**T**opological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 14, 1999, 183–192

# SELECTION APPROACH TO MULTIVALUED SEPARATION THEOREMS

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Abstract. A selection problem for convex-valued mappings is studied. Two general results, so called "sandwich" theorems, are proved.

## 1. Introduction

The main result of the present article are the following "sandwich" theorems.

Theorem 1. *Let $F : X \to Y$ be a lower semicontinuous convex-valued and closed-valued mapping of a paracompact space $X$ into a complete metric space $Y$ with a complete convex structure and let $G : X \to Y$ be an upper semicontinuous compact-valued selection of $F$. Then there exists a continuous convex-valued and compact-valued mapping $H : X \to Y$ such that, for all $x \in X$,*

$$G(x) \subset H(x) \subset F(x).$$

Theorem 1'. *If in Theorem* 1 *space $X$ is perfectly normal and space $Y$ is separable then there exists a sequence $\{H_n\}_{n \in \mathbb{N}}$ of continuous compact-valued*

1991 *Mathematics Subject Classification.* Primary 54C60, 54C65, 52A01; Secondary 28B20, 54B20.

*Key words and phrases.* Multivalued mapping, selection, approximation, convex structure, compact exponenta.

The authors was supported by RFFI grant 99-01-00009.

*and convex-valued mappings* $H_n : X \to Y$ *such that, for all* $x \in X$ *and* $n \in \mathbb{N}$*,*

$$\begin{aligned} G(x) &\subset H_n(x) \subset F(x), \\ H_n(x) &\subset H_{n+1}(x), \\ F(x) &= \mathrm{Cl}\Big\{ \bigcup_{n=1}^{\infty} H_n(x) \Big\}. \end{aligned}$$

Let us quickly quote all we know about similar facts in the case of the Banach space range $Y$ endowed by the standard convexity structure induced by the linearity structure of $Y$.

For mappings $F(x) = [-\infty, f(x)]$ and $G(x) = [-\infty, g(x)]$ induced by singlevalued mappings $f$ and $g$ and for $X = Y = \mathbb{R}$, Theorem 1 is a corollary of classical Baire results (see [2]). For analogous $F$ and $G$, $Y = \mathbb{R}$ and for $X$ being a normal, countably paracompact space this theorem coincides with Dowker theorem [6]. For compact-valued $F$ and $G$ and for $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ a version of Theorem 1 was obtained by Zaremba [21]. Hukuhara [7] proved for $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$ and for compact-valued $G$ and $F(x) \equiv Y$ the existence of a countable set of continuous separation mappings. For compact-valued $F$ and $G$, for metric space $X$ and for $Y = \mathbb{R}^m$ Theorems 1 and 1' are the Aseev's result [1]. For metric space $X$ and for a separable real Banach space $Y$ Theorems 1 and 1' were proved by De Blasi (see [3]) under the additional assumption (and with an additional conclusion) that the values $F(x)$, $x \in X$, (respectively, the values of $H(x)$) are bounded subsets of $Y$ with non-empty interior. He also used a metric version of semicontinuity. For similar results see also theorem of Choban and Ipate [4], where in such terms a characterization of perfect normality was given. A "measurably"-parametric version of Aseev theorem for mappings $F, G : X \times Z \to Y$ which are semicontinuous in the first argument and are measurable in the second argument was proved by Kucia and Novak in [8]. A parametric version of Hukuhara theorem was proved by Srivastava in [17].

Our approach has no intersections with methods of the above mentioned papers. A key ingriedient of our proof is the following Michael–Curtis selection theorem for a convex-valued mapping into a metric space with a suitable convex structure (see [10] and [5]).

Theorem 2. *Let* $M$ *be a complete metric space with a convex structure. Then every lower semicontinuous convex-valued and closed-valued mapping from a paracompact space into* $M$ *admits a continuous singlevalued selection.*

After preparing this paper V. Gutev kindly informed us that G. Nepomnjashchiĭ in [12] proved a more general (purely topological) version of Theorem 1. He used another technique which is clear from the fact that in his result mapping

$F$ is assumed to be connected-valued and with the family $\{F(x)\}_{x\in X}$ of values to be equi-locally connected.

## 2. Notation and definitions

As for a definition of convex structure, we use a more modern (and simple) Curtis version of the original Michael definition. Let $\Delta^n$ be the standard unit simplex with $n$ vertexes in $n$-dimensional Euclidean space, $\dim \Delta^n = n-1$.

Definition 3. A convex structure on a metric space $(M,\varrho)$ is defined as a sequence $\{(M_n,k_n)\}_{n=1}^{\infty}$, of pairs where $M_n$ is a subset of $n$th Cartesian power $M^n$ and $k_n$ is a mapping $k_n : M_n \times \Delta^n \to M$ with the following properties:

(a) $k_n(x,x,\dots,x;t_1,t_2,\dots,t_n) = x$ for all $x \in M$ and $(t_1,\dots,t_n) \in \Delta^n$,

(b) if $x \in M_n$, then $\partial_i x \in M_{n-1}$ and, moreover, if additionally $t_i = 0$ for a $t \in \Delta^n$, then $k_n(x,t) = k_{n-1}(\partial_i x, \partial_i t)$, where $\partial_i$ is the usual $i$th boundary operator,

(c) for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $n \in \mathbb{N}$ and for all $x, y \in M_n$ the inequalities $\varrho(x_i,y_i) < \delta$, $1 \le i \le n$, imply that $\varrho(k_n(x,t),k_n(y,t)) < \varepsilon$ for all $t \in \Delta^n$.

In the original Michael definition property (c) was stated in its non-uniform variant. It turns out that such uniform restriction allows to omit some point of Michael definition which doesn't hold for convex structures constructed below.

For a given convex structure on a metric space $(M,\varrho)$ a subset $C \subset M$ is said to be *convex* if for every $x = (x_1,\dots,x_n) \in C^n$ we have that $x \in M_n$ and $k_n(x,t) \in C$ for all $t \in \Delta^n$. We also say that a subset $Z \subset M$ is *admissible* if its $n$th power $Z^n$ lies in $M_n$ for every natural $n$. In applications property (c) is the most difficult to verify. But it automatically holds in the case of a linear metric space with convex open balls.

Definition 4. The *convex hull* $\mathrm{conv}(Z)$ of an admissible set $Z$ is defined as the set $\{k_n(x,t) \mid n \in \mathbb{N},\ x \in Z^n,\ t \in \Delta^n\}$. A convex structure in a metric space $(M,\varrho)$ is said to be a complete convex structure if the closure of the convex hull of every admissible subcompact of the space $M$ is compact.

Recall that a multivalued mapping $F$ of a topological space $X$ into a topological space $Y$ is said to be: *lower semicontinuous* if for every nonempty open subset $U \subset Y$ the following subset is open in space $X$

$$F^{-1}(U) = \{x \in X \mid F(x) \cap U \neq \emptyset\},$$

and *upper semicontinuous* if for every nonempty open subset $U \subset Y$ the following subset is open in space $X$

$$F_{-1}(U) = \{x \in X \mid F(x) \subset U\}.$$

*Continuity* of a compact-valued mapping into a metric space $(Y, \varrho)$ below means continuity with respect to the Hausdorff distance topology in the set $\exp(Y)$ of all compact subsets of $Y$. Recall that the Hausdorff distance $h_\varrho(A, B)$ between two compacta $A$ and $B$ is defined as the infimum of the set of all positive $\varepsilon$ for which $A$ lies in open $\varepsilon$-neighbourhood $D_\varrho(B, \varepsilon)$ of $B$ and symmetrically, $B$ is a subset of $D_\varrho(A, \varepsilon)$. It is a well-known fact, that $(\exp(Y), h_\varrho)$ is a complete metric space whenever $(Y, \varrho)$ is complete metric space.

## 3. Proof of Theorem 1

**Sketch.** We apply Theorem 2 in the following manner. Let $M = \exp\operatorname{conv}(Y)$ be the set of all nonempty convex subcompacta of a given complete metric space $(Y, \varrho)$ with a complete convex structure $\{(Y_n, k_n)\}$. Such set $M$ will be topologizied by the Hausdorff metric $h_\varrho$. Clearly, $M$ is a closed subset of the complete metric space $(\exp(Y), h_\varrho)$, i.e. of the compact exponent of $Y$. Hence $(M, h_\varrho)$ is a complete metric space, too. Our crucial technical observation is that the complete convex stucture $\{(Y_n, k_n)\}$ naturally induces some suitable (generally, noncomplete) convex structure $\{(M_n, K_n)\}$ in the complete metric space $(M, h_\varrho)$.

After this, to every point $x \in X$ one can naturally associate the set $\Phi(x)$ of all convex (with respect to $\{(Y_n, k_n)\}$) subcompacta of the convex closed set $F(x)$ which contains the convex compact set $G(x)$. So, such corresponding $\Phi$ can be regarded as a multivalued mapping from $X$ into $(M, h_\varrho)$ and we apply Theorem 2 exactly to the mapping $\Phi$. In reverse direction, a singlevalued continuous selection of mapping $\Phi : X \to M$ can be regarded as a multivalued mapping from $X$ into $Y$. Clearly, such a selection automatically will be a required continuous convex-valued and compact-valued mapping which separates a given semicontinuous mappings $F$ and $G$.

**Details.** First, we define a convex structure $\{(M_n, K_n)\}$ in the space $(M, h_\varrho)$. For every natural $n$ we set:

$$M_n = \left\{(A_1, \dots, A_n) \in M^n \;\middle|\; \bigcup A_j \text{ lies in some convex closed subset of } Y\right\},$$
$$K_n(A_1, \dots, A_n; t_1, \dots, t_n)$$
$$= \operatorname{Cl}(\operatorname{conv}(\{k_n(a_1, \dots, a_n; t_1, \dots, t_n) \mid a_1 \in A_1, \dots, a_n \in A_n\})).$$

Lemma 5. *The sequence of pairs $(M_n, K_n)$ is a well-defined convex structure in the space $(M, h_\varrho)$.*

Proof. Let $C$ be a closed convex subset of $Y$ and let us (for a fixed $n \in \mathbb{N}$) consider a convex subcompacta $A_1, \dots, A_n$ of $C$ and a point $t = (t_1, \dots, t_n) \in \Delta^n$. Then the set

$$\{k_n(a_1, \dots, a_n; t_1, \dots, t_n) \mid a_1 \in A_1, \dots, a_n \in A_n\}$$

is a compact as a continuous image of the compactum $A_1 \times \ldots \times A_n$. Hence, $K_n(A_1, \ldots, A_n; t_1, \ldots, t_n)$ is a subcompact of $C$ thanks to the completeness of the convex structure of space $Y$. Convexity of $K_n(A_1, \ldots, A_n; t_1, \ldots, t_n)$ follows from the general observation that the closure of a convex set $S$ is a convex set, too, whenever $S$ is a subset of a closed convex set $C$. The last statement is a corollary of property (c) of the convex structure $\{(Y_n, k_n)\}$. Therefore $K_n(M_n \times \Delta^n) \subset M$, i.e. the sequence of pairs $\{(M_n, K_n)\}$ is well-defined.

(a) If $A_1 = A_2 = A_3 = \ldots = A_n \in M$, then the set

$$\{k_n(a_1, \ldots, a_n; t_1, \ldots, t_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\},$$

coincides with $A$ due to the convexity of $A$ and due to property (a) for the convex structure in $Y$. Hence

$$\mathrm{Cl}(\mathrm{conv}(\{k_n(a_1, \ldots, a_n; t_1, \ldots, t_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\})),$$

coincides with $A$ due to the convexity and to the compactness of $A$.

(b) If $(A_1, \ldots, A_n) \in M_n$ then $\partial_i((A_1, \ldots, A_n)) \in M_{n-1}$ because $(\bigcup A_j) \setminus A_i \subset (\bigcup A_j) \subset C$ for some closed convex $C \subset Y$. If, additionally, $t \in \Delta^n$ with $t_i = 0$ then

$$\begin{aligned} &K_n(A_1, \ldots, A_n; t_1, \ldots, t_n) \\ &\quad = \mathrm{Cl}(\mathrm{conv}(\{k_n(a_1, \ldots, a_n; t_1, \ldots, t_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\})) \\ &\quad = \mathrm{Cl}(\mathrm{conv}(\{k_{n-1}(\partial_i(a_1, \ldots, a_n); \partial_i(t_1, \ldots, t_n)) \mid a_1 \in A_1, \ldots, a_n \in A_n\})) \\ &\quad = K_{n-1}(\partial_i(A_1, \ldots, A_n); \partial_i(t_1, \ldots, t_n)). \end{aligned}$$

(c) For a fixed positive $\varepsilon$ let $\delta = \delta(\varepsilon)$ be a positive number from property (c) for the convex structure in $Y$. Let us show that $\sigma(\varepsilon) = \delta(\delta(\varepsilon/2))$ works for the above constructed convex structure in $M$. So, let $h_\varrho(A_j, B_j) < \sigma$ for all $j = 1, \ldots, n$. We want to verify, that for every $t = (t_j) \in \Delta^n$ the inequality

$$h_\varrho(K_n((A_j); t), K_n((B_j); t)) < \varepsilon$$

holds. Take a point $y \in K_n((A_j); t)$. There exists a convergent to $y$ sequence $\{y_l\}$ of points $y_l$ from

$$\mathrm{conv}(\{k_n(a_1, \ldots, a_n; t_1, \ldots, t_n) \mid a_1 \in A_1, \ldots, a_n \in A_n\}).$$

Let $y_l = k_m(z_1, \ldots, z_m; \tau_1, \ldots, \tau_m)$ for some $m = m(l) \in \mathbb{N}$, for some $\tau = (\tau_1, \ldots, \tau_m) \in \Delta^m$ and some $z_i \in k_n(A_1 \times \ldots \times A_n; t)$. The inequalities $h_\varrho(A_j, B_j) < \sigma$ imply that sets $A_j$ lie in the $D_\varrho(B_j, \sigma)$, i.e. each point of $A_j$ is $\sigma$-close to a point of $B_j$, $j = 1, \ldots n$. Following the defintion of the function $\delta = \delta(\varepsilon)$, we find that each point $z_i$ is $\delta(\varepsilon/2)$-close to a point $w_i \in k_n(B_1 \times \ldots \times B_n; t)$, $i = 1, \ldots, m(l)$. Using once more the definition of the function $\delta = \delta(\varepsilon)$, we see that every point $y_l$ is $(\varepsilon/2)$-close to the point $u_l = k_m(w_1, \ldots, w_m; \tau_1, \ldots, \tau_m)$,

$l \in \mathbb{N}$. Hence, the sequence $\{y_l\}$ lies in the open $(\varepsilon/2)$-neighbourhood (with respect to $\varrho$) of the set

$$\operatorname{conv}(\{k_n(b_1, \dots, b_n; t_1, \dots, t_n) \mid b_1 \in B_1, \dots, b_n \in B_n\}).$$

So, $y = \lim y_l$ lies in the open $\varepsilon$-neighbourhood of this set, i.e.

$$y \in D_\varrho(K_n((B_j); t), \varepsilon) \quad \text{and} \quad K_n((A_j); t) \subset D_\varrho(K_n((B_j); t), \varepsilon).$$

The inverse implication can be proved by the symmetric consideration. Lemma 5 is proved. □

Now, we return to the properties of mapping $\Phi : X \to M$ defined in the sketch section. It is easy to verify the convexity (with respect to convex structure $\{(M_n, K_n)\}$) and closedness (in $M$) of the values $\Phi(x)$ of mapping $\Phi$. The nonemptness of the values $\Phi(x)$ follows from the obvious fact that $G(x) \in \Phi(x)$. So, we must verify only the lower semicontinuity of $\Phi$.

Lemma 6. *Under the assumption of Theorem* 1, *let for* $x \in X$

$$\Phi(x) = \{A \subset Y \mid A \text{ be a convex compact and } G(x) \subset A \subset F(x)\}.$$

*Then* $\Phi$ *is a lower semicontinuous mapping from* $X$ *into the metric space* $(M, h_\varrho)$.

Proof. Pick a point $x \in X$ and pick an element $A \in \Phi(x)$. We must show that $x$ is an interior point of the preimage

$$\Phi^{-1}(D_h(A, \varepsilon)) = \{x' \in X \mid \Phi(x') \cap D_h(A, \varepsilon) \neq \emptyset\},$$

for every positive $\varepsilon$. Here $D_h(A, \varepsilon)$ denotes an open $\varepsilon$-ball centered at $A$ in the metric space $(M, h_\varrho)$.

Find a finite $\sigma$-net $y_1, \dots, y_n$ in the compact set $A$ with $G(x) \subset A \subset F(x)$ where $\sigma = \delta(\varepsilon)/2$ and let

$$U(x) = G_{-1}(D_\varrho(G(x), \sigma)) \cap \left[\bigcap_{i=1}^{n} F^{-1}(D_\varrho(y_i, \sigma))\right].$$

The set $U(x)$ is an open neighbourhood of the point $x$ due to the upper semicontinuity of mapping $G$ and due to the lower semicontinuity of mapping $F$. For every $x' \in U(x)$ in the set $F(x')$ there exist points $z_1, \dots, z_n$ such that $\varrho(z_i, y_i) < \sigma$. Let us consider the following subset of the set $F(x')$

$$K' = G(x') \cup \{z_1, \dots, z_n\},$$

and define the following convex subcompactum of the set $F(x')$

$$A' = \operatorname{Cl}(\operatorname{conv}(K')).$$

Note, that here we once more use the completness of the convex stucture in $Y$.

Clearly, $G(x') \subset A' \subset F(x')$, i.e. $A' \in \Phi(x')$. By the construction $G(x') \subset D_\varrho(G(x), \sigma) \subset D_\varrho(A, \sigma)$ and $\varrho(z_i, y_i) < \sigma$ where $y_i \in A$. Hence $K' \subset D_\varrho(A, \sigma) \subset D_\varrho(A, 2\sigma) = D_\varrho(A, \delta(\varepsilon))$ and

$$A' = \mathrm{Cl}(\mathrm{conv}(K')) \subset D_\varrho(\mathrm{Cl}(\mathrm{conv}(A)), \varepsilon) = D_\varrho(A, \varepsilon),$$

thanks to property (c) of the convex structure in $Y$.

On the other hand, each point $y \in A$ is $\sigma$-close to some point $y_i$. Hence, y is $2\sigma$-close to the point $z_i \in K' \subset A'$. So, $A \subset D_\varrho(A', 2\sigma) \subset D_\varrho(A', \varepsilon)$ because one can assume that $\delta(\varepsilon) \leq \varepsilon$. Finally, we see that $h_\varrho(A, A') < \varepsilon$, i.e. that the set $\Phi(x')$ meets the open $\varepsilon$-ball $D_h(A, \varepsilon)$ in the space $(M, h_\varrho)$. Lemma 6 and Theorem 1 are proved. □

## 4. Applications and remarks

(a) For zero-dimensional domains the analogous appoach immediately gives the following "sandwich" theorem.

Theorem 7. *Let $F_0 : X_0 \to Y$ be a lower semicontinuous complete-valued mapping of a zero-dimensional* (*in* dim-*sense*) *paracompact space $X_0$ into a metric space $Y$ and let $G_0 : X_0 \to Y$ be an upper semicontinuous compact-valued selection of $F_0$. Then there exists a continuous compact-valued mapping $H_0 : X \to Y$ such that*

$$G_0(x) \subset H_0(x) \subset F_0(x),$$

*for all $x \in X$.*

For $n$-dimensional domains and $UV^n$-valued mappings see [16].

(b) After Theorem 7 an alternative appoach to the proof of Theorem 1 arises in the spirit of the paper [14]. We represent the paracompact domain $X$ as an image of some zero-dimensional paracompact $X_0$ under some Milutin mapping $m : X_0 \to X$. Let $\nu$ be an associated with $m$ mapping from $X$ into the space $P(X_0)$ of all probabilistic measures on $X_0$. Namely, support of the measure $\nu_x$ is a subset of the preimage $m^{-1}(x)$ which is a compact subset of $X_0$, $x \in X$. After this, Theorem 7 works for the mappings $F_0 = F \circ m$ and $G_0 = G \circ m$. The final step is the integration procedure, i.e. for $x \in X$ one can put

$$H(x) = \int H_0(z)\, d\nu_x(z), \quad z \in m^{-1}(x).$$

The problem here is to find a "true" definition for integrals of continuous convex-valued and compact-valued mappings. One of the possible ways is to define it as the closure of the set of integrals of all continuous sinlevalued selections of the mapping $H_0$. The other approach is to construct such an integral directly in the space $M$ (= the convex compact exponent of $Y$) thanks to the observation that the construction of integrals of singlevalued continuous mappings with compact domains in the case of Banach $Y$ can be performed directly as in the case of mappings into a locally convex topological space, [15]. But in both variants the verification of continuity of $H$ is a non-trivial job. That is why we use the Michael–Curtis Theorem 2 about selections of mappings with axiomatically defined convex structures in ranges.

(c) Some remarks about continuous multivalued approximations. Let us consider the case of perfectly normal domains and separable ranges of multivalued mappings. Clearly, the separability of $Y$ implies the separability of its compact exponent $\exp(Y)$ and the separability of its convex compact exponent $M = \operatorname{expconv}(Y)$. So, a "word by word" repetition of the proof of Lemma 5.2 from [9] gives the existence of a countable set $\{\varphi_i\}$ of continuous singlevalued selections of mapping $\Phi$ which are dense in $\Phi$ in the sense that $\{\varphi_i(x)\}$ is a dense subset of $\Phi(x)$ for every $x \in X$. Going back to space $Y$, we find a countable set $\{S_i\}$ of continuous convex-valued and compact-valued mappings which "densely" separate the given semicontinuous mappings $F$ and $G$. One can easily obtain an increasing sequence of mappings of such type. Namely, if $H_i(x) = \mathrm{Cl}(\operatorname{conv}\{S_1(x) \cup \ldots \cup S_i(x)\})$, $x \in X$, $i = 1, 2, \ldots$ then

$$\begin{gathered} H_1(x) \subset \ldots \subset H_i(x) \subset \cdots \subset F(x), \\ F(x) = \mathrm{Cl}\{H_1(x) \cup \ldots \cup H_i(x) \cup \ldots\}, \end{gathered}$$

and all mappings $H_i$, $i \in \mathbb{N}$, are continuous compact-valued and convex-valued. In other words, we prove Theorem 1' and our approach gives the new proofs of Aseev, De Blasi, Choban and Ipate approximaion theorems for paracompact (generally, non-metric) domains. Generalizations of measurable-parametric versions of sandwich theorems are also true.

For a metric domains one can use another Michael's "density" selection theorem [11] and prove the folowing sandwich theorem.

Theorem 8. *Let $F : X \to Y$ be a lower semicontinuous convex-valued and closed-valued mapping of a metric space $X$ into a complete metric space $Y$ with complete convex structure and let $G : X \to Y$ be an upper semicontinuous convex-valued and compact-valued selection of $F$. Then for every infinite cardinal $\alpha$ there exists a family $\Gamma$ (with cardinality $\leq \alpha$) of continuous convex-valued and compact-valued mappings $H_\gamma : X \to Y$ such that $G(x) \subset H_\gamma(x) \subset F(x)$ for all $x \in X$, and such that $F(x) = \mathrm{Cl}(\bigcup\{H_\gamma(x) \mid \gamma \in \Gamma\})$, whenever $F(x)$ has a dense subset of cardinality $\leq \alpha$.*

Note, that the intersection of two continuous convex-valued mappings can be non lower semicontinuous. Hence our approach doesn't automatically give the theorems about approximations of upper semicontinuous mappings by a sequence of continuous mappings.

(d) Every selection theorem gives (as a special case) some extension theorem. So, as a corollary of our approach we obtain the following "multivalued Dugundgji" theorem.

Theorem 9. *Every continuous compact-valued and convex-valued mapping from a closed subset $Z$ of a paracompactum $X$ into a complete metric space $Y$ with complete convex structure such that $Y$ is convex, admits a continuous compact-valued and convex-valued extension over the whole space $X$.*

Theorem 9 admits a generalization in the spirit of Tolstonogov's theorem [18], i.e. one can assume that in Theorem 9 the partial multivalued mapping and its extension are selections of a given continuous compact-valued and convex-valued mapping from $X$ into $Y$. In [18] such theorem was proved for a metric domain and for a Banach space ranges. Note, also, that Theorem 9 is a corollary of the result of G. Nepomnjashchiĭ [13].

(e) There exists an alternative approach to the notion of convex structure in topological spaces. Such approach is essentialy due to Van de Vel, and the important difference with Michael–Curtis approach is the absence of real parameters describing convex combinations (see [19] and [20]). In these papers Van de Vel proved that a "complete" metric convex structure on a space $Y$ induces a suitable metric convex structure on the convex exponent of space $Y$. Hence the analogs of Theorems 1 and 1' hold for such type of convex structures. Moreover, in [20] exactly the above selection approach was used for finding a continuous $\varepsilon$-enlargement of an upper-semicontinuous mapping. We emphasize that the whole range space $Y$ must be convex in Van de Vel approach and that it can be non-convex in Michael–Curtis approach.

We finish by observing that there are other notions of convex structures. But the question when a "good" convex structure on $Y$ implies the existence of a "good" convex structure on $\operatorname{expconv} Y$ is still open.

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*Manuscript received July 21, 1997*

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