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BIFURCATION OF PERIODIC SOLUTIONS IN SYMMETRIC MODELS OF SUSPENSION BRIDGES

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Abstract. We consider a nonlinear model for time-periodic oscillations of a suspension bridge. Under some additional restrictive assumptions we describe our model by a standard bifurcation scheme which allows us to use global bifurcation theorems and make some new conclusions.

1. Introduction

In this paper we try to enrich the known facts concerning the theory of time-periodic oscillations of suspension bridges. Before we formulate exactly the problem we work with, we would like to explain some circumstances and facts which motivated us.

As a starting point for our thoughts we used the model of Lazer and McKenna [8] who described suspension bridge as a one-dimensional bending beam with simply supported ends, suspended by nonlinear cables:

$$\begin{gathered} u_{tt} + \alpha^2 u_{xxxx} + \beta u_t + bu^+ = W(x) + \varepsilon f(x,t), \\ (x,t) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(\pm\frac{\pi}{2}, t\right) = u_{xx}\left(\pm\frac{\pi}{2}, t\right) = 0, \quad u(x,t) = u(x,t+2\pi). \end{gathered} \tag{1}$$

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This model says that the displacement $u(x,t)$ of the roadbed (measured as positive in the downward direction) is influenced by the weight $W(x)$ of the roadbed, by some external forces $\varepsilon f(x,t)$, and by the presence of supporting cable-stays which act as one-sided springs: they obey Hooke's law with the spring constant b if they are stretched, but they have no influence if they are compressed. Here $u^+(x,t)=\max\{u(x,t),0\}$, α^2 and β are the constants coming from the elasticity and the damping, respectively.

In spite of the fact that this description neglects the torsional motion and omits the presence of the main cable and the side parts with towers, it is rather realistic, as can be seen from several numerical experiments (see e.g. [6], [8]).

As for the results concerning model (1) without other simplifications, we can cite the work of Berkovits, Drábek, Leinfelder, Mustonen and Tajčová [1] who proved that under the assumption that the external force $\varepsilon f(x,t)$ is sufficiently small and the damping term βu_t is present the equation keeps the linear character and has a unique solution which represents small oscillations around the equilibrium. A little bit different result can be found in paper [11] by Tajčová where the existence of a unique solution is proved for an arbitrary right hand side but with rather restrictive assumptions on the bridge parameters: $b<\min\{\alpha^2,\beta\}$. In other words, this says that the stiffness of the cable-stays must be small with respect to other parameters.

Other, perhaps more interesting results, can be obtained after some additional simplifications of the model. If we neglect the damping term, i.e. put $\beta=0$, add the symmetry conditions and "normalize" the problem in some sense, we obtain the following model

$$
\begin{gathered}
u_{tt}+u_{xxxx}+bu^+=1+\varepsilon f(x,t) \quad \text{in } \left(-\frac{\pi}{2},\frac{\pi}{2}\right)\times\mathbb{R},\\
u\left(\pm\frac{\pi}{2},t\right)=u_{xx}\left(\pm\frac{\pi}{2},t\right)=0,\\
u(x,t)=u(-x,t)=u(x,-t)=u(x,t+\pi).
\end{gathered}
\tag{2}
$$

This description of a suspension bridge was used by McKenna and Walter in [9] and they showed that if the parameter b crosses a certain eigenvalue of a related eigenvalue problem, an additional solution appears. In particular, they proved that for $-1<b<3$, problem (2) has a unique solution, however for $3<b<15$ and ε small enough another solution exists.

This result was extended at first by Choi, Jung and McKenna [4] who obtained the existence of at least three solutions for $3<b<15$ by a variational reduction method, and then by Humphreys and McKenna [7] who showed that for $15<b<15+\eta$, $\eta>0$, at least four solutions exist. Moreover, additional solutions tend to have large amplitudes.

These results hint that the number of solutions could increase with respect to the number of crossed eigenvalues. That is why we decided to formulate the problem (2) "in the language of bifurcation theory" and to explain this phenomenon from this point of view. Actually, our feeling was encouraged by the existence and multiplicity result for another, more simplified model.

In fact, Lazer and McKenna [8] suggested considering the right hand side in a special form $W(x) + \varepsilon f(x,t) = \cos x + \varepsilon f(t) \cos x$ and expected the solution to have a similar character $u(x,t) = y(t) \cos x$. If we put these relations into equation (2), we obtain the following ODE model

$$\begin{gathered} y'' + y + by^+ = 1 + \varepsilon f(t), \\ y(t) = y(-t) = y(t+\pi). \end{gathered} \tag{3}$$

In [8] we can find a theorem which, indeed, says that the number of solutions increases as b crosses the eigenvalues corresponding to the linear part of equation (3). Unfortunately, the solution set is not specified in more details.

In particular, it follows from our results that the multiple solutions in (2) and (3) exist not because of the perturbation terms $\varepsilon f(x,t)$ and $\varepsilon f(t)$, respectively, but because of the absence of the damping term βu_t.

Main results. As for (3) with $\varepsilon = 0$ our result is really sharp. Indeed, we show that there is the sequence $b_m = 4m^2 - 1$, $m \in N \cup \{0\}$, such that (3) with $\varepsilon = 0$ has exactly $2m+1$ solutions if $b \in (b_m, b_{m+1})$. Moreover, the set of all solutions is described in a rather explicit form (see Theorem 3.1, Corollary 3.1 and Figure 2 for details).

Concerning (2) with $\varepsilon = 0$, our results are weaker. Roughly speaking, we can prove that the multiple solutions exist for some values of $b \geq 3$ (see Theorems 2.1 and Corollary 2.1). Also in this case we provide some qualitative information about the solution set (see Theorem 2.2).

2. PDE-problem

We study the following problem

$$\begin{gathered} u_{tt} + u_{xxxx} + bu^+ = 1 \quad \text{in } \Big(-\frac{\pi}{2}, \frac{\pi}{2} \Big) \times \mathbb{R}, \\ u\Big(-\frac{\pi}{2}, t \Big) = u\Big(\frac{\pi}{2}, t \Big) = u_{xx}\Big(-\frac{\pi}{2}, t \Big) = u_{xx}\Big(\frac{\pi}{2}, t \Big) = 0, \\ u(x,t) = u(-x,t) = u(x,-t) = u(x,t+\pi). \end{gathered} \tag{4}$$

The last conditions in equations (4) say that we are looking for even solutions in x and t, and π-periodic in t.

Let us denote by Ω the domain $(-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ and let $\mathcal{D}$ stand for all C^∞-functions $\psi : [-\pi/2, \pi/2] \times \mathbb{R} \to \mathbb{R}$ satisfying the conditions from (4).

Set $H := \{u \in L^2(\Omega) :\ u \text{ even in } x \text{ and } t\}$ with a standard L^2-norm $\|\cdot\|$ and standard inner product $\langle\,\cdot\,,\,\cdot\,\rangle$, and consider a nonlinear function $f = f(u,x,t) : \mathbb{R} \times (-\pi/2,\pi/2) \times \mathbb{R} \to \mathbb{R}$ such that the following implication holds true for the restrictions of u and f (denoted again by u and f, respectively):

$$u \in H \Rightarrow f(u,x,t) \in H.$$

A function $u : (-\pi/2,\pi/2)\times\mathbb{R} \to \mathbb{R}$ is then called *a weak solution* of the problem

$$\begin{gathered} u_{tt} + u_{xxxx} = f(u,x,t) \quad \text{in } \Big(-\frac{\pi}{2},\frac{\pi}{2}\Big) \times \mathbb{R}, \\ u\Big(-\frac{\pi}{2},t\Big) = u\Big(\frac{\pi}{2},t\Big) = u_{xx}\Big(-\frac{\pi}{2},t\Big) = u_{xx}\Big(\frac{\pi}{2},t\Big) = 0, \\ u(x,t) = u(-x,t) = u(x,-t) = u(x,t+\pi), \end{gathered}$$

if and only if

$$\int_\Omega u(\psi_{tt} + \psi_{xxxx}) = \int_\Omega f(u,\,\cdot\,,\,\cdot\,)\psi \quad \text{for all } \psi \in \mathcal{D},$$

and the restriction of u belongs to H. Note that $1 - bu^+ \in H$ for any $u \in H$. Hence the weak solution of problem (4) is well defined.

The set of functions $\{\varphi_{mn}\}$, $m,n = 0,1,\ldots$, defined by

$$\begin{aligned} \varphi_{mn} &= \frac{2}{\pi}\cos 2mt \cos(2n+1)x, \quad && m > 0,\ n \ge 0, \\ \varphi_{0n} &= \frac{\sqrt{2}}{\pi}\cos(2n+1)x, && n \ge 0, \end{aligned}$$

forms an orthonormal basis in the Hilbert space H. Each $u \in H$ has a representation

$$u = \sum_{m=0}^\infty \sum_{n=0}^\infty u_{mn}\varphi_{mn},$$

with $u_{mn} = \langle u, \varphi_{mn}\rangle$. The *abstract realization* of the *beam operator* $u \mapsto u_{tt} + u_{xxxx}$ with the boundary conditions from (4) is then the linear operator $L : \operatorname{dom}(L) \subset H \to H$ defined by

$$Lu = \sum_{m=0}^\infty \sum_{n=0}^\infty \big[(2n+1)^4 - 4m^2\big]\, u_{mn}\varphi_{mn},$$

where $\operatorname{dom}(L) = \{u \in H :\ \sum_{m,n}\big[(2n+1)^4 - 4m^2\big]^2 |u_{mn}|^2 < \infty\}$. Then L is a linear densely defined and symmetric operator. A function $u \in H$ is then a weak solution of problem (4) if and only if

$$Lu + bu^+ = 1 \quad \text{with } u \in \operatorname{dom}(L). \tag{5}$$

Using Fourier representations one can see that L is also closed and selfadjoint operator onto H and, moreover, $L^{-1} : H \to H$ is well defined and compact operator. The spectrum of L, $\sigma(L)$, consists of eigenvalues $\lambda_{mn} = (2n+1)^4 - 4m^2$ with the corresponding eigenfunctions φ_{mn} (see [13], [2]). The eigenvalues can be ordered into an increasing sequence

$$\ldots < \lambda_{30} = -35 < \lambda_{51} = -19 < \lambda_{20} = -15 < \lambda_{10} = -3 < \lambda_{00} = 1 < \ldots,$$

and it can be seen that e.g. eigenvalues belonging to the interval $[-35, 1]$ are simple.

Let us consider $\lambda \in \mathbb{R}$, $\lambda \notin \sigma(L)$. Then the operator $\lambda I - L : H \to H$ is invertible, the inverse $(\lambda I - L)^{-1}$ is linear, compact and using Fourier representation we can estimate its operator norm as follows

$$\|(\lambda I - L)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, \sigma(L))} \tag{6}$$

(see [12]).

Let $\widetilde{H}$ denote all functions from H which are independent of t and $\widetilde{L}$ stand for L restricted on $\widetilde{H}$. Then $\sigma(\widetilde{L}) = \{\lambda_{0n} : \ n = 0, 1, \ldots\}$, $\widetilde{L}$ is a closed and selfadjoint operator, $\widetilde{L}^{-1}$ is compact, and for $\lambda \in \mathbb{R}$, $\lambda \notin \sigma(\widetilde{L})$, $(\lambda I - \widetilde{L})^{-1} : \widetilde{H} \to \widetilde{H}$ is linear, compact and the estimate

$$\|(\lambda I - \widetilde{L})^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, \sigma(\widetilde{L}))} \tag{7}$$

holds. In particular, for $\lambda < 1$, we have $\operatorname{dist}(\lambda, \sigma(\widetilde{L})) = 1 - \lambda$.

Our first lemma concerns the solvability of equation (4).

Lemma 2.1. *The condition $b > -1$ is the necessary condition for the solvability of equation* (4).

Proof. It follows from (5) that

$$(Lu, \varphi_{00}) + b(u^+, \varphi_{00}) = (1, \varphi_{00}).$$

If we use the fact that L is self adjoint, $L\varphi_{00} = \varphi_{00}$ and the decomposition $u = u^+ - u^-$, we can transform it to the form

$$(1 + b)(u^+, \varphi_{00}) = (1, \varphi_{00}) + (u^-, \varphi_{00}).$$

Since φ_{00} is strictly positive in Ω, we see that $b > -1$. $\square$

Using the result of Lemma 4 in [9], we formulate the following assertion.

Lemma 2.2. *For $b > -1$, problem (4) has a unique weak stationary solution $u_b(x,t) = y_b(x)$, $(x,t) \in \Omega$, where $y_b = y_b(x)$ is the classical solution of*

$$y^{IV} + by^+ = 1, \quad in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),$$
$$y\left(-\frac{\pi}{2}\right) = y\left(\frac{\pi}{2}\right) = y''\left(-\frac{\pi}{2}\right) = y''\left(\frac{\pi}{2}\right) = 0.$$

Moreover, $y_b > 0$ for $x \in (-\pi/2, \pi/2)$ and $y_b'(-\pi/2) > 0$, $y_b'(\pi/2) < 0$. In particular, for b from any compact set $J \subset (-1,\infty)$ there exists $\varepsilon = \varepsilon(J) > 0$ such that $y_b'(-\pi/2) > \varepsilon$, $y_b'(\pi/2) < -\varepsilon$.

Due to Lemma 2.2 the mapping $B : (-1,\infty) \to H$, $B : b \mapsto u_b$ is well defined.

Lemma 2.3. *The mapping $B : (-1,\infty) \to H$ is continuous.*

Proof. Since u_b does not depend on t and $u_b(x,t) > 0$ for any $(x,t) \in \Omega$ and $b > -1$, u_b solves the following operator equation

$$\widetilde{L}u_b + bu_b = 1 \quad \text{in } \widetilde{H}. \tag{8}$$

Let now $b > -1$ be fixed and $b_n \to b$. We can assume, without loss of generality, that

$$b_n + 1 > \frac{b+1}{2}. \tag{9}$$

Let $u_{b_n} = u_{b_n}(x,t)$ be the corresponding stationary solution, i.e.

$$\widetilde{L}u_{b_n} + b_n u_{b_n} = 1. \tag{10}$$

Then we can rewrite (8), (10) in the following way

$$u_b = (bI + \widetilde{L})^{-1}(1), \tag{11}$$
$$u_{b_n} = (b_n I + \widetilde{L})^{-1}(1), \tag{12}$$

or, for example, in the way

$$u_b = (b_n I + \widetilde{L})^{-1}(1 + (b_n - b)u_b).$$

Then, using (7), (9), (11) and (12), we get

$$\begin{aligned}\|u_{b_n} - u_b\| &= \|(b_n I + \widetilde{L})^{-1}(b - b_n)u_b\| \le \|(b_n I + \widetilde{L})^{-1}\| \, |b - b_n| \, \|u_b\| \\ &\le \frac{1}{1+b_n}|b - b_n| \, \|(bI + \widetilde{L})^{-1}(1)\| \le \frac{\pi|b - b_n|}{(1+b_n)(1+b)} < \frac{2\pi|b - b_n|}{(b+1)^2}.\end{aligned}$$

This implies $u_{b_n} \to u_b$ (i.e. strongly) in H. □

Now, we are ready to give an equivalent formulation of (5). For $u \in H$ let $u := u_b + w$, u_b from Lemma 2.2, $w \in H$. Then (5) reads as

$$L(u_b + w) + b(u_b + w) + b(u_b + w)^- = 1,$$

and if we realize that $Lu_b + bu_b = 1$ then we end up with

$$Lw + bw + b(u_b + w)^- = 0. \tag{13}$$

Applying L^{-1} on both sides of (13) we obtain

$$w + bL^{-1}w + bL^{-1}(u_b + w)^- = 0. \tag{14}$$

Lemma 2.4. *The operator $N : (-1,\infty) \times H \to H$ defined by $N(b,w) := bL^{-1}(u_b+w)^-$ is compact. Moreover, given any compact subinterval J of $(-1,\infty)$ the limit*

$$\lim_{\|w\|\to 0} \frac{N(b,w)}{\|w\|} = 0$$

is uniform with respect to $b \in J$.

Proof. The compactness follows from the compactness of L^{-1}, continuity of $B : b \mapsto u_b$ and $u \mapsto u^+$ from H into H.

Let us take sequences $\{b_k\} \subset J$ and $\{w_k\} \subset H$ such that $\|w_k\| \to 0$. We will use the following notation

$$\begin{aligned} u_k &:= \left(\frac{u_{b_k}}{\|w_k\|} + \frac{w_k}{\|w_k\|}\right)^- \qquad (\geq 0),\\ \widetilde{w}_k &:= \frac{w_k}{\|w_k\|},\\ \mathcal{A}_k(x,t) &:= \{(x,t) \in \Omega :\ u_k(x,t) \neq 0\}. \end{aligned}$$

Since u_{b_k} is a stationary solution which is strictly positive for all $(x,t) \in \Omega$ and $\frac{\partial}{\partial x} u_{b_k}(-\pi/2,t) > \varepsilon$, $\frac{\partial}{\partial x} u_{b_k}(\pi/2,t) < -\varepsilon$, $\varepsilon = \varepsilon(J) > 0$ (see Lemma 2.2), we can conclude that

$$\operatorname{meas} \mathcal{A}_k(x,t) \to 0.$$

Moreover, we have

$$\|u_k\| \leq \|\widetilde{w}_k\| = 1,$$

thus the sequence $\{u_k\}$ is bounded and we can pass to a suitable subsequence — let us call it again u_k — such that

$$u_k \rightharpoonup u_0 \quad \text{(i.e. weakly) in } H.$$

This means that

$$\int_\Omega u_k \varphi \to \int_\Omega u_0 \varphi \quad \text{for all } \varphi \in H. \tag{15}$$

In particular, $u_k \geq 0$ a.e. in Ω implies that $u_0 \geq 0$ a.e. in Ω. Let us suppose that $u_0 > 0$ on some set $\mathcal{A}$, $\operatorname{meas} \mathcal{A} = \delta > 0$. Then we can take a subsequence $\{u_j\}$ of $\{u_k\}$ such that

$$\operatorname{meas} \mathcal{A}_j \leq \frac{\delta}{2^{j+1}}.$$

Then

$$\operatorname{meas} \bigcup_j \mathcal{A}_j \leq \sum_j \operatorname{meas} \mathcal{A}_j \leq \frac{1}{2} \operatorname{meas} \mathcal{A} = \frac{\delta}{2},$$

and hence

$$\operatorname{meas} \left(\mathcal{A} \setminus \bigcup \mathcal{A}_j \right) \geq \frac{\delta}{2} > 0.$$

Now, if we take $\varphi = \chi_{\mathcal{A} \setminus \bigcup \mathcal{A}_j}$ (the characteristic function of the set $\mathcal{A} \setminus \bigcup \mathcal{A}_j$) in the relation(15), we obtain

$$\underbrace{\int_\Omega u_j \chi_{\mathcal{A} \setminus \bigcup \mathcal{A}_j}}_{=0} \to \underbrace{\int_\Omega u_0 \chi_{\mathcal{A} \setminus \bigcup \mathcal{A}_j}}_{>0},$$

which is a contradiction. Thus $u_0 = 0$ a.e. on the whole domain Ω and so

$$u_k \rightharpoonup 0 \quad \text{in } H. \tag{16}$$

In considerations above, set now $b_k = b \in J$ fixed. Then since operator L^{-1} is compact and $L^{-1}(0) = 0$, we can pass from a weak to strong convergence and obtain

$$L^{-1}(u_k) \to L^{-1}(0) = 0 \quad \text{in } H,$$

and hence we can conclude that

$$\frac{bL^{-1}(u_{b_k} + w_k)^-}{\|w_k\|} \to 0 \quad \text{in } H \text{ as } \|w_k\| \to 0.$$

It remains to prove that this limit is uniform with respect to $b \in J$. We argue via contradiction. If this is not the case, there would be sequences $\{b_k\} \subset J$, $\{w_k\} \subset H$, $\|w_k\| \to 0$, such that

$$\frac{\|b_k L^{-1}(u_{b_k} + w_k)^-\|}{\|w_k\|} \geq \eta > 0.$$

The compactness of L^{-1} implies that $u_k = (u_{b_k}/\|w_k\| + \widetilde{w}_k)^-$ cannot approach zero weakly, which contradicts (16). $\square$

Due to Lemma 2.4 the operator equation (4) represents the classical bifurcation scheme in H. Moreover, since some of the eigenvalues λ_{mn} of operator L are simple, we can use global Rabinowitz theorem [10], or Dancer theorem [5], and formulate the following assertion.

However, first of all, let us remind Dancer's notation of the bifurcation branches emanating from $(-\lambda_{mn}, 0)$ in the direction of the eigenfunctions $\pm\varphi_{mn}$.

Definition 2.1. Let the space $\mathbb{R} \times H$ be equipped with the norm

$$\|(b, w)\| = (|b|^2 + \|w\|^2)^{1/2}, \quad (b, w) \in \mathbb{R} \times H. \tag{17}$$

Let us denote by

$$S := \overline{\{(b,w) \in \mathbb{R} \times H : \ (b,w) \text{ solves } (14), \ w \not\equiv 0\}}$$

the closure of the set of nontrivial solutions and by C_{mn} its maximal connected subset containing the point $(-\lambda_{mn}, 0)$.

Now, for $\varepsilon \in (0,1)$ let

$$\begin{aligned} K_\varepsilon^+ &:= \{(b,w) \in \mathbb{R} \times H : \ \langle \varphi_{mn}, w \rangle > \varepsilon \|w\|\}, \\ K_\varepsilon^- &:= \{(b,w) \in \mathbb{R} \times H : \ -\langle \varphi_{mn}, w \rangle > \varepsilon \|w\|\}, \end{aligned}$$

and $B_r(b,w)$ be the ball in $\mathbb{R} \times H$ centred at (b,w) with radius r.

Since there exists $r_0 = r_0(\varepsilon) > 0$ such that

$$(S \setminus \{(-\lambda_{mn}, 0)\}) \cap \overline{B_{r_0}(-\lambda_{mn}, 0)} \subset K_\varepsilon^+ \cup K_\varepsilon^-,$$

(see Remark 2.1 below), we can define for all $r \in (0, r_0]$ the following sets

$$\begin{aligned} D_{mn}^+(r) &:= \{(-\lambda_{mn}, 0)\} \cup (S \cap \overline{B_r(-\lambda_{mn}, 0)} \cap K_\varepsilon^+), \\ D_{mn}^-(r) &:= \{(-\lambda_{mn}, 0)\} \cup (S \cap \overline{B_r(-\lambda_{mn}, 0)} \cap K_\varepsilon^-). \end{aligned}$$

Further, let $C_{mn}^+(r)$ denotes the component of $\overline{C_{mn} \setminus D_{mn}^-(r)}$ containing the point $(-\lambda_{mn}, 0)$, and $C_{mn}^-(r)$ denotes the component of $\overline{C_{mn} \setminus D_{mn}^+(r)}$ containing the point $(-\lambda_{mn}, 0)$.

Finally, we can define

$$C_{mn}^+ := \overline{\bigcup_{r \le r_0} C_{mn}^+(r)}, \qquad C_{mn}^- := \overline{\bigcup_{r \le r_0} C_{mn}^-(r)}.$$

Both sets C_{mn}^+ and C_{mn}^- are connected, independent on ε and $C_{mn}^+ \cup C_{mn}^- = C_{mn}$.

We thus have the following global bifurcation result.

Theorem 2.1. *Every $b = -\lambda_{mn}$, where $\lambda_{mn} < 1$ has an odd multiplicity, is a point of global bifurcation of* (14)*, such that there exists a continuum of solutions C_{mn}, $(-\lambda_{mn}, 0) \in C_{mn}$ which is either unbounded in $\mathbb{R} \times H$, or meets another point $(-\lambda, 0)$, where $\lambda_{mn} \neq \lambda \in \sigma(L)$. Moreover,*

$$\mathrm{proj}_{\mathbb{R}} C_{mn} \subset (-1, \infty),$$

where $\mathrm{proj}_{\mathbb{R}} C_{mn} := \{b \in (-1, \infty) : \ (b,w) \in C_{mn}\}$*. In addition, for λ_{mn} simple, C_{mn} contains two subcontinua C_{mn}^+, C_{mn}^- bifurcating from the point $(-\lambda_{mn}, 0)$ in the direction of the corresponding eigenfunctions φ_{mn}, and $-\varphi_{mn}$, respectively. Both continua $C_{mn}^\pm$ are either unbounded in $\mathbb{R} \times H$, or*

$$C_{mn}^+ \cap C_{mn}^- \neq \{(-\lambda_{mn}, 0)\}.$$

Remark 2.1. By the standard argument also every bifurcation point of (14) coincides with a certain characteristic value of $-L^{-1}$, i.e. a certain eigenvalue of $-L$.

Let us investigate the behaviour of $C^{\pm}_{mn}$ more carefully.

Lemma 2.5. *Let $b \in (-1,3)$. Then equation* (5) *has a unique solution $u = u_b$ with u_b defined in Lemma* 2.2.

Proof. Equation (5) can be written in the equivalent form

$$Lu + \varepsilon u + bu^+ = 1 + \varepsilon u,$$

or, choosing ε such that $-\varepsilon \notin \sigma(L)$, as

$$u = (\varepsilon I + L)^{-1}(1 + \varepsilon u - bu^+).$$

If we denote the operator of the right hand side by G, i.e. $G(u) := (\varepsilon I + L)^{-1}(1 + \varepsilon u - bu^+)$, we can make the following estimate

$$\begin{aligned} \|G(u_1) - G(u_2)\| &= \|(\varepsilon I + L)^{-1}((\varepsilon u_1 - bu_1^+) - (\varepsilon u_2 - bu_2^+))\| \\ &\le \|(\varepsilon I + L)^{-1}\| \ \max\{|\varepsilon|, |\varepsilon - b|\} \ \|u_1 - u_2\| \\ &\le \frac{1}{\operatorname{dist}(-\varepsilon, \sigma(L))} \ \max\{|\varepsilon|, |\varepsilon - b|\} \ \|u_1 - u_2\|. \end{aligned}$$

For $b \in (-1,3)$ we can take $\varepsilon = 1$. Then $\operatorname{dist}(-\varepsilon, \sigma(L)) = 2$ and $\max\{|\varepsilon|, |\varepsilon - b|\} < 2$ and hence

$$\|G(u_1) - G(u_2)\| \le K\|u_1 - u_2\|, \quad \text{where } K = \frac{\max\{|\varepsilon|, |\varepsilon - b|\}}{\operatorname{dist}(-\varepsilon, \sigma(L))} < 1.$$

So, we can see that for $b \in (-1,3)$ and $\varepsilon = 1$ operator G is contractive and thus equation (5) must have a unique solution. $\square$

Let us define for $p, r \in \mathbb{N} \cup \{0\}$

$$H^{p,r} := \left\{ u \in H : \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} ((2n+1)^{2p} + (2m)^{2r})|(u, \varphi_{mn})|^2 < \infty \right\}.$$

We endow $H^{p,r}$ with the norm

$$\|u\|_{p,r} = \left(\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} ((2n+1)^{2p} + (2m)^{2r})|(u, \varphi_{mn})|^2 \right)^{1/2}.$$

Then $H^{0,0} = H$. Let $C^{p,r}(\overline{\Omega})$ be the space of all functions $v \in H$ that have continuous derivatives up to order p in x and up to order r in t. We endow this space with the norm

$$\|v\|_{C^{p,r}} = \sum_{0 \le \alpha \le p} \sup_{\overline{\Omega}} |\partial_x^\alpha v(x,t)| + \sum_{0 \le \beta \le r} \sup_{\overline{\Omega}} |\partial_t^\beta v(x,t)|.$$

The following continuous imbedding is a consequence of Lemma 2.2 in [1]:

$$H^{p,r} \hookrightarrow C^{\alpha,\beta},$$

provided

$$\max\left\{\frac{\alpha+1/2}{p}+\frac{1}{2r}, \frac{\beta+1/2}{r}+\frac{1}{2p}\right\}<1,$$

(cf. [13]). In particular, $H^{3,2} \hookrightarrow C^{1,1}$.

LEMMA 2.6. *Assume that* (b_k, w_k) *satisfy* (13), *the sequence* $\{b_k\}$ *is bounded and* $\|w_k\| \to 0$. *Then*

$$\|w_k\|_{C^{1,1}} \to 0.$$

PROOF. Let us realize first that

$$L^{-1} : H^{p,r} \to H^{p+2,r+1} \quad \text{is continuous.} \tag{18}$$

Indeed, for any $u, h \in H$ we can use Fourier representation

$$u = \sum u_{mn}\varphi_{mn}, \quad h = \sum h_{mn}\varphi_{mn},$$

and rewrite the equation $Lu = h$ (i.e. $u = L^{-1}h$) into the equivalent form

$$\sum_{m,n}[(2n+1)^4-(2m)^2]u_{mn}\varphi_{mn} = \sum_{m,n} h_{mn}\varphi_{mn}.$$

Hence we obtain that

$$u_{mn} = \frac{h_{mn}}{(2n+1)^4-(2m)^2},$$

and thus

$$\begin{aligned}
\|u\|^2_{H^{p+2,r+1}} &= \sum_{m,n}[(2n+1)^{2p+4}+(2m)^{2r+2}]|u_{mn}|^2\\
&= \sum_{m,n}\frac{(2n+1)^{2p+4}+(2m)^{2r+2}}{[(2n+1)^4-(2m)^2]^2}|h_{mn}|^2\\
&\le \sum_{m,n}\frac{(2n+1)^4+(2m)^2}{[(2n+1)^4-(2m)^2]^2}[(2n+1)^{2p}+(2m)^{2r}]|h_{mn}|^2.
\end{aligned}$$

And since

$$\begin{aligned}
\frac{(2n+1)^4+(2m)^2}{[(2n+1)^4-(2m)^2]^2} &\le \frac{[(2n+1)^2+2m]^2}{[(2n+1)^2-2m]^2[(2n+1)^2+2m]^2}\\
&\le \frac{1}{\min\limits_{m,n}[(2n+1)^2-2m]^2} = 1,
\end{aligned}$$

we can conclude that

$$\|u\|^2_{H^{p+2,r+1}} \le \|h\|^2_{H^{p,r}}. \tag{19}$$

Now, since (13) is equivalent to

$$w = -bL^{-1}w - bL^{-1}(u_b + w)^-,$$

then $w \in H$ implies $w \in H^{2,1}$. Then $u_b + w \in H^{2,1}$ and so $(u_b + w)^- \in H^{1,1}$ (see [1]). Applying (18) once again we get $w \in H^{3,2} \hookrightarrow C^{1,1}$.

Let $\{b_k\}$ be a bounded sequence and u_{b_k} be stationary solutions of (4) with $b = b_k$. It follows from the proof of Lemma 4 in [9] that $\|u_{b_k}|_{\mathcal{A}_k}\|_{H^{1,1}} \to 0$ if $\operatorname{meas} \mathcal{A}_k \to 0$.

Let now (b_k, w_k) satisfy assumptions of Lemma 2.6, i.e.

$$w_k = -b_k L^{-1} w_k - b_k L^{-1}(u_{b_k} + w_k)^-. \tag{20}$$

Then clearly $\|w_k\| \to 0$ implies $\|(u_{b_k} + w_k)^-\| \to 0$. So we get from (20) and (18) that $\|w_k\|_{H^{2,1}} \to 0$. If $\mathcal{A}_k$ has the same meaning as in the proof of Lemma 2.4, we have $\operatorname{meas}\mathcal{A}_k \to 0$. The comment above yields

$$\|(u_{b_k} + w_k)^-\|_{H^{1,1}} \leq \|u_{b_k}|_{\mathcal{A}_k}\|_{H^{1,1}} + \|w_k\|_{H^{2,1}} \to 0.$$

Applying (20) and (18) again we get

$$\|w_k\|_{H^{3,2}} \to 0.$$

The assertion now follows from the imbedding $H^{3,2} \hookrightarrow C^{1,1}$. $\square$

Lemma 2.7. *Let J be a compact interval in $(-1, 19)$. Then there exists a constant $c = c(J) > 0$ such that for any $b \in J$ we have $\|w\| \leq c$, where w is a solution of* (14).

Proof. Let us suppose that there exists a sequence $\{(b_k, w_k)\}$ of solutions of (14), such that $\|w_k\| \to \infty$ and $b_k \to b$, $b \in J \subset (-1, 19)$. This means

$$w_k + b_k L^{-1} w_k + b_k L^{-1}(u_{b_k} + w_k)^- = 0.$$

If we divide this equation by $\|w_k\|$ and denote $\widetilde{w}_k := w_k/\|w_k\|$, $\|\widetilde{w}_k\| = 1$, we obtain

$$\widetilde{w}_k + b_k L^{-1}\widetilde{w}_k + b_k L^{-1}\left(\frac{u_{b_k}}{\|w_k\|} + \widetilde{w}_k\right)^- = 0.$$

Due to the compactness of operator L^{-1}, passing to the limit results in the relation

$$\widetilde{w}_k \to \widetilde{w} \quad \text{in } H,$$

and

$$\widetilde{w} + bL^{-1}\widetilde{w} + bL^{-1}\widetilde{w}^- = 0,$$

which is equivalent to

$$L\widetilde{w} + b\widetilde{w}^+ = 0.$$

Using the result of [9] (actually, inspecting carefully the assumptions of Lemma 1 in [9] one can see that it holds for all $b \in (-1, 19)$), we can conclude that for

$b \in (-1, 19)$ this equation has only a trivial solution. But this contradicts the fact that $\|\widetilde{w}\| = \|\widetilde{w}_k\| = 1$. $\square$

Theorem 2.2. *Let C_{mn} be the set from Theorem 2.1. Then*

$$\operatorname{proj}_{\mathbb{R}} C_{mn} \subset [3, \infty). \tag{21}$$

In addition,

$$\operatorname{proj}_{\mathbb{R}} C_{1\,0} \supset [3, 15]. \tag{22}$$

Moreover, for any $\lambda_{mn} < 1$ simple, there exists $s = s(\lambda_{mn})$ such that $(b, w) \in C_{mn} \cap B_s(-\lambda_{mn}, 0)$ implies $b = -\lambda_{mn}$ and $w = c\varphi_{mn}$ with some $c \in \mathbb{R}$ small enough.

Proof. The fact (21) follows from Lemma 2.5 and the relation (22) is a consequence of Lemma 2.7. The second part follows from Lemma 2.6 since for $s(\lambda_{mn})$ small enough every $(b, w) \in C_{mn} \cap B_s(-\lambda_{mn}, 0)$ satisfies $(u_b + w)^- = 0$ in Ω. $\square$

The first part of the previous assertion says that the "first" branch C_{10} emanates "to the right" from 3 and any other branch cannot cross the value $b = 3$. The second part of the assertion expresses the fact that every branch C_{mn} near the point $(-\lambda_{mn}, 0)$, where λ_{mn} is simple, consists of two one dimensional continua $C^{\pm}_{mn}$ which contain the positive (and negative) multiples of the corresponding eigenfunction φ_{mn} (see Figure 1 for the possible shape of the bifurcation diagram).

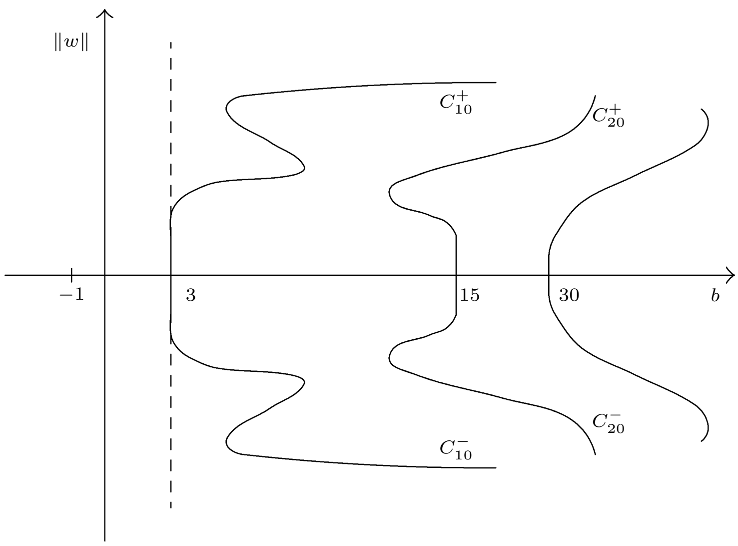

Figure 1. The possible shape of the bifurcation diagram of equation (14).

In particular, combining Lemmas 2.1, 2.5, 2.7 and Theorems 2.1, 2.2 we get the following result.

Corollary 2.1. *Problem* (4) *has no solutions for* $b \le -1$, *one unique* (*positive and stationary*) *solution for* $b \in (-1, 3)$ *and at least two different* (*one positive and stationary and the other one changing sign in* Ω) *solutions for* $b \in (3, 15)$.

Open problem. Unfortunately, we are not able to characterize more the behaviour of the branches $C^{\pm}_{mn}$. In particular, we cannot exclude that C_{mn} is bounded and we have no information about the behaviour in the eigenvalues with an even multiplicity. Although it follows from our result that multiple solutions of (4) occur for some values of parameter $b \ge 3$, an *open problem* consists in proving that this is the case for any $b \ge 3$.

If we want to obtain more information, we can simplify the situation and, instead of the constant right hand side in equation (4), consider the right hand side of the form $\cos x$. This corresponds to the situation when the weight of the suspension bridge is not constant but it is described by the cosinus function.

3. Restriction to ODE

Now, we will consider the problem

$$\begin{gathered} u_{tt} + u_{xxxx} + bu^+ = \cos x \quad \text{in } \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\ u\left(-\frac{\pi}{2}, t\right) = u\left(\frac{\pi}{2}, t\right) = u_{xx}\left(-\frac{\pi}{2}, t\right) = u_{xx}\left(\frac{\pi}{2}, t\right) = 0, \\ u(x,t) = u(-x,t) = u(x,-t) = u(x,t+\pi). \end{gathered} \tag{23}$$

This allows us to suppose that the solution has a similar form as the right hand side, i.e. $u(x,t) = y(t)\cos x$. If we put it into equation (23) and realize that the function $\cos x$ is positive for all $x \in (-\pi/2, \pi/2)$, we can simplify our problem to the following one

$$\begin{gathered} y''(t) + y(t) + by^+(t) = 1, \\ y(t) = y(-t) = y(t+\pi). \end{gathered} \tag{24}$$

Now we can repeat the ideas from Section 2 to get the same results.

Again, all eigenvalues of operator $L : \mathrm{dom}(L) \subset X \to X$, where $X := \{y \in C([-\pi/2, \pi/2]) : \ y \text{ periodic and even}\}$, $\mathrm{dom}(L) := X \cap C^2([-\pi/2, \pi/2])$, $Ly = y'' + y$, are given by

$$\lambda_m = 1 - 4m^2, \quad m \in \mathbb{N} \cup \{0\},$$

and they are all simple. The corresponding eigenfunctions are given by $\varphi_m(t) = \cos 2mt$. Operator L has a compact inverse $L^{-1} : X \to X$ and problem (24) is thus equivalent to the operator equation

$$Ly + by^+ = 1. \tag{25}$$

First of all, this equation is solvable only for $b > -1$ for the same reason as in the case of the partial differential equation.

For $b > -1$ equation (25) has a unique stationary solution $y_b = 1/(b+1)$. So, if we denote $y(t) = y_b + w(t)$, we can again transform our equation into an equivalent form

$$w + bL^{-1}w + bL^{-1}(y_b + w)^- = 0. \tag{26}$$

Note that, for any $b > -1$, we have

$$N(b, w) = 0,$$

for any $w \in X$, $\|w\| \le 1/(b+1)$, where $N(b, w) := bL^{-1}(y_b+w)^-$ and $\|\cdot\|$ denotes the usual maximum norm in X. Hence the operator equation (26) represents again a classical bifurcation scheme in X. However, since the problem is now one dimensional, we can get more precise result.

Let $v \in X$ have a finite number of zero points in $[-\pi/2, \pi/2]$, $v(-\pi/2) \ne 0$. We say that a function $u \in X$ *keeps the nodal properties* of v if the number of zeros of u coincides with that of v and $\operatorname{sgn} v(-\pi/2) = \operatorname{sgn} u(-\pi/2)$.

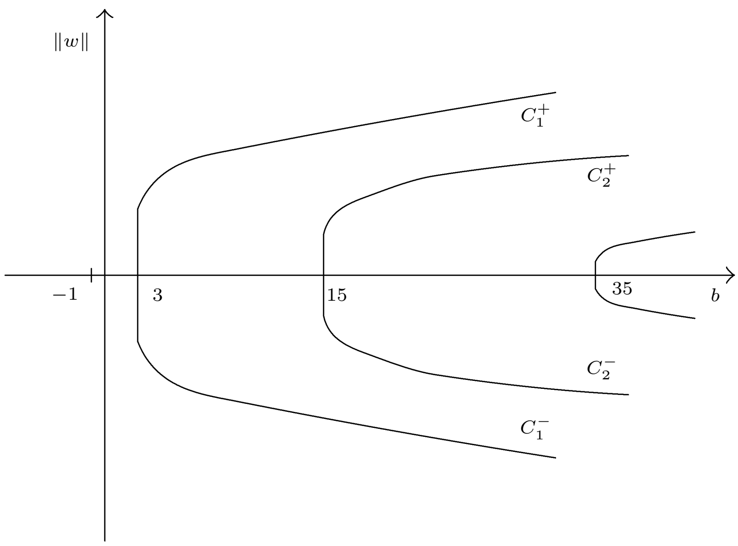

Figure 2. The shape of the bifurcation diagram of equation (26).

Theorem 3.1. *Let $b = -\lambda_m$, $m \geq 1$. Then there exists a continuum of solutions $C_m \subset \mathbb{R} \times X$ of* (26) *such that $(b, 0) \in C_m$. Moreover, $C_m = C_m^+ \cup C_m^-$, $C_m^+ \cap C_m^- = \{(-\lambda_m, 0)\}$, $\mathrm{proj}_{\mathbb{R}} C_m^{\pm} = [-\lambda_m, \infty)$ and for any $m \in \mathbb{N}$ there exists $s = s(m) > 0$ such that $(b, w) \in C_m^{\pm} \cap B_s(-\lambda_m, 0)$ implies $b = -\lambda_m$ and $w = c\varphi_m(t)$ with some $c \in \mathbb{R}$ sufficiently small. Both continua $C_m^{\pm}$ are one-dimensional and for any $m \geq 1$ and $b \in (-\lambda_m, \infty)$ there exists unique couple $w^{(\pm)} \in X$ such that $(b, w^{(\pm)}) \in C_m^{\pm}$ and $w^{(\pm)}$ keeps the nodal properties of $\pm\varphi_m$. In particular, for $m, \widetilde{m} \geq 1$, $m \neq \widetilde{m}$, we have $C_{\widetilde{m}}^+ \cap C_m^{\pm} = \emptyset$, $C_{\widetilde{m}}^- \cap C_m^{\pm} = \emptyset$ (see Figure* 2 *for the shape of the bifurcation diagram).*

Proof. By the same reasoning based on a direct application of Rabinowitz [10] and Dancer [5] theorems as in Section 2, we show that every $-\lambda_m$, $m \geq 1$, is the point of global bifurcation of (26). The fact that in a small neighbourhood of $(-\lambda_m, 0)$ the set C_m is formed by elements $(-\lambda_m, c\varphi_m)$, $|c|$ small, follows directly from the above mentioned property of operator N. In order to investigate further properties of C_m, $C_m^{\pm}$, we need some other lemmas.

Lemma 3.1.

(i) *Problem* (24) *has for $b \in (-1, 3)$ a unique positive stationary solution $y_b = 1/(b+1)$.*

(ii) *The problem*

$$
\begin{gathered}
y'' + y + by^+ = 0,\\
y(t) = y(-t) = y(t + \pi)
\end{gathered}
\tag{27}
$$

has for $b \in (-1, \infty)$ only the trivial solution $y \equiv 0$ in $[-\pi/2, \pi/2]$.

Proof. (i) The first assertion follows from the same argument based on contraction principle as in Lemma 2.5. (ii) Any nontrivial periodic solution of equation (27) has a period $\pi/\sqrt{b+1} + \pi > \pi$ (cf. [9]). □

In particular, it follows from Lemma 3.1(i) that $\mathrm{proj}_{\mathbb{R}} C_m \subset [3, \infty)$.

Lemma 3.2. *Let $J \subset (-1, \infty)$ be a compact set. Then for any $b \in J$, w a solution of* (26), *we have $\|w\| < K$, where $K = K(J)$ depends only on J.*

Proof. Considering a sequence $\{(b_k, w_k)\}$ of solutions of (26) such that $\|w_k\| \to \infty$, $b_k \to b$, $b \in J \subset (-1, \infty)$, we derive as in the proof of Lemma 2.7 that $\widetilde{w} \in X$, $\|\widetilde{w}\| = 1$ solves the equation

$$
L\widetilde{w} + b\widetilde{w}^+ = 0.
$$

This contradicts Lemma 3.1(ii). □

LEMMA 3.3. *Let* $(b, w) \in C_m$ *for some* $m \geq 1$. *Then* $y = y_b + w$ *is a classical solution of equation* (26). *In particular,* $w \in C^1([-\pi/2, \pi/2]) \cap X$. *Moreover, if* $(b_k, w_k) \to (b, w)$ *in* $\mathbb{R} \times X$ *then* $\|w_k - w\|_{C^1} \to 0$.

PROOF. The first part is a consequence of the regularity of the solution of one dimensional boundary value problem with continuous data. The second part follows from (26) and the fact that L^{-1} is continuous from X into $\operatorname{dom}(L)$. □

Assume that for some $m \geq 1$, we have $(-\lambda_{\widetilde{m}}, 0) \in C_m$, $\widetilde{m} \neq m$, $\widetilde{m} \geq 1$, (i.e. the alternative of Rabinowitz [10]), or $C_m^+ \cap C_m^- \neq \{(-\lambda_m, 0)\}$ (the alternative of Dancer [5]). In both cases it follows from Lemma 3.3 that there exists $(b, w) \in C_m$ and $t_0 \in [-\pi/2, \pi/2]$ such that $w(t_0) = w'(t_0) = 0$. Since w solves the equation

$$w'' + (b+1)w + b(y_b + w)^- = 0,$$

the uniqueness theorem for the initial value problem (see e.g. [3]) implies $w \equiv 0$, a contradiction. In particular, it follows from here and [10] and [5] that all $C_m^\pm$ are unbounded and have no joint points besides $\{(-\lambda_m, 0)\} = C_m^+ \cap C_m^-$. Lemma 3.2 implies that these sets are unbounded in b, i.e. $\operatorname{proj}_{\mathbb{R}} C_m^\pm \supset [-\lambda_m, \infty)$.

Now, we will try to find the explicit form of the solutions of (25), which means to solve the equation

$$w'' + (b+1)w + b\left(\frac{1}{b+1} + w\right)^- = 0, \tag{28}$$

with the conditions w even and π-periodic. We know that for any $b > -1$ this equation has a trivial solution $w \equiv 0$, and for $b = -\lambda_m = 4m^2 - 1$, $m \in \mathbb{N} \cup \{0\}$, there is a set of solutions of the form $w = c\varphi_m$, where φ_m is an eigenfunction of L associated with the eigenvalue λ_m and c is an arbitrary real constant such that $c\varphi_m + 1/(b+1) > 0$.

From the previous considerations we know that there exist some other solutions and their number increases with respect to parameter b. These solutions must be of such a form that the term $1/(b+1) + w$ changes sign in the interval $(-\pi/2, \pi/2)$.

The function $w > -1/(b+1)$ is a solution of the equation

$$w'' + (b+1)w = 0,$$

and thus

$$w = w_1 = A \sin t\sqrt{b+1} + B \cos t\sqrt{b+1}, \tag{29}$$

and the function $w < -1/(b+1)$ solves the equation

$$w'' + w = \frac{b}{b+1},$$

and thus

$$w = w_2 = \frac{b}{b+1} + C\sin t + D\cos t. \tag{30}$$

Let us study the first couple of branches $C_1^\pm$, i.e. the continua of solutions emanating from the point $(b,w) = (3,0)$. All the solutions along both these branches must keep the nodal properties of the first eigenfunction $\varphi_1 = \cos 2t$, or $-\varphi_1 = -\cos 2t$, respectively, i.e. they have exactly two zero points in the interval $(-\pi/2, \pi/2)$. Moreover, these solutions are even and π-periodic.

So, each of the two branches is characterized by one of the following conditions

$$\text{(i)} \qquad w(0) = w_1(0), \quad w\bigg(\pm\frac{\pi}{2}\bigg) = w_2\bigg(\pm\frac{\pi}{2}\bigg),$$

$$\text{(ii)} \qquad w(0) = w_2(0), \quad w\bigg(\pm\frac{\pi}{2}\bigg) = w_1\bigg(\pm\frac{\pi}{2}\bigg).$$

Let us investigate the first case in more details. From the fact that the solution must be even and π-periodic, we obtain that

$$w_1(t) = B\cos t\sqrt{b+1}, \qquad w_2(t) = \frac{b}{b+1} + C\sin t,$$

where B and C are real parameters. Since the solution must be of the class C^1, we ask whether there exists a point $t_0 \in (0, \frac{\pi}{2})$ (and due to the symmetry also a point $-t_0$) such that

$$w_1(t_0) = w_2(t_0) = -\frac{1}{b+1}, \qquad w_1'(t_0) = w_2'(t_0),$$

and whether it is unique. These conditions lead to the relations

$$\begin{gathered} B\cos t_0\sqrt{b+1} = -\frac{1}{b+1}, \\ C\sin t_0 = -1, \\ \tan t_0\sqrt{b+1}\ \tan t_0 = -\sqrt{b+1}. \end{gathered}$$

The last equation has for any $b \in \big(4m^2-1, 4(m+1)^2-1\big]$, $m \in \mathbb{N}\cup\{0\}$, exactly m solutions $(t_0)_i \in (0, \pi/2)$. Moreover, for any $i = 1, \dots, m$ we have

$$(t_0)_i \in \left(\frac{\pi}{2\sqrt{b+1}}(2i-1), \frac{\pi}{2\sqrt{b+1}}2i\right).$$

We can see that only the first point $(t_0)_1$ can fulfill the conditions

$$\begin{aligned} w_1(t) &> -\frac{1}{b+1} \quad \text{for all } t \in (0, t_0), \\ w_2(t) &< -\frac{1}{b+1} \quad \text{for all } t \in \bigg(t_0, \frac{\pi}{2}\bigg). \end{aligned}$$

Moreover, the existence of a unique point $t_0 \in (0, \pi/2)$ determines the unique values of parameters B and C. And since $B = w(0) > 0$ and $C + b/(b+1) = w(\pm\pi/2) < 0$, we can conclude that for a given b there exist only one value $w(0)$ and one value $w(\pm\pi/2)$ such that we can construct a smooth symmetric even function $w(t)$ with exactly two zero points in the interval $(-\pi/2, \pi/2)$, and which solves equation (28). It has the following form

$$w(t) = \begin{cases} \dfrac{b}{b+1} + \dfrac{\sin t}{\sin t_0} & \text{for all } t \in [-\pi/2, -t_0]\,, \\ -\dfrac{1}{b+1} \dfrac{\cos\sqrt{b+1}t}{\cos\sqrt{b+1}t_0} & \text{for all } t \in [-t_0, t_0]\,, \\ \dfrac{b}{b+1} - \dfrac{\sin t}{\sin t_0} & \text{for all } t \in [t_0, \pi/2]\,. \end{cases}$$

As for the branch C_1^-, it can be seen that the corresponding solutions $(b, w^{(-)})$ are given by $w^{(-)}(t) = w^{(+)}(t - \pi/2)$, where $w^{(+)}$ denotes the above mentioned solution on the branch C_1^+ for the same b.

Further, if we realize that the solutions belonging to branches $C_m^{\pm}$ must be in fact π/m-periodic, we can repeat the previous discussion for the interval $(-\pi/2m, \pi/2m)$, $m \in \mathbb{N}$, and conclude by the same way that for a given b and m there exists a unique couple of solutions $w_m^{(\pm)}$ of the equation (28), which are symmetric, π-periodic and $(b, w_m^{(\pm)}) \in C_m^{\pm}$.

This completes the proof of Theorem 3.1. □

Corollary 3.1. *For any* $b \in (-\lambda_m, -\lambda_{m+1})$, $m \in \mathbb{N} \cup \{0\}$, *there exist precisely* $(2m+1)$ *solutions of* (24).

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