Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 21, 2003, 41–51

NON-RADIAL SOLUTIONS WITH ORTHOGONAL SUBGROUP INVARIANCE FOR SEMILINEAR DIRICHLET PROBLEMS

Ryuji Kajikiya

Abstract. A semilinear elliptic equation, $-\Delta u = \lambda f(u)$, is studied in a ball with the Dirichlet boundary condition. For a closed subgroup G of the orthogonal group, it is proved that the number of non-radial G invariant solutions diverges to infinity as λ tends to ∞ if G is not transitive on the unit sphere.

1. Introduction

We study the multiple existence of non-radial solutions for a semilinear elliptic equation

$$-\Delta u = \lambda f(u), \quad x \in \Omega, \tag{1.1}$$

$$u = 0, \quad x \in \partial\Omega, \tag{1.2}$$

$$u(gx) = u(x), \quad x \in \Omega,\ g \in G, \tag{1.3}$$

where $\Omega \equiv \{x \in \mathbb{R}^n : |x| < R\}$, $n \geq 2$ and G is a closed subgroup of the orthogonal group $O(n)$ and $\lambda > 0$ is a parameter. We deal with the nonlinear term like as $f(u) = u - |u|^{p-1}u$ with $p > 1$ or $f(u) = \sin u$ and prove that

2000 *Mathematics Subject Classification.* 35J20, 35J60.

Key words and phrases. Semilinear elliptic equation, group invariant solution, non-radial solution, variational method.

Supported in part by the Grant-in-Aid for Scientific Research (C) (No. 12640197), Ministry of Education, Culture, Sports, Science and Technology.

the number of non-radial solutions of (1.1)–(1.3) diverges to ∞ as $\lambda \to \infty$. Since $g \in G$ is an orthogonal matrix and Ω is a ball, (1.3) makes sense. We call a solution of (1.1)–(1.3) a G *invariant solution*. It is clear that a radially symmetric solution is G invariant for any G. In this paper, we study non-radial G invariant solutions. Since G is a closed subgroup of $O(n)$, it is an isometric linear transformation group on the unit sphere S^{n-1},

$$S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}.$$

A group G is said to be *transitive* on S^{n-1} if for any $x, y \in S^{n-1}$ there exists a $g \in G$ such that $gx = y$. We suppose the assumption below.

Assumption (A). There exists a constant $a > 0$ such that $f(s)$ is defined on $[-a, a]$ and satisfies the following conditions:

(A1) $f(s) > 0$ for $s \in (0, a)$ and $f(a) = 0$,
(A2) f is odd and Lipschitz continuous on $[-a, a]$,
(A3) $f(s)$ is non-decreasing in a neighborhood of $s = 0$,
(A4) $f(s)/s$ is strictly decreasing in $(0, a)$.

Theorem 1. *Suppose that Assumption* (A) *holds. Then the following assertions are equivalent:*

(i) *G is not transitive on S^{n-1},*
(ii) *for each $k \in \mathbb{N}$, there exists a $\lambda_k > 0$ such that for $\lambda > \lambda_k$,* (1.1)–(1.3) *has at least k non-radial G invariant solutions whose L^∞-norms are less than or equal to a.*

If G is transitive, then a G invariant solution becomes a radially symmetric solution. Therefore the assertion (ii) in Theorem 1 implies (i). What a kind of G is transitive on S^{n-1}? The answer is due to Montgomery–Samelson ([6]) and Borel ([1]) as follows.

Theorem 0 ([1], [6]). *Let $n \geq 2$ and G be a connected closed subgroup of $SO(n)$. Then the following are equivalent:*

(i) *G is transitive on S^{n-1},*
(ii) *G is $O(n)$-conjugate to one of the following groups: $SO(n)$; $SU(m)$, $U(m)$ $(n = 2m)$; $Sp(m)$, $Sp(m)Sp(1)$, $Sp(m)U(1)$ $(n = 4m)$; $Spin(9)$ $(n = 16)$; $Spin(7)$ $(n = 8)$; G_2 $(n = 7)$.*

When G is not necessarily connected, G is transitive if and only if the connected component of G which has a unit matrix is $O(n)$-conjugate to one of the Lie groups listed in (ii) of Theorem 0.

Under Assumption (A), $f(s)/s$ has a finite limit as $s \to 0$, i.e. $f'(0) = \lim_{s\to 0} f(s)/s$ exists and $0 < f'(0) < \infty$. If f is locally Lipschitz continuous in

$[-a, a] \setminus \{0\}$ and $f'(0) = \infty$, then for each $\lambda > 0$ fixed, (1.1)–(1.3) has infinitely many non-radial G invariant solutions $\{u_k\}$ such that the $C^2(\overline{\Omega})$ norm of u_k converges to zero as $k \to \infty$. This result has been proved in my paper [5].

If $f(u) = |u|^{p-1}u$ with $1 < p < (n+2)/(n-2)$, then (1.1)–(1.3) has infinitely many non-radial G invariant solutions $\{u_k\}$ such that the $C^2(\overline{\Omega})$ norm of u_k diverges to ∞ as $k \to \infty$. This is proved in my earlier paper [3].

2. Examples

In this section we give some examples of $f(u)$ and G.

Example 2.1. Examples of $f(u)$ which satisfies Assumption (A): $f(u) = u - |u|^{p-1}u$ with $1 < p < \infty$, $f(u) = \sin u$, $f(u) = -u \log(|u| + 1/2)$.

Example 2.2. Let G be the n dimensional symmetric group, i.e.

$$G = \{g \in O(n) : \text{ each element of } g \text{ is equal to 1 or 0}\}.$$

This is a finite group, and so it is not transitive. Theorem 1 shows that for each $k \in \mathbb{N}$, if $\lambda > 0$ is sufficiently large, (1.1)–(1.3) has at least k solutions u_i $(1 \le i \le k)$ which are non-radial and satisfy

$$u_i(x_1, \cdots, x_n) = u_i(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \quad \text{for any permutation } \sigma.$$

Example 2.3. Let $1 \le m < n$ and set

$$G = O(m) \times O(n-m) = \left\{ \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} : g \in O(m),\ h \in O(n-m) \right\}.$$

For each $k \in \mathbb{N}$, when $\lambda > 0$ is sufficiently large, (1.1)–(1.3) has at least k solutions u_i $(1 \le i \le k)$ which are non-radial and satisfy $u_i = u_i(|x'|, |x''|)$ for $x' \in \mathbb{R}^m$, $x'' \in \mathbb{R}^{n-m}$ with $|(x', x'')| < R$.

Recall that $f'(0) = \lim_{s \to 0} f(s)/s$ exists and $0 < f'(0) < \infty$ because of Assumption (A). We can assume without loss of generality that Ω is a unit ball and the following condition (A5) holds.

(A5) $f'(0) = 1$ and $f(s) = 0$ for $|s| \ge a$.

Indeed, let the radius of Ω be R. For a solution u of (1.1)–(1.3), we set $v(x) = u(Rx)$, $\mu = \lambda R^2 f'(0)$ and $h(u) = f(u)/f'(0)$. Then v satisfies

$$\begin{aligned} -\Delta v &= \mu h(v), && |x| < 1, \\ v &= 0, && |x| = 1, \\ v(gx) &= v(x), && |x| < 1,\ g \in G. \end{aligned}$$

Since h satisfies Assumption (A) and $h'(0) = 1$, we may assume that Ω is a unit ball and $f'(0) = 1$. We define $\widetilde{f}(s) = f(s)$ for $|s| \le a$ and $\widetilde{f}(s) = 0$ for $|s| > a$. Instead of (1.1), we consider

$$-\Delta u = \lambda \widetilde{f}(u), \quad x \in \Omega. \tag{2.1}$$

Let u be any solution of (2.1), (1.2) and (1.3). Set $v(x) = u(x) - a$ and

$$D = \{x \in \Omega : u(x) > a\}.$$

If $D \neq \emptyset$, then v satisfies

$$-\Delta v = 0 \quad (x \in D), \quad v = 0 \ (x \in \partial D),$$

which proves that $v \equiv 0$ in D. This is impossible. Hence $D = \emptyset$ and $u(x) \le a$ for $x \in \Omega$. The same way as above proves that $u(x) \ge -a$ for $x \in \Omega$. Consequently, any solution u of (2.1), (1.2) and (1.3) becomes a solution of (1.1)–(1.3). When G is not transitive, we have only to prove (ii) of Theorem 1 for $\widetilde{f}$ in place of f. Therefore, we may assume that (A5) holds. Hereafter we always assume that Assumption (A) with (A5) holds, Ω is a unit ball and G is not transitive on S^{n-1}.

3. G invariant critical values

In this section, we construct G invariant critical values $\{\alpha_k(\lambda)\}$ and estimate them. We define a functional $I_\lambda(u)$ by

$$I_\lambda(u) = \int_\Omega \left(\frac{1}{2}|\nabla u|^2 - \lambda F(u) \right) dx, \quad \text{where } F(u) = \int_0^u f(t)\,dt.$$

Set

$$H_0^1(\Omega, G) \equiv \{u \in H_0^1(\Omega) : u(gx) = u(x) \ (g \in G, x \in \Omega)\}.$$

We restrict the domain of the definition for $I_\lambda(\,\cdot\,)$ to $H_0^1(\Omega, G)$. Then $I_\lambda(u)$ is of the C^1 class. The equation $I'_\lambda(u) = 0$ at $u \in H_0^1(\Omega, G)$ means

$$I'_\lambda(u)v \equiv \int_\Omega (\nabla u \nabla v - \lambda f(u)v)\,dx = 0 \quad \text{for } v \in H_0^1(\Omega, G).$$

If this equation holds for any $v \in H_0^1(\Omega, G)$, then it remains valid for any $v \in H_0^1(\Omega)$ also. For the proof, see [3, Lemma 6.2] or [7]. Hence a critical point $u \in H_0^1(\Omega, G)$ of $I_\lambda(\,\cdot\,)$ becomes a weak solution of (1.1)–(1.3). Moreover, it belongs to $C^2(\overline{\Omega})$ by the elliptic regularity theorem.

DEFINITION 3.1. A real number c is called a G *invariant critical value* if there exists a $u \in H_0^1(\Omega, G)$ such that $I'_\lambda(u) = 0$ and $I_\lambda(u) = c$. In this definition, if u is radially symmetric, then c is called a *radially symmetric critical value.*

DEFINITION 3.2. Let X be a real Banach space and A a closed *symmetric* subset of X, i.e. $u \in A$ implies $-u \in A$. Suppose that $0 \notin A$. Then we define a *genus* $\gamma(A)$ of A by the smallest integer k such that there exists an odd continuous mapping from A to $\mathbb{R}^k \setminus \{0\}$. If there does not exist such a k, we define $\gamma(A) = \infty$. For the empty set, we set $\gamma(\emptyset) = 0$.

DEFINITION 3.3. Let Γ_k denote the family of closed symmetric subsets A of $H_0^1(\Omega, G)$ such that $0 \notin A$ and $\gamma(A) \geq k$. Define $\alpha_k(\lambda)$ by

$$\alpha_k(\lambda) = \inf_{A \in \Gamma_k} \sup_{u \in A} I_\lambda(u).$$

DEFINITION 3.4. We define

$$m = m(G) = \max\{\dim G(x) : x \in S^{n-1}\}, \tag{3.1}$$

$$G(x) = \{gx : g \in G\} \quad \text{for } x \in S^{n-1}. \tag{3.2}$$

Since G is a closed subgroup of $O(n)$, $G(x)$ is a closed submanifold of S^{n-1}. Since G is not transitive, it follows that $0 \leq m \leq n-2$. We prove that $\{\alpha_k(\lambda)\}$ are G invariant critical values and we estimate them from above in the next proposition.

PROPOSITION 3.5. *The following assertions hold.*

(i) *If $\alpha_k(\lambda) < 0$, then each $\alpha_i(\lambda)$ with $1 \leq i \leq k$ is a G invariant critical value.*

(ii) *If $\alpha_k(\lambda) = \alpha_{k+1}(\lambda) = \ldots = \alpha_{k+p}(\lambda) < 0$, then $\gamma(K) \geq p+1$. Here*

$$K = \{u \in H_0^1(\Omega, G) : I'_\lambda(u) = 0,\ I_\lambda(u) = \alpha_k(\lambda)\}.$$

(iii) *Fix ν in $(2, \infty)$. Then there exists a constant $C > 0$ independent of k and λ such that*

$$\alpha_k(\lambda) \leq \inf_{t \geq 0} \{Ck^{2/(n-m)} t^2 - \lambda |\Omega| F(t) + \lambda C k^{(\nu-2)/2} t^\nu\}$$

for any $k \in \mathbb{N}$. Here $|\Omega|$ denotes the volume of Ω and m is defined by Definition 3.4.

(iv) *There exists a positive constant A such that if $\lambda \geq Ak^{2/(n-m)}$, then $\alpha_k(\lambda) < 0$.*

To prove this proposition, we need the Palais–Smale condition for $I_\lambda(\,\cdot\,)$.

LEMMA 3.6 ([8, Theorem 2.32], [5, Lemma 3.6]). *For each $\lambda > 0$ fixed, $I_\lambda(\,\cdot\,)$ is bounded from below and satisfies the Palais–Smale condition.*

The following lemma is crucial to obtain Proposition 3.5.

LEMMA 3.7. *For each positive integer k, there exists a closed symmetric subset A_k of $H_0^1(\Omega, G) \cap C(\overline{\Omega})$ satisfying $0 \notin A$ such that*

(i) $\gamma(A_k) = k + 1$,
(ii) $\|\nabla u\|_2 = 1$ *for* $u \in A_k$,
(iii) $C_1 k^{-1/(n-m)} \le \|u\|_1$ *for* $u \in A_k$,
(iv) $\|u\|_\nu \le C_\nu k^{(n-m-2)/2(n-m)-1/\nu}$ *for* $u \in A_k$ *and* $\nu \in (2, \infty)$.

Here m is the integer defined by Definition 3.4 *and constants $C_1, C_\nu > 0$ are independent of u and k.*

The proof of Lemma 3.7 is based on the next lemma.

LEMMA 3.8. *Let m be defined by* (3.1). *For each $k \in \mathbb{N}$ there exist functions ϕ_i $(1 \le i \le 2k)$ such that $\phi_i \in H_0^1(\Omega, G) \cap C(\overline{\Omega})$, $\|\nabla \phi_i\|_2 = 1$,*

$$\operatorname{supp} \phi_i \cap \operatorname{supp} \phi_j = \emptyset \quad \textit{if } i \ne j, \tag{3.3}$$

$$A_\nu k^{(n-m-2)/2(n-m)-1/\nu} \le \|\phi_i\|_\nu \le B_\nu k^{(n-m-2)/2(n-m)-1/\nu}, \tag{3.4}$$

for $1 \le i \le 2k$ and $\nu \ge 1$. Here $\operatorname{supp} \phi$ denotes the support of a function ϕ. A_ν, B_ν are positive constants which depend only on ν.

This lemma has already been proved in [5, Lemma 4.2] except for (3.4). However, observing the proof of [5, Lemma 4.2], we can obtain (3.4) also.

PROOF OF LEMMA 3.7. For a positive integer k, let ϕ_i $(1 \le i \le 2k)$ be determined in Lemma 3.8. Then we define sets A_k and B_k as follows:

$$A_k = \left\{ \sum_{i=1}^{2k} t_i \phi_i : (t_1, \dots, t_{2k}) \in B_k \right\}.$$

Let B_k be a set of points $(t_1, \dots, t_{2k}) \in \mathbb{R}^{2k}$ such that $\sum_{i=1}^{2k} t_i^2 = 1$ and there exist at least k elements t_i $(i = r_1, \dots, r_k)$ satisfying $|t_i| \ge 1/\sqrt{2k}$ for $i = r_1, \dots, r_k$. Then A_k satisfies the assertions (i), (ii) and (iii) of Lemma 3.7. This has been proved in [5, Lemma 3.9].

We show (iv). Let $u = \sum_{i=1}^{2k} t_i \phi_i \in A_k$ and $\nu > 2$. Since $|t_i| \le 1$ and $\nu > 2$, we have $|t_i|^\nu \le |t_i|^2$ for $1 \le i \le 2k$, and so

$$\sum_{i=1}^{2k} |t_i|^\nu \le \sum_{i=1}^{2k} |t_i|^2 = 1.$$

This inequality together with (3.3) and (3.4) shows

$$\|u\|_\nu^\nu = \sum_{i=1}^{2k} |t_i|^\nu \|\phi_i\|_\nu^\nu \le C k^{((n-m-2)\nu/2(n-m))-1}.$$

The proof is complete. $\square$

PROOF OF PROPOSITION 3.5. Let A_k be defined by Lemma 3.7. Since $A_k \in \Gamma_{k+1} \subset \Gamma_k$, the set Γ_k is non-empty and $\alpha_k(\lambda) \le \alpha_{k+1}(\lambda) \le \sup_{A_k} I_\lambda(u) < \infty$. Since $I_\lambda(u)$ is bounded from below, it holds that $\alpha_k(\lambda) > -\infty$. Consequently, $\alpha_k(\lambda)$ is well-defined and satisfies

$$-\infty < \alpha_1(\lambda) \le \alpha_2(\lambda) \le \ldots \le \alpha_k(\lambda) < \infty.$$

For the proof of (i) and (ii), see [2] or [8, p. 53]. We show (iii). Fix $\nu \in (2, \infty)$. We set $G(t) \equiv F(t) + C|t|^\nu$ and prove that if $C > 0$ is sufficiently large, $G(t)$ is convex in $[0, \infty)$. It is sufficient to prove that $G'(t)$ is nondecreasing for $t \ge 0$. By (A3), $f(t)$ is non-decreasing in $[0, \delta]$ with a certain $\delta > 0$, and hence so is $G'(t)$. Let L be the Lipschitz constant of $f(t)$. For $\delta \le t_1 < t_2 < \infty$, there is a $\xi \in (t_1, t_2)$ by the mean value theorem such that

$$t_2^{\nu-1} - t_1^{\nu-1} = (\nu-1)\xi^{\nu-2}(t_2 - t_1) \ge (\nu-1)\delta^{\nu-2}(t_2 - t_1).$$

This proves that

$$\begin{aligned} G'(t_2) - G'(t_1) &= f(t_2) - f(t_1) + C\nu t_2^{\nu-1} - C\nu t_1^{\nu-1} \\ &\ge -L(t_2 - t_1) + C\nu(\nu-1)\delta^{\nu-2}(t_2 - t_1) > 0, \end{aligned}$$

provided that C is chosen so large that $C\nu(\nu-1)\delta^{\nu-2} - L > 0$. Therefore $G'(t)$ is non-decreasing for $t \ge 0$. Since $G(t)$ is even and convex, Jensen's inequality gives

$$\frac{1}{|\Omega|}\int_\Omega G(u)dx = \frac{1}{|\Omega|}\int_\Omega G(|u|)\,dx \ge G(|\Omega|^{-1}\|u\|_1).$$

This proves

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\|\nabla u\|_2^2 - \lambda\int_\Omega F(u)\,dx \\ &\le \frac{1}{2}\|\nabla u\|_2^2 - \lambda|\Omega|G(|\Omega|^{-1}\|u\|_1) + \lambda C\|u\|_\nu^\nu \\ &\le \frac{1}{2}\|\nabla u\|_2^2 - \lambda|\Omega|F(|\Omega|^{-1}\|u\|_1) + \lambda C\|u\|_\nu^\nu. \end{aligned} \tag{3.5}$$

Let A_k be defined by Lemma 3.7. We set

$$sA_k = \{su : u \in A_k\} \quad \text{for } s > 0.$$

Lemma 3.7 and (3.5) prove

$$I_\lambda(u) \le J(s) \quad \text{for } u \in sA_k \text{ and } s > 0, \tag{3.6}$$

where

$$J(s) \equiv s^2/2 - \lambda|\Omega|F(|\Omega|^{-1}C_1 k^{-1/(n-m)}s) + \lambda C k^{((n-m-2)\nu/2(n-m))-1}s^\nu.$$

Since $sA_k \in \Gamma_{k+1}$, inequality (3.6) yields

$$\alpha_k(\lambda) \le \alpha_{k+1}(\lambda) \le \sup_{u \in sA_k} I_\lambda(u) \le J(s),$$

which proves

$$\alpha_k(\lambda) \le \inf_{s>0} J(s).$$

Setting $t = C_1|\Omega|^{-1}k^{-1/(n-m)}s$ in the above inequality, we obtain (iii).

We show (iv). Observe the inequality in the assertion (iii), i.e.,

$$\alpha_k(\lambda) \le \inf_{t\ge 0} K(t),$$
$$K(t) \equiv Ck^{2/(n-m)}t^2 - \lambda|\Omega|F(t) + \lambda Ck^{(\nu-2)/2}t^\nu.$$

Since $\lim_{t\to 0} F(t)/t^2 = f'(0)/2 = 1/2$ and $\nu > 2$, we have

$$\lim_{t\to 0} K(t)/t^2 = Ck^{2/(n-m)} - \lambda|\Omega|/2 < 0,$$

provided that $\lambda \ge 4C|\Omega|^{-1}k^{2/(n-m)}$. Hence, for $s > 0$ sufficiently small, it follows that

$$\alpha_k(\lambda) \le \inf_{t\ge 0} K(t) \le K(s) < 0 \quad \text{if } \lambda \ge 4C|\Omega|^{-1}k^{2/(n-m)}.$$

The proof is complete. □

4. Proof of Theorem 1

In this section we show the uniqueness of radially symmetric nodal solutions and use this fact to prove Theorem 1. For a radially symmetric solution $u = u(r)$, $r = |x|$, the problem (1.1) with (1.2) is reduced to

$$u'' + \frac{n-1}{r}u' + \lambda f(u) = 0, \tag{4.1}$$

$$u'(0) = 0, \qquad u(1) = 0. \tag{4.2}$$

Definition 4.1. Let μ_k and ϕ_k denote the k-th eigenvalue and eigenfunction of the problem,

$$u'' + \frac{n-1}{r}u' + \mu u = 0, \tag{4.3}$$

with the boundary condition (4.2). We may assume that $\phi_k(0) = 1$.

It is well-known that $\{\mu_k\}$ is strictly increasing and μ_k/k^2 converges to π^2 as $k \to \infty$ (refer to [5, Proof of Proposition 6.1]).

Proposition 4.2. *Suppose that k and λ satisfy $\mu_k < \lambda \le \mu_{k+1}$. Then for each integer $j \in [1,k]$, there exists a unique solution $u_j(r)$ of* (4.1) *and* (4.2) *which has exactly j zeros in $[0,1]$ and satisfies $u(0) > 0$. Moreover, the set of all solutions to* (4.1), (4.2) *consists of $\pm u_j(r)$ for $1 \le j \le k$ and the zero solution.*

To prove this proposition, we consider the initial condition,

$$u'(0) = 0, \quad u(0) = \xi. \tag{4.4}$$

Lemma 4.3 ([4, Theorems 1 and 2]). *For $\xi \in (0,a)$, the problem* (4.1) *with* (4.4) *has a unique global solution $u(r) = u(r,\xi)$ defined on $[0,\infty)$. Furthermore, it holds that $|u(r,\xi)| \le \xi$ for $r \ge 0$. There exist unbounded sequences $\{z_k(\xi,\lambda)\}$ and $\{t_k(\xi,\lambda)\}$ such that*

$$\begin{gathered} u(z_k) = 0, \quad u'(t_k) = 0, \\ 0 < z_1 < t_1 < z_2 < t_2 < \dots \nearrow \infty, \end{gathered}$$

$$\begin{aligned} u(r) > 0 \quad &\text{for } r \in (0,z_1), \quad (-1)^k u(r) > 0 \quad &&\text{for } r \in (z_k, z_{k+1}), \\ u'(r) < 0 \quad &\text{for } r \in (0,t_1), \quad (-1)^{k+1} u'(r) > 0 \quad &&\text{for } r \in (t_k, t_{k+1}), \end{aligned}$$

where $z_k = z_k(\xi,\lambda)$ and $t_k = t_k(\xi,\lambda)$. Moreover, for $\lambda > 0$ fixed, each $z_k(\xi,\lambda)$ is strictly increasing with respect to $\xi \in (0,a)$ and satisfies

$$\lim_{\xi \to a-0} z_k(\xi,\lambda) = \infty \quad \text{for any } k \in \mathbb{N}.$$

Lemma 4.4. *Let μ_k and $z_k(\xi,\lambda)$ be defined by Definition* 4.1 *and Lemma* 4.3*, respectively.*

(i) *If $\mu_k < \lambda$, then $z_k(\xi,\lambda) < 1$ for $\xi > 0$ sufficiently small.*
(ii) *If $\lambda \le \mu_{k+1}$, then $z_{k+1}(\xi,\lambda) > 1$ for all $\xi \in (0,a)$.*

Proof. We show (i). Suppose that $\mu_k < \lambda$. Since $f'(0) = 1$, there exists a $\delta > 0$ such that $\lambda f(s)/s > \mu_k$ for $|s| \le \delta$. Since $|u(r,\xi)| \le \xi$ for $r \ge 0$ by Lemma 4.3, we have

$$\lambda f(u(r,\xi))/u(r,\xi) > \mu_k \quad \text{for } 0 < \xi \le \delta \text{ and } r \ge 0.$$

Compare two equations below,

$$(r^{n-1}u')' + \lambda r^{n-1}\frac{f(u)}{u}u = 0, \quad u'(0) = 0,\ u(0) = \xi, \tag{4.5}$$

$$(r^{n-1}\phi_k')' + \mu_k r^{n-1}\phi_k = 0, \quad \phi_k'(0) = 0,\ \phi_k(0) = 1. \tag{4.6}$$

Recall that the k-th zero of ϕ_k is equal to $r = 1$. Then Sturm's comparison theorem means that $z_k(\xi,\lambda) < 1$.

We show (ii). Suppose that $\lambda \le \mu_{k+1}$. Since $f(s)/s$ is strictly decreasing in $(0,a)$ and $f(s)$ is odd in $[-a,a]$, we get

$$\lambda f(u(r,\xi))/u(r,\xi) \le \lambda f'(0) = \lambda \le \mu_{k+1} \quad \text{for } r \ge 0,\ \xi \in (0,a),$$

and moreover $\lambda f(u(r,\xi))/u(r,\xi) \not\equiv \mu_{k+1}$. Compare (4.5) with the equation below,

$$(r^{n-1}\phi_{k+1}')' + \mu_{k+1} r^{n-1}\phi_{k+1} = 0.$$

Then Sturm's comparison theorem proves that $1 < z_{k+1}(\xi,\lambda)$ for any $\xi \in (0,a)$. The proof is complete. □

Proof of Proposition 4.2. Let $\mu_k < \lambda \le \mu_{k+1}$. Fix an integer $j \in [1,k]$. Then Lemmas 4.3 and 4.4 mean the following facts.

Fact 1. $z_j(\xi,\lambda) < z_k(\xi,\lambda) < 1$ for $\xi > 0$ sufficiently small.

Fact 2. $z_j(\xi,\lambda)$ is strictly increasing with respect to $\xi \in (0,a)$.

Fact 3. $\lim_{\xi\to a-0} z_j(\xi,\lambda) = \infty$.

By these facts, there exists a unique $\xi_j \in (0,a)$ such that $z_j(\xi_j,\lambda) = 1$. We set $u_j(r) \equiv u(r,\xi_j)$, which is a solution of (4.1), (4.2) having exactly j zeros in $[0,1]$ and satisfies $u(0) > 0$. Since $z_{k+1}(\xi,\lambda) > 1$ for all $\xi \in (0,a)$ by Lemma 4.4 (ii), no solution $u \not\equiv 0$ of (4.1), (4.2) has more zeros than k in the interval $[0,1]$. Therefore, all solutions of (4.1), (4.2) consist of $\pm u_j(r)$ for $1 \le j \le k$ and the zero solution. The proof is complete. □

Proof of Theorem 1. It is clear that (ii) implies (i) in Theorem 1. We prove the converse. Suppose that (i) holds. On the contrary, assume that (ii) is false. That is, there exist a sequence $\{\lambda_k\}$ and a positive integer k_0 such that $\{\lambda_k\}$ diverges to ∞ as $k \to \infty$ and (1.1)–(1.3) with $\lambda = \lambda_k$ has at most k_0 solutions which are non-radial and G invariant.

Fix $k \in \mathbb{N}$ arbitrarily. Let i be an integer satisfying

$$Ai^{2/(n-m)} \le \lambda_k < A(i+1)^{2/(n-m)}. \tag{4.7}$$

Here A is a positive constant defined by Proposition 3.5(iv). Then $\alpha_i(\lambda_k)$ is defined by Definition 3.3 and it is negative by Proposition 3.5(iv). Hence, each $\alpha_p(\lambda_k)$ $(1 \le p \le i)$ is a G invariant critical value because of Proposition 3.5(i).

Define an integer j by $\mu_j < \lambda_k \le \mu_{j+1}$. Here μ_j denotes the j-th eigenvalue of (4.2) and (4.3). Since μ_j/j^2 converges to π^2 as $j \to \infty$, there exists a constant $C > 0$ independent of j and k such that

$$j \le C\sqrt{\lambda_k}. \tag{4.8}$$

Proposition 4.2 asserts that all of radially symmetric solutions of (1.1)–(1.3) are $\pm u_p$ for $1 \le p \le j$ and the zero solution. Here u_p is the p-nodal solution of (4.1) and (4.2). If $\alpha_p(\lambda_k) = \alpha_{p+1}(\lambda_k)$ with some $p \in [1, i-1]$, then Proposition 3.5(ii) shows that $\gamma(K) \ge 2$, where

$$K = \{u \in H_0^1(\Omega, G) : I'_{\lambda_k}(u) = 0, I_{\lambda_k}(u) = \alpha_p(\lambda_k)\}.$$

Therefore K is infinite because of the definition of the genus. Since the set of radially symmetric solutions, $\pm u_p(r)$ with $1 \le p \le j$ and the zero solution, is finite, K has infinitely many G invariant non-radial solutions. This conclusion contradicts our assumption that (1.1)–(1.3) with $\lambda = \lambda_k$ has at most k_0 solutions which are non-radial and G invariant. Therefore, we deduce that $\alpha_p(\lambda_k) < \alpha_{p+1}(\lambda_k)$ for $1 \le p \le i-1$. Since the number of non-radial G invariant solutions is at most k_0 by our assumption, there exist integers $\nu_1, \dots, \nu_l$ with $l \le k_0$ such that each $\alpha_p(\lambda_k)$ with $1 \le p \le i$ except for $\alpha_{\nu_1}(\lambda_k), \dots, \alpha_{\nu_l}(\lambda_k)$ is a radially symmetric critical value. We set

$$\begin{aligned} P &= \{\alpha_p(\lambda_k) : 1 \le p \le i, p \ne \nu_q (1 \le q \le l)\}, \\ Q &= \{\beta_p : 1 \le p \le j\}, \quad \text{where} \quad \beta_p = I_{\lambda_k}(u_p). \end{aligned}$$

Then it follows that $P \subset Q$, and so $i - k_0 \le \#P \le \#Q \le j$. Combining this inequality, (4.7) and (4.8), we obtain a constant $C > 0$ independent of k such that

$$\lambda_k^{(n-m)/2} \le C\lambda_k^{1/2} + C \quad \text{for all } k \in \mathbb{N}.$$

Since $\lim_{k\to\infty} \lambda_k = \infty$, we have a contradiction because of $0 \le m \le n - 2$. The proof is complete. □

REFERENCES

[1] A. BOREL, *Le plan projectif des octaves et les sphères comme espaces homogènes*, C. R. Acad. Sci. Paris Sér. **230** (1950), 1378–1380.

[2] D. C. CLARK, *A variant of the Lusternik–Schnirelman theory*, Indiana Univ. Math. J. **22** (1972), 65–74.

[3] R. KAJIKIYA, *Orthogonal group invariant solutions of the Emden–Fowler equation*, Nonlinear Anal. **44** (2001), 845–896.

[4] ______, *Necessary and sufficient condition for existence and uniqueness of nodal solutions to sublinear elliptic equations*, Adv. Differential Equations (to appear).

[5] ______, *Multiple existence of non-radial solutions with group invariance for sublinear elliptic equations*, preprint.

[6] D. MONTGOMERY AND H. SAMELSON, *Transformation groups of spheres*, Ann. of Math. **44** (1943), 454–470.

[7] R. S. PALAIS, *The principle of symmetric criticality*, Comm. Math. Phys. **69** (1979), 19–30.

[8] P. H. RABINOWITZ, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math., vol. 65, Amer. Math. Soc., Providence, 1986.

Manuscript received July 10, 2001

RYUJI KAJIKIYA
Nagasaki Institute of Applied Science
536 Aba-machi
Nagasaki 851-0193, JAPAN

E-mail address: kajikiya@bas.nias.ac.jp