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MULTIPLE PERIODIC SOLUTIONS FOR AUTONOMOUS CONSERVATIVE SYSTEMS

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Abstract. We consider an autonomous conservative second order system defined by a potential which admits a connected set Γ of critical points at level zero. We prove the existence of multiple periodic solutions of large period which are located near Γ.

1. Introduction and statement of the results

In the present paper, we deal with an autonomous conservative second order Hamiltonian system defined by a potential V which admits a connected set Γ of critical points at level zero. We are interested in finding periodic orbits with large period T, namely T-periodic solutions to

$$\ddot{x}(t) = V'(x(t)) \quad \text{in } \mathbb{R}, \tag{P}$$

where $V \in C^1(\mathbb{R}^2, \mathbb{R})$ and V' denotes the gradient of V.

Plainly, any point in Γ is a trivial solution to (P); as a consequence, looking for nontrivial solutions to (P) requires avoiding the constant solutions at level zero.

When the problem has a suitable symmetry, namely V is even, a $\mathbb{Z}_2$-version of Ljusternik–Schnirelman theory can be applied. If Γ is a circle or an annulus in

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the plane, it has a finite $\mathbb{Z}_2$-index; as far as large period solutions are concerned, this allows avoiding trivial solutions. We refer to Remark 5.2 for some details on such an approach.

In the general case, (P) has still an intrinsic symmetry, i.e. it is S^1-invariant. Nevertheless, in the situations we are interested in the S^1-index of Γ (see [2]) is not finite, hence an S^1-version of Ljusternik–Schnirelman theory cannot be successfully applied.

In order to prove existence of multiple nontrivial periodic solutions of large period to (P), without evenness assumptions on V, our idea consists in regarding (P) as a problem with a singular potential. Indeed, a simple change of variable proves that T-periodic solutions for (P) correspond to 1–periodic solutions for

$$(\mathrm{P}_T) \qquad \ddot{x}(t) = T^2 V'(x(t)) \quad \text{in } \mathbb{R}.$$

Since, for any point $\xi \in \mathbb{R}^2$ with $V(\xi) > 0$, the nonlinear term $T^2 V(\xi)$ becomes larger and larger as T increases, we can say that (P_T) looks like a problem with singular potential. As a consequence, we obtain our multiplicity result by a standard minimization argument in a class of functions which wind around some point ξ with $V(\xi) > 0$. Let us recall that existence and multiplicity of periodic solutions with singular potential have been extensively investigated by many authors; for instance, see [1], [4], [9] for strong-force potential, [7], [12], [13] for weak-force potential and references therein.

Before stating our main results, we need some notations and definitions. We say that $\Gamma \subset \mathbb{R}^2$ is an *admissible set* if it is bounded and its complementary Γ^c has exactly two connected components. Remark that, by its very definition, an admissible set is not simply connected in the plane.

For any admissible set Γ we denote by $\mathcal{B}$ and $\mathcal{U}$, respectively, the bounded and the unbounded connected component of Γ^c. Moreover, for any $r > 0$ we set $\mathcal{B}_r = \{\xi \in \mathcal{B} : \operatorname{dist}(\xi, \Gamma) \le r\}$, $\mathcal{U}_r = \{\xi \in \mathcal{U} : \operatorname{dist}(\xi, \Gamma) \le r\}$.

THEOREM 1.1. *Let Γ be an admissible set such that $V \equiv 0$ on Γ. Assume that for some $r > 0$*

$$(1.1) \qquad V(x) > 0 \quad \textit{for any } x \in \mathcal{B}_r \cup \mathcal{U}_r.$$

Moreover, assume that Γ contains the support of a noncontractible closed curve of class H^1. Then, for any $k \in \mathbb{N}$ there exists $T_k > 0$ such that (P) *has at least k distinct nontrivial T-periodic solutions, for any $T > T_k$.*

As a model situation, one can consider $V(x_1, x_2) = (|x|^2 - 1)^2 e^{x_1}$, $x = (x_1, x_2)$; in this case, $\Gamma = S^1$ and all the assumptions in Theorem 1.1 are fulfilled for any $0 < r < 1$.

We point out that in Theorem 1.1 no assumptions on the behaviour of the potential at infinity are required. In our approach, the existence of nontrivial

T-periodic solutions of (P) depends only on the behaviour of the potential in a neighbourhood of Γ.

As concerns the hypothesis about the existence of a noncontractible closed curve of class H^1 with support in Γ, let us note that it is satisfied in many situations. For example, it holds if Γ is diffeomorphic to S^1 or to an annulus in the plane; more in general, if there exists a piecewise differentiable homeomorphism between S^1 or an annulus and Γ. Moreover such an hypothesis is plainly satisfied if $\overline{\mathcal{B}} \cap \overline{\mathcal{U}} = \emptyset$; roughly speaking, this means that Γ is a "thick" set. The existence of such a curve in Γ is needed for technical reasons; as it will be clear in Section 3, we use it to obtain a uniform control on the minima of the action functional associated to (P_T). In this sense, such a curve can be considered as a "comparison term".

In the next theorem we consider a particular case, namely we assume that the exterior boundary of Γ (that is, $\Gamma \cap \overline{\mathcal{U}}$) is a circle in the plane. Such a geometrical hypothesis has two main consequences. Firstly, it guarantees the existence of a comparison term, in the sense of the previous remark. Secondly, it allows an easy modification of V in $\mathcal{U}$ such that the solutions of the modified equation still solve (P_T). As a result, the behaviour of V in the unbounded component of Γ^c is not influent, and assumption (1.1) in Theorem 1.1 can be weakened.

Theorem 1.2. *Let Γ be an admissible set such that $V \equiv V' \equiv 0$ on Γ. Assume that $\Gamma \cap \overline{\mathcal{U}}$ is a circle in the plane and*

$$V(x) > 0 \quad \textit{for any } x \in \mathcal{B}_r. \tag{1.2}$$

Then, for any $k \in \mathbb{N}$ there exists $T_k > 0$ such that (P) *has at least k distinct nontrivial T–periodic solutions, for any $T > T_k$.*

Let us remark that in Theorem 1.1 the condition $V' \equiv 0$ is obviously satisfied. In Theorem 1.2, V can exhibit any behaviour outside Γ, provided it is flat on Γ.

Finally, in this paper we describe the qualitative behaviour of T-periodic solutions to (P) as the period becomes larger and larger. Precisely, we show that if x_T is any solution found via Theorem 1.1 or 1.2, then as $T \to \infty$ x_T approaches a curve x_0, such that $x_0(t) \in \Gamma$ for any t (we refer to Remark 5.1 for details). As a consequence, the long period periodic solutions of (P) are located near Γ. Since each point in Γ is a constant periodic solution of (P), in some sense we can say that we obtain a "Birkhoff–Lewis" type result (see the classical paper [5] and also [3] and references therein).

2. Variational setting and preliminaries

Let us introduce some notations. Let H be the Sobolev space of the absolutely continuous 1-periodic curves in $\mathbb{R}^2$ with square integrable derivative, endowed with the usual norm $\|\cdot\|$. The action functional associated with problem (P_T) is

$$f_T(x) = \frac{1}{2}\int_0^1 |\dot{x}(t)|^2 dt + T^2 \int_0^1 V(x(t))\, dt, \quad x \in H.$$

Remark 2.1. For any $T > 0$, f_T is weakly lower semicontinuous in H, as it is the sum of a weakly lower semicontinuous functional (the quadratic term) and a weakly continuous one. Indeed, $x_n \rightharpoonup x$ in H implies $x_n \to x$ uniformly and the continuity of V yields $\int_0^1 V(x_n(t))\, dt \to \int_0^1 V(x(t))\, dt$ as $n \to \infty$.

We aim to find critical points of f_T as minima in suitable homotopy classes and to this end we shall adapt the arguments in [6].

Let $r > 0$ be fixed as in Theorem 1.1. Let $0 < \rho < r$ and define

$$\Lambda = \{x \in H : \mathrm{dist}(x(t), \Gamma) < \rho \text{ for all } t \in [0,1]\};$$

plainly, Λ is an open subset of H whose boundary is $\partial\Lambda = \{x \in H :$ exists $t \in [0,1]$ s.t. $\mathrm{dist}(x(t), \Gamma) = \rho\}$. Any $x \in \Lambda$ is a closed curve in the plane that does not cross any point in the set $\mathcal{B} \setminus \mathcal{B}_r$ (cf. Section 1 for notations); therefore, we can consider the winding number of x around any such point, which we shall denote by $\mathrm{Ind}(x)$. Finally, for any $q \in \mathbb{Z}$ define

$$\Lambda_q = \{x \in \Lambda : \mathrm{Ind}(x) = q\}.$$

Let $\gamma \in H$ be a noncontractible closed curve such that $\gamma([0,1]) \subset \Gamma$; the existence of such a curve is one of the assumptions in Theorem 1.1. With no restrictions, we can suppose $\mathrm{Ind}(\gamma_{|[0,1]}) = 1$. Plainly, the curve $\gamma_q(t) = \gamma(qt)$, $t \in [0,1]$, belongs to Λ_q. The main properties of Λ_q are collected in the following lemma.

Lemma 2.2. *For any $q \in \mathbb{Z}$, Λ_q is an open connected subset of Λ. Furthermore, Λ_q is weakly closed in Λ.*

Proof. Let $q \in \mathbb{Z}$. The first assertions follow from the elementary properties of the winding number. To prove the last statement, let $\{x_n\} \subset \Lambda_q$ and $x \in \Lambda$ be such that x_n weakly converges to x in Λ. Choose $\xi \in \mathcal{B} \setminus \mathcal{B}_r$; plainly, $|x(t) - \xi| > r - \rho$ for any t. On the other hand, as x_n converges to x uniformly in $[0,1]$, for n large we have $|x_n(t) - x(t)| < r - \rho$ for any t; thus $\mathrm{Ind}(x_n) = \mathrm{Ind}(x)$ for n large, whence $x \in \Lambda_q$. □

The next result is analogous to the well known Gordon Lemma [8], concerning singular strong-force potential.

Lemma 2.3. *For any $M > 0$ there exists $\overline{T} > 0$ such that the sublevel set $f_T^M = \{x \in \Lambda : f_T(x) \le M\}$ is weakly sequentially compact, for any $T \ge \overline{T}$.*

Proof. By contradiction, assume that there exist $M > 0$, $T_k \to \infty$ and a sequence $\{x_{n,k}\} \subset f_{T_k}^M$ which has no subsequences weakly converging in $f_{T_k}^M$. As

$\{x_{n,k}\}$ is bounded in H, it certainly has a subsequence, still denoted by $\{x_{n,k}\}$, which converges to some x_k weakly in H, strongly in L^2 and uniformly in $[0,1]$. By weak lower semicontinuity, $f_{T_k}(x_{n,k}) \leq M$, so necessarily $x_k \in \partial\Lambda$. As a result, there exists $t_k \in [0,1]$ such that $\operatorname{dist}(x_k(t_k),\Gamma) = \rho$. Let $0 < \rho_1 < \rho$ be such that $0 < \alpha \leq V(\xi)$ if $\rho_1 \leq \operatorname{dist}(\xi,\Gamma) \leq \rho$. For any $t \in [0,1]$ we have

$$|x_k(t) - x_k(t_k)| \leq c|t - t_k|^{1/2}\|\dot{x}_k\|_2. \tag{2.1}$$

Since $\|x_k\|^2 \leq \liminf_{n\to\infty} \|x_{n,k}\|^2 \leq \|x_k\|_2^2 + 2M$, we get $\|\dot{x}_k\|_2^2 \leq 2M$. Then (2.1) implies $\rho_1 \leq \operatorname{dist}(x_k(t),\Gamma) \leq \rho$ for any $t \in [t_k - \delta, t_k + \delta]$, for some $\delta > 0$. Hence

$$\begin{aligned} M &\geq T_k^2 \int_0^1 V(x_k(t))\,dt \geq T_k^2 \int_{t_k-\delta}^{t_k+\delta} V(x_k(t))\,dt \\ &\geq 2\delta T_k^2 \min_{[t_k-\delta,t_k+\delta]} V(x_k(t)) \geq \delta\alpha T_k^2 \end{aligned}$$

which yields a contradiction as $k \to \infty$. □

3. Proof of Theorem 1.1

The curve γ we introduced in Section 2 plays a relevant role in the following result.

Proposition 3.1. *For any $q \in \mathbb{Z}$, there exists $T_q > 0$ such that f_T attains its infimum in Λ_q, for any $T \geq T_q$.*

Proof. Let $q \in \mathbb{Z}$; as in Section 2, let $\gamma_q(t) = \gamma(qt)$. Let $M_q \equiv f_T(\gamma_q) = (q^2/2)\int_0^1 |\dot{\gamma}(t)|^2\,dt$. Let $\overline{T}_q > 0$ be associated with $2M_q > 0$ as in Lemma 2.3. Let $T \geq \overline{T}_q$ and let $\{x_{n,T}\} \subset \Lambda_q$ be a minimizing sequence, namely $f_T(x_{n,T}) \to \inf_{\Lambda_q} f_T$ as $n \to \infty$. For n sufficiently large $f_T(x_{n,T}) \leq 2M_q$. By Lemmas 2.2 and 2.3, $\{x_{n,T}\}$ weakly converges to some $x_T \in \Lambda_q$. Finally, Remark 2.1 gives $f_T(x_T) \leq \liminf_{n\to\infty} f_T(x_{n,T}) = \inf_{\Lambda_q} f_T$, whence x_T is a minimum point for f_T in Λ_q. □

Proof of Theorem 1.1. Let $k \in \mathbb{N}$ and $0 < q_1 < \ldots < q_k$. By Proposition 3.1, for any $j = 1,\ldots,k$ there exists $T_{q_j} > 0$ such that for any $T \geq T_{q_j}$ the functional f_T has a critical point x_{q_j} with $\operatorname{Ind}(x_{q_j}) = q_j$. Let $\overline{T} = \max\{T_{q_1},\ldots,T_{q_k}\}$. Then, for any $T \geq \overline{T}$ the functional f_T has k critical points x_{q_j}, $j = 1,\ldots,k$ with $\operatorname{Ind}(x_{q_j}) = q_j$. Therefore there exist k nontrivial distinct periodic solutions of (P_T), for any $T \geq \overline{T}$. □

4. Proof of Theorem 1.2

Without restrictions, in Theorem 1.2 we can assume $\Gamma \cap \overline{\mathcal{U}} = S^1$. Before proving the theorem, some remarks are in order. In the proof of Theorem 1.1, the key point consists in proving that, for large T, the H^1-weak limit x of any sequence $\{x_n\}$ in Λ, satisfying $f_T(x_n) \le \mathit{const}$, is again in Λ (this is substantially the meaning of Lemma 2.3). Roughly speaking, $x \in \partial\Lambda$ if for some t either $x(t) \in \mathcal{B}$ and $\operatorname{dist}(x(t),\Gamma) = \rho$ or $x(t) \in \mathcal{U}$ and the same equality holds. In order to prove that none of these cases can occur, we take advantage of the behaviour of V around Γ, as prescribed in (1.1). In Theorem 1.2, the behaviour of V around Γ is prescribed only on the bounded component of Γ^c. Therefore, we are no more able to prevent the case $x(t) \in \mathcal{U}$ and $\operatorname{dist}(x(t),\Gamma) = \rho$. To overcome such a difficulty, we look for critical points of f_T in a set different from Λ. Precisely, let $r > 0$ be fixed as in Theorem 1.2, let $0 < \rho < r$ and $\widetilde{\Gamma} = \{\xi \in \mathcal{B} : \operatorname{dist}(\xi,\Gamma) < \rho\} \cup \Gamma \cup \mathcal{U}$; we define

$$\widetilde{\Lambda} = \{x \in H : x(t) \in \widetilde{\Gamma} \text{ for all } t \in [0,1]\}.$$

Again, $\widetilde{\Lambda}$ is an open subset of H whose boundary is $\partial\widetilde{\Lambda} = \{x \in H :$ exists $t \in [0,1]$ s.t. $x(t) \in \mathcal{B},\ \operatorname{dist}(x(t),\Gamma) = \rho\}$. Exactly as before, we can define $\operatorname{Ind}(x)$ and consider $\widetilde{\Lambda}_q = \{x \in \widetilde{\Lambda} : \operatorname{Ind}(x) = q\}$, for $q \in \mathbb{Z}$. Plainly, Lemma 2.2 still holds for any $\widetilde{\Lambda}_q$; let us just point out that every such set is nonempty because it contains the curve $\exp(i2\pi qt)$, $t \in [0,1]$.

As concerns Lemma 2.3, in the present situation a crucial ingredient is missing. Indeed, in the proof of Lemma 2.3 we use the fact that any sequence $\{x_n\} \subset \Lambda$ satisfying $f_T(x_n) \le C$, for some $C > 0$, is bounded in H. This follows from the very definition of the action functional and the fact that Λ is bounded in L^∞, hence in L^2. This last assertion does not hold for $\widetilde{\Lambda}$. To get round this difficulty, we replace V by a coercive potential in such a way that the critical points of the modified action functional are still critical points of f_T. At this point, the assumption $\Gamma \cap \overline{\mathcal{U}} = S^1$ is used to construct the modified potential; a similar modification is used in [10].

Lemma 4.1. *Let $\widetilde{V} \in C^1(\mathbb{R}^2,\mathbb{R}_+)$ be such that $\widetilde{V}(x) = V(x)$ for $|x| \le 1$ and $(\widetilde{V}'(x),x) > 0$ for any $|x| > 1$. For any $x \in H$ let*

$$\widetilde{f}_T(x) = \frac{1}{2}\int_0^1 |\dot{x}(t)|^2\,dt + T^2\int_0^1 \widetilde{V}(x(t))\,dt.$$

Then any $x \in H$ critical point for $\widetilde{f}_T$ is a critical point for f_T.

Proof. Let $x \in H$ be a critical point for $\widetilde{f}_T$ and let us define

$$x_0(t) = \begin{cases} x(t) & \text{if } |x(t)| \le 1, \\ x(t)/|x(t)| & \text{if } |x(t)| > 1. \end{cases}$$

We have

$$0 = \langle \widetilde{f}_T'(x), x - x_0\rangle = \int_0^1 (\dot{x}, \dot{x} - \dot{x}_0)\,dt + T^2\int_0^1 (\widetilde{V}'(x), x - x_0)\,dt. \tag{4.1}$$

Furthermore,

$$\begin{aligned}(4.2)\qquad \int_0^1 (\dot{x}, \dot{x} - \dot{x}_0)\,dt &= \int_{|x(t)|>1} (\dot{x}, \dot{x} - \dot{x}_0)\,dt \\ &= \int_{|x(t)|>1} \left(\dot{x}, \dot{x} - \frac{\dot{x}}{|x|} + \frac{(\dot{x}, x)}{|x|^3} x\right) dt \\ &= \int_{|x(t)|>1} \left(\left(1 - \frac{1}{|x|}\right)|\dot{x}|^2 + \frac{(\dot{x}, x)^2}{|x|^3}\right) dt \geq 0.\end{aligned}$$

If $x \not\equiv x_0$, then

$$\int_0^1 (\widetilde{V}'(x), x - x_0)\,dt = \int_{|x(t)|>1} \left(1 - \frac{1}{|x|}\right) (\widetilde{V}'(x), x)\,dt > 0;$$

the last inequality, (4.1) and (4.2) give a contradiction. □

It is very easy to give examples of C^1 maps $\widetilde{V}$ satisfying the hypotheses of Lemma 4.1. For instance, one can consider

$$\widetilde{V}(x) = \begin{cases} V(x) & \text{for } |x| \leq 1, \\ 2|x|^3 - 3|x|^2 + 1 & \text{for } |x| > 1, \end{cases}$$

note that $\widetilde{V}$ is C^1 because $V \equiv V' \equiv 0$ on S^1.

By Lemma 4.1, we can now assume V coercive at infinity; moreover, any critical point x_T of the modified functional satisfies $\|x_T\|_\infty \leq 1$. At this point, with minor changes in the proof of Lemma 2.3 we obtain

LEMMA 4.2. *For any $M > 0$ there exists $\overline{T} > 0$ such that the sublevel set $\{x \in \widetilde{\Lambda} : f_T(x) \leq M\}$ is weakly sequentially compact, for any $T \geq \overline{T}$.*

As a result, we can easily prove

PROPOSITION 4.3. *For any $q \in \mathbb{Z}$, there exists $T_q > 0$ such that f_T attains its infimum in $\widetilde{\Lambda}_q$, for any $T \geq T_q$.*

Let us point out that in the proof of Proposition 4.3 the natural comparison term $\exp(i2\pi qt)$, $t \in [0, 1]$, can be used. Finally, arguing exactly as in the proof of Theorem 1.1 we can conclude the proof of Theorem 1.2.

5. Final remarks

In the following remark we describe the qualitative behaviour of the T-periodic solutions found via Theorem 1.1 and 1.2 as the period increases.

REMARK 5.1. Let $q \in \mathbb{Z}$ and $T_q > 0$ be as in Proposition 3.1 (respectively, in Proposition 4.3). For any $T \geq T_q$, let x_T be a minimum point for f_T in Λ_q (respectively in $\widetilde{\Lambda}_q$). As the minimum level of the action functional is bounded by a positive constant $C = C(q)$ which depends on q and is independent of T, the same holds for $\|x_T\|$. Then there exists a function x_0 such that $x_T \to x_0$, as $T \to \infty$, weakly in H and uniformly in $[0,1]$. We claim that $x_0(t) \in \Gamma$ for any $t \in [0,1]$. By contradiction, assume $\operatorname{dist}(x_0(t),\Gamma) > \beta > 0$ for any t in some interval $[t_1,t_2] \subset [0,1]$. Therefore, for T large, $\operatorname{dist}(x_T(t),\Gamma) \geq \beta$ in $[t_1,t_2]$; on the other hand, by construction, $\operatorname{dist}(x_T(t),\Gamma) \leq \rho$. If we set $m = \min\{V(\xi) : \xi \in \mathbb{R}^2,\ \beta \leq \operatorname{dist}(\xi,\Gamma) \leq \rho\}$ (respectively $m = \min\{V(\xi) : |\xi| \leq 1,\ \beta \leq \operatorname{dist}(\xi,\Gamma) \leq \rho\}$), then $m > 0$ and

$$C \geq T^2 \int_0^1 V(x_T(t))\,dt \geq T^2 \int_{t_1}^{t_2} V(x_T(t))\,dt \geq mT^2(t_2 - t_1)$$

for any T large, a contradiction.

Remark 5.2. As we noted in the Introduction, when V has a suitable symmetry a $\mathbb{Z}_2$-version of Ljusternik–Schnirelman theory can be easily applied to find multiple solutions to (P_T). Let us give an example, referring to [2,11], where similar situations are dealt with, for some details. For the sake of simplicity, assume that $V \geq 0$ in $\mathbb{R}^2$ and $V(\xi) = 0$ iff $|\xi| = 1$. Assume $V(-\xi) = V(\xi)$ for any $|\xi| \leq 1$. Furthermore, suppose that V is of class C^2 around the origin, $\xi = 0$ is a local maximum of V and the Hessian matrix $V''(0)$ has at least one nonzero eigenvalue. Let us explicitly remark that the assumptions on V involve only the unit disc; indeed, as shown in Section 4 we can modify V outside the unit circle in such a way that the action functional f_T satisfies Palais–Smale condition. Let Σ be the set of closed, symmetric subset of H which do not contain 0; for $A \in \Sigma$, let $\gamma(A)$ be the $\mathbb{Z}_2$-genus of A. Standard theory implies that the inf-max levels defined by

$$c_q = \inf_{\substack{A\in\Sigma \\ \gamma(A)\geq q}} \max_{x\in A} f_T(x), \quad q \geq 1,$$

are critical levels for f_T. To distinguish c_q from the trivial critical levels 0 and $f_T(0)$, some remarks on the genus of suitable sets are in order. Firstly, observe that $c_q > 0$ for any $q \geq 3$; indeed, if $0 = c_1 = \ldots = c_l$ for some l, then l is less than the genus of the set of critical points of f_T at level 0, that is $\gamma(S^1) = 2$. Secondly, let L_T be the self–adjoint realization in $L^2(0,1)$ of the operator $x \mapsto -\ddot{x} + T^2V''(0)x$, with 1-periodicity conditions, and Z_T be the subspace of H spanned by the nonconstant eigenfunctions of L_T corresponding to strictly negative eigenvalues. It is easy to see that there exists $\delta(T) > 0$ such that $\sup\{f_T(x) : x \in Z_T \cap \partial B_{\delta(T)}\} < T^2V(0) = f_T(0)$. Remark that $q_T \equiv \gamma(Z_T \cap \partial B_{\delta(T)}) = \dim Z_T$; as a consequence, q_T increases to $+\infty$ as T increases to $+\infty$. Therefore, for any fixed integer k, there exists $T_k > 0$ such that for any $T > T_k$ we have $0 < c_3 \leq c_{2+k} \leq c_{q_T} < f_T(0)$, which implies that (P_T) has at least k nontrivial periodic solutions. Let us finally remark that this result still holds if the plane is replaced by $\mathbb{R}^m$, $m \geq 2$.

References

[1] A. Ambrosetti and V. Coti Zelati, *Periodic solutions of singular dynamical systems*, Nato Adv. Sci. Inst. Ser., Periodic Solutions of Hamiltonian Systems and Related Topics (P.H: Rabinowitz et al., eds.), vol. 209, Reidel, Dordrecht, 1987, pp. 1–10.

[2] V. Benci, *A geometrical index for the group S^1 and some applications to the study of periodic solutions of ordinary differential equations*, Comm. Pure Appl. Math. **34** (1981), 393–432.

[3] V. Benci and D. Fortunato, *A "Birkhoff–Lewis" type result for non autonomous differential equations*, Proc. ELAM VIII Lecture Notes in Mathematics **1324** (1988), Springer-Verlag, 85–96.

[4] V. Benci and F. Giannoni, *Periodic solutions of prescribed energy for a class of Hamiltonian systems with singular potential*, J. Differential Equations **157** (1991), 568–576.

[5] G. D. Birkhoff and D. C. Lewis, *On the periodic motions near a given periodic motion of a dynamical system*, Ann. Mat. Pura Appl. **12** (1933), 117–133.

[6] S. Cingolani, M. Lazzo and J. F. Padial, *Multiple radial solutions for a class of elliptic systems with singular nonlinearities*, Ann. Mat. Pura Appl. IV **175** (1998), 365–373.

[7] V. Coti Zelati, *Periodic solutions for a class of planar singular dynamical systems*, J. Math. Pures Appl. **68** (1989), 109–119.

[8] W.B. Gordon, *Conservative dynamical systems involving strong forces*, Trans. Amer. Math. Soc. **204** (1975), 113–135.

[9] C. Greco, *Periodic solutions of a class of singular Hamiltonian systems*, Nonlinear Anal. **12** (1988), 259–269.

[10] D. Passaseo, *Multiplicity of critical points for some functionals related to the minimal surfaces problem*, Calc. Var. Partial Differential Equations **6** (1998), 105–121.

[11] P. Rabinowitz, *Periodic and heteroclinic orbits for a periodic hamiltonian system*, Ann. Inst. H. Poincaré Anal. Non Linéaire **6** (1989), 331–346.

[12] E. Serra and S. Terracini, *Noncollision solutions to some singular minimization problems with Keplerian–like potentials*, Nonlinear Anal. **22** (1994), 45–62.

[13] S. Terracini, *An homotopical index and multiplicity of periodic solutions to dynamical systems with singular potential*, J. Differential Equations **98** (1992), 197–225.

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