Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 14, 1999, 239–255

SOME RECENT RESULTS ON THIN DOMAIN PROBLEMS

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Dedicated to the memory of Julisz P. Schauder

Abstract. Let Ω be an arbitrary smooth bounded domain in $\mathbb{R}^2$ and $\varepsilon > 0$ be arbitrary. Write (x, y) for a generic point of $\mathbb{R}^2$. Squeeze Ω by the factor ε in the y-direction to obtain the squeezed domain $\Omega_\varepsilon = \{(x, \varepsilon y) \mid (x, y) \in \Omega\}$. Consider the following reaction-diffusion equation on Ω_ε:

$$(\mathrm{E}_\varepsilon)\qquad \begin{aligned} &u_t = \Delta u + f(u), && t > 0,\ (x, y) \in \Omega_\varepsilon \\ &\partial_{\nu_\varepsilon} u = 0, && t > 0,\ (x, y) \in \partial\Omega_\varepsilon. \end{aligned}$$

Here, ν_ε is the exterior normal vector field on $\partial\Omega_\varepsilon$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a nonlinearity satisfying some growth and dissipativeness conditions ensuring that (E_ε) generates a semiflow π_ε on $H^1(\Omega_\varepsilon)$ with a global attractor $\mathcal{A}_\varepsilon$. In this paper we report on some recent results concerning the asymptotic behavior of the equations (E_ε) as $\varepsilon \to 0$.

1. Limit dynamics on squeezed domains

Let Ω be an arbitrary smooth bounded domain in $\mathbb{R}^2$ and $\varepsilon > 0$ be arbitrary. Write (x, y) for a generic point of $\mathbb{R}^2$. Given $\varepsilon > 0$ squeeze Ω by the factor ε in the y-direction to obtain the squeezed domain Ω_ε. More precisely, define the map

$$T_\varepsilon\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad (x, y) \mapsto (x, \varepsilon y)$$

and set $\Omega_\varepsilon := T_\varepsilon(\Omega)$. Consider the following reaction-diffusion equation on Ω_ε:

$$(1.1)\qquad \begin{aligned} &u_t = \Delta u + f(u), && t > 0,\ (x, y) \in \Omega_\varepsilon \\ &\partial_{\nu_\varepsilon} u = 0, && t > 0,\ (x, y) \in \partial\Omega_\varepsilon. \end{aligned}$$

1991 *Mathematics Subject Classification.* 35B10, 35B42.
Key words and phrases. General behaviour of solutions, intertial manifolds.

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Here, ν_ε is the exterior normal vector field on $\partial\Omega_\varepsilon$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a C^1-nonlinearity of polynomial growth, that is

$$|f'(s)| \leq C(1 + |s|^\beta) \quad \text{for } s \in \mathbb{R}, \tag{1.2}$$

where C and $\beta \in [0, \infty[$ are arbitrary real constants. In addition, suppose that f is dissipative in the sense that

$$\limsup_{|s|\to\infty} f(s)/s \leq -\delta_0 \quad \text{for some } \delta_0 > 0. \tag{1.3}$$

These hypotheses imply that (1.1) generates a semiflow $\widetilde{\pi}_\varepsilon = \widetilde{\pi}_{\varepsilon,f}$ on $H^1(\Omega_\varepsilon)$ which has a global attractor $\widetilde{\mathcal{A}}_\varepsilon = \widetilde{\mathcal{A}}_{\varepsilon,f}$.

As $\varepsilon \to 0$ the thin domain Ω_ε degenerates to a one-dimensional interval.

One may ask what happens in the limit to the family $(\widetilde{\pi}_\varepsilon)_{\varepsilon>0}$ of semiflows and to the family $(\widetilde{\mathcal{A}}_\varepsilon)_{\varepsilon>0}$ of attractors. Is there a limit semiflow and a corresponding limit attractor?

This problem was first considered by Hale and Raugel in [7] for the case when the domain Ω is the *ordinate set* of a smooth positive function g defined on an interval $[a, b]$, i.e.

$$\Omega = \{(x, y) \mid a < x < b \text{ and } 0 < y < g(x)\}.$$

The authors prove that, in this case, there exists a limit semiflow $\widetilde{\pi}_0$, which is defined by the one-dimensional boundary value problem

$$\begin{aligned} &u_t = (1/g)(gu_x)_x + f(u), && t > 0,\ x \in\,]a, b[\,, \\ &u_x = 0, && t > 0,\ x = a, b. \end{aligned} \tag{1.4}$$

Moreover, $\widetilde{\pi}_0$ has a global attractor $\widetilde{\mathcal{A}}_0$ and, in some sense, the family $(\widetilde{\mathcal{A}}_\varepsilon)_{\varepsilon\geq 0}$ is upper-semicontinuous at $\varepsilon = 0$.

Hale and Raugel also prove that one can modify the nonlinearity f in such a way that each modified semiflow $\widetilde{\pi}'_\varepsilon$ possesses an invariant (inertial) C^1-manifold $\widetilde{\mathcal{M}}_\varepsilon$ of some fixed dimension ν which includes the attractor $\widetilde{\mathcal{A}}_\varepsilon$ of the original semiflow $\widetilde{\pi}_\varepsilon$. The semiflows $\widetilde{\pi}_\varepsilon$ and $\widetilde{\pi}'_\varepsilon$ coincide on the attractor $\widetilde{\mathcal{A}}_\varepsilon$.

Moreover, as $\varepsilon \to 0$, the reduced flow on $\widetilde{\mathcal{M}}_\varepsilon$ converges in the C^1-sense to the reduced flow on $\widetilde{\mathcal{M}}_0$.

If the domain Ω is not the ordinate set of some function (e.g. if Ω has holes or different horizontal branches) then (1.4) can no longer be a limiting equation for (1.1). Nevertheless, as it was proved in [12] the family $\widetilde{\pi}_\varepsilon$ still has a limit semiflow. Moreover, there exists a limit global attractor and the upper-semicontinuity result continues to hold.

In order to describe the main results of [12] we first transfer the family (1.1) to boundary value problems on the fixed domain Ω. More explicitly, we use the linear isomorphism $\Phi_\varepsilon\colon H^1(\Omega_\varepsilon) \to H^1(\Omega)$, $u \mapsto u \circ T_\varepsilon$, to transform problem (1.1) to the equivalent problem

$$
\begin{aligned}
&u_t = u_{xx} + \frac{1}{\varepsilon^2} u_{yy} + f(u), && t > 0,\ (x,y) \in \Omega,\\
&u_x \nu_1 + \frac{1}{\varepsilon^2} u_y \nu_2 = 0, && t > 0,\ (x,y) \in \partial\Omega.
\end{aligned}
\tag{1.5}
$$

on Ω. Here, $\nu = (\nu_1, \nu_2)$ is the exterior normal vector field on $\partial\Omega$.

Note that equation (1.5) can be written in the abstract form

$$\dot u + A_\varepsilon u = \widehat{f}(u)$$

where $\widehat{f}\colon H^1(\Omega) \to L^2(\Omega)$, $u \mapsto f \circ u$, is the Nemitski operator generated by the function f, and A_ε is the linear operator defined by

$$A_\varepsilon u = -u_{xx} - \frac{1}{\varepsilon^2} u_{yy} \in L^2(\Omega) \quad \text{for } u \in H^2(\Omega) \text{ with } u_x\nu_1 + \frac{1}{\varepsilon^2} u_y \nu_2 = 0 \text{ on } \partial\Omega.$$

Equation (1.5) defines a semiflow $\pi_\varepsilon = \pi_{\varepsilon,\widehat{f}}$ on $H^1(\Omega)$ which is equivalent to $\widetilde{\pi}_\varepsilon$ and has the global attractor $\mathcal{A}_\varepsilon := \Phi_\varepsilon(\widetilde{\mathcal{A}}_\varepsilon)$, consisting of the orbits of all full bounded solutions of (1.5).

The operator A_ε is, in the usual way, induced by the following bilinear form

$$a_\varepsilon(u,v) := \int_\Omega \Big(u_x v_x + \frac{1}{\varepsilon^2} u_y v_y\Big)\, dx\, dy, \quad u, v \in H^1(\Omega).$$

Notice that, for every fixed $\varepsilon > 0$ and $u \in H^1(\Omega)$, the formula

$$|u|_\varepsilon = (a_\varepsilon(u,u) + |u|^2_{L^2(\Omega)})^{1/2}$$

defines a norm on $H^1(\Omega)$ which is equivalent to $|\cdot|_{H^1(\Omega)}$. However, $|u|_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ whenever $u_y \neq 0$ in $L^2(\Omega)$. In fact, we see that for $u \in H^1(\Omega)$

$$\lim_{\varepsilon \to 0^+} a_\varepsilon(u,u) = \begin{cases} \displaystyle\int_\Omega u_x^2\, dx\, dy & \text{if } u_y = 0,\\ \infty & \text{otherwise.} \end{cases}$$

Thus the family $a_\varepsilon(u,u)$, $\varepsilon > 0$, of real numbers has a finite limit (as $\varepsilon \to 0$) if and only if $u \in H^1_s(\Omega)$, where we define the closed linear subspace $H^1_s(\Omega)$ of $H^1(\Omega)$ by

$$H^1_s(\Omega) := \{u \in H^1(\Omega) \mid u_y = 0\}.$$

The corresponding limit bilinear form is given by the formula:

$$a_0(u,v) := \int_\Omega u_x v_x\, dx\, dy, \quad u, v \in H^1_s(\Omega). \tag{1.6}$$

The form a_0 uniquely determines a densely defined selfadjoint linear operator

$$A_0\colon D(A_0) \subset H^1_s(\Omega) \to L^2_s(\Omega)$$

by the usual formula

$$a_0(u,v) = \langle A_0 u, v\rangle_{L^2(\Omega)} \quad \text{for } u \in D(A_0) \text{ and } v \in H^1_s(\Omega).$$

Here, the linear space $L^2_s(\Omega)$ is defined as the closure of $H^1_s(\Omega)$ in the L^2-norm. It follows that the Nemitski operator $\widehat{f}$ maps the space $H^1_s(\Omega)$ into $L^2_s(\Omega)$. Consequently the abstract parabolic equation

$$\dot{u} = -A_0 u + \widehat{f}(u) \tag{1.7}$$

defines a semiflow $\pi_0 = \pi_{0,\widehat{f}}$ on the space $H^1_s(\Omega)$. This is the limit semiflow of the family π_ε. In fact, the following results are proved in [12]:

Theorem 1.1. *Let $(\varepsilon_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers convergent to zero and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $L^2(\Omega)$ converging in the norm of $L^2(\Omega)$ to some $u_0 \in L^2_s(\Omega)$. Moreover, let $(t_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive numbers converging to some positive number t_0. Then*

$$|e^{-t_n A_{\varepsilon_n}} u_n - e^{-t_0 A_0} u_0|_{\varepsilon_n} \to 0 \quad \textit{as } n \to \infty.$$

If, in addition, $u_n \in H^1(\Omega)$ for every $n \in \mathbb{N}$ and if $u_0 \in H^1_s(\Omega)$, then

$$|u_n \pi_{\varepsilon_n} t_n - u_0 \pi_0 t_0|_{\varepsilon_n} \to 0 \quad \textit{as } n \to \infty.$$

The limit semiflow π_0 possesses a global attractor $\mathcal{A}_0$. The upper-semicontinuity result alluded to above reads as follows:

Theorem 1.2. *The family of attractors $(\mathcal{A}_\varepsilon)_{\varepsilon\in[0,1]}$ is upper-semicontinuous at $\varepsilon = 0$ with respect to the family of norms $|\cdot|_\varepsilon$. This means that*

$$\lim_{\varepsilon\to 0^+} \sup_{u\in\mathcal{A}_\varepsilon} \inf_{v\in\mathcal{A}_0} |u - v|_\varepsilon = 0.$$

In particular, there exists an $\varepsilon_1 > 0$ and an open bounded set U in $H^1(\Omega)$ including all the attractors $\mathcal{A}_\varepsilon$, $\varepsilon \in [0,\varepsilon_1]$.

The definition of the linear operator A_0, as given above, is not very explicit. However, as it is shown in [12], there is a large class of the so-called *nicely decomposed* domains on which A_0 can be characterized as a system of one-dimensional second order linear differential operators, coupled to each other by certain compatibility and Kirchhoff type balance conditions. In this case, the abstract limit equation (1.7) is equivalent to a parabolic equation on a finite graph (cf. Section 2 below).

Let $(\lambda_\nu)_{\nu\in\mathbb{N}}$ be the nondecreasing sequence of the eigenvalues of the limit operator A_0 (each of the eigenvalues being repeated according to its multiplicity). Assume that the sequence $(\lambda_\nu)_{\nu\in\mathbb{N}}$ satisfies the following gap condition:

$$\limsup_{\nu\to\infty} \frac{\lambda_{\nu+1} - \lambda_\nu}{{\lambda_\nu}^{1/2}} > 0. \tag{1.8}$$

In [13] it is shown that under hypothesis (1.8) there is an $\varepsilon_0 > 0$, $\varepsilon_0 \le \varepsilon_1$, and there exists a family $\mathcal{M}_\varepsilon$, $0 \le \varepsilon \le \varepsilon_0$ of (inertial) C^1-manifolds of some finite dimension ν such that, whenever $0 \le \varepsilon \le \varepsilon_0$, then $\mathcal{A}_\varepsilon \subset \mathcal{M}_\varepsilon$ and the manifold $\mathcal{M}_\varepsilon$ is locally invariant relative to the semiflow π_ε on the neighbourhood U of the attractor $\mathcal{A}_\varepsilon$. Furthermore, as $\varepsilon \to 0$, the reduced flow on the manifold $\mathcal{M}_\varepsilon$ converges in the C^1-sense to the reduced flow on $\mathcal{M}_0$. It is also proved that the gap condition (1.8) is satisfied on nicely decomposed domains satisfying a natural additional condition. In particular, our inertial manifold theorem contains, as a special case, the inertial manifold theorem of Hale and Raugel and it even improves the latter.

Let us discuss in some detail the construction of the inertial manifolds given in the proof of Theorem 1.3 below. We apply the method of functions of exponential growth, used before by many researchers (cf. [5], [15] and the references contained in these papers). First we choose an open set U in $H^1(\Omega)$ which includes all the attractors $\mathcal{A}_\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$, $\varepsilon_0 > 0$ small. Then we modify the Nemitski operator $\widehat{f}$ (rather than the *function* f) by finding a globally Lipschitzian map $g\colon H^1(\Omega) \to L^2(\Omega)$ with $\widehat{f}(u) = g(u)$ for $u \in U$. For fixed $\varepsilon \in [0, \varepsilon_0]$ we seek an invariant manifold $\mathcal{M}_\varepsilon$ for the modified semiflow $\pi_{\varepsilon,g}$ in the form $\mathcal{M}_\varepsilon = \Lambda_\varepsilon(\mathbb{R}^\nu)$, where $\Lambda_\varepsilon\colon \mathbb{R}^\nu \to H^1(\Omega)$ is a map obtained from the contraction mapping principle applied to a properly defined nonlinear operator Γ_ε defined on a certain space of maps $y\colon]-\infty, 0] \to H^1(\Omega)$ of exponential growth. If the operator Γ_ε is a contraction then the map Λ_ε is well-defined and $\mathcal{A}_\varepsilon \subset \mathcal{M}_\varepsilon$. It follows that $\mathcal{M}_\varepsilon$ is invariant with respect to solutions of the original semiflow $\pi_{\varepsilon,\widehat{f}}$ as long as these solutions stay in the open set U. One can even find an open set $V \subset \mathbb{R}^\nu$ such that for $\varepsilon \in [0, \varepsilon_0]$ the set $\Lambda_\varepsilon(V)$ is positively invariant with respect to $\pi_{\varepsilon,\widehat{f}}$ and $\mathcal{A}_\varepsilon \subset \Lambda_\varepsilon(V) \subset U$.

The only problem is that, under the usual norm $|\cdot|_\varepsilon$ on $H^1(\Omega)$, the operator Γ_ε is *not* a contraction. In fact, the gap condition (1.8), which is the best possible, does not yield gaps which are large enough to counterbalance the Lipschitz constant of the given Nemitski operator. At this point we use an ingenious idea due to Brunovský and Tereščák (see Theorem 4.1 in [3] and its proof) and, given positive numbers l and L introduce an equivalent norm

$$\|u\|_\varepsilon = L|u|_{L^2} + l|u|_\varepsilon$$

on $H^1(\Omega)$. Similarly, as in [3], we seek to choose the constants l and L in such a way that the operator Γ_ε is a uniform contraction with respect to the norm $\|\cdot\|_\varepsilon$. That this is possible is due to the Gagliardo–Nirenberg inequality combined with some linear estimates obtained from the variation-of-constants formula.

In order to state the inertial manifold theorem we need some notation. For every $\varepsilon \in [0, 1]$ let $(\lambda_{\varepsilon,j})_{j\in\mathbb{N}}$ be the repeated sequence of the eigenvalues of A_ε and let $(w_{\varepsilon,j})_{j\in\mathbb{N}}$ be a corresponding complete orthonormal sequence of eigenvectors. For every $\varepsilon \in [0,1]$ and every $\nu \in \mathbb{N}$ let $E_{\varepsilon,\nu}\colon \mathbb{R}^\nu \to L^2(\Omega)$ be defined by

$$E_{\varepsilon,\nu}\xi := \sum_{j=1}^{\nu} \xi_j w_{\varepsilon,j}, \quad \xi \in \mathbb{R}^\nu.$$

Furthermore, if $\varepsilon > 0$ (respectively, $\varepsilon = 0$), let $P_{\varepsilon,\nu}\colon L^2(\Omega) \to L^2(\Omega)$ (respectively, $P_{\varepsilon,\nu}\colon L^2_s(\Omega) \to L^2_s(\Omega)$) be the orthogonal projection of $L^2(\Omega)$ (respectively, $L^2_s(\Omega)$) onto the the span of the vectors $w_{\varepsilon,j}$, $j = 1,\ldots,\nu$.

Theorem 1.3. *Suppose that $f \in C^1(\mathbb{R} \to \mathbb{R})$ satisfies the growth and dissipativeness conditions* (1.2) *and* (1.3). *Suppose the eigenvalues of A_0 satisfy the gap condition* (1.8). *Then there are an $\varepsilon_0 > 0$ and an open bounded set $U \subset H^1(\Omega)$ such that for every $\varepsilon \in [0,\varepsilon_0[$ the attractor $\mathcal{A}_\varepsilon$ of the semiflow $\pi_{\varepsilon,\widehat{f}}$ lies in U.*

Furthermore, there exists a globally Lipschitzian map $g \in C^1(H^1(\Omega) \to L^2(\Omega))$ with $g(u) = \widehat{f}(u)$ for $u \in U$.

Besides, there is a positive integer ν and for every $\varepsilon \in [0,\varepsilon_0[$ there is a map $\Lambda_\varepsilon \in C^1(\mathbb{R}^\nu \to H^1(\Omega))$ if $\varepsilon > 0$ and $\Lambda_\varepsilon \in C^1(\mathbb{R}^\nu \to H^1_s(\Omega))$ if $\varepsilon = 0$ such that

$$P_{\varepsilon,\nu} \circ \Lambda_\varepsilon = E_{\varepsilon,\nu} \tag{1.9}$$

and $\Lambda_\varepsilon(\mathbb{R}^\nu)$ is an invariant manifold with respect to the semiflow $\pi_{\varepsilon,g}$.

Finally, there is an open set $V \subset \mathbb{R}^\nu$ such that, for every $\varepsilon \in [0,\varepsilon_0[$,

$$\mathcal{A}_\varepsilon \subset \Lambda_\varepsilon(V) \subset U$$

and the set $\Lambda_\varepsilon(V)$ is positively invariant with respect to the semiflow $\pi_{\varepsilon,\widehat{f}}$.

The reduced equation on $\Lambda_\varepsilon(\mathbb{R}^\nu)$ takes the form

$$\dot{\xi} = v_\varepsilon(\xi), \quad \xi \in \mathbb{R}^\nu, \tag{1.10}$$

where

$$v_\varepsilon\colon \mathbb{R}^\nu \to \mathbb{R}^\nu, \quad \xi \mapsto -A_\varepsilon E_{\varepsilon,\nu}\xi + P_{\varepsilon,\nu} g(\Lambda_\varepsilon(\xi)).$$

Moreover, whenever $\varepsilon_n \to 0^+$ and $\xi_n \to \xi_0$ in $\mathbb{R}^\nu$, then

$$|\Lambda_{\varepsilon_n}(\xi_n) - \Lambda_0(\xi_0)|_{\varepsilon_n} + \sum_{j=1}^{\nu} |\partial_j \Lambda_{\varepsilon_n}(\xi_n) - \partial_j \Lambda_0(\xi_0)|_{\varepsilon_n} \to 0 \tag{1.11}$$

and

$$|v_{\varepsilon_n}(\xi_n) - v_0(\xi_0)|_{\mathbb{R}^\nu} + \sum_{j=1}^{\nu} |\partial_j v_{\varepsilon_n}(\xi_n) - \partial_j v_0(\xi_0)|_{\mathbb{R}^\nu} \to 0. \tag{1.12}$$

The reader is referred to [13] for a detailed proof of Theorem 1.3.

2. Nicely decomposed domains

In this section we report on some results from [12] and [13] which characterize the function spaces $H^1_s(\Omega)$ and $L^2_s(\Omega)$ and the domain of the limit operator $A = A_0$ on nicely decomposed domains. Moreover, we state the additional conditions which ensure that the eigenvalues of the operator A satisfy the gap condition described in the previous section.

For the reader's convenience we will first recall the definition of a nicely decomposed domain.

We say that an open set $\Omega \subset \mathbb{R}^2$ *has connected vertical sections* if for every $x \in \mathbb{R}$ the x-section Ω_x is connected. Such a section is, of course, nonempty if and only if $x \in P(\Omega)$, where $P: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto x$ is the projection onto the first component. Note that, given a nonempty bounded domain Ω in $\mathbb{R}^2$, $J_\Omega := P(\Omega)$ is a nonempty bounded open interval in $\mathbb{R}$, that is $J_\Omega =]a_\Omega, b_\Omega[$, where $-\infty < a_\Omega < b_\Omega < \infty$.

Given $a \in \mathbb{R}$ and $\delta \in]0, \infty[$ we set

$$I_\delta(a) :=]a - \delta, a + \delta[\,, \quad I^-_\delta(a) :=]a - \delta, a[\quad \text{and} \quad I^+_\delta(a) :=]a, a + \delta[.$$

Definition 2.1. Let Ω, Ω_1 and Ω_2 be nonempty bounded domains in $\mathbb{R}^2$. Set $a_i := a_{\Omega_i}$ and $b_i := b_{\Omega_i}$, $i = 1, 2$. Given $c \in \mathbb{R}$ we say that Ω_1 *joins* Ω_2 *at* c *in* Ω if the following properties hold:

(1) $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \{c\} \times [\beta, \gamma]$ where $\beta = \beta_{\Omega_1,\Omega_2}$ and $\gamma = \gamma_{\Omega_1,\Omega_2}$ are some real numbers with $\beta < \gamma$,
(2) $c = a_{\Omega_2} = b_{\Omega_1}$,
(3) $\{c\} \times]\beta, \gamma[\subset \Omega$,
(4) whenever $d \in]\beta, \gamma[$, then there is a $\delta = \delta(d) > 0$ with the property that

$$I_\delta(d) \subset]\beta, \gamma[\quad \text{and} \quad I^-_\delta(c) \times I_\delta(d) \subset \Omega_1, \quad I^+_\delta(c) \times I_\delta(d) \subset \Omega_2.$$

We say that Ω_1 and Ω_2 *join at* c *in* Ω if Ω_1 joins Ω_2 at c in Ω or Ω_2 joins Ω_1 at c in Ω.

Definition 2.2 (cf. Figure 1). Assume that $\Omega \subset \mathbb{R} \times \mathbb{R}$ is a nonempty bounded open domain with Lipschitz boundary. Let $P: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(x, y) \mapsto x$ be the projection onto the first variable. A *nice decomposition of* Ω is a collection $\Omega_1, \ldots, \Omega_r$ of nonempty pairwise disjoint open connected subsets of Ω with connected vertical sections such that, defining $J_k := J_{\Omega_k}$, $a_k := a_{\Omega_k}$, $b_k := b_{\Omega_k}$, $k = 1, \ldots, r$, the following properties are satisfied:

(1) $\Omega \setminus (\bigcup_{k=1}^r \Omega_k) \subset Z$, where $Z := \bigcup_{l=1}^r (\{a_l, b_l\} \times \mathbb{R})$,
(2) whenever $k = 1, \ldots, r$ then $\partial\Omega_k \subset \partial\Omega \cup (\{a_k, b_k\} \times \mathbb{R})$ and for $c \in \{a_k, b_k\}$ $\partial\Omega_k \cap (\{c\} \times \mathbb{R}) = \{c\} \times I$, where I is a compact (possibly degenerate) interval in $\mathbb{R}$,

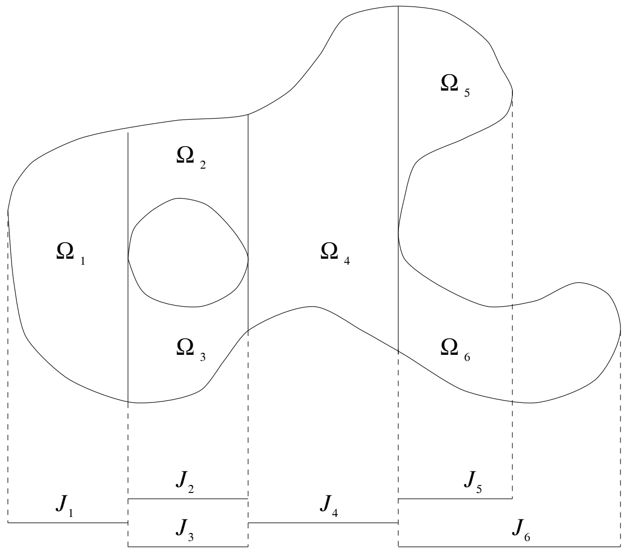

Figure 1

(3) whenever $k, l = 1, \ldots, r$, $k \neq l$ and $(c,d) \in \overline{\Omega}_k \cap \overline{\Omega}_l$ is arbitrary then either Ω_k and Ω_l join at c in Ω or else there is an $m \in \{1, \ldots, r\}$ such that Ω_k and Ω_m join at c in Ω and Ω_l and Ω_m join at c in Ω,

(4) for every $k = 1, \ldots, r$ the function $p_k : J_k \to]0, \infty[$, $x \mapsto \mu_1((\Omega_k)_x)$, is such that $1/p_k \in L^1(J_k)$.

Remarks. (1) Definition 2.2 says that, up to a set of measure zero, contained in a set Z of finitely many vertical lines, Ω can be decomposed into the finitely many domains Ω_k, $k = 1, \ldots, r$ in such a way that at Z the various sets Ω_k and Ω_l "join" in a nice way. Points of $\overline{\Omega} \cap Z$ are, intuitively speaking, those at which connected components of the vertical sections Ω_x bifurcate.

(2) Let $R_1, \ldots, R_s$ be closed bounded intervals in $\mathbb{R}^2$, with nonempty interior; let Ω_R be the interior of $\bigcup_{k=1}^{s} R_k$. Then any connected component Ω of Ω_R is a nicely decomposable domain with Lipschitz boundary.

(3) In Section 3 below we will show that all real analytic domains are nicely decomposable.

Finally, given a nice decomposition $\Omega_1, \ldots, \Omega_r$ of Ω, we set

$$E := \bigcup_{k=1}^{r} ((\{a_k, b_k\} \times \mathbb{R}) \cap \partial\Omega_k).$$

As we already said, our goal is to give a detailed description of the spaces $H^1_s(\Omega)$ and $L^2_s(\Omega)$ when Ω is a nicely decomposed domain. We begin our description by first considering the simpler case of an open set O with connected vertical sections. Below, such a role will be played by the sets Ω_k occurring in the nice decomposition of Ω. We do not assume that O has Lipschitz boundary, since the sets Ω_k occurring in the nice decomposition of Ω in general do not have this property.

Let $P\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$, $(x,y)\mapsto x$ be the projection onto the first variable. Let $J := P(O)$ and assume for simplicity that $J =]0,1[$. Define the function $p\colon J\to]0,\infty[$ by $x\mapsto \mu_1(\Omega_x)$. In [12] it is shown that, if $u\in L^2(O)$ satisfies $u_y = 0$ in the distributional sense, then there is a null set S in $\mathbb{R}^2$ and a function $v\in L^1_{\mathrm{loc}}(J)$ such that $u(x,y) = v(x)$ for every $(x,y)\in O\setminus S$. Moreover, $p^{1/2}v\in L^2(J)$. If $u\in H^1(O)$ then $v'\in L^1_{\mathrm{loc}}(J)$ and we can choose the null set S so that $u(x,y) = v(x)$ and $u_x(x,y) = v'(x)$ for every $(x,y)\in\Omega\setminus S$. Moreover, $p^{1/2}v'\in L^2(J)$ and we can choose the function v to be absolutely continuous on J.

Now, since O is open and bounded, it is easy to see that the function p satisfies the following hypothesis:

(A) $p\in L^\infty(0,1)$ and for every ε, $0<\varepsilon<1-\varepsilon$, there exists $\delta>0$ such that $p(x)>\delta$ in $]\varepsilon,1-\varepsilon[$.

Given an arbitrary function p satisfying hypothesis (A), note that $p(x)>0$ for $0<x<1$. Therefore the following linear spaces

$$H(p) := \{u\in L^1_{\mathrm{loc}}(0,1)\mid p^{1/2}u\in L^2(0,1)\}$$

and

$$V(p) := \{u\in L^1_{\mathrm{loc}}(0,1)\mid u'\in L^1_{\mathrm{loc}}(0,1), p^{1/2}u\in L^2(0,1), p^{1/2}u'\in L^2(0,1)\}$$

are well-defined. Define on $H(p)$ and $V(p)$ the scalar products

$$\langle u,v\rangle_{H(p)} := \int_0^1 p(x)u(x)v(x)\,dx$$

and

$$\langle u,v\rangle_{V(p)} := \int_0^1 p(x)u'(x)v'(x)\,dx + \int_0^1 p(x)u(x)v(x)\,dx.$$

It is easy to check that these products define Hilbert space structures on $H(p)$ and $V(p)$.

Now define the mapping

$$\iota\colon L^2_s(O)\to H(p),\quad u\mapsto v,$$

where v is the function $v\colon J\to\mathbb{R}$ such that $u(x,y) = v(x)$ almost everywhere in O. It turns out that ι is a well-defined isometry of $L^2_s(O)$ onto $H(p)$. Moreover, ι restricts to an isometry of $H^1_s(O)$ onto $V(p)$.

In [13] the following result is proved:

Proposition 2.3. *Assume that the function p satisfies* (A) *and* $(1/p) \in L^1(0,1)$. *Let $u \in V(p)$. Then there exists a function $v \in C^0([0,1])$ such that $u = v$ almost everywhere in $]0,1[$. Moreover, the imbedding $V(p) \hookrightarrow C^0([0,1])$ (and hence the imbedding $V(p) \hookrightarrow H(p)$) is compact.*

Now we consider the full nicely decomposed domain Ω. For $k = 1, \ldots, r$ let us define the linear spaces

$$H_k := \{u \in L^1_{\mathrm{loc}}(a_k, b_k) \mid p_k^{1/2} u \in L^2(a_k, b_k)\}$$

and

$$V_k := \{u \in H_k \mid u' \in L^1_{\mathrm{loc}}(a_k, b_k),\ p_k^{1/2} u' \in L^2(a_k, b_k)\}.$$

We have seen that H_k and V_k, with the scalar products

$$\langle u, v\rangle_{H_k} := \int_{a_k}^{b_k} p_k(x) u(x) v(x)\, dx$$

and

$$\langle u, v\rangle_{V_k} := \int_{a_k}^{b_k} p_k(x) u'(x) v'(x)\, dx + \int_{a_k}^{b_k} p_k(x) u(x) v(x)\, dx$$

respectively, are Hilbert spaces and that the imbedding $V_k \hookrightarrow H_k$ is dense and compact. Moreover, consider the following bilinear forms on V_k:

$$\begin{aligned} a_k(u,v) &:= \int_{a_k}^{b_k} p_k(x) u'(x) v'(x)\, dx, \\ b_k(u,v) &:= \int_{a_k}^{b_k} p_k(x) u(x) v(x)\, dx. \end{aligned}$$

The fact that we use the same symbols "a_k" and "b_k" to denote both a bilinear form and the endpoints of the interval J_k will not lead to confusion.

Define the product spaces

$$H_\oplus := H_1 \oplus \ldots \oplus H_r := \{[u] = (u_1, \ldots, u_r) \mid u_k \in H_k,\ k = 1, \ldots, r\}$$

and

$$V_\oplus := V_1 \oplus \ldots \oplus V_r := \{[u] = (u_1, \ldots, u_r) \mid u_k \in V_k,\ k = 1, \ldots, r\},$$

with the scalar products

$$\langle [u], [v]\rangle_{H_\oplus} := \sum_{k=1}^r \langle u_k, v_k\rangle_{H_k} \quad \text{and} \quad \langle [u], [v]\rangle_{V_\oplus} := \sum_{k=1}^r \langle u_k, v_k\rangle_{V_k},$$

respectively. It is easy to check that $H_\oplus$ and $V_\oplus$ are Hilbert spaces and that the imbedding $V_\oplus \hookrightarrow H_\oplus$ is dense and compact.

Furthermore, consider the following bilinear forms on $V_\oplus$:

$$a_\oplus([u], [v]) := \sum_{k=1}^r a_k(u_k, v_k) \quad \text{and} \quad b_\oplus([u], [v]) := \sum_{k=1}^r b_k(u_k, v_k).$$

Note that $b_\oplus$ is just the restriction to $V_\oplus \times V_\oplus$ of the scalar product $\langle\,\cdot\,,\,\cdot\,\rangle_{H_\oplus}$.

For $k = 1, \ldots, r$, let $\iota_k := \iota_{p_k}$. Define the map

$$\iota_\oplus : L^2_s(\Omega) \to H_\oplus, \quad \iota_\oplus u := (\iota_1(u|_{\Omega_1}), \ldots, \iota_r(u|_{\Omega_r})).$$

It follows that $\iota_\oplus$ is an isometry of $L^2_s(\Omega)$ into $H_\oplus$ and that $\iota_\oplus$ restricts to an isometry of $H^1_s(\Omega)$ into $V_\oplus$. Set

$$V_\oplus^\sim := \{[u] \in V_\oplus \mid u_k(b_k) = u_l(a_l) \text{ whenever } b_k = a_l = c \text{ and } \Omega_k \text{ and } \Omega_l \text{ join at } c\}.$$

The following result characterizes the spaces $H^1_s(\Omega)$ and $L^2_s(\Omega)$:

Proposition 2.4. *The following properties hold:*

(1) $\iota_\oplus(L^2_s(\Omega)) = H_\oplus$,
(2) $\iota_\oplus(H^1_s(\Omega)) = V_\oplus^\sim$.

Next we consider the limit operator $A = A_0$, which, as we know, is generated by the bilinear form a_0 defined in formula (1.6). Let $a_\oplus^\sim$ be the restriction of $a_\oplus$ to $V_\oplus^\sim \times V_\oplus^\sim$ and let $A_\oplus^\sim$ be the self-adjoint operator generated by $a_\oplus^\sim$ in $H_\oplus$. If $u \in D(A)$, then, for all $v \in H^1_s(\Omega)$,

$$\langle Au, v\rangle_{L^2_s(\Omega)} = a_0(u, v) = a_\oplus^\sim(\iota_\oplus u, \iota_\oplus v).$$

On the other hand

$$\langle Au, v\rangle_{L^2_s(\Omega)} = \langle \iota_\oplus Au, \iota_\oplus v\rangle_{H_\oplus}.$$

It follows that

$$a_\oplus^\sim(\iota_\oplus u, \iota_\oplus v) = \langle \iota_\oplus Au, \iota_\oplus v\rangle_{H_\oplus}$$

for all $v \in H^1_s(\Omega)$, so $\iota_\oplus u \in D(A_\oplus^\sim)$ and $A_\oplus^\sim \iota_\oplus u = \iota_\oplus Au$. Similarly, one can prove that, whenever $[u] \in D(A_\oplus^\sim)$, then $\iota_\oplus^{-1}[u] \in D(A)$, and $A\iota_\oplus^{-1}[u] = \iota_\oplus^{-1} A_\oplus^\sim [u]$. This means that $\iota_\oplus$ restricts to an isometry of $D(A)$ onto $D(A_\oplus^\sim)$ and that $A = \iota_\oplus^{-1} A_\oplus^\sim \iota_\oplus$.

For $k = 1, \ldots, r$, let us define the spaces

$$Z_k := \{u \in V_k \mid (p_k u')' \in L^1_{\mathrm{loc}}(a_k, b_k),\ p_k^{-1/2}(p_k u')' \in L^2(a_k, b_k)\}.$$

Moreover, set $Z_\oplus := Z_1 \oplus \ldots \oplus Z_r$. Then we obtain the following characterization of the domain $D(A_\oplus^\sim)$ of A:

Theorem 2.5. *Let $A_\oplus^\sim$ be the self-adjoint operator generated by the bilinear form $a_\oplus^\sim$. Then $D(A_\oplus^\sim) = Z_\oplus^\sim$, where $Z_\oplus^\sim$ is the subspace of $Z_\oplus$ consisting of all $[u] = (u_1, \ldots, u_k)$ satisfying the following properties:*

(1) *$u_k(b_k) = u_l(a_l)$ whenever $b_k = a_l = c$ and Ω_k and Ω_l join at c;*

(2) *whenever* Γ *is a connected component of* E (*necessarily of the form* $\Gamma = \{c\} \times I$, *where* $c \in \bigcup_{k=1}^{r} \{a_k, b_k\}$ *and* I *is an interval*) *then*

$$\sum_{k \in \sigma_+} (p_k {u_k}')(c) = \sum_{k \in \sigma_-} (p_k {u_k}')(c).$$

Here, $\sigma_+ = \sigma_+(\Gamma)$ *is the set of all* k *such that* $\overline{\Omega}_k \cap (\{b_k\} \times \mathbb{R}) \subset \Gamma$ *and so* $b_k = c$, *while* $\sigma_- = \sigma_-(\Gamma)$ *is the set of all* k *such that* $\overline{\Omega}_k \cap (\{a_k\} \times \mathbb{R}) \subset \Gamma$ *and so* $a_k = c$.

Moreover, for $[u] \in Z_{\oplus}^{\sim}$, $A_{\oplus}^{\sim}[u] = (p_1^{-1}(p_1 u_r')', \ldots, p_r^{-1}(p_r u_r')')$.

If Ω is nicely decomposable, then, due to the isometry $\iota_{\oplus}$ and in view of Theorem 2.5, the abstract equation (1.7) is equivalent to the following system of "concrete" one-dimensional reaction-diffusion equations:

$$\partial_t u_k = (1/p_k)(p_k {u_k}')' + f(u_k) \quad \text{on }]a_k, b_k[\text{ for } k = 1, \ldots, r,$$

with compatibility conditions

$$u_k(c) = u_l(c)$$

whenever $b_k = a_l = c$ and Ω_k and Ω_l join at c, and Kirchhoff type balance conditions

$$\sum_{k \in \sigma_+(\Gamma)} (p_k {u_k}')(c) = \sum_{k \in \sigma_-(\Gamma)} (p_k {u_k}')(c)$$

whenever $\Gamma = \{c\} \times I$ is a connected component of E.

As it is explained in [12], such a system can be interpreted as a reaction-diffusion equation on an appropriate finite graph.

Now assume that the nicely decomposed domain Ω satisfies the following additional hypothesis:

(C) For every $k = 1, \ldots, r$ one of the following conditions is satisfied:

(1) there exist two constants α_k and β_k, $0 < \alpha_k \leq \beta_k$, such that $\alpha_k \leq p_k(x) \leq \beta_k$ in $]a_k, b_k[$;

(2) there exists a function $q_k \in C^0([a_k, b_k]) \cap C^2(]a_k, b_k])$, with $q(x) > 0$, $q'(x) \geq 0$ and $q''(x) \leq 0$ on $]a_k, b_k]$, $q(a_k) = 0$ and $(1/q_k) \in L^1(a_k, b_k)$, and there exist two constants α_k and β_k, $0 < \alpha_k \leq \beta_k$, such that $\alpha_k q_k(x) \leq p_k(x) \leq \beta_k q_k(x)$ in $]a_k, b_k[$;

(3) there exists a function $q_k \in C^0([a_k, b_k]) \cap C^2([a_k, b_k[)$, with $q(x) > 0$, $q'(x) \leq 0$ and $q''(x) \leq 0$ on $[a_k, b_k[$, $q(b_k) = 0$ and $(1/q_k) \in L^1(a_k, b_k)$, and there exist two constants α_k and β_k, $0 < \alpha_k \leq \beta_k$, such that $\alpha_k q_k(x) \leq p_k(x) \leq \beta_k q_k(x)$ in $]a_k, b_k[$.

Remark. The technical condition (C) is general enough to cover all the classes of nicely decomposable domains discussed in the remarks following Definition 2.2.

The following result is proved in [13]:

Theorem 2.6. *Let $\Omega \subset \mathbb{R}^2$ be a nicely decomposed domain and assume condition* (C) *is satisfied. Let $(\lambda_\nu)_{\nu\in\mathbb{N}}$ be the repeated sequence of eigenvalues of the limit operator A_0. Then the gap condition* (1.8) *is satisfied.*

The proof this result is quite technical and is based on comparison arguments and on the min-max characterization of the eigenvalues of selfadjoint operators. The reader is referred to [13] for details. As we said before, this result enables us to construct inertial manifolds for the semiflows π_ε, for $\varepsilon \geq 0$ small enough.

3. An example: analytic domains

Let $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be a real analytic function, and assume that $\phi(x,y) \to \infty$ as $|(x,y)| \to \infty$. Assume that 0 is a regular value of ϕ and consider the sublevel set

$$\Omega_\phi := \{(x,y) \in \mathbb{R}^2 \mid \phi(x,y) < 0\}.$$

The set Ω_ϕ is open, bounded, and has analytic boundary. Finally, let Ω be any connected component of Ω_ϕ. In this section we shall prove that such a domain Ω admits a nice decomposition. We give only the main ideas of the proof and leave the details to the reader (cf. also Figure 1).

The starting point is the following proposition, which is an easy consequence of the analyticity of ϕ and the implicit function theorem.

Proposition 3.1. *Let $(x_0,y_0) \in \mathbb{R}^2$ be such that $\phi(x_0,y_0) = 0$, $\phi_y(x_0,y_0) = 0$. Then there exist $\delta, \eta > 0$ such that $(\phi(x,y), \phi_y(x,y)) \neq (0,0)$ for all $(x,y) \in I_\eta(x_0) \times I_\delta(y_0)$ with $(x,y) \neq (x_0,y_0)$.*

Since Ω is bounded, it follows immediately from Proposition 3.1 that there is only a finite number of points $(x,y) \in \partial\Omega$ such that $\phi(x,y) = 0$ and $\phi_y(x,y) = 0$. Set

$$\begin{aligned} X := \{x \in \mathbb{R} \mid {} & \text{there exists } y \in \mathbb{R} \\ & \text{with } \phi(x,y) = 0,\ \phi_y(x,y) = 0 \text{ and } (x,y) \in \partial\Omega\}. \end{aligned}$$

Then X is finite, and we can write

$$X = \{x_0, \ldots, x_s\} \quad \text{with } x_0 < \ldots < x_s.$$

Let $P(\Omega) := \,]a,b[$. It is easy to check that $a = x_0$ and $b = x_s$. The open domain Ω is therefore contained in the strip $]x_0, x_s[\, \times \mathbb{R}$. Let us define

$$Z := \bigcup_{i=0}^{s} (\{x_i\} \times \mathbb{R}).$$

Consider the strips $S_i :=]x_{i-1}, x_i[\times \mathbb{R}$ for $i = 1, \ldots, s$. Then

$$\Omega \setminus Z \subset \bigcup_{i=1}^{s} S_i.$$

Let us fix $i = 1, \ldots, s$ and consider $\Omega \cap S_i$. Let $(\overline{x}, \overline{y}) \in \partial\Omega \cap S_i$. Then $\phi(\overline{x}, \overline{y}) = 0$ and $\phi_y(\overline{x}, \overline{y}) \neq 0$. The implicit function theorem and the analiticity of ϕ imply that there exists a unique analytic function $f:]x_{i-1}, x_i[\to \mathbb{R}$ such that $f(\overline{x}) = \overline{y}$ and $\phi(x, f(x)) \equiv 0$ on $]x_{i-1}, x_i[$. Furthermore, the function f can be extended to a continuous function, again denoted by f, on the closed interval $[x_{i-1}, x_i]$.

Now we fix $\xi_i \in]x_{i-1}, x_i[$ arbitrarily and we set

$$Y_i := \{y \in \mathbb{R} \mid \phi(\xi_i, y) = 0 \text{ and } (\xi_i, y) \in \partial\Omega\}.$$

At such points we have $\phi_y(\xi_i, y) \neq 0$, so Y_i is finite. Let us write

$$Y_i = \{y_{i,1}, \ldots, y_{i,t(i)}\} \quad \text{with } y_{i,1} < \ldots < y_{i,t(i)}.$$

Thus for every $j = 1, \ldots, t(i)$ there exists a unique continuous function

$$f_{i,j}: [x_{i-1}, x_i] \to \mathbb{R},$$

which is in fact analytic on $]x_{i-1}, x_i[$, and satisfies $f_{i,j}(\xi_i) = y_{i,j}$ and $\phi(x, f_{i,j}(x)) \equiv 0$ on $[x_{i-1}, x_i]$. Moreover,

$$f_{i,1}(x) < \ldots < f_{i,t(i)}(x) \quad \text{for all } x \in]x_{i-1}, x_i[$$

and

$$\begin{cases} \phi_y(x, f_{i,j}(x)) < 0 & \text{for all } x \in]x_{i-1}, x_i[\text{ if } j \text{ is odd,} \\ \phi_y(x, f_{i,j}(x)) > 0 & \text{for all } x \in]x_{i-1}, x_i[\text{ if } j \text{ is even.} \end{cases}$$

It follows that $t(i)$ is even for every $i = 1, \ldots, s$, and

$$\Omega \cap S_i = \bigcup_{j \in D_i} \Omega_{i,j} \quad \text{for } i = 1, \ldots, s,$$

where

$$D_i := \{1, 3, \ldots, t(i) - 1\} \quad \text{for } i = 1, \ldots, s$$

and

$$\Omega_{i,j} := \{(x, y) \in S_i \mid f_{i,j}(x) < y < f_{i,j+1}(x)\}$$

for $i = 1, \ldots, s$ and $j \in D_i$.

The family $(\Omega_{i,j})_{(i,j)}$, where $i = 1, \ldots, s$ and $j \in D_i$, is a nice decomposition of the set Ω. The proof of properties (1), (2) and (3) in Definition 2.2 depends only on the implicit function theorem and does not present particular difficulties. Property (4) is a little more delicate and requires some estimates for the first order derivatives of implicitly defined functions. In the same way one can prove that the domain Ω satisfies condition (C), so the eigenvalues of the operator A_0 on Ω satisfy the gap condition (1.8). The details are left to the reader.

Remarks. (1) Open sets with C^∞ boundary need not be nicely decomposable. As an example, consider the curve

$$\gamma_1(t) := (\phi(t)\sin(1/t), t) \quad \text{for } t \in [0,1],$$

where

$$\phi(t) := \begin{cases} e^{(-1/t^2)} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Moreover, choose a C^∞-imbedded curve $\gamma_2\colon [1,2] \to \mathbb{R}^2$ such that

$$\gamma_2(1) = (e^{-1}\sin(1), 1), \quad \gamma_2(2) = (e^{-1}\sin(-1), -1)$$

and such that the curve $\gamma\colon [-1,2] \to \mathbb{R}^2$ defined by

$$\gamma(t) := \begin{cases} \gamma_1(t) & \text{for } t \in [-1,1], \\ \gamma_2(t) & \text{for } t \in [1,2], \end{cases}$$

is a C^∞-imbedded curve. Then γ is a Jordan curve, and the corresponding bounded region Ω is an open set with C^∞ boundary. Clearly, this domain does not admit any nice decomposition.

(2) On the other hand, let $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function such that $\phi(x,y) \to \infty$ as $|(x,y)| \to \infty$. Assume that 0 is a regular value of ϕ. Let $\Omega_\phi := \{(x,y) \mid \phi(x,y) < 0\}$ and let Ω be any connected component of Ω_ϕ. Moreover, suppose that, whenever $\phi(x_0,y_0) = 0$, $\phi_y(x_0,y_0) = 0$ and $(x_0,y_0) \in \partial\Omega$, then $\phi_{yy}(x_0,y_0) \neq 0$. Then Ω is nicely decomposable. The proof is similar as in the case of analytic domains.

4. Concluding remarks

It is straightforward to generalize some of the preceding results to domains of dimension higher than two. In fact, assume that Ω is a bounded domain in $\mathbb{R}^{M+N} \cong \mathbb{R}^M \times \mathbb{R}^N$ with Lipschitz boundary. Define, for $\varepsilon > 0$, the squeezing operator

$$T_\varepsilon\colon \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^M, \quad (x,y) \mapsto (x, \varepsilon y) \tag{4.1}$$

and set $\Omega_\varepsilon := T_\varepsilon(\Omega)$. We can then consider, as before, the boundary value problem (1.1). With some obvious notational changes, we can define the operators A_ε, $\varepsilon > 0$, the function spaces $H^1_s(\Omega)$ and $L^2_s(\Omega)$ and the limit operator A_0. If the growth exponent β of the function f satisfies appropriate restrictions then we can also define the semiflows $\pi_{\varepsilon,\widehat{f}}$, $\varepsilon > 0$, and the limit semiflow $\pi_{0,\widehat{f}}$. Theorems 1.1 and 1.2 continue to hold in this more general setting (see [12] for details).

The transformation T_ε defined in (4.1) is an example of flat squeezing of the space $\mathbb{R}^{M+N}$ "toward" an M-dimensional linear subspace. One can consider, much more generally, a squeezing transformation of an open subset of $\mathbb{R}^{M+N}$ toward an arbitrary (curved) smooth M-dimensional submanifold S. More specifically, if S is orientable, then there is a system $\nu_k: S \to \mathbb{R}^{M+N}$, $k = 1, \ldots, N$ of linearly independent smooth vectorfields of norm one which are normal to S. Moreover, there is a (tubular) neighbourhood U of S in $\mathbb{R}^{M+N}$ and smooth maps $\phi: U \to S$ and $\alpha_k: U \to \mathbb{R}$, $k = 1, \ldots, N$, such that ϕ represents the "normal" projection of U onto S and $\alpha_k(z)$, $z \in U$ gives the "distance" between z and $\phi(z)$ in the $\nu_k(\phi(z))$-direction. Note that

$$z \equiv \phi(z) + \sum_{k=1}^{N} \alpha_k(z)\nu_k(\phi(z)), \quad z \in U.$$

We can now define, for $\varepsilon > 0$, the squeezing map $T_\varepsilon: U \to \mathbb{R}^{M+N}$ by

$$T_\varepsilon(z) := \phi(z) + \varepsilon \sum_{k=1}^{N} \alpha_k(z)\nu_k(\phi(z)), \quad z \in U. \tag{4.2}$$

If Ω is a bounded domain in $\mathbb{R}^{M+N}$ with Lipschitz boundary and $\Omega \subset U$, then set

$$\Omega_\varepsilon := T_\varepsilon(\Omega).$$

Note that formula (4.2) reduces to formula (4.1) in the special case $U = \mathbb{R}^{M+N} \cong \mathbb{R}^M \times \mathbb{R}^N$, $S = \mathbb{R}^M \times \{0\}$, $\phi(x,y) \equiv (x,0)$, $\nu_k(x) \equiv e_{M+k}$ and $\alpha_k(x,y) \equiv y_k$, $k = 1, \ldots, N$. Here, $e_1, \ldots, e_{M+N}$ is the canonical basis of $\mathbb{R}^{M+N}$.

The simplest nonflat example is obtained by letting S be the M-dimensional unit sphere in $\mathbb{R}^{M+1}$. A squeezing transformation toward S can then be defined by (4.2) by setting $U = \mathbb{R}^{M+1} \setminus \{0\}$, $\phi(z) \equiv z/\|z\|$, $\alpha_1(z) \equiv \|z\| - 1$, $z \in U$ and $\nu_1(z) \equiv z$, $z \in S$. Here $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^{M+1}$.

Again one can consider the boundary value problems (1.1) on Ω_ε. One can now translate these problems to an equivalent family of boundary value problems on the fixed domain Ω. One can define the analogues of the spaces $H^1_s(\Omega)$ and $L^2_s(\Omega)$, the limit operator A_0 and one can prove, in this general setting, most of the results described in this paper. The details are presented in the recent work [11] of M. Rinaldi and the present authors.

Some applications of the Conley index to thin domain problems are contained in the recent paper [4] of M. Carbinatto and the second author.

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Manuscript received November 15, 1999

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