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CRITICAL SUPERLINEAR AMBROSETTI–PRODI PROBLEMS

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Abstract. We consider the existence of multiple solutions for problem (1.1) below with either $\lambda \neq \lambda$ or $\lambda = \lambda_1$, where λ_k, $k = 1, 2, \ldots$ are eigenvalues of $(-\Delta, H_0^1(\Omega))$. The local bifurcation from $\lambda = \lambda_k$ is also investigated.

1. Introduction

The main purpose of this work is to investigate the existence of multiple solutions of the critical superlinear problem

$$-\Delta u = \lambda u + u_+^{2^*-1} + f(x) \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $2^* = 2N/(N-2)$, $N \geq 3$ is the critical Sobolev exponent, and $\lambda > 0$ is a constant. u^+ denotes the positive part of $u : u^+(x) = \max\{u(x), 0\}$.

This problem belongs to a class of problems which are known as the Ambrosetti–Prodi type. Due to the important role of the Ambrosetti–Prodi result [2] in subsequent research and for completeness we state it next. Let $g : \mathbb{R} \to \mathbb{R}$ be a C^2-function such that $g''(s) > 0$ for all $s \in \mathbb{R}$ and

$$0 < \lim_{s \to -\infty} g'(s) < \lambda_1 < \lim_{s \to \infty} g'(s) < \lambda_2,$$

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where λ_1 and λ_2 are the first and second eigenvalues of $(-\Delta, H_0^1(\Omega))$. They consider the following boundary value problem

$$-\Delta u = g(u) + f(x) \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a $C^{2,\alpha}$ boundary $\partial\Omega$. Then, there is a C^1 manifold M in $C^{0,\alpha}(\overline{\Omega})$, which splits the space into two open sets O_0 and O_2 with the following properties

(i) if $f \in O_0$, problem (1.2) has no solution,
(ii) if $f \in M$, problem (1.2) has exactly one solution,
(iii) if $f \in O_2$, problem (1.2) has exactly two solutions.

A solution here means a function $u \in C^{2,\alpha}(\overline{\Omega})$.

After this work, several authors have extended this result in different directions. The literature on this problem is quite extensive; even risking the possibility of omitting some important work, we mention the following papers [1], [3], [4], [12], [17], [18], etc.

The above result shows the role that the location of the limits

$$g_- = \lim_{s\to-\infty} \frac{g(s)}{s}, \quad g_+ = \lim_{s\to\infty} \frac{g(s)}{s} \tag{1.3}$$

with respect to the spectrum of $(-\Delta, H_0^1(\Omega))$ plays in the question of existence of solutions for problem (1.2). Indeed, the Ambrosetti–Prodi's result contrasts with the well-known fact that if $g_\pm$ are strictly between two consecutive eigenvalues, or both $g_\pm$ are strictly less than λ_1, then problem (1.2) is solvable for all f. (We are assuming that f is locally Lipschizian, and then solutions are in $C^{2,\alpha}(\Omega)\cap C^0(\overline{\Omega})$). So the interesting cases are when the interval (g_-, g_+) contains eigenvalues. Problems with this feature are called problems of the Ambrosetti–Prodi type, or problems with jumping nonlinearities in a terminology introduced by Fučik, see [17]. These Ambrosetti–Prodi type problems can be seen as a question of characterizing (or at least, describing part of) the range of a perturbation of a linear operator (say, $-\Delta$) by some nonlinear operator (say $Nu := -g(x,u)$, which in our case is $g(x,u) := \lambda u + u_+^{2^*-1}$). We can distinguish three different types of Ambrosetti–Prodi problems.

In type I, we have $g_- < \lambda_1 < g_+$, where g_- could be $-\infty$, and g_+ could be ∞. We write $f = t\phi_1 + h$, where $t \in \mathbb{R}$, ϕ_1 is a first eigenfunction of $(-\Delta, H_0^1(\Omega))$ with $\phi_1 > 0$ and $\int_\Omega \phi_1^2\, dx = 1$, and $\int_\Omega h\phi_1\, dx = 0$. Then we can prove that in this case there is a t_0 such that if $t < t_0$, problem (1.2) has at least one solution. Such a result holds under more general assumptions. Namely g can depend also on x, and the first limit in (1.3) can be replaced by limsup. Similarly the second limit can be replaced by liminf. See, for instance, the survey paper [16].

Type II is when g_- and g_+ are finite, with the interval (g_-, g_+) containing eigenvalues. These problems are called asymptotically linear. They have been extensively studied by Lazer–McKenna, see for instance [20]. In the treatment of this problem, via Topological and Variational Methods, it has appeared in an essential way the so-called Fučik spectrum [17].

Type III is when g_- is between two consecutive eigenvalues and $g_+ = \infty$. These are superlinear problems with a crossing of all but a finite number of eigenvalues. In this case one can prove that there is a $t_0 \in \mathbb{R}$ such that problem (1.2) with $f = t\phi_1 + h$ has a negative solution for $t > t_0$. These problems have been treated in [25], and [15].

We remark that existence of a first solution for problems of type I and III does not require any growth at $\pm\infty$. So subcritical, critical or supercritical problems are treated. Observe that the reason is that: (i) in type I, one can find a subsolution and a supersolution, and then a solution of problem (1.2) comes either by the Monotone Iteration Method if, for instance, the derivative of g is bounded, or by some Variational Methods after an appropriate truncation of the nonlinearity; (ii) in the case of type III we truncate the nonlinearity g for $s > 0$, getting a function $\widetilde{g}$ in such a way that g_- and $\lim_{s\to\infty} \widetilde{g}(s)/s$ are between the same pair of consecutive eigenvalues.

The importance of the growth of g at infinite comes when one tries to get a second solution. The reason being that in order to have the functional associated to Equation (1.2)

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega G(u)\,dx - \int_\Omega fu\,dx$$

well defined in $H_0^1(\Omega)$, one has to require that

$$|g(s)| \le C|s|^p + C,$$

where $1 \le p \le 2^* - 1$. The subcritical case $p < 2^* - 1$ has been discussed by several authors mentioned before. Recently, Deng [13] considered problem (1.2) with a nonlinearity of the type $g(u) = |u|^{2^*-1} + k(u)$, where k is a lower perturbation of the expression with the critical exponent. This problem belongs to an Ambrosetti–Prodi problem of type I. In this case, the variational tool is the Mountain Pass Theorem.

Our problem stated in the beginning of this Introduction is of type I if $\lambda < \lambda_1$ and of type III if $\lambda > \lambda_1$. In order to get a second solution, we have to recourse to a Linking Theorem. Both the geometry of functional associated to equation (1.2) and the determination of the levels where a (PS) condition fails are much more involved in type III than in type I. All along this paper we write the non-homogeneous term in the form $f = t\phi_1 + h$, where $h \perp \phi_1$ in the L^2-sense. Let $(0 <)\lambda_1 < \lambda_2 \le \lambda_3 \le \dots$ be the eigenvalues of $-\Delta$ subject to Dirichlet data, with corresponding eigenfunctions $\phi_1, \phi_2, \dots$. In Section 2, we prove the following result.

Theorem 1.1 (I. Existence of a negative solution).

(i) *If* $0 < \lambda < \lambda_1$ *and given* $h \in L^2$, *then there exists a* $t_0 = t_0(h) < 0$ *such that if* $t < t_0$, *Problem* (1.1) *has a negative solution* u_t.

(ii) *If* $\lambda > \lambda_1$, *and given* $h \in L^2$, *such that* $h \in \ker(-\Delta - \lambda)^\perp$ *in the case that* λ *is an eigenvalue, then there exists* $t_0 = t_0(h) > 0$ *such that if* $t > t_0$, *Problem* (1.1) *has a negative solution* u_t.

(II. Existence of a second solution). *If, in addition to either of the hypotheses above, one assumes that* λ *is not an eigenvalue of* $(-\Delta, H_0^1(\Omega))$ *and the dimension* $N > 6$, *Problem* (1.1) *has a second solution.*

Although the methods used here are essentially the same as for problems of Brézis–Nirenberg type, namely

$$-\Delta u = |u|^{2^*-2}u + g(x,u) \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega, \tag{1.4}$$

where $g(x,0) = 0$ and g is some perturbation of lower order of the critical power, the technicalities have some new features. Indeed, for problem (1.4) the first solution is $u \equiv 0$, and from there one builds up the variational approach. In case of (1.2), the first solution $u_t \neq 0$ and the translation of the functional to be centered at u_t introduces nonhomogeneities which are delicate to handle.

When one of the limits g_- or g_+ is equal to an eigenvalue, we have a resonant problem. The solvability of (1.2) in this situation requires usually some additional conditions on g, like the Landesman–Lazer condition, see [20]. In Section 3 we discuss a case of resonance at $\lambda = \lambda_1$, where such a condition does not hold. Namely, the following result is proved.

Theorem 1.2. *Suppose* $\lambda = \lambda_1$. *Then there is an* $\varepsilon > 0$ *such that if* $\|f\|_{L^2} < \varepsilon$, *then* (1.1) *has a solution.*

Finally in Section 4, we discuss local bifurcation at $\lambda = \lambda_k$, $k > 1$. Using the theory of bifurcation for variational problems as developed by Böhme [5] and Marino [21], we can handle eigenvalues of any algebraic multiplicity, and prove the next result.

Theorem 1.3. *Let* $h \in \ker(-\Delta - \lambda_k)^\perp$ *with* $k > 1$. *In the space* $\mathbb{R} \times H_0^1(\Omega)$, *let* $(\lambda, u_t(\lambda))$ *for* λ *near* λ_k *be the line of negative solutions of* (1.1) *obtained in Theorem* 1.1. *Then* $(\lambda_k, u_t(\lambda_k))$ *is a point of bifurcation.*

2. The proof of Theorem 1.1

We write $f(x) = t\phi_1(x) + h(x)$, where ϕ_1 is the first eigenfunction of $-\Delta$, $\phi_1 \perp h$ in L^2-sense. We first prove that (1.1) has a negative solution u_t. Indeed, all negative solutions of (1.1) satisfies

$$-\Delta u = \lambda u + t\phi_1 + h. \tag{2.1}$$

If λ is an eigenvalue of $(-\Delta, H_0^1(\Omega))$, we suppose that $h \in \ker(-\Delta - \lambda)^\perp$. Then the problem

$$-\Delta u = \lambda u + h \quad \text{in } \Omega,\ u = 0 \text{ on } \partial\Omega \tag{2.2}$$

has a solution u_0. Consequently, the function $w = u_t - u_0$, where u_t is some solution of (2.1), is a solution of

$$-\Delta w = \lambda w + t\phi_1 \quad \text{in } \Omega,\ w = 0 \text{ on } \partial\Omega. \tag{2.3}$$

Problem (2.3) has a unique solution $w = \beta\phi_1$ where $\beta = t/(\lambda_1 - \lambda)$. Since we look for $u_t \le 0$, it follows that: (i) for $\lambda < \lambda_1$, we obtain such u_t for $t < 0$ and large, which comes from a negative β; (ii) for $\lambda > \lambda_1$, we obtain such u_t for $t > 0$ and large, which comes also from a negative β. So $u_t = w + u_0$ is the solution of (2.1) which we are looking for.

To find a second solution u of (1.1), we set $u = v + u_t$, and then v satisfies

$$-\Delta v = \lambda v + (v + u_t)_+^{2^*-1} \quad \text{in } \Omega,\ v = 0 \text{ on } \partial\Omega. \tag{2.4}$$

So the second solution of (1.1) is obtained by finding a nontrivial solution v of (2.4). Using variational methods we look for a critical point of the functional

$$I(v) = \frac{1}{2}\int_\Omega (|\nabla v|^2 - \lambda v^2)\,dx - \frac{1}{2^*}\int_\Omega (v + u_t)_+^{2^*}\,dx$$

defined in $E = H_0^1(\Omega)$. We use a Linking Theorem without Palais–Smale condition, see Theorem 4.3 in [22], or Theorem 5.1 in [14].

Suppose $\lambda > 0$ is not an eigenvalue of $(-\Delta, H_0^1(\Omega))$. We assume $\lambda \in (\lambda_k, \lambda_{k+1})$ from now on. The other case $0 < \lambda < \lambda_1$ can be treated in a similar and simpler way, using the Mountain Pass Theorem. Let us denote

$$E^- = \begin{cases} \emptyset & \text{if } \lambda \in (0, \lambda_1), \\ \operatorname{span}\{\phi_1, \dots, \phi_k\}, & \text{otherwise,} \end{cases}$$

and $E^+ = (E^-)^\perp$.

Let $S_\rho = \partial B_\rho \cap E^+$ and $Q = [0, Re] \oplus (\overline{B}_r \cap E^-)$, $e \in E^+$, where $\rho > 0$, $R > 0$ and $r > 0$ will be determined later and in a way that

$$I\,|_{S_\rho} \ge \alpha > 0, \quad \rho < R, \tag{2.5}$$

$$I\,|_{\partial Q} < \alpha, \tag{2.6}$$

$$\max_{\overline{Q}} I < \frac{1}{N} S^{N/2}, \tag{2.7}$$

where S is the best Sobolev constant. Inequalities (2.5)–(2.6) will give the geometry of the functional I required by the Linking Theorem of Rabinowitz [24]. We will use it in the version without the assumption of Palais–Smale, see Theorem 4.3 in [22] or Theorem 5.1 in [14]. For that matter, condition (2.7) is used to prove that the solution obtained as a weak limit of a (PS)-sequence at the minimax level is not a trivial one.

Lemma 2.1. *There exist* $\rho_0 > 0$ *and a function* $\alpha > 0$, $\alpha : [0, \rho_0] \to \mathbb{R}$ *such that*

$$I(v) \geq \alpha(\rho) \quad \text{for all } v \in S_\rho = \partial B_\rho \cap E^+.$$

Explicitly, we have

$$\rho_0 = \left\{ S^{N/(N-2)} \left(1 - \frac{\lambda}{\lambda_{k+1}} \right) \right\}^{(N-2)/4},$$

$$\alpha(\rho) = \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{k+1}} \right) \rho^2 - \frac{1}{2^*} S^{-N/(N-2)} \rho^{2^*}$$

and the maximum value of $\alpha(\rho)$

$$\widehat{\alpha} = \frac{1}{N} S^{N/2} \left(1 - \frac{\lambda}{\lambda_{k+1}} \right)^{N/2}$$

is assumed at

$$\widehat{\rho} = \left(1 - \frac{\lambda}{\lambda_{k+1}} \right)^{(N-2)/4} S^{N/4}.$$

Proof. Using the fact that $u_t < 0$ and the variational characterization of λ_{k+1} we get

$$I(v) \geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{k+1}} \right) \int_\Omega |\nabla v|^2 \, dx - \frac{1}{2^*} \int_\Omega v_+^{2^*} \, dx.$$

By Sobolev imbedding we obtain

$$\begin{aligned} I(v) &\geq \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{k+1}} \right) \int_\Omega |\nabla v|^2 \, dx - \frac{1}{2^*} S^{-N/(N-2)} \left(\int_\Omega |\nabla v|^2 \, dx \right)^{2^*/2} \\ &= \frac{1}{2} \left(1 - \frac{\lambda}{\lambda_{k+1}} \right) \rho^2 - \frac{1}{2^*} S^{-N/(N-2)} \rho^{2^*}. \end{aligned}$$

The result follows by maximizing the function defined by the last equality. □

The best Sobolev constant S used above is defined by

$$S = \inf\{ \|\nabla u\|_2^2 / \|u\|_{2^*}^2 : u \neq 0, \ u \in H^1(\mathbb{R}^N) \} \tag{2.8}$$

which is assumed by the functions

$$\psi_\varepsilon(x) = \left(\frac{\varepsilon \sqrt{N(N-2)}}{\varepsilon^2 + |x|^2} \right)^{(N-2)/2}, \quad \varepsilon > 0. \tag{2.9}$$

Let $\xi \in C_c^1(\mathbb{R}^N)$ be a function such that $\xi(x) = 1$ on $B_{1/2}(0)$, $\xi(x) = 0$ on $\mathbb{R}^N \setminus B_1(0)$ and $0 \leq \xi(x) \leq 1$ on $\mathbb{R}^N$. We may assume $B_1(0) \subset \Omega$. Let $\phi_\varepsilon(x) = \xi(x)\psi_\varepsilon(x)$, then we have following estimates.

Lemma 2.2. ([8])

$$\|\nabla\phi_\varepsilon\|_2^2 = S^{N/2} + o(\varepsilon^{N-2}), \tag{2.10}$$

$$\|\phi_\varepsilon\|_{2^*}^{2^*} = S^{N/2} + o(\varepsilon^{N}), \tag{2.11}$$

$$\|\phi_\varepsilon\|_2^2 = \begin{cases} K_1\varepsilon^2 + o(\varepsilon^{N-2}) & \text{if } N \ge 5, \\ K_1\varepsilon^2|\log\varepsilon^2| + o(\varepsilon^2) & \text{if } N = 4, \end{cases} \tag{2.12}$$

$$\|\phi_\varepsilon\|_1 \le K_2\varepsilon^{(N+2)/2}, \tag{2.13}$$

and

$$\|\phi_\varepsilon\|_{2^*-1}^{2^*-1} \le K_3\varepsilon^{(N-2)/2}, \tag{2.14}$$

where $K_1 > 0$, $K_2 > 0$ and $K_3 > 0$ are constants.

Denote by $P_\pm$ the orthogonal projections of E onto $E^\pm$ respectively. Using arguments as in [11], we can prove the following lemma.

Lemma 2.3.

$$\left|\int_\Omega [(P_+\phi_\varepsilon)^{2^*} - \phi_\varepsilon^{2^*}]\,dx\right| \le C\varepsilon^{N-2}, \tag{2.15}$$

$$\left|\int_\Omega (|\nabla\phi_\varepsilon|^2 - |\nabla(P_+\phi_\varepsilon)|^2)\,dx\right| \le C\varepsilon^{N-2}, \tag{2.16}$$

$$\|P_+\phi_\varepsilon\|_{2^*-1}^{2^*-1} \le C\varepsilon^{(N-2)/2}, \tag{2.17}$$

$$\|P_+\phi_\varepsilon\|_1 \le C\varepsilon^{(N+2)/2}, \tag{2.18}$$

and

$$\|P_-\phi_\varepsilon\|_\infty \le C\varepsilon^{(N-2)/2}. \tag{2.19}$$

Define for any fixed $K > 0$ the set $\Omega_{\varepsilon,K} = \{x \in \Omega : P_+\phi_\varepsilon(x) > K\}$. By (2.19) we know that

$$P_+\phi_\varepsilon(0) = \phi_\varepsilon - P_-\phi_\varepsilon(0) \ge C\varepsilon^{-(N-2)/2} - \|P_-\phi_\varepsilon\|_\infty \ge C\varepsilon^{-(N-2)/2}$$

which implies $P_+\phi_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$. By the continuity of $P_+\phi_\varepsilon$, there exists $\delta > 0$ such that $B_\delta(0) \subset \Omega_{\varepsilon,K}$. Therefore we have a result as follows.

Lemma 2.4.

$$\int_{\Omega_{\varepsilon,K}} |P_+\phi_\varepsilon|^{2^*}\,dx = \int_\Omega \phi_\varepsilon^{2^*}\,dx + O(\varepsilon^{N-2}), \tag{2.20}$$

$$\int_{\Omega_{\varepsilon,K}} |P_+\phi_\varepsilon|^{2^*-1}\,dx = \int_\Omega \phi_\varepsilon^{2^*-1}\,dx + O(\varepsilon^{(N+2)/2}), \tag{2.21}$$

and

$$\int_{\Omega_{\varepsilon,K}} |P_+\phi_\varepsilon|\,dx = \int_\Omega \phi_\varepsilon\,dx + O(\varepsilon^{N}). \tag{2.22}$$

Lemma 2.5. *Let $u, v \in L^p(\Omega)$ with $2 \le p \le 2^*$. If $\omega \subset \Omega$ and $u + v > 0$ on ω, then*

$$\left| \int_\omega (u+v)^p \, dx - \int_\omega |u|^p \, dx - \int_\omega |v|^p \, dx \right| \le C \int_\omega (|u|^{p-1}|v| + |u||v|^{p-1}) \, dx, \tag{2.23}$$

where C depends only on p.

Proof. By the Fundamental Theorem of Calculus the left side of (2.23) is equal to

$$\left| p \int_0^1 d\tau \int_\omega [|v + \tau u|^{p-2}(v + \tau u) - |\tau u|^{p-2} \tau u] u \, dx \right|,$$

which by its turn is equal to, using the mean value theorem

$$p(p-1) \left| \int_0^1 d\tau \int \omega |\tau u + v\theta(x)|^{p-2} uv \, dx \right|, \quad 0 < \theta(x) < 1.$$

This last expression can be estimated by

$$C \int_0^1 d\tau \int_\omega (\tau^{p-2}|u|^{p-1}|v| + |u||v|^{p-1}) \, dx \le C \int_\omega (|u|^{p-2}|v| + |u||v|^{p-1}) \, dx. \quad \square$$

Lemma 2.6. *Let A, B, C and α be positive numbers. Consider the function*

$$\Phi_\varepsilon(s) = \frac{1}{2} s^2 A - \frac{1}{2^*} s^{2^*} B + s^{2^*} \varepsilon^\alpha C, \quad s > 0.$$

Then

$$s_\varepsilon = \left(\frac{A}{B - 2^* \varepsilon^\alpha C} \right)^{1/(2^*-2)}$$

is the point where Φ_ε achieves its maximum. Write $s_\varepsilon = (1 + t_\varepsilon) s_0$, where $s_0 = (A/B)^{1/(2^*-2)}$ is the point at which Φ_0 achieves its maximum. Then $t_\varepsilon = O(\varepsilon^\alpha)$, and

$$\Phi_\varepsilon(s) \le \Phi_\varepsilon(s_\varepsilon) = \frac{1}{2} \left(\frac{A^N}{B^{N-2}} \right)^{1/2} + O(\varepsilon^\alpha).$$

Proof. It is clear that Φ_ε achieves its maximum at s_ε and s_ε satisfies

$$s_\varepsilon A - s_\varepsilon^{2^*-1} B + 2^* C \varepsilon^\alpha s_\varepsilon^{2^*-1} = 0. \tag{2.24}$$

This implies

$$s_\varepsilon \ge s_0. \tag{2.25}$$

Writing $s_\varepsilon = (1 + t_\varepsilon) s_0$, we derive from (2.24) that

$$s_\varepsilon \to s_0, \quad t_\varepsilon \to 0 \quad \text{as } \varepsilon \to 0 \tag{2.26}$$

and

$$(1 + t_\varepsilon) s_0 A - (1 + t_\varepsilon)^{2^*-1} s_0^{2^*-1} B + 2^* C \varepsilon^\alpha (1 + t_\varepsilon)^{2^*-1} s_0^{2^*-1} = 0. \tag{2.27}$$

That is

$$\left(\frac{A^{2^*-1}}{B}\right)^{1/(2^*-2)}[(1+t_\varepsilon)-(1+t_\varepsilon)^{2^*-1}]+2^*C\varepsilon^\alpha(1+t_\varepsilon)^{2^*-1}s_0^{2^*-1}=0.$$

Expanding for t_ε we obtain

$$\left[\frac{4}{N-2}t_\varepsilon+o(t_\varepsilon)\right]\left(\frac{A^{2^*-1}}{B}\right)^{1/(2^*-2)}=2^*C\varepsilon^\alpha(1+t_\varepsilon)^{2^*-1}s_0^{2^*-1}. \tag{2.28}$$

Hence

$$t_\varepsilon=O(\varepsilon^\alpha). \tag{2.29}$$

□

Our aim is to choose Q and ρ such that (2.5), (2.6) and (2.7) hold. So choose e as a function of ε: $e_\varepsilon=P_+\phi_\varepsilon$.

Lemma 2.7. *There exist $r_0>0$, $R_0>0$, and $\varepsilon_0>0$ such that for $r\geq r_0$, $R\geq R_0$ and $0<\varepsilon\leq\varepsilon_0$ we have*

$$I|_{\partial Q}<\alpha,$$

where $\alpha>0$ is determined in Lemma 2.1.

Proof. We may write $\partial Q=\Gamma_1\cup\Gamma_2\cup\Gamma_3$ with

$$\begin{aligned}&\Gamma_1=\overline{B}_r\cap E^-,\\&\Gamma_2=\{v\in E:v=w+se_\varepsilon,\ w\in E^-,\ \|w\|=r,\ 0\leq s\leq R\},\\&\Gamma_3=\{v\in E:v=w+Re_\varepsilon,\ w\in E^-\cap B_r(0)\}.\end{aligned}$$

We will show that on each Γ_i, we have $I|_{\Gamma_i}<\alpha$, $i=1,2,3$.

For any $v\in E^-$ we have

$$\int_\Omega|\nabla v|^2\,dx\leq\lambda_k\int_\Omega v^2\,dx. \tag{2.30}$$

So, for $v\in\Gamma_1$,

$$I(v)=\frac12\left(1-\frac{\lambda}{\lambda_k}\right)\int_\Omega|\nabla v|^2\,dx-\frac{1}{2^*}\int_\Omega(v+u_t)_+^{2^*}\,dx\leq 0.$$

For $v\in\Gamma_2$, we distinguish two cases.

Define $\delta^2=\sup_{0<\varepsilon\leq1}\int_\Omega|\nabla e_\varepsilon|^2\,dx$. If $0\leq s\leq s_0:=\sqrt{2\widehat{\alpha}}/\delta$, then

$$\begin{aligned}I(v)\leq\frac12\left(1-\frac{\lambda}{\lambda_k}\right)r^2+\frac12 s^2\int_\Omega|\nabla e_\varepsilon|^2\,dx-\frac{1}{2^*}\int_\Omega(w+se_\varepsilon+u_t)_+^{2^*}\,dx&\\ \leq\frac12 s^2\delta^2<\widehat{\alpha}.&\end{aligned}$$

If $s\geq s_0=\sqrt{2\widehat{\alpha}}/\delta$, denote

$$K=\sup\left\{\left\|\frac{w+u_t}{s}\right\|_{L^\infty}:s_0\leq s\leq R,\ \|w\|_E=r,\ w\in E^-\right\}.$$

K is independent of R. Since $P_+\phi_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, there exists $\varepsilon_0' > 0$ such that for all ε, $0 < \varepsilon < \varepsilon_0'$ and $s \ge s_0$

$$\Omega_\varepsilon = \{x \in \Omega : e_\varepsilon(x) > K\} \ne \emptyset.$$

Whence by Lemma 2.5

$$\begin{aligned}(2.31)\qquad &\int_\Omega \left(e_\varepsilon + \frac{w+u_t}{s}\right)_+^{2^*} dx \ge \int_{\Omega_\varepsilon} \left(e_\varepsilon + \frac{w+u_t}{s}\right)^{2^*} dx \\ &\ge \int_{\Omega_\varepsilon} |e_\varepsilon|^{2^*}\,dx + \int_{\Omega_\varepsilon} \left|\frac{w+u_t}{s}\right|^{2^*} dx \\ &\quad - C\int_{\Omega_\varepsilon} \left(|e_\varepsilon|^{2^*-1}\left|\frac{w+u_t}{s}\right| + |e_\varepsilon|\left|\frac{w+u_t}{s}\right|^{2^*-1}\right) dx \\ &\ge \int_{\Omega_\varepsilon} |e_\varepsilon|^{2^*}\,dx + \int_{\Omega_\varepsilon} \left|\frac{w+u_t}{s}\right|^{2^*} dx - C(\|e_\varepsilon\|_{L^{2^*-1}(\Omega_\varepsilon)}^{2^*-1} + \|e_\varepsilon\|_{L^1(\Omega_\varepsilon)}).\end{aligned}$$

By Lemmas 2.2, 2.3 and 2.4 and (2.31) we obtain

$$\begin{aligned}(2.32)\qquad I(v) &\le \frac12\left(1 - \frac{\lambda}{\lambda_k}\right)r^2 + \frac12 s^2 S^{N/2} - \frac{s^{2^*}}{2^*}S^{N/2} + Cs^{2^*}\varepsilon^{(N-2)/2} \\ &:= \frac12\left(1 - \frac{\lambda}{\lambda_k}\right)r^2 + \Phi_\varepsilon(s).\end{aligned}$$

Applying Lemma 2.6 to $\Phi_\varepsilon(s)$, we obtain

$$I(v) \le \frac12\left(1 - \frac{\lambda}{\lambda_k}\right)r^2 + \frac1N S^{N/2} + O(\varepsilon^{(N-2)/2}).$$

We may choose $r > 0$ such that $I(v) < 0$. This determines r_0.

For $v \in \Gamma_3$ we have $v = w + Re_\varepsilon$, $w \in E^- \cap B_r(0)$ and

$$I(v) \le \frac12\left(1 - \frac{\lambda}{\lambda_k}\right)\|w\|_E^2 + \frac12 R^2 \int_\Omega |\nabla e_\varepsilon|^2\,dx - \frac{1}{2^*}R^{2^*}\int_\Omega \left(e_\varepsilon + \frac{u_t + w}{R}\right)_+^{2^*} dx.$$

By the boundedness of w and u_t, there exists $K > 0$ such that

$$\|w + u_t\|_{L^\infty(\Omega)} \le K.$$

Again since $P_+\phi_\varepsilon(0) \to \infty$ as $\varepsilon \to 0$, there exists $\varepsilon_0 > 0$ (take $\varepsilon_0 < \varepsilon_0'$) such that if $0 < \varepsilon < \varepsilon_0$, $P_+\phi_\varepsilon(0) > 2K$. Given $\varepsilon > 0$, $0 < \varepsilon < \varepsilon_0$, there exist $R_0 = R_0(\varepsilon)$, $\eta = \eta(\varepsilon)$ such that

$$\left|\left\{x \in \Omega : e_\varepsilon + \frac{w+u_t}{R} > 1\right\}\right| \ge \eta > 0$$

for all $R > R_0$. Hence we find $\varepsilon_0, R_0 > 0$ such that if $\varepsilon < \varepsilon_0$, and $R > R_0$, we have $I(v) \le 0$ for $v \in \Gamma_3$. $\square$

LEMMA 2.8.

$$\max_{\overline{Q}} I < \frac{1}{N} S^{N/2}. \tag{2.33}$$

PROOF. Let us fix $\varepsilon < \varepsilon_0$, so that the geometry of the Linking Theorem holds. For $w + se_\varepsilon \in Q$, we have

$$\begin{aligned} I(w+se_\varepsilon) = {}& \frac{1}{2}\int_\Omega (|\nabla w|^2 - \lambda w^2)\,dx \\ &+ \frac{1}{2}s^2 \int_\Omega (|\nabla e_\varepsilon|^2 - \lambda e_\varepsilon^2)\,dx - \frac{1}{2^*}\int_\Omega (w+se_\varepsilon+u_t)_+^{2^*}\,dx. \end{aligned} \tag{2.34}$$

With the same notations and arguments as in the proof of Lemma 2.7, if $s < s_0$ we have

$$I(w+se_\varepsilon) \le \frac{1}{2}s^2 \int_\Omega |\nabla e_\varepsilon|^2\,dx = \frac{1}{2}s^2\delta^2 < \frac{1}{N}S^{N/2}. \tag{2.35}$$

If $s \ge s_0$, using (2.31), we deduce as (2.32) that

$$\begin{aligned} I(w+se_\varepsilon) \le {}& \frac{1}{2}s^2 \int_\Omega (|\nabla e_\varepsilon|^2 - \lambda e_\varepsilon^2)\,dx \\ &- \frac{1}{2^*}s^{2^*} \int_{\Omega_\varepsilon} e_\varepsilon^{2^*}\,dx + Cs^{2^*}\varepsilon^{(N-2)/2} := \Phi_\varepsilon(s). \end{aligned} \tag{2.36}$$

An application of Lemma 2.6 to $\Phi_\varepsilon(s)$ yields

$$I(w+se_\varepsilon) = \frac{1}{N}\bigg[\int_\Omega (|\nabla e_\varepsilon|^2 - \lambda e_\varepsilon^2)\,dx\bigg]^{N/2}\bigg(\int_{\Omega_\varepsilon} e_\varepsilon^{2^*}\,dx\bigg)^{-(N-2)/2} + O(\varepsilon^{(N-2)/2}).$$

Using the estimates in Lemmas 2.2, 2.3 and 2.4 on e_ε we get

$$I(w+se_\varepsilon) \le \frac{1}{N}S^{N/2} - \frac{1}{2}\lambda \begin{cases} O(\varepsilon^2) & \text{if } N \ge 5 \\ O(\varepsilon^2|\log\varepsilon^2|) & \text{if } N = 4 \end{cases} + O(\varepsilon^{(N-2)/2}).$$

If $N > 6$, i.e. $2 < (N-2)/2$, the result follows by choosing $\varepsilon > 0$ sufficiently small. $\square$

PROOF OF THEOREM 1.1. It remains to prove the existence of a second solution of (1.1), i.e. a nontrivial solution of (2.4). Using the Linking Theorem, Lemmas 2.1 and 2.7, there exists $\{v_n\} \subset E$ such that

$$I(v_n) = \frac{1}{2}\int_\Omega (|\nabla v_n|^2 - \lambda v_n^2)\,dx - \frac{1}{2^*}\int_\Omega (v_n+u_t)_+^{2^*}\,dx = c + o(1), \tag{2.37}$$

$$\langle I'(v_n),\phi\rangle = \int_\Omega (\nabla v_n \nabla\phi - \lambda v_n\phi)\,dx - \int_\Omega (v_n+u_t)_+^{2^*-1}\phi\,dx = o(1)\|\phi\|_E, \tag{2.38}$$

for all $\phi \in H_0^1(\Omega)$, where c is the minimax level in the Linking Theorem with $e_\varepsilon = P_+\phi_\varepsilon$, and $\varepsilon < \varepsilon_0$ sufficientlly small in order to have the validity of Lemmas 2.7 and 2.8, and Q as above.

First we prove that $\{v_n\}$ is bounded in E. It follows from (2.37) and (2.38)

$$\frac{1}{N}\int_\Omega (v_n+u_t)_+^{2^*}\,dx - \frac{1}{2}\int_\Omega (v_n+u_t)_+^{2^*-1}u_t\,dx \le c+\varepsilon_n\|v_n\|_E+o(1)$$

where $\varepsilon_n \to \infty$ as $n\to\infty$. It implies

$$\int_\Omega (v_n+u_t)_+^{2^*} \le c+\varepsilon_n\|v_n\|_E+o(1) \tag{2.39}$$

since $u_t \le 0$. Writing $v_n = v_n^+ + v_n^-$, with $v_n^\pm \in E^\pm$ we get from (2.37)–(2.38), using Hölder and Young inequalies that

$$\begin{aligned}\bigg(1-\frac{\lambda}{\lambda_{k+1}}\bigg)\|v_n^+\|_E^2 \le{}& \int_\Omega (v_n+u_t)_+^{2^*-1}v_n^+\,dx+\varepsilon\|v_n^+\| \\ \le{}& \varepsilon\bigg(\int_\Omega |v_n^+|^{2^*}\,dx\bigg)^{2/2^*} \\ &+ C_\varepsilon\bigg(\int_\Omega (v_n+u_t)_+^{2^*}\,dx\bigg)^{2(2^*-1)/2^*} + \varepsilon_n\|v_n^+\|_E \\ \le{}& \varepsilon\bigg(\int_\Omega |v_n^+|^{2^*}\,dx\bigg)^{2/2^*} + C_\varepsilon + \varepsilon_n(\|v_n\|_E^{(N+2)/N}+\|v_n^+\|_E).\end{aligned}$$

So we obtain

$$\|v_n^+\|_E^2 \le C+\varepsilon_n(\|v_n\|_E^{(N+2)/N}+\|v_n^+\|_E).$$

In the same way, we have

$$\|v_n^-\|_E^2 \le C+\varepsilon_n(\|v_n\|_E^{(N+2)/N}+\|v_n^-\|_E).$$

Consequently, $\|v_n\|_E \le C$. Hence we may assume

$$\begin{aligned} &v_n \to v \quad \text{weakly in } H_0^1(\Omega),\\ &v_n \to v \quad \text{in } L^q(\Omega),\ 2\le q<2^*,\\ &v_n \to v \quad \text{a.e. in } \Omega,\end{aligned} \tag{2.40}$$

as $n\to\infty$. It follows that v is a weak solution of

$$-\Delta v = \lambda v + (v+u_t)_+^{2^*-1} \tag{2.41}$$

which implies

$$\int_\Omega (|\nabla v|^2-\lambda v^2)\,dx - \int_\Omega (v+u_t)_+^{2^*}\,dx + \int_\Omega (v+u_t)_+^{2^*-1}u_t\,dx = 0. \tag{2.42}$$

By Brézis–Lieb Lemma [7]

$$\int_\Omega (v_n+u_t)_+^{2^*}\,dx = \int_\Omega (v_n-v)_+^{2^*}\,dx + \int_\Omega (v+u_t)_+^{2^*}\,dx + o(1). \tag{2.43}$$

Hence, using (2.43),

$$I(v_n) = I(v) + \frac{1}{2}\int_\Omega |\nabla(v_n-v)|^2\,dx - \frac{1}{2^*}\int_\Omega (v_n-v)_+^{2^*}\,dx + o(1), \tag{2.44}$$

and similarly, by (2.41),

$$\langle I'(v_n), v_n\rangle = \int_\Omega |\nabla(v_n - v)|^2\,dx - \int_\Omega (v_n - v)_+^{2^*}\,dx - \int_\Omega (v_n - v)_+^{2^*-1} u_t\,dx + o(1). \tag{2.45}$$

Since $\int_\Omega (v_n - v)_+^{2^*-1} u_t\,dx \to 0$ as $n \to \infty$, it yields

$$\int_\Omega |\nabla(v_n - v)|^2\,dx = \int_\Omega (v_n - v)_+^{2^*}\,dx + o(1). \tag{2.46}$$

Let $w_n = v_n - v$ and

$$\lim_{n\to\infty} \int_\Omega |\nabla w_n|^2 = k \ge 0. \tag{2.47}$$

If $k = 0$, then $v_n \to v$ strongly in $H_0^1(\Omega)$ as $n \to \infty$, then $\alpha \le c = I(v)$, v is a nontrivial solution of (2.4).

If $k > 0$, we claim that $v \ne 0$. Indeed, using (2.46) and the Sobolev inequality we obtain

$$\|w_n\|_{H_0^1}^2 \ge S\bigg(\int_\Omega |w_n|^{2^*}\,dx\bigg)^{2/2^*} \ge S\bigg(\int_\Omega (w_n)_+^{2^*}\,dx\bigg)^{2/2^*} \ge S\bigg[\int_\Omega |\nabla w_n|^2\,dx + o(1)\bigg]^{2/2^*} \tag{2.48}$$

which gives

$$k \ge S k^{(N-2)/N} \quad \text{i.e. } k \ge S^{N/2}. \tag{2.49}$$

From (2.44), (2.46) and (2.49), if $v \equiv 0$ we have

$$c + o(1) = \frac{k}{N} \ge \frac{1}{N} S^{N/2}.$$

It contradicts to the statement of Lemma 2.8. Therefore $v \not\equiv 0$. By (2.41) we know v is not negative. □

3. Existence of solutions for the case $\lambda = \lambda_1$

We consider

$$\begin{cases} -\Delta u = \lambda_1 u + u_+^{2^*-1} + f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

A necessary condition for the solvability of (3.1) is given by

$$\int_\Omega f\phi_1\,dx < 0, \tag{3.2}$$

where ϕ_1 is the first eigenfunction of $-\Delta$. Although one expects that (3.2) would be a sufficient condition for the solvability of (3.1), we have not been able to prove it. Indeed, we require in addition that f has small L^2-norm. Let $E^- = \operatorname{span}\{\phi_1\}$ and $E^+ = (E^-)^\perp$. For any $u \in E$ there are $t \in \mathbb{R}$ and $v \in E$ such that $u = t\phi_1 + v$. The functional $I : E \to \mathbb{R}$ associated with equation (3.1) can be written as

$$I(u) = \frac{1}{2}\int_\Omega [|\nabla v|^2 - \lambda_1 v^2]\,dx - \frac{1}{2^*}\int_\Omega (v + t\phi_1)_+^{2^*}\,dx - \int_\Omega f(v + t\phi_1)\,dx,$$

where $u = v + t\phi_1$, $t = \int_\Omega u\phi_1\,dx$.

Lemma 3.1. *For any given $v \in E^+$, the functional I is bounded above in E^-.*

Proof. Given $v \in E^+$, let us define the real-valued function

$$g(t) = I(v + t\phi_1). \tag{3.3}$$

For $t < 0$ we have

$$g(t) \le \frac{1}{2}\int_\Omega [|\nabla v|^2 - \lambda_1 v^2]\,dx + \|f\|_{L^2}\|v\|_{L^2}.$$

For $t > 0$, we claim

$$\lim_{t\to\infty}\left\{\frac{1}{2^*}\int_\Omega (v + t\phi_1)_+^{2^*}\,dx + \int_\Omega f(v + t\phi_1)\,dx\right\} = \infty \tag{3.4}$$

which completes the proof, since g is continuous.

To prove (3.4) we proceed as follows: let $a = \max\{\phi_1(x) : x \in \Omega\}$. Choose $\Omega_0 \subset\subset \Omega$ such that $\phi_1(x) > a/2$ for $x \in \Omega_0$. By Lusin's theorem, given any $\delta > 0$ (choose $\delta = |\Omega_0|/2$) there exists a continuous function $h(x)$ in Ω_0 such that

$$\operatorname{meas}\{x : h(x) \ne v(x)\} < \delta.$$

So the set $G = \{x : h(x) = v(x)\}$ has measure greater than $|\Omega_0| - \delta$. Let $M = \sup\{|v(x)| : x \in G\}$. Then, for $x \in G$ we have

$$\phi_1(x) + \frac{v(x)}{t} \ge \frac{a}{2} - \frac{M}{t} \ge \frac{a}{4}$$

if $t \ge t_0 := 4M/a$. So there is $\eta > 0$ such that

$$\int_\Omega \left(\phi_1 + \frac{v}{t}\right)_+^{2^*} dx \ge \eta \quad \text{for } t \ge t_0.$$

Then the first term in (3.4) is larger than Ct^{2^*} which proves the claim. □

Next we claim that, for each $v \in E^+$, there is a unique $t(v)$ such that

$$g(t(v)) = \max\{g(t) : t \in \mathbb{R}\}. \tag{3.5}$$

At a point t_0 of maximum we have $g'(t_0) = 0$, i.e.

$$g'(t_0) = -\int_\Omega (t_0\phi_1 + v)_+^{2^*-1}\phi_1\,dx - \int_\Omega f\phi_1\,dx = 0. \tag{3.6}$$

If $t \geq t_0$, $g'(t) \leq g'(t_0)$ and if $t \leq t_0$, $g'(t) \geq g'(t_0)$, hence

$$g(t_0) = \max_{\{t \in \mathbb{R}\}} g(t), \text{ i.e. } I(t\phi_1 + v) \leq I(t_0\phi_1 + v).$$

The second derivative of g is given by

$$g''(t) = -\int_\Omega (t\phi_1 + v)_+^{2^*-2} \phi_1^2 \, dx \leq 0$$

which says that g is concave. So the set of maxima is a closed interval, and we show it is a single point. At a point t_0 of maximum $g''(t_0)$ cannot be 0. Indeed, if this were the case, then $(t_0\phi_1 + v)_+ = 0$, which would imply by (3.6) that $\int_\Omega f\phi_1 \, dx = 0$, a contradiction. So g is strictly concave at t_0. This also proves, as a consequence of the Implicit Function Theorem that the mapping

$$v \in E^+ \to t(v) \in \mathbb{R}$$

is continuous and differentiable. Therefore

$$I(t\phi_1 + v) \leq I(t(v)\phi_1 + v) \quad \text{if } t \neq t(v) \tag{3.7}$$

and from (3.6) we have

$$\int_\Omega (v + t(v)\phi_1)_+^{2^*-1} \phi_1 + \int_\Omega f\phi_1 = 0, \quad \text{for all } v \in E^+. \tag{3.8}$$

The relation (3.6) for $v = 0$ gives

$$\int_\Omega (t(0)\phi_1)_+^{2^*-1} \phi_1 \, dx = -\int_\Omega f\phi_1 \, dx \tag{3.9}$$

and the function $g(t)$ in this case is

$$-\frac{1}{2^*} \int_\Omega (t\phi_1)_+^{2^*} \, dx - t \int_\Omega f\phi_1 \, dx \tag{3.10}$$

which shows that $t(0)$ has to be greater than 0. So the relation (3.9) can be written as

$$t(0)^{2^*-1} \int_\Omega \phi_1^{2^*} \, dx = -\int_\Omega f\phi_1 \, dx. \tag{3.11}$$

Let us introduce the notations

$$A = -\int_\Omega f\phi_1 \, dx, \quad B = \int_\Omega \phi_1^{2^*} \, dx. \tag{3.12}$$

Our next step is to show that the functional $F : E^+ \to \mathbb{R}$ given by

$$F(v) = I(v + t(v)\phi_1)$$

has a minimum in the interior of certain ball B_ρ centered at the origin.

It is easy to see that

$$F(0) = \frac{N+2}{2N} \frac{A^{2N/(N+2)}}{B^{(N-2)/(N+2)}}, \tag{3.13}$$

and next we estimate $F(v)$:

$$F(v) = \frac{1}{2}\int_\Omega [|\nabla v|^2 - \lambda_1 v^2]\,dx - \frac{1}{2^*}\int_\Omega (v+t(v)\phi_1)_+^{2^*}\,dx - \int_\Omega f(v+t(v)\phi_1)\,dx. \tag{3.14}$$

Let

$$M_1 = \frac{1}{N+1}\lambda_2^{-N/4} S^{N/4}\left(\frac{N}{N+2}\right)^{(N-2)/4}(\lambda_2-\lambda_1)^{(N+2)/4}, \tag{3.15}$$

$$M_2 = \min\left\{\left(\frac{2}{N+2}\right)^{(N+2)/2N} S^{(N+2)/4}, \left(\frac{2}{N+2}\right)^{(N+2)/2N}\|\phi_1\|_{2^*}\left[\frac{N}{N+2}\left(1-\frac{\lambda_1}{\lambda_2}\right)S\right]^{(N+2)/4}\right\}. \tag{3.16}$$

In addition to (3.2), we suppose that f satisfies

$$\|f\|_2 \le M_1 \quad \text{and} \quad -\int_\Omega f\phi_1\,dx < M_2. \tag{3.17}$$

Lemma 3.2. *Suppose* (3.2) *and* (3.17), *there is an* $\alpha > 0$ *such that*

$$F(v) \ge \alpha > F(0) \tag{3.18}$$

provided that $\|v\|_E = \rho_0$ *with* $\rho_0 = [\frac{N}{N+2}(1-\frac{\lambda_1}{\lambda_2})]^{(N-2)/4}S^{N/4}$.

Proof. It follows from (3.6) and the inequality

$$\int_\Omega |\nabla v|^2\,dx \ge \lambda_2\int_\Omega v^2\,dx, \quad \text{for all } v\in E^+$$

that

$$\begin{aligned} F(v) \ge I(v) &= \frac{1}{2}\int_\Omega (|\nabla v|^2 - \lambda_1 v^2)\,dx - \frac{1}{2^*}\int_\Omega v_+^{2^*}\,dx - \int_\Omega fv\,dx \\ &\ge \frac{1}{2}\left(1-\frac{\lambda_1}{\lambda_2}\right)\int_\Omega |\nabla v|^2\,dx - \frac{1}{2^*}\int_\Omega |v|^{2^*}\,dx - \|f\|_2\|v\|_2. \end{aligned} \tag{3.19}$$

Using Sobolev inequality and (3.19) we obtain

$$F(v) \ge \frac{1}{2}\left(1-\frac{\lambda_1}{\lambda_2}\right)\rho - \frac{1}{2^*}S^{-N/(N-2)}\rho^{2^*} - \|f\|_2\lambda_2^{-1/2}\rho, \tag{3.20}$$

where S is the best Sobolev constant and $\rho = (\int_\Omega |\nabla v|^2\,dx)^{1/2}$. Consider the real function

$$k(\rho) = \frac{1}{2}a\rho^2 - \frac{1}{2^*}b\rho^{2^*} - c\rho := \rho j(\rho).$$

The maximum point ρ of $j(\rho)$ on $\mathbb{R}_+$ satisfies

$$j'(\rho_0) = \frac{1}{2}a - \frac{2^*-1}{2^*}b\rho_0^{2^*-2} = 0.$$

Then we have

$$\rho_0 = \left[\frac{N}{N+2}\left(1-\frac{\lambda_1}{\lambda_2}\right)\right]^{(N-2)/4} S^{N/4}$$

and

$$k(\rho_0) = \rho_0\left[\frac{2}{N+2}\left(\frac{N}{(N+2)b}\right)^{(N-2)/4} a^{(N+2)/4} - c\right]. \tag{3.21}$$

With $a = 1-\lambda_1/\lambda_2$, $b = S^{-N/(N-2)}$ and $c = \|f\|_2\lambda_2^{-1/2}$ in (3.21) and by the assumption (3.17) we obtain

$$F(v) \geq \frac{\rho_0}{N+2}\left[\frac{N}{(N+2)b}\right]^{(N-2)/4} a^{(N+2)/4} \tag{3.22}$$

if $\|v\|_E = \rho_0$. (3.22) and (3.17) imply $F(v) > F(0)$ provided that $\|v\|_E = \rho_0$. The proof is complete. □

It follows from (3.17) that

$$F(0) < \frac{1}{N}S^{N/2}. \tag{3.23}$$

We consider the problem

$$m := \min\{F(v) : v \in B_{\rho_0}\}. \tag{3.24}$$

Lemma 3.3. *Problem* (3.1) *has a nontrivial solution* $v_0 \in B_{\rho_0}$.

Proof. By (3.23) we have

$$m < \frac{1}{N}S^{N/2}. \tag{3.25}$$

Let $\{v_n\}$ be a minimizing sequence of (3.24). Since $\|v_n\|_E \leq \rho_0$ we may assume

$$\begin{aligned} v_n &\to v_0 \quad \text{weakly in } E,\\ v_n &\to v_0 \quad \text{in } L^q(\Omega),\ 2 \leq q < 2^*,\\ v_n &\to v_0 \quad \text{a.e. in } \Omega, \end{aligned} \tag{3.26}$$

as $n \to \infty$. The weak continuity of norm gives

$$\|v_0\|_E \leq \lim_{n\to\infty} \|v_n\|_E \leq \rho_0. \tag{3.27}$$

By the Ekeland's variational principle, we may assume that

$$F(v_n) \to m, \quad F'(v_n) \to 0 \tag{3.28}$$

as $n \to \infty$. Because of

$$F'(v_n) \to 0 \Leftrightarrow J'(v_n + t(v_n)\phi_1) \to 0 \tag{3.29}$$

as $n \to \infty$, we have

$$\tag{3.30} \frac{1}{2}\int_\Omega (|\nabla v_n|^2 - \lambda_1 v_n^2)\,dx - \frac{1}{2^*}\int_\Omega (v + t(v_n)\phi_1)_+^{2^*}\,dx - \int_\Omega f(v_n + t(v_n)\phi_1)\,dx = m + o(1)$$

and

$$\tag{3.31} \int_\Omega (|\nabla v_n|^2 - \lambda_1 v_n^2)\,dx - \int_\Omega (v_n + t(v_n)\phi_1)_+^{2^*-1} v_n\,dx - \int_\Omega f v_n\,dx = o(1).$$

By the weak convergence we know that v_0 satisfies

$$\tag{3.32} -\Delta v = \lambda_1 v + (v + t(v)\phi_1)_+^{2^*-1} + f,$$

and then

$$\tag{3.33} \int_\Omega (|\nabla v_0|^2 - \lambda_1 v_0^2 - (v_0 + t(v_0)\phi_1)_+^{2^*-1} v_0 - f v_0)\,dx = 0,$$

$$\tag{3.34} \int_\Omega [(v_0 + t(v_0)\phi_1)_+^{2^*-1}\phi_1 + f\phi_1]\,dx = 0.$$

The proof will be complete if we may show $v_0 \not\equiv 0$. First we claim that

$$\tag{3.35} \lim_{n\to\infty} t(v_n) = t(v_0).$$

If not, we would have $\lim_n t(v_n) = t_1 \neq t(v_0)$. By (3.6)

$$\int_\Omega (v_n + t(v_n)\phi_1)_+^{2^*-1}\phi_1\,dx = -\int_\Omega f\phi_1\,dx = \int_\Omega (v_0 + t(v_0)\phi_1)_+^{2^*-1}\phi_1\,dx,$$

it follows

$$\int_\Omega (v_0 + t_1\phi_1)_+^{2^*-1}\phi_1\,dx = \int_\Omega (v_0 + t(v_0)\phi_1)_+^{2^*-1}\phi_1\,dx$$

giving a contradiction. Letting $w_n = v_n - v_0$. By (3.30), (3.31) and Brézis–Lieb Lemma, we obtain

$$\tag{3.36} \frac{1}{2}\int_\Omega |\nabla w_n|^2\,dx - \frac{1}{2^*}\int_\Omega (w_n)_+^{2^*}\,dx + \frac{1}{2}\int_\Omega (|\nabla v_0|^2 - \lambda_1 v_0^2)\,dx - \frac{1}{2^*}\int_\Omega (v_0 + t(v_0)\phi_1)_+^{2^*}\,dx - \int_\Omega f(v_0 + t(v_0)\phi_1)\,dx = m + o(1),$$

i.e.

$$\tag{3.37} F(v_0) + \frac{1}{2}\int_\Omega |\nabla w_n|^2\,dx - \frac{1}{2^*}\int_\Omega (w_n)_+^{2^*}\,dx = m + o(1).$$

Similarly by (3.31), (3.34) and Brézis–Lieb Lemma we deduce

$$\int_\Omega |\nabla w_n|^2\,dx - \int_\Omega (w_n)_+^{2^*}\,dx - \int_\Omega (v_0 + t(v_0)\phi_1)_+^{2^*}\,dx + \int_\Omega (|\nabla v_0|^2 - \lambda_1 v_0^2)\,dx - \int_\Omega f(v_0 + t(v_0)\phi_1)\,dx = o(1),$$

namely

$$\int_\Omega |\nabla w_n|^2\,dx - \int_\Omega (w_n)_+^{2^*}\,dx = o(1). \tag{3.38}$$

Let $\lim_{n\to\infty}\int_\Omega |\nabla w_n|^2\,dx = k \geq 0$. If $k = 0$, we have done. If $k > 0$, by the Sobolev inequality

$$\int_\Omega |\nabla w_n|^2\,dx \geq S\bigg(\int_\Omega (w_n)_+^{2^*}\,dx\bigg)^{2/2^*}. \tag{3.39}$$

Taking the limit in (3.39) we obtain by (3.38) and (3.39) that

$$k \geq Sk^{(N-2)/N},$$

i.e.

$$k \geq S^{N/2}. \tag{3.40}$$

It yields by (3.37), (3.39) and (3.40)

$$\frac{1}{N}S^{N/2} > m \geq F(v_0) + \frac{1}{N}k \geq F(v_0) + \frac{1}{N}S^{N/2}.$$

So $F(v_0) < 0$ and $v_0 \not\equiv 0$. Since $F(v) \geq \alpha > 0$ if $\|v\|_E = \rho_0$, we have $v_0 \in B_{\rho_0}$. The proof is complete. □

4. Bifurcations at $\lambda = \lambda_k$

In this section we discuss the bifurcation of the set of solutions of (1.1). Let $u_t(\lambda) = u_t$ be the negative solution obtained in Section 2. If $f = t\phi_1 + h$ and $h \in \ker(-\Delta - \lambda)^\perp$, $u_t(\lambda)$ is well defined for all $\lambda \neq \lambda_1$. In the case $\lambda = \lambda_k$, $k \neq 1$, the set of solutions of (1.1) bifurcating from $(\lambda_k, u_t(\lambda_k))$ is equivalent to the set of solutions of (2.4) bifurcating from $(\lambda_k, 0)$. Let

$$E^- = \operatorname{span}\{\phi_1, \ldots, \phi_k\}, \quad E^+ = (E^-)^\perp.$$

Now we state a bifurcation result.

Proposition 4.1. *Every eigenvalue λ_k of $-\Delta$ gives rise to a bifurcation point of* $(\lambda_k, 0)$ *of* (2.4). *As a result, we obtain Theorem* 1.3.

Proof. The conclusion follows from an abstract bifurcation theorem due to Böhme [5] and Marino [21], see also Theorem 11.4 in [24]. Let $\chi(\xi) \in C^\infty(\mathbb{R},\mathbb{R})$ satisfy $\chi(\xi) = 1$ for $|\xi| \leq 1$, $\chi(\xi) = 0$ and $|\xi| \geq 2$, and $0 \leq \chi(\xi) \leq 1$ for all ξ. Define

$$g(\lambda,\xi) = \chi(\xi)(\xi + u_t(\lambda))_+^{2^*-1} + (1 - \chi(\xi)).$$

Then $g \in C^1$, and $g(\lambda,\xi) = o(|\xi|)$ for λ bounded. Set

$$\Phi(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega G(\lambda, v)\,dx$$

with $u \in E := W_0^{1,2}(\Omega)$, where $G(\lambda, v) = \int_0^v g(\lambda, t)\,dt$. It is standard to show that $\Phi \in C^2$. A critical point u of Φ on the manifold $M := \{u \in E : \int_\Omega |u|^2\,dx = r^2\}$ is a weak solution of

$$-\Delta u - g(\lambda, u) = \gamma u$$

for some Lagrange multiplier γ. Define the operator L by

$$(Lv, \phi) = \int_\Omega \nabla v \nabla \phi\,dx$$

and H by

$$H(v)\phi = \int_\Omega g(\lambda, v)\phi\,dx$$

for $\phi \in E$. For any ν satisfies $2 < \nu < 2^*$ and $\omega := \{x \in \Omega : v(x) \geq 2\}$ with $v \in E$, we have

$$\int_\Omega |v|^\nu\,dx \geq 2^\nu \operatorname{meas} \omega.$$

Hence

$$| H(v)\phi | \leq \int_{\Omega/\omega} |v|^{2^*-1}|\phi|\,dx + \int_\omega |\phi|\,dx \leq C\|v\|^\nu \|\phi\|_E.$$

It concludes

$$\|H(v)\| = o(\|v\|).$$

So by Theorem 11.4 in [24], each eigenvalue of $-\Delta$ provides a bifurcation point of

$$-\Delta v - g(\lambda, v) = \lambda v. \tag{4.1}$$

Since $g(\lambda, v) = o(|v|)$ and λ is bounded, it follows from (4.1) that

$$\|v\|_E \leq C\|v\|_{L^2(\Omega)} = Cr.$$

Arguments from elliptic regularity theory [6] show if r is small enough,

$$\|v\|_{L^\infty(\Omega)} < 1 \quad \text{and} \quad g(\lambda, v) = (v + u_t(\lambda))_+^{2^*-1}.$$

The proof is complete. □

Next, we show that the bifurcation branch bends locally to the left.

Proposition 4.2. *If $(\lambda, v(\lambda)), v(\lambda) \neq 0$, is a solution of* (2.4) *such that $\lambda \to \lambda_k$, $k \neq 1$, $v(\lambda) \to 0$, then $\lambda < \lambda_k$. Consequently, if $h \in \ker(-\Delta - \lambda_k)^\perp$ and $(\lambda, u(\lambda)), u(\lambda) \neq 0$, is a solution of* (1.1) *such that $\lambda \to \lambda_k$, $k \neq 1$ and $u(\lambda) \to u_t(\lambda_k)$, then $\lambda < \lambda_k$.*

Proof. Let $u = v + w$ be a solution of (2.4) with $v \in E^-$ and $w \in E^+$. Multiplying (2.4) by $w - v$ and integrating by part, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla w|^2 - |\nabla v|^2)\,dx &= \int_{\mathbb{R}^N} [\lambda u + (u + u_t(\lambda))_+^{2^*-1}]\,dx \\ &= \int_{\mathbb{R}^N} [\lambda(w^2 - v^2) + (v + w + u_t(\lambda))_+^{2^*-1}]\,dx.\end{aligned} \tag{4.2}$$

It follows

$$(4.3)\quad \left(1-\frac{\lambda}{\lambda_{k+1}}\right)\int_{\mathbb{R}^N}|\nabla w|^2\,dx-\left(1-\frac{\lambda}{\lambda_k}\right)\int_{\mathbb{R}^N}|\nabla v|^2\,dx \le \int_{\mathbb{R}^N}(v+w+u_t(\lambda))_+^{2^*-1}(w-v)\,dx.$$

By the convexity of the function $(v+w+u_t(\lambda))_+^{2^*-1}$ and since u_t is negative

$$(4.4)\quad \begin{aligned}\int_{\mathbb{R}^N}&(v+w+u_t(\lambda))_+^{2^*-1}(w-v)\,dx\\ &=\int_{\mathbb{R}^N}(v+w+u_t(\lambda))_+^{2^*-1}(2w-u)\,dx\\ &\le\int_{\mathbb{R}^N}(2w+u_t(\lambda))_+^{2^*}\,dx-\int_{\mathbb{R}^N}(u+u_t(\lambda))_+^{2^*}\,dx\\ &\le\int_{\mathbb{R}^N}(2w+u_t(\lambda))_+^{2^*}\,dx\le\int_{\mathbb{R}^N}|2w|^{2^*}\,dx\le C\|w\|_E^{2^*}.\end{aligned}$$

(4.3) and (4.4) imply

$$(4.5)\quad \left[\left(1-\frac{\lambda}{\lambda_{k+1}}\right)-C\|w\|_E^{2^*-2}\right]\|w\|_E^2-\left(1-\frac{\lambda}{\lambda_k}\right)\|v\|_E^2\le 0.$$

Suppose by contradiction that $\lambda\ge\lambda_k$. Since $\lambda/\lambda_k-1>0$ and $u=v+w\ne 0$ we must have $w\ne 0$. Hence

$$(4.6)\quad \left(1-\frac{\lambda}{\lambda_{k+1}}\right)\le C\|w\|_E^{2^*-2}\le C\|u\|_E^{2^*-2}.$$

It yields a contradition when we let $\lambda\to\lambda_k$. □

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