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FROM THE SCHAUDER FIXED-POINT THEOREM TO THE APPLIED MULTIVALUED NIELSEN THEORY

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Dedicated to the memory of Juliusz P. Schauder

Abstract. Starting from the famous Schauder fixed-point theorem, we present some Lefschetz-like and Nielsen-like generalizations for certain admissible (multivalued) self-maps on metric ANR-spaces. These fixed-point principles are applied for obtaining the existence and multiplicity results for boundary value problems.

1. Schauder's lines

Everybody knows the second version (Satz II in [17]) of the celebrated *Schauder fixed-point theorem*. Its original form reads in German as follows.

Theorem 1 (J. Schauder). *In einem "B"-Raume sei eine konvexe und abgeschlossene Menge gegeben. Die stetige Funktionaloperation $F(x)$ bilde H auf sich sebst ab. Ferner sei die Menge $F(H) \subset H$ kompakt. Dann ist ein Fixpunkt vorhanden.*

One can readily recognize three characteristic features in Theorem 1:

(i) a subset H of a Banach space is *convex* and *closed*,

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(ii) $F : H \to H$, i.e. F is a *self-map*,

(iii) this self-map is *continuous* and *compact*, i.e. $\overline{F(H)}$ is compact.

The influence of this theorem to modern mathematical analysis, in particular the fixed-point theory and its application to boundary value problems for differential equations, is enormous (see e.g. [9], [12], [18] and the references therein).

Keeping in mind condition (ii), our main goal will be to improve Theorem 1, when replacing some conditions in (i) and (iii) by those being less restrictive. Thus, we shall not deal here e.g. with the degree theories or, more generally, fixed-point index theories. On the other hand, since in all our situations the Lefschetz number can be always considered as a normalization property for fixed-point indices of self-maps, our first interest is naturally related to a suitable variant of the *Lefschetz fixed-point theorem* (see e.g. [8], [11] and the references therein). An essential step in this direction has been done by A. Granas in [4] (cf. also [15]), when dropping out condition (i).

Theorem 2 (A. Granas). *Let $F : X \to X$ be a compact* (*continuous*) *self-map on a* (*metric*) ANR*-space X. Then F is a Lefschetz mapping and the nontrivial generalized Lefschetz number, $\Lambda(F) \neq 0$, implies that F has a fixed-point. If, in particular, X is an* AR*-space, then $\Lambda(F) = 1$, by which the associated fixed-point set is nonempty, i.e.* $\mathrm{Fix}(F) \neq \emptyset$.

We would like to proceed furthermore in this way, but before it is necessary to recall some notions like ANR-spaces, AR-spaces, Lefschetz maps, a Lefschetz number, etc.

2. Lefschetz-type theorems

In the entire text, all topological spaces are metric and by a space we mean a metric space.

Definition 1. A space Y is called an *absolute retract* (AR) or an *absolute neighbourhood retract* (ANR) if, for any metrizable X and any closed $A \subset X$, each continuous $f : A \to Y$ is extendable over X or over an open neighbourhood U of A in X, respectively.

According to the well-known theorem of K. Borsuk (cf. [12]), a metrizable space is an AR or an ANR if and only if it is a retract of some normed space or of an open subset of some normed space, respectively.

Thus, since any normed space is known to be homologically trivial (cf. [11]), any AR-space must be homologically trivial as well.

The definition of a *Lefschetz map* is based on the notion of a *Leray endomorphism* (for its definition and more details, see e.g. [8], [11], [13]). Let H be the Čech homology functor with compact carriers and coefficients in the field of rational numbers $\mathbb{Q}$. For a metric space X, consider the graded vector space $H(X) = \{H_q(X)\}$. For a continuous map $f : X \to Y$, $H(f)$ is the induced linear map $f_* = \{f_q\}$, where $f_q : H_q(X) \to H_q(Y)$.

Definition 2. A continuous mapping $f : X \to X$ is called *Lefschetz map* (w.r.t. H) if $f_* : H(X) \to H(X)$ is a Leray endomorphism. For such an f, we define the (generalized) Lefschetz number $\Lambda(f)$ of f by putting $\Lambda(f) = \Lambda(f_*)$.

For more details and the most important properties of the Lefschetz number (including the invariance under homotopy) — see again [8], [11]. Let us only note that, in view of the homological triviality of an AR-space, any continuous self-map on an AR is Lefschetz with $\Lambda(f) = 1$.

Now, we would like to pass directly to multivalued analysis, where the fundamental role is played by *admissible maps* in the sense of [11].

Definition 3. An upper-semi-continuous (u.s.c.) map with nonempty compact values is called *admissible* if it possesses a multivalued selector which can be composed by *acyclic maps*, i.e. u.s.c. maps with nonempty compact acyclic values.

Let us recall that a set is *acyclic* (w.r.t. any continuous theory of cohomology) if it is homologically same as a one point space.

It is known (see [11], [12]) that for any admissible map, say $\varphi : X \multimap Z$ (the symbol "$\multimap$" means, as usual, $X \to 2^Z \setminus \{\emptyset\}$), we can associate a (*selected*) *pair* $(p, q) \subset \varphi$ of single-valued continuous maps p, q such that $X \xleftarrow{p} Y \xrightarrow{q} Z$, provided

(i) p is a Vietoris map, i.e. it is proper and, for every $x \in X$, $p^{-1}(x)$ is acyclic,

(ii) $q(p^{-1}(x)) \subset \varphi(x)$, for every $x \in X$.

Observe that if φ is compact and $(p, q) \subset \varphi$, then so is q.

So, an equivalent definition (to Definition 3) is the one, where we assume the existence of a selected pair (p, q) of an u.s.c. map with nonempty compact values.

In the sequel, all multivalued maps are assumed to be admissible. For them, a multivalued version of Definition 2 can be given as follows.

Definition 4. An admissible map $\varphi : X \multimap X$ is called a *Lefschetz map* if, for each selected pair $(p, q) \subset \varphi$, the given linear map $q_* p_*^{-1} : H(X) \to H(X)$ is a Leray endomorphism. For such a φ, we define the (generalized) *Lefschetz set* $\Lambda(\varphi)$ of φ by putting $\Lambda(\varphi) = \{\Lambda(q_* p_*^{-1}) \mid (p, q) \subset \varphi\}$. If, in particular, φ is an acyclic Lefschetz map, then the set $\Lambda(\varphi)$ becomes a singleton.

Since we would like to deal also with not necessarily compact admissible maps (in order to avoid condition (iii) above), we need still another family of locally compact operators with only certain amount of compactness, so called CAC-*maps.*

In the single-valued case, particular types of these maps have been studied e.g. by F. E. Browder, R. Nussbaum, R. S. Palais, W. V. Petryshyn, etc.

Some of these particular cases (eventually compact maps $\subset$ asymptotically compact maps = core maps = Palais maps) are involved in the class of compact attractions or, equivalently (cf. [16]) compact absorbing contractions (shortly, CAC-maps). This last class has been defined and systematically investigated also for multivalued maps in [10] (cf. [13], [16] and the references therein).

In a multivalued setting, analogies exist for admissible maps (cf. [10]) with the same hierarchy (cf. [16]). As usual, an admissible map $\varphi : X \multimap Y$ is called *locally compact* if, for each $x \in X$, there is an open subset V of X such that $x \in V$ and the restriction $\varphi|_V$ of φ to V is compact.

Definition 5. An admissible locally compact map $\varphi : X \multimap X$ is said to be a *compact absorbing contraction* (CAC) if there exists an open subset U of X such that $\overline{\varphi(u)}$ is a compact subset of U and $X \subset \bigcup_{i=0}^{\infty} \varphi^{-i}(U)$.

In [10], the following (multivalued) generalization of Theorem 2 has been proved.

Theorem 3. *Let $\varphi : X \multimap X$ be a* (*multivalued*) CAC-*map on a* (*metric*) ANR-*space X. Then φ is a Lefschetz mapping and the nontrivial* (*well defined*) *generalized Lefschetz set, $\Lambda(\varphi) \neq \{0\}$, implies that φ has a fixed-point, i.e. there exists $\widehat{x} \in \varphi(\widehat{x})$. If, in particular, X is an* AR-*space, then $\Lambda(\varphi) = \{1\}$, by which* $\mathrm{Fix}(\varphi) \neq \emptyset$.

Now, consider the boundary value problem

$$(1) \qquad \begin{cases} x' \in F(t,x), & \text{for a.a. } t \in J, \\ x \in S, \end{cases}$$

where J is a given (possibly infinite) real interval, $F : J \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an (upper) Carathéodory map and S is a subset of $AC_{\mathrm{loc}}(J, \mathbb{R}^n)$.

Let us recall that by a *Carathéodory map F* we understand that:

(i) the set of values of F is nonempty, compact and convex for all $(t,x) \in J \times \mathbb{R}^n$;
(ii) $F(t, \cdot\,)$ is upper-semi-continuous (u.s.c.) for a.a. $t \in J$;
(iii) $F(\,\cdot\,, x)$ is measurable for all $x \in \mathbb{R}^n$, i.e. for any open $U \subset \mathbb{R}^n$ and every $x \in \mathbb{R}^n$ the set $\{t \in J \mid F(\,\cdot\,, x) \cap U \neq \emptyset\}$ is measurable.

By a *solution* $x(t)$ of (1), we always mean a locally absolutely continuous function $x(t) \in \mathrm{AC}_{\mathrm{loc}}(J, \mathbb{R}^n)$ satisfying (1) for a.a. $t \in J$, i.e. the one in the sense of Carathéodory.

As an application of Theorem 3 to (1), we can give the following theorem which can be deduced from the results in [4] (cf. Corollary 2.34) or in [3] (cf. Theorem 3).

Theorem 4. *Consider problem* (1). *Let* $G : J \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ *be a Carathéodory map such that* $G(t,c,c) \subset F(t,c)$, *for all* $(t,c) \in J \times \mathbb{R}^n$. *Assume that*

(i) *there exists a neighbourhood retract* Q *in* $C(J,\mathbb{R}^n)$ *such that the associated linearized problem*

$$\begin{cases} x' \in G(t,x,q(t)), & \text{for a.a. } t \in J, \\ x \in S \cap Q, \end{cases} \tag{2}$$

has an R_δ*-set of solutions* $T(q)$, *for each* $q \in Q$, *and* $\Lambda(T) \neq 0$;

(ii) *there exists a locally integrable function* $\alpha : J \to \mathbb{R}$ *such that*

$$|G(t,x(t),q(t))| \le \alpha(t), \quad \text{a.e. in } J,$$

for any pair $(q,x) \in \Gamma_T$ (*i.e. from the graph* Γ *of* T);

(iii) $T(Q)$ *is bounded in* $C(J,\mathbb{R}^n)$ *and* $\overline{T(Q)} \subset S$.

Then problem (1) *has a solution.*

Let us recall that by an R_δ*-set* we mean the one homeomorphic to the intersection of a decreasing sequence of compact AR-spaces and by an R_δ*-map* the u.s.c. one with R_δ-values (cf. [12]).

3. Nielsen-type theorems

In [10], it has been shown that, for any (multivalued) CAC-self-map, or its associated selected pair (p,q), on an ANR-space X, the Lefschetz number $\Lambda(p,q) \in \mathbb{Z}$ is well-defined and $\Lambda(p,q) \neq 0$ implies the existence of a coincidence point z of the pair (p,q), i.e. $p(z) = q(z)$.

As we have demonstrated in [5], a lower estimate of the number of coincidence points for admissible maps is, in the frame of the multivalued Nielsen theory, rather appropriate than the one of fixed-points. In fact, one can deal with fixed-points only for R_δ-maps (see [1]), but no more with their compositions.

With this respect, the *multivalued Nielsen theory* (suitable for applications below) concerns a lower estimate of coincidence points. Its central notion, the *Nielsen number* $N(p,q)$, can be defined only for certain classes of CAC-self-maps on ANR-spaces X with suitable properties (see [5], [6]).

As in the single-valued case (cf. [2], [7], [8]), the definition of a related Nielsen number can be done in two stages: at first, the set of coincidence points is split into disjoint (Nielsen) classes and then the essential classes are defined (i.e. those with nontrivial associated Lefschetz number). Since all of these seems to be rather technical, we restrict ourselves here only to saying that in [6] we have introduced the *H-Nielsen number* $N_H(p,q)$ *as the number of essential Nielsen classes of* (p,q) modulo a normal subgroup H of $\pi_1 X$.

Of course, $N_H(p,q)$ is a homotopy invariant w.r.t. an appropriate class of homotopy. For more details — see [6] (cf. also [1], [2], [5]), where the following statement has been proved.

Theorem 5. *A* (*multivalued*) CAC-*self-map* $\varphi : X \multimap X$ *on a* (*metric*) *connected* ANR-*space has at least* $N_H(\varphi)$ *coincidence points, provided*

(i) φ *can be composed by an* R_δ*-mapping and a continuous single-valued function,*

(ii) X *has a finitely generated abelian fundamental group or, if* φ *is just an* R_δ*-mapping,* X *is compact.*

If, in particular, $\varphi : X \to X$ *is a single-valued* (*continuous*) CAC-*self-map on a* (*metric*) ANR-*space, then it has at least* $N(\varphi)$ *fixed-points.*

In order to apply Theorem 5 to problem (1) as in [6], it is convenient to employ the following definition.

Definition 6. We say that the mapping $T : Q \multimap U$ is *retractible onto* Q, where U is an open subset of $C(J,\mathbb{R}^n)$ containing Q if there is a retraction $r : U \to Q$ and $p \in U\backslash Q$, $r(p) = q$ implies that $p \notin T(q)$.

Theorem 6. *Consider problem* (1). *Let* $G : J\times\mathbb{R}^n\times\mathbb{R}^n \multimap \mathbb{R}^n$ *be a Carathéodory map such that* $G(t,c,c) \subset F(t,c)$, *for all* $(t,c) \in J\times\mathbb{R}^n$. *Assume that*

(i) *there exists a closed connected subset* Q *of* $C(J,\mathbb{R}^n)$ *with a finitely generated abelian fundamental group such that the associated linearized problem*

$$\begin{cases} x' \in G(t,x,q(t)), & \text{for a.a. } t \in J, \\ x \in S, \end{cases} \tag{3}$$

has for every $q \in Q$, *a* (*nonempty*) R_δ*-set of solutions* $T(q)$;

(ii) *the operator* $T : Q \multimap U$, *related to problem* (3), *is retractible onto* Q *with a retraction* r (*in the sense of Definition* 6);

(iii) *there exists a locally integrable function* $\alpha : J \to \mathbb{R}$ *such that*

$$|G(t,x(t),q(t))| \le \alpha(t), \quad \text{a.e. in } J,$$

for any pair $(q,x) \in \Gamma_T$ (*i.e. from the graph* Γ *of* T);

(iv) $T(Q)$ *is bounded in* $C(J,\mathbb{R}^n)$ *and* $\overline{T(Q)} \subset S$.

Then problem (1) *admits at least* $N_H(r|_{T(Q)}\circ T)$ *solutions belonging to* Q.

Now, consider still the boundary value problem

$$\begin{cases} x' + A(t)x \in F(t,x), & \text{for a.a. } t \in I, \\ Lx = \Theta,\ L : C(I,\mathbb{R}^n) \to \mathbb{R}^n, \end{cases} \tag{4}$$

on a compact interval I, where $A \in C(I, \mathbb{R}^n \times \mathbb{R}^n)$ is a (single-valued) continuous $(n{\times}n)$-matrix and $F : \mathbb{R}^n \multimap \mathbb{R}^n$ is an (upper-)Carathéodory product-measurable mapping.

For this problem, Theorem 6 takes the following particular form (see [6]), which will be suitable for a concrete example below.

THEOREM 7. *Consider problem* (4) *and the associated linear homogeneous problem*

$$\begin{cases} x' + A(t)x = 0, \\ Lx = 0. \end{cases} \tag{5}$$

Let

$$|F(t,x)| \le \mu(t)(|x| + 1)$$

hold for all $(t,x) \in I \times \mathbb{R}^n$, *where* $\mu : I \to [0,\infty)$ *is a suitable Lebesgue-integrable bounded function. Furthermore, let* $L : C(I,\mathbb{R}^n) \to \mathbb{R}^n$ *be a linear operator such that problem* (5) *has only the trivial solution on* I. *Then the original problem* (4) *has* $N_H(r|_{T(Q)} \circ T)$ *solutions, provided there exists a closed connected subset* Q *of* $C(I,\mathbb{R}^n)$ *with a finitely generated abelian fundamental group such that*

(i) $T(Q)$ *is bounded;*
(ii) $T(\,\cdot\,)$ *is retractible onto* Q *with a retraction* r *in the sense of Definition* 6*;*
(iii) $\overline{T(Q)} \subset \{x \in C(I,\mathbb{R}^n) \mid Lx = \Theta\}$*;*

where $T(q)$ *denotes the set of* (*existing*) *solutions to the linearized problem*

$$\begin{cases} x' + A(t)x \in F(t,q(t)), & \textit{for a.a. } t \in I, \\ Lx = \Theta. \end{cases} \tag{6}$$

REMARK 1. In the single-valued case (for differential equations), a unique solvability of the associated linearized problems allows us to consider more general problems than (4), even on noncompact intervals (cf. [2]).

4. Nontrivial example showing the power of former theorems

Consider the (upper) Carathéodory system

$$\begin{cases} x' + ax \in e(t,x,y)y^{1/m} + g(t,x,y), \\ y' + by \in f(t,x,y)x^{1/n} + h(t,x,y), \end{cases} \tag{7}$$

where a, b are constants with $ab > 0$; m, n are odd integers with $\min(m,n) \ge 3$ and $e(t,x,y) \equiv e(t+\omega,x,y)$, $f(t,x,y) \equiv f(t+\omega,x,y)$, $g(t,x,y) \equiv g(t+\omega,x,y)$, $h(t,x,y) \equiv h(t+\omega,x,y)$ are product-measurable.

Let suitable positive constants E_0, F_0, G, H exist such that

$$|e(t,x,y)| \le E_0, \quad |f(t,x,y)| \le F_0, \quad |g(t,x,y)| \le G, \quad |h(t,x,y)| \le H,$$

hold for a.a. $t \in [0, \omega]$ and all $(x, y) \in \mathbb{R}^2$. Furthermore, assume the existence of positive constants e_0, f_0, δ_1, δ_2 such that

$$0 < e_0 \le e(t, x, y) \tag{8}$$

for $x \ge -\delta_1$, $y \ge \delta_2$ and a.a. t as well as for $x \le \delta_1$, $y \le -\delta_2$ and a.a. t, jointly with

$$0 < f_0 \le f(t, x, y) \tag{9}$$

for $x \ge \delta_1$, $y \le \delta_2$ and a.a. t as well as for $x \le -\delta_1$, $y \ge -\delta_2$ and a.a. t.

Another possibility is that (8) holds for $x \le \delta_1$, $y \ge \delta_2$ and a.a. t as well as for $x \ge -\delta_1$, $y \le -\delta_2$ and a.a. t and that (9) holds at the same time for $x \ge \delta_1$, $y \ge -\delta_2$ and a.a. t as well as for $x \le -\delta_1$, $y \le \delta_2$ and a.a. t.

- We have shown in [1], [6] that, under the above assumptions, system (7) admits (according to Theorem 7) *at least two ω-periodic solutions*, provided δ_1, δ_2 satisfy

$$\begin{cases} \dfrac{1}{|a|}|e_0 \delta_2^{1/m} - G| \ge \delta_1 > \left(\dfrac{H}{f_0}\right)^n, \\ \dfrac{1}{|b|}|f_0 \delta_1^{1/n} - H| \ge \delta_2 > \left(\dfrac{G}{e_0}\right)^m. \end{cases} \tag{10}$$

- If still

$$\sup_{t \in [0,\omega]} \mathrm{ess}\Big(\sup_{(x,y) \in B} |g(t, x, y)|\Big) < G \quad \text{and} \quad \sup_{t \in [0,\omega]} \mathrm{ess}\Big(\sup_{(x,y) \in B} |h(t, x, y)|\Big) < H, \tag{11}$$

  where $B = \{(x, y) \in \mathbb{R}^2 \mid (|x| = \delta_1 \wedge |y| \le \delta_2) \vee (|x| \le \delta_1 \wedge |y| = \delta_2)\}$, then we can prove, using the additivity, excision and existence properties of an appropriate fixed-point index (cf. [1], [6] and the references therein), *the third ω-periodic solution* of (7).

This result cannot be obtained in any other way. As a concrete example of (7) satisfying (8)–(11), which demonstrates the necessity of applying the Nielsen theory, we can give

$$\begin{cases} x' + x = e(x, y) y^{1/3} + g(t, x, y), \\ y' + y = f(x, y) x^{1/5} + h(t, x, y), \end{cases} \tag{12}$$

where ($\delta_1 = 10^{-4}$, $\delta_2 = 10^{-2}$, $R = 100$)

$$e(x, y) = \begin{cases} 10, & \text{for } (x, y) \in \mathbb{R}^2 \backslash \{(-R < x < -\delta_1 \wedge \delta_2 < y < R) \\ & \qquad \vee (\delta_1 < x < R \wedge -R < y < -\delta_2)\}, \\ 0, & \text{for } (x, y) \in \left\{ \left(\dfrac{-R - \delta_1}{2}, \dfrac{R + \delta_2}{2} \right), \left(\dfrac{R + \delta_1}{2}, \dfrac{-R - \delta_2}{2} \right) \right\}, \\ \multicolumn{2}{l}{\text{any continuous extension with } |e(x, y)| \le 10, \text{ otherwise},} \end{cases}$$

$$f(x,y)=\begin{cases}10, & \text{for } (x,y)\in\mathbb{R}^2\backslash\{(\delta_1<x<R\wedge\delta_2<y<R)\\ & \qquad\vee(-R<x<-\delta_1\wedge -R<y<-\delta_2)\},\\ 0, & \text{for } (x,y)\in\left\{\left(\frac{R+\delta_1}{2},\frac{R+\delta_2}{2}\right),\left(\frac{-R-\delta_1}{2},\frac{-R-\delta_2}{2}\right)\right\},\\ \text{any continuous extension with } |f(x,y)|\le 10, \text{otherwise,}\end{cases}$$

$$g(t,x,y)\equiv g(t+\omega,x,y),\qquad G\le 10^{1/3}-10^{-4}(\doteq 2.154),$$
$$\sup_{t\in[0,\omega]}\operatorname{ess}(\sup_{(x,y)\in B}|g(t,x,y)|)<10^{1/3}-10^{-4},$$
$$h(t,x,y)\equiv h(t+\omega,x,y),\qquad H\le 10^{1/5}-10^{-2}(\doteq 1.574),$$
$$\sup_{t\in[0,\omega]}\operatorname{ess}(\sup_{(x,y)\in B}|h(t,x,y)|)<10^{1/5}-10^{-2},$$
$$B=\{(x,y)\in\mathbb{R}^2|(|x|=\delta_1\wedge|y|\le\delta_2)\vee(|x|\le\delta_1\wedge|y|=\delta_2)\},$$

and

$$g(0,x,y)=10^{1/3}-10^{-4}\quad\text{for } (x,y)\in P,$$
$$h(0,x,y)=10^{1/5}-10^{-2}\quad\text{for } (x,y)\in P,$$
$$P=\left\{(x,y)\in\mathbb{R}^2\,\middle|\,\left(|x|=\delta_1\wedge|y|=\frac{R+\delta_2}{2}\right)\vee\left(|x|=\frac{R+\delta_1}{2}\wedge|y|=\delta_2\right)\right\}.$$

REMARK 2. Applying Theorem 4, *only one ω-periodic solution* of system (7) can be guaranteed, under the same assumptions, by means of the nontrivial Lefschetz number. Using the additivity, excision and existence properties of a fixed-point index, one can still prove *the second ω-periodic solution of* (7).

REMARK 3. On the basis of the Schauder fixed-point theorem (Theorem 1) one can get (in the single-valued case) *only one ω-periodic solution* of system (7).

REMARK 4. If, additionally, conditions (8), (9) hold for a.a. $t\in[0,\omega]$ and all $(x,y)\in\mathbb{R}^2$, and if the sharp inequalities take place in (10), then at least three ω-periodic solutions of (7) can be also deduced by means of the fixed-point index technique or (in the single-valued case) at least two ω-periodic solutions, when applying the Schauder fixed-point theorem.

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