det a'_{1,v}(t_0,u_0,v_0)\neq 0,$$

(A2) *every solution* $x \in W$ *of* (5.1), (5.2) *satisfies condition* (5.3),

(A3) *boundary value problems* (5.4_λ) *do not have solutions on the boundary of* $W(\lambda)$ *for all* $\lambda \in (0,1]$;

(A4) *the nonoriented degree for the map* $d\colon \Omega \to \mathbb{R}^n$, $d(u) = a_1(0,u,0)$, *is well-posed on the set* $\Omega = \{u \in \mathbb{R}^n : (0,u,0) \in V_0\}$.

Then the index of the solution set of equation (4.5_λ) *is well-posed for each* $\lambda \in (0,1]$ *and*

$$\operatorname{ind}_2(f_\omega(\,\cdot\,,1), W_{0,\omega}, 0) = \operatorname{ind}_2((f_1(\lambda), l_1(\lambda)), W(\lambda), 0) = \deg_2(d, \Omega, 0).$$

We separate the proof of the theorem into some lemmas. Introduce the notations:

$$\begin{aligned} C_\omega([0,\omega],\mathbb{R}^n) &= \{x(t) \in C([0,\omega],\mathbb{R}^n) : x(0) = x(\omega)\},\\ C^1_\omega([0,\omega],\mathbb{R}^n) &= \{x(t) \in C^1([0,\omega],\mathbb{R}^n) : x(0) = x(\omega),\ x'(0) = x'(\omega)\}. \end{aligned}$$

We assume that $C^1_\omega([0,\omega],\mathbb{R}^n) = C^1_\omega(\mathbb{R}^n)$ and $W_\omega = W \cap C^1_\omega([0,\omega],\mathbb{R}^n)$.

Lemma 5.2. *Let the function* a_1 *satisfy all conditions of Theorem* 5.1. *If* $y \in C_\omega(\mathbb{R}^n)$ *is sufficiently close to zero, every solution* $x \in W_0$ *of the equation*

$$a_1(t, x(t), x'(t)) = y(t), \quad t \in [0,\omega], \tag{5.6}$$

which satisfies (5.2), *belongs to* W_ω.

Proof. Suppose the contrary, i.e. for every index k there exists $y_k \in C_\omega(\mathbb{R}^n)$ such that $\|y_k\| < 1/k$, and $x_k \in W$ which is a solution of the equation

$$a_1(t, x(t), x'(t)) = y_k(t),$$

satisfying $x(0) = x(\omega), x'(0) \neq x'(\omega)$. Repeating the arguments of Lemma 4.2, it is easy to show that the set of solutions $x_k(t)$ is precompact. So, without loss of generality we may assume that the sequence $\{x_k\}$ converges in $C^1([0,\omega],\mathbb{R}^n)$ and its limit is x_0. It is clear that x_0 is a solution of (5.1), satisfying (5.2).

Note that the function $a_1(0,u,v) - w$ satisfies the conditions of the Implicit Function Theorem (see [15]) with respect to v in some neighbourhood of the point $(u_0, v_0, w_0) = (x_0(0), x_0'(0), 0)$. Therefore the equation $a_1(0,u,v) - w = 0$ has a unique solution $v = v(u,w)$ near the point (u_0, w_0) and $v_0 = v(u_0, w_0)$. However, since

$$a_1(0, x_k(0), x_k'(0)) = y_k(0), \quad a_1(0, x_k(0), x_k'(\omega)) = y_k(0)$$

and $x_k(0) \to x_0(0)$, $x_k'(0) \to x_0'(0)$, $x_k'(\omega) \to x_0'(\omega)$, $y_k(0) \to 0$, as $k \to \infty$, then $(x_k(0), x_k'(0), y_k(0))$ and $(x_k(0), x_k'(\omega), y_k(0))$ are different solutions, which are sufficiently close to $(x_0(0), x_0'(0), 0)$. The contradiction proves the lemma. □

We introduce the maps

$$f\colon C^1([0,\omega],\mathbb{R}^n)\to C([0,\omega],\mathbb{R}^n),\quad f(x)(t)=a_1(t,x(t),x'(t)),$$
$$l\colon C^1([0,\omega],\mathbb{R}^n)\to \mathbb{R}^n,\qquad x\mapsto \frac{x(\omega)-x(0)}{\omega}.$$

Lemma 5.3. *Let the function a_1 satisfy the conditions of Theorem* 5.1 *and $x_0\in W_{0,\omega}$ be a solution of* (5.1). *The point x_0 is a regular point for $f_\omega(\,\cdot\,,1)$ if and only if it is a regular point for the map (f,l).*

Proof. An arbitrary function $z\in C^1([0,\omega],\mathbb{R}^n)$ can be uniquely extended as $z(t)=\overline{z}(t)+z_1t+z_2t^2$ with $\overline{z}\in C^1_\omega([0,\omega],\mathbb{R}^n)$, $z_1,z_2\in\mathbb{R}^n$:

$$z_1=\frac{z(\omega)-z(0)}{\omega}-\frac{z'(\omega)-z'(0)}{2}\quad\text{and}\quad z_2=\frac{z'(\omega)-z'(0)}{2\omega}.$$

The above extension gives the natural isomorphism p_1:

$$p_1\colon C^1([0,\omega],\mathbb{R}^n)\to C^1_\omega([0,\omega],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n.$$

Analogously the function $y\in C([0,\omega],\mathbb{R}^n)$ can be extended as $y(t)=\overline{y}(t)+y_1t$ with $\overline{y}\in C_\omega([0,\omega],\mathbb{R}^n)$, $y_1\in\mathbb{R}^n: y_1=(y(\omega)-y(0))/\omega$. This gives the natural isomorphism

$$p_0\colon C([0,\omega],\mathbb{R}^n)\to C_\omega([0,\omega],\mathbb{R}^n)\times\mathbb{R}^n.$$

The map (f,l) is regular at the point x_0, if the Frechét derivative $(f'(x_0),l)$ is an invertible operator. This operator is invertible if the operator

$$L=(p_0,I)\cdot(f'(x_0),l)\cdot p_1^{-1}\colon C^1_\omega([0,\omega],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n\to C_\omega([0,\omega],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n$$

is invertible. So it is sufficient to prove that the linear operators L and $L_\omega=f'_\omega(x_0,1)$ are invertible simultaneously.

We introduce the projector

$$\pi\colon C_\omega([0,\omega],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n\to C_\omega([0,\omega],\mathbb{R}^n)\times\mathbb{R}^n\times\mathbb{R}^n,$$
$$\pi(\overline{z},z_1,z_2)=(\overline{z},0,0).$$

Note that $\pi\cdot L(\overline{z},0,0)=L(\overline{z},0,0)=L_\omega(\overline{z})$ and $(I-\pi)\cdot L(\overline{z},0,0)=0$ for all $\overline{z}\in C^1_\omega([0,\omega],\mathbb{R}^n)$. Then

$$L=\begin{pmatrix}L_\omega & \pi\cdot L\\ 0 & (I-\pi)\cdot L\end{pmatrix}. \tag{5.7}$$

Hence, if L is invertible, then the map L_ω is invertible too.

Let the map L_ω be invertible. From (5.7) it is clear that the invertibility of $(I-\pi)\cdot L\colon\{0\}\times\mathbb{R}^n\times\mathbb{R}^n\to\{0\}\times\mathbb{R}^n\times\mathbb{R}^n$ implies the invertibility of L.

The matrix of the map $(I-\pi)\cdot L$ is

$$\begin{pmatrix}f'_u(x_0) & \omega f'_u(x_0)+2f'_v(x_0)\\ I & I\omega\end{pmatrix},$$

where $f'_u(x_0), f'_v(x_0)$ are defined similarly to the maps F'_u, F'_v from the proof of Lemma 4.3. Due to the hypothesis of our lemma we have:

$$\det\begin{pmatrix} f'_u(x_0) & \omega f'_u(x_0) + 2f'_v(x_0) \\ I & I\omega \end{pmatrix} = \det\begin{pmatrix} f'_u(x_0) & 2f'_v(x_0) \\ I\omega & 0 \end{pmatrix} = 2(-1)^n \det f'_v(x_0) \neq 0.$$

Consequently the operator $(I - \pi) \cdot L$ is invertible and therefore the operator L is also invertible. The lemma is proved. □

The above statements allow us to evaluate the nonoriented index of the solution set for the equation $(f, l)(x) = (0, 0)$ instead of the equation $f_\omega(x, 1) = 0$.

To compare the values of the index for different λ, we consider a new family of boundary value problems

$$(5.8_\lambda) \qquad \begin{aligned} a_1\bigg(\lambda t, y(t), \frac{1}{\lambda} y'(t)\bigg) &= 0, \quad t \in [0, \omega],\ \lambda \in (0, 1], \\ \frac{y(\omega) - y(0)}{\omega} &= 0. \end{aligned}$$

Denote by $V(\lambda)$ the set

$$V(\lambda) = \{(t, u, v) \in [0, \omega] \times \mathbb{R}^n \times \mathbb{R}^n : (\lambda t, u, v/\lambda) \in V_0\}$$

and by W_λ the set

$$W_\lambda = \{y \in C^1([0, \omega], \mathbb{R}^n) : (t, y(t), y'(t)) \in V(\lambda),\ \text{for all } t \in [0, \omega]\}.$$

Let

$$\Phi(\lambda)\colon W_\lambda \to C([0, \omega], \mathbb{R}^n), \quad y \mapsto a_1\bigg(\lambda t, y(t), \frac{1}{\lambda} y'(t)\bigg).$$

Then the operator equations

$$(5.9_\lambda) \qquad (\Phi(\lambda), l)(y) = (0, 0), \quad \lambda \in (0, 1],$$

are equivalent to the boundary value problems (5.8_λ).

LEMMA 5.4. *Let the function a_1 satisfy conditions* (A1'), A3 *of Theorem* 5.1. *Then for every $\lambda \in (0, 1]$ the index of the solution set of equation* (5.9_λ) *on W_λ is well-posed, the values of the index for all λ are equal to each other and*

$$(5.10) \qquad \operatorname{ind}_2((f_1(\lambda), l_1(\lambda)), W(\lambda), 0) = \operatorname{ind}_2((\Phi(\lambda), l)), W_\lambda, 0).$$

PROOF. Since

$$\begin{aligned} (a_1(\lambda t, u, v/\lambda))'_v &= (1/\lambda) a'_{1,v}(\lambda t, u, v/\lambda), \\ \det a'_{1,v}(\lambda t, u, v/\lambda &\neq 0 \quad \text{at } (\lambda t, u, v/\lambda) \in D, \end{aligned}$$

then similarly to Lemma 4.3 we can prove that for every fixed $\lambda \in (0, 1]$ the map $(\Phi(\lambda), l)$ is Fredholm on some neighbourhood of each solution y of (5.8_λ).

The index of the solution set of equation (5.9_λ) on the set W_λ is well-posed if the equation has no solutions on the boundary of W_λ. Note, that the substitution $t = \lambda\tau$ in $x(t)$ gives $y(\tau) = x(\lambda\tau)$ and equation (5.5_λ) transforms into (5.8_λ), every solution $x(t)$ of the first equation corresponds to a solution $y(\tau)$ of the last equation. Moreover, $W(\lambda)$ transforms into W_λ. By (A3) equation (5.5_λ) has no solution on the boundary of $W(\lambda)$, this implies that (5.8_λ) also has no solution on the boundary of W_λ. Therefore the index of solution set $\operatorname{ind}_2((\Phi(\lambda), l), W_\lambda, 0)$ is well-posed for every $\lambda \in (0, 1]$.

Without loss of generality we assume that (0,0) is a regular value of the Fredholm map $(\Phi(\lambda), l)$ on some neighbourhood of the solution set of (5.8_λ). Then the number of solutions of equation (5.8_λ) is finite and its residue class mod 2 is equal to the index of solution set $\operatorname{ind}_2((\Phi(\lambda), l), W_\lambda, 0)$. Equation ($5.5_\lambda$) has the same number of solutions and its residue class mod 2 is equal to the index of solution set $\operatorname{ind}_2((f_1(\lambda), l_1(\lambda)), W(\lambda), 0)$. Hence, for all $\lambda \in (0, 1]$,

$$\operatorname{ind}_2((f_1(\lambda), l_1(\lambda)), W(\lambda), 0) = \operatorname{ind}_2((\Phi(\lambda), l), W_\lambda, 0).$$

The domain W_λ is transforming continuously while λ changes. The continuous map $(\Phi(\lambda), l)$ determines the continuous homotopy of Φ_0C^1-maps on some neighbourhood of the solution set of each equation (5.8_{λ_0}), $\lambda_0 \in (0, 1]$, for λ sufficiently close to λ_0. Then from the definition of the index and the homotopy invariance property for Fredholm maps we get coincidence of the indices $\operatorname{ind}_2((\Phi(\lambda), l), W_\lambda, 0)$ with different $\lambda \in (0, 1]$. □

For $\lambda = \lambda_0 \in (0, 1]$, sufficiently small, we consider the family of boundary value problems

$$(5.11_\eta) \qquad \begin{aligned} &a_1(t, x(t), x'(t)) = 0 \quad \text{for } t \in [0, \lambda_0\omega],\\ &\begin{cases} \dfrac{x(\eta\lambda_0\omega) - x(0)}{\eta\lambda_0\omega} = 0 & \text{for } \eta \in (0, 1],\\ x'(0) = 0 & \text{for } \eta = 0, \end{cases} \end{aligned}$$

on $W(\lambda_0)$. We consider

$$l_2(\eta)\colon W(\lambda_0) \to \mathbb{R}^n, \quad l_2(\eta)x = \begin{cases} \dfrac{x(\eta\lambda_0\omega) - x(0)}{\eta\lambda_0\omega} & \text{if } \eta \in (0, 1],\\ x'(0) & \text{if } \eta = 0. \end{cases}$$

We obtain the family of operator equations

$$(5.12_\eta) \qquad (f_1(\lambda_0), l_2(\eta))(x) = (0, 0), \quad \eta \in [0, 1].$$

Lemma 5.5. *Let the function a_1 satisfy conditions* (A1'), (A4) *of Theorem* 5.1. *Given λ_0, sufficiently small, the index of the solution set of each equation* (5.12_η) *is well-posed on $W(\lambda_0)$, the values of the index coincide for all*

$\eta \in [0,1]$ and the equality

$$\mathrm{ind}_2((f_1(\lambda_0), l_2(\eta)), W(\lambda_0), 0) = \deg_2(d, \Omega, 0) \tag{5.13}$$

holds, where $d\colon \Omega \to \mathbb{R}^n$, $d(u) = a_1(0,u,0)$ and $\Omega = \{u \in \mathbb{R}^n : (0,u,0) \in V_0\}$.

Proof. Repeating arguments of the proof of Lemma 4.3, it is easy to show that the map $(f_1(\lambda_0), l_2(\eta))$ is the $\Phi_0 C^1$-map on some neighbourhood of the solution set of equation (5.12_η) for all $\eta \in [0,1]$, moreover this map is continuous in η.

Applying Lemma 4.2 to the equation

$$a_1(t, x(t), x'(t)) = 0, t \in [0, \lambda_0\omega], \tag{5.14}$$

it is easy to show that the set of solutions (x,η) of the family of equations (4.12_η) is compact in $W(\lambda_0) \times [0,1]$.

Let us show that equation (5.12_η) has no solutions on the boundary of $W(\lambda_0)$ for $\eta = 0$, and so, for η sufficiently small.

If $\eta = 0$, we get the initial-value problem

$$\begin{aligned} a_1(t, x(t), x'(t)) &= 0, \quad t \in [0, \lambda_0\omega], \\ x'(0) &= 0. \end{aligned}$$

If $t = 0$, we have the equation

$$a_1(0,u,0) = 0. \tag{5.15}$$

By virtue of (A4) of Theorem 5.1 it has no solution on the boundary of Ω.

Let u_0 be a solution of equation (5.15). Using the Implicit Function Theorem (see [15]) for the equation $a_1(t,u,v) = 0$ on some neighbourhood of the point $(0,u_0,0)$, we obtain a continuous function $v = v(t,u)$ with continuous derivative v'_u. Then the equation (5.14) in the neighbourhood of the point $(0,u_0,0)$ is replaced by the equation $x'(t) = v(t,x(t))$. It has the unique solution $x(t)$, satisfying the initial condition $x(0) = u_0$. For t from some segment $[0,t_1]$ the points $(t,x(t),x'(t))$ are contained in V_0 and they do not belong to the boundary of V_0.

As V_0 is bounded and $(t,x(t),x'(t)) \in V_0$ for all $t \in [0,t_1]$, then $x'(t)$ is bounded. For every solution u of (5.15) the initial values $(0,u,0)$ are contained inside V_0, so for every t in some interval $[0,t_1]$ all points $(t,x(t),x'(t))$ are contained inside V_0 for every solution $x(t)$ to (5.12_η) corresponding to $\eta = 0$.

If $\lambda_0 < t_1/\omega$, the equation (5.12_η) has no solutions on the boundary of $W(\lambda_0)$ for $\eta = 0$, and therefore for small η, i.e. for $\eta \in [0,\eta_0]$.

Let us choose $\lambda_0 = \min\{t_1/\omega, \eta_0/\omega\}$. It is easy to see, that equations (5.12_η) for $\eta \in [0,1]$ have no solutions on the boundary of $W(\lambda_0)$. Then for every equation the index of solution set is well-posed on $W(\lambda_0)$ and these indices are equal. Hence

$$\mathrm{ind}_2((f_1(\lambda_0), l_2(\eta)), W(\lambda_0), 0) = \mathrm{ind}_2((f_1(\lambda_0), l_2(0)), W(\lambda_0), 0).$$

Without loss of generality we assume that 0 is a regular value of the map $(f_1(\lambda_0), l_2(0))$, considered on some neighbourhood of the solution set of the equation (5.12_η) for $\eta = 0$. We also assume that 0 is the regular value of the map d on Ω. Then the equation (5.15) has a finite number of solutions $u_1, \dots, u_s$. Each solution u_i determines a unique solution $x_i(t)$ of equation (5.14), such that $x_i(0) = u_i$. The choice of λ_0 allows us to extend the solution on the segment $[0, \lambda_0\omega]$ and $(t, x(t), x'(t)) \in V_0$ for all $t \in [0, \lambda_0\omega]$. Hence, equations (5.15) and (5.12_η) at $\eta = 0$ have the same number s of solutions. By the definition of the index of a solution set and the nonoriented degree of Fredholm maps of a zero index we have

$$\mathrm{ind}_2((f_1(\lambda_0), l_2(\eta)), W(\lambda_0), 0) = s (\mathrm{mod}\, 2), \quad \deg_2(d, \Omega, 0) = s (\mathrm{mod}\, 2).$$

The lemma is proved. □

To complete the proof of Theorem 5.1, let us write down the equalities for indices of the solution set that we have obtained above:

$$\begin{aligned} \mathrm{ind}_2(f_\omega(\,\cdot\,, 1), W_{0,\omega}, 0) &= \mathrm{ind}_2((f_1(\lambda), l_1(\lambda)), W(\lambda), 0) \\ &= \mathrm{ind}_2((f_1(\lambda_0), l_1(\lambda_0)), W(\lambda_0), 0) \\ &= \mathrm{ind}_2((f_1(\lambda_0), l_2(1)), W(\lambda_0), 0) \\ &= \mathrm{ind}_2((f_1(\lambda_0), l_2(0)), W(\lambda_0), 0) = \deg_2(d, \Omega, 0). \end{aligned}$$

Theorem 5.1 is proved. □

6. Existence of periodic solutions of equation (4.1)

Now we are in position to present the main result of this paper. We suppose that the differential equation

$$a_0(t, x(t), x(t-\tau), x'(t), x'(t-\tau)) = b_0(t, x(t), x(t-\tau), x'(t), x'(t-\tau))$$

is contained in the family of equations

$$(4.1_\lambda)\quad a(t, x(t), x(t-\tau), x'(t), x'(t-\tau), \lambda) = b(t, x(t), x(t-\tau), x'(t), x'(t-\tau), \lambda).$$

Theorem 6.1. *Let the continuous functions $A(t, u, v, \lambda)$ and $B(t, u, v, \lambda)$ satisfy the following conditions:*

- (A1) *for any $\lambda_0 \in [0, 1]$ and $(t_0, u_0, v_0) \in D$ there exists a neighbourhood of $(t_0, u_0, v_0, \lambda_0)$ where the function $A(t, u, v, \lambda)$ has continuous derivatives A'_u, A'_v, A'_λ and $\det A'_v(t_0, u_0, v_0, \lambda_0) \neq 0$,* (B1) *for any* $\lambda_0 \in [0,1]$ *and* $(t_0, u_0, v_0) \in D_{\lambda_0}$ *there exists an* $\varepsilon > 0$ *such that*

$$|B(t_0, u, v, \lambda) - B(t_0, u, \overline{v}, \lambda)| \le c|A'_v(t_0, u_0, v_0, \lambda_0)(v - \overline{v})|$$

for all $\lambda : |\lambda - \lambda_0| < \varepsilon$, *and* $(t_0, u, v), (t_0, u, \overline{v})$ *from the* ε*-neighbourhood of* (t_0, u_0, v_0), *where* $c < 1$ *is a constant,*

(C1) *the equations of family* (4.1_λ), $\lambda \in [0,1]$ *do not have solutions on the boundary of* $W_{0,\omega}$,

(A2) *every solution* $x \in W_0$ *of* (5.1), (5.2) *satisfies condition* (5.3),

(A3) *the boundary value problems* (5.4_λ) *do not have solutions on the boundary of* $W(\lambda)$ *for all* $\lambda \in (0,1]$,

(A4') *the non-oriented degree on the set* $\Omega = \{u \in \mathbb{R}^n : (0, u, 0) \in V_0\}$ *for the map*

$$d : \Omega \to \mathbb{R}^n, \quad d(u) = a_1(0, u, 0),$$

differs from zero.

Then equation (4.1) *has at least one solution of* $W_{0,\omega}$.

Proof. The existence of ω-periodic solution for equation (4.1) is equivalent to the solvability of operator equation (4.2)

$$f_0(x) - g_0(x) = 0.$$

From Theorems 4.1 and 5.1 it follows that in conditions of Theorem 6.1 the index of the solution set $\operatorname{ind}_2(f_0 - g_0, W_{0,\omega}, 0)$ of equation (4.2) is well-posed, and using the homotopy invariance property of the index, we obtain the following equality

$$\begin{aligned} \operatorname{ind}_2(f_0 - g_0, W_{0,\omega}, 0) &= \operatorname{ind}_2(f_\omega(\,\cdot\,, 1), W_{0,\omega}, 0) \\ &= \operatorname{ind}_2((f_1(\lambda), l_1(\lambda)), W(\lambda), 0) = \deg_2(d, \Omega, 0). \end{aligned}$$

From assumption (A4') we have $\deg_2(d, \Omega, 0) \neq 0$, and therefore

$$\operatorname{ind}_2(f_0 - g_0, W_{0,\omega}, 0) \neq 0.$$

Now applying Property 2.5 of the index of the solution set we obtain that equation (4.1) has at least one solution in $W_{0,\omega}$ and the conclusion of the theorem follows. □

Remark 6.2. If we have

$$a(t, x(t), x(t-\tau), x'(t), x'(t-\tau), 1) = a_1(x(t), x'(t)), \tag{6.1}$$

then condition (A1) implies condition (A2), therefore we can exclude condition (A2) from the hypotheses of the theorem.

Applying the Implicit Function Theorem in some neighbourhood of the graph of any solution $x(t)$ of problem (5.1)–(5.2) we can rewrite this equation in the form $x'(t) = \varphi(x(t))$. Therefore condition (5.3) is fulfilled.

Remark 6.3. It may be conjectured that if equality (6.1) holds, condition (A3) is superfluous among assumptions of the theorem.

Example. We consider the existence of ω-periodic solutions of the differential equation

$$\begin{aligned}\mu(1+(y''(t))^2)y''(t)-(y'(t))^{12}-2y(t)\\ =\varphi(t,y(t),y(t-\tau),y'(t),y'(t-\tau),y''(t),y''(t-\tau)).\end{aligned}\tag{6.2}$$

Let the continuous function φ be ω-periodic with respect to the first variable and satisfy the following conditions

(a) for all $(t,u_1,u_2,v_1,v_2)\in[0,\omega]\times[-1/2,1/2]\times[-1/2,1/2]\times[-M_1,M_1]\times[-M_1,M_1]$ with $M_1=2^{1/6}\mathrm{tg}(2^{7/6}/\mu^{1/3})$ and any $w_1,w_2,\overline{w}_1,\overline{w}_2\in\mathbb{R}$ such that $|w_1|,|w_2|,|\overline{w}_1|,|\overline{w}_2|<M_2+\delta$, $\delta>0$, $M_2=(M_1^{12}+2)^{1/3}/\mu^{1/3}$, the following inequality holds

$$\begin{aligned}|\varphi(t,u_1,u_2,v_1,v_2,w_1,w_2)-\varphi(t,u_1,u_2,v_1,v_2,\overline{w}_1,\overline{w}_2)|\\ \le c(|w_1-\overline{w}_1|+|w_2-\overline{w}_2|)\end{aligned}$$

where $c<c_0=\min\{1,2\mu\}$ is a certain constant,

(b) for any $(t,u_1,u_2,v_1,v_2,w_1,w_2)$ such that $t\in[0,\omega]$, $|u_1|,|u_2|\le 1/2$, $|v_1|,|v_2|\le M_1$, $|w_1|,|w_2|\le M_2$

$$|\varphi(t,u_1,u_2,v_1,v_2,w_1,w_2)|<1.$$

Applying Theorem 6.1 we shall show that equation (6.2) has an ω-periodic solution $y(t)$ such that

$$(t,y(t),y'(t),y''(t))\in\mathbb{R}\times[-1/2,1/2]\times[-M_1,M_1]\times[-M_2,M_2]$$

for all $t\in\mathbb{R}$.

Reduce equation (6.2) to the system of first order differential equations in the following way. Let $x_1(t)=y(t)$, $x_2(t)=y'(t)$ and $x(t)=(x_1(t),x_2(t))$. Then the above-mentioned system has the form

$$\begin{cases}x_1'(t)-x_2(t)=0,\\ \mu(1+(x_2'(t))^2)x_2'(t)-(x_2(t))^{12}-2x_1(t)\\ \qquad=\varphi(t,x_1(t),x_1(t-\tau),x_2(t),x_2(t-\tau),\ x_2'(t),x_2'(t-\tau)).\end{cases}\tag{6.3}$$

Introduce the notations

$$u=(u_1,u_2),\quad v=(v_1,v_2),\quad \overline{u}=(\overline{u}_1,\overline{u}_2),\quad \overline{v}=(\overline{v}_1,\overline{v}_2),$$
$$a_0(u,v)=(v_1-u_2,\mu(1+v_2^2)v_2-u_2^{12}-2u_1),$$
$$b_0(t,u,\overline{u},v,\overline{v})=(0,\varphi(t,u_1,\overline{u}_1,u_2,\overline{u}_2,v_2,\overline{v}_2)).$$

Then system (6.3) can be rewritten as the equation

$$a_0(x(t),x'(t)) = b_0(t,x(t),x(t-\tau),x'(t),x'(t-\tau)). \tag{6.4}$$

Denote by V_0 the set

$$V_0 = \{(t,u,v) \in [0,\omega]\times\mathbb{R}^2\times\mathbb{R}^2 : |u_1| < 1/2,\ |u_2|,|v_1| < M_1,\ |v_2| < M_2\}.$$

Consider the following family of differential equations

$$a_0(x(t),x'(t)) = \lambda b_0(t,x(t),x(t-\tau),x'(t),x'(t-\tau)). \tag{6.4$_\lambda$}$$

For the sake of simplicity we assume $\tau = \omega/2$. Then the maps the A and B have the form

$$A(u,\overline{u},v,\overline{v}) = \begin{pmatrix} v_1 - u_2 \\ \mu(1+v_2^2)v_2 - u_2^{12} - 2u_1 \\ \overline{v}_1 - \overline{u}_2 \\ \mu(1+\overline{v}_2^2)\overline{v}_2 - \overline{u}_2^{12} - 2\overline{u}_1 \end{pmatrix},$$

$$B(t,u,\overline{u},v,\overline{v}) = \begin{pmatrix} 0 \\ \varphi(t,u_1,\overline{u}_1,u_2,\overline{u}_2,v_2,\overline{v}_2)) \\ 0 \\ \varphi(t,\overline{u}_1,u_1,\overline{u}_2,u_2,\overline{v}_2,v_2) \end{pmatrix}.$$

Let us show that the system of equations (6.3) satisfies all conditions of Theorem 6.1 on an appropriate domain.

(1) The function A has continuous derivatives A'_u, A'_v:

$$A'_{v,\overline{v}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu(1+3v_2^2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu(1+3\overline{v}_2^2) \end{pmatrix},$$

$$\det A'_{v,\overline{v}} = \mu^2(1+3v_2^2)(1+3\overline{v}_2^2) \neq 0.$$

(2) For all $(t,u,\overline{u},v,\overline{v}),(t,u,\overline{u},w,\overline{w})$ the inequality

$$\begin{aligned} |B(t,u,\overline{u}&,v,\overline{v}) - B(t,u,\overline{u},w,\overline{w})| \\ &= |\varphi(t,u_1,\overline{u}_1,u_2,\overline{u}_2,v_2,\overline{v}_2) - \varphi(t,u_1,\overline{u}_1,u_2,\overline{u}_2,w_2,\overline{w}_2)| \\ &\leq c(|v_2-w_2| + |\overline{v}_2 - \overline{w}_2|) \leq 2c|(v,w)-(\overline{v},\overline{w})|, \end{aligned}$$

holds. Moreover, for any $v,\overline{v}\in\mathbb{R}^2$ and $w,\overline{w}\in\mathbb{R}^2$ we have

$$\begin{aligned} |A'_{v,\overline{v}}&(e,\overline{e})(v-w,\overline{v}-\overline{w})| \\ &= |(v_1-w_1,\mu(1+3e_2^2)(v_2-w_2)),(\overline{v}_1-\overline{w}_1,\mu(1+3\overline{e}_2^2)(\overline{v}_2-\overline{w}_2))| \\ &\geq c_0|(v,w)-(\overline{v},\overline{w})|, \end{aligned}$$

where $c_0 = \min\{1, 2\mu\}$. Therefore

$$|B(t,u,\overline{u},v,\overline{v}) - B(t,u,\overline{u},w,\overline{w})| \le \frac{2c}{c_0}|A'_{v,\overline{v}}(e,\overline{e})(v-w,\overline{v}-\overline{w})|$$

for all $(t,u,\overline{u},v,\overline{v}),(t,u,\overline{u},w,\overline{w}) \in \widetilde{V}$ and $2c/c_0 < 1$.

(3) Notice that on the domain $\Omega = \{u \in \mathbb{R}^2 : |u_1| < 1/2,\ |u_2| < M_1\}$ the map

$$d\colon \Omega \to \mathbb{R}^2, \quad d(u) = (-u_2, -u_2^{12} - 2u_1),$$

is invertible and $d(0) = 0$. Therefore $\deg_2(d,\Omega,0) = 1$.

(4) We shall show that equations $(6.4_\lambda), \lambda \in [0,1]$, have no solutions on the boundary of $W_{0,\omega}$. Moreover, we shall show, that the boundary value problems (5.4_λ) have no solutions on the boundary of $W(\lambda)$ for all $\lambda \in (0,1]$.

Assume that the graph of a certain solution $(x_1(t), x_2(t))$ attains the boundary V_0 at a point t_0. Then either $|x_1(t_0)| = 1/2$, or $|x_2(t_0)| = |x_1'(t_0)| = M_1$, or $|x_2(t_0)| = M_2$. We consider each case separately.

Let $|x_1(t_0)| = 1/2$, then t_0 is a point of a local extremum of the function $x_1(t)$. If $t_0 \in (0,\omega)$, then $x_1'(t_0) = x_2(t_0) = 0$. As $x_1(t)$ is a solution of equation (6.4_λ), at the point t_0 we have

$$\begin{aligned}\mu(1 + (x_1''(t_0))^2)&x_1''(t_0) - 2x_1(t_0)\\ &= \lambda\varphi(t_0, x_1(t_0), x_1(t_0-\omega/2), 0, x_1'(t_0-\omega/2), x_1''(t_0), x_1''(t_0-\omega/2)).\end{aligned}$$

Hence, if $x_1(t_0) = 1/2$, then $\mu(1 + (x"_1(t_0))^2)x"_1(t_0) > 0$ and $x"_1(t_0) > 0$, what is impossible, since the point t_0 is a maximum point of the function. For $x_1(t_0) = -1/2$ we have $x_1''(t_0) < 0$, that also leads to a contradiction.

If $t_0 = 0$ or $t_0 = \omega$, the condition $x_1'(0) = x_1'(\omega)$ is fulfilled only in the case $x_1'(0) = x_1'(\omega) = 0$. Otherwise in some neighbourhood of either 0 or ω there are values $t_0 \in (0,\omega)$ such that $|x_1(t_0)| > 1/2$, and then $x \notin W_0$. The case $x_1'(t_0) = 0$ is considered above and leads to a contradiction.

If $t_0 = 0$ or $t_0 = \omega$ the requirement $x_1'(0) = x_1'(\omega)$ is fulfilled only in the case $x_1'(0) = x_1'(\omega) = 0$.

Let $|x_1'(t_0)| = |x_2(t_0)| = M_1$. From the condition $x_1(0) = x_1(\omega)$ we get the existence of a point $t^* \in (0,\omega)$, such that $x_1'(t^*) = 0$. Let us consider one of the possible cases $x_1'(t_0) = M_1$, $t_0 > t^*$ and $x_1'(t) > 0$ on $[t_0, t^*]$. Since $x_1(t)$ is a solution of equation (6.4_λ), then

$$\mu|x_1''(t)|^3 \le \mu\ (1 + (x_1''(t))^2)|x_1''(t)| \le (x_1'(t))^{12} + 2|x_1(t)| + 1 \le (x_1'(t))^{12} + 2.$$

Therefore $|x_1''(t)| < (1/\mu^{1/3})(|x_1''(t)|)^4 + 2^{1/3})$. Further,

$$-x_1''(t) < \frac{1}{\mu^{1/3}}(|x_1''(t)|)^{+}2^{1/3}),$$

$$-\frac{x_1''(t)}{|x_1''(t)|)^4+2^{1/3}} < \frac{1}{\mu^{1/3}},$$
$$-\frac{x_1'(t)x_1''(t)}{|x_1''(t)|)^4+2^{1/3}} < \frac{1}{\mu^{1/3}}x_1'(t).$$

Integrating over the interval $[t_0,t^*]$, we obtain

$$-\int_{t_0}^{t^*}\frac{x_1'(t)x_1''(t)}{|x_1''(t)|)^4+2^{1/3}}\,dx < \frac{1}{\mu^{1/3}}(x_1(t^*)-x(t_0)),$$
$$\int_0^{x_1(t_0)}\frac{sds}{s^4+2^{1/3}} < \frac{1}{\mu^{1/3}},$$
$$\frac{1}{2^{7/6}}\arctan\left(\frac{x_1'(t_0)}{2^{1/6}}\right) < \frac{1}{\mu^{1/3}}.$$

We get a contradiction, since $x_1'(t_0)=M_1=2^{1/6}\mathrm{tg}(2^{7/6}/\mu^{1/3})$.

The case $x_1'(t_0)=-M_1$ and another disposition of the point t^* with respect to the point t_0 can be investigated analogously and lead to a contradiction. Hence $|x_1'(t)|=|x_2(t)|<M_1$ for all t.

From equation (6.4_λ) and the estimates $|x_1(t)|<1/2$, $|x_1'(t)|=|x_2(t)|<M_1$, $|\varphi|<1$ we obtain the inequality

$$\mu|x_1''(t)|^3\le\mu(1+(x_1''(t))^2)|x_1''(t)|\le(x_1'(t))^{12}+2|x_1(t)|+1\le M_1^{12}+2.$$

Hence $|x_1"(t)|<(M_1^{12}+2)^{1/3}/\mu^{1/3}$ and the equality $|x_2'(t_0)|=|x_1''(t_0)|=M_2$ is impossible.

Thus we have shown that conditions (C1), (A3) of Theorem 6.1 are satisfied. From Remark 6.2 it follows that (A2) is satisfied. From Theorem 6.1 it follows that equation (6.2) has an ω-periodic solution.

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Manuscript received January 15, 2001

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