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QUASILINEAR PARABOLIC EQUATIONS WITH NONLINEAR MONOTONE BOUNDARY CONDITIONS

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Abstract. Of concern is the following quasilinear parabolic equation with a nonlinear monotone boundary condition:

$$(*)\qquad \begin{cases} u_t(x,t) = \dfrac{\partial \alpha(x,u_x)}{\partial x} + g(x,u), \quad (x,t) \in (0,1) \times (0,\infty), \\ (\alpha(0,u_x(0,t)), -\alpha(1,u_x(1,t))) \in \beta(u(0,t),u(1,t)), \\ u(x,0) = u_0(x). \end{cases}$$

Here β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, which contains the origin $(0,0)$. It is showed that (*) has a unique strong solution u, with the property that

$$\sup_{t\in[0,T]} \|u(x,t)\|_{C^{1+\nu}[0,1]}$$

is uniformly bounded for $0 < \nu < 1$ and finite $T > 0$.

1. Introduction

We consider the following parabolic equation

$$(1)\qquad \begin{cases} u_t(x,t) = \dfrac{\partial \alpha(x,u_x)}{\partial x} + g(x,u), \quad (x,t) \in (0,1) \times (0,\infty), \\ (\alpha(0,u_x(0,t)), -\alpha(1,u_x(1,t))) \in \beta(u(0,t),u(1,t)), \\ u(x,0) = u_0(x), \end{cases}$$

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where β is a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$, containing the origin $(0,0)$. We apply the evolution equation theory [1]–[5], [8], [14], [16], [17] to show that (1) has a unique strong solution. Finally, a difference scheme from the method of lines [11], [20] is employed to obtain a strong solution u, which coincides with the solution from the evolution equation theory and has the property:

$$\sup_{t\in[0,T]} \|u(x,t)\|_{C^{1+\nu}[0,1]}$$

is uniformly bounded for $0<\nu<1$ and finite $T>0$.

When $\alpha(x,\xi)=\sigma(x)\xi$, a case in [18] follows, where a more general linear equation of order $2n$ is considered and many other nice results are obtained. When $\beta(x,y)=(\beta_0 x,\beta_1 y)$ and β_0 and β_1 are maximal monotone graphs in $\mathbb{R}$, containing the origin, we obtain a case in [9]. Both [18] and [9] use the evolution equation theory. Elliptic problems corresponding to (1) are studied in [21], [22] with less nonlinearity. Nonlinear monotone boundary conditions of this sort in (1) are very general, from which follows all the traditional ones, such as Dirichlet, Neumann, Robin, and periodic; the derivation of these results can be seen in e.g. [17], [18], [21], [22].

There are many ways to tackle parabolic problems. The traditional one for solving quasilinear equations with linear boundarey conditions is detailed quite well in [13]. Linear evolution equation (operator semigroup) approach is used in e.g. [6], [15] and the nonlinear counterpart is applied in e.g. [1]–[5], [8], [9], [14], [16]–[18].

The nonlinear evolution equation (operator semigroup) approach is to rewrite (1) as an abstract ODE

$$\frac{du}{dt}=Au, \quad u(0)=u_0 \tag{2}$$

in a Banach space $(X,\|\cdot\|)$. If the nonlinear operator A satisfies conditions:

(i) *Dissipativity condition.* $\|u-v\|\le\|(u-v)-\lambda(Au-Av)\|$ for $\lambda>0$ and $u,v\in D(A)$.

(ii) *Range condition.* The range of $(I-\lambda A)\supset D(A)$ for small $\lambda>0$,

then A generates a nonlinear operator semigroup

$$T(t)u_0\equiv\lim_{n\to\infty}\left(I-\frac{t}{n}A\right)^{-n}u_0$$

for $u_0\in D(A)$ by the Crandall–Liggett theorem [5] or the Komura theorem [12] in the case of Hilbert spaces, and $u(t)\equiv T(t)u_0$ for $u_0\in D(A)$ is the unique generalized solution to (2). The notion of a generalized solution is due to Benilan [2]. When X is reflexive, u is a strong solution which satisfies (2) for almost every t. If A satisfies (i) and

(iii) The range of $(I-\lambda A)=X$ for small $\lambda>0$,

A is called m-dissipative.

The method of lines [11], [20] is to time-discretize (2) and construct the Rothe's functions. In doing so, some crucial apriori estimates need to be derived.

The rest of this paper is organized as follows. Section 2 contains some basic assumptions and preliminary results. The proof by the evolution equation (operator semigroup) approach is given in Section 3 and Section 4 deals with the the difference scheme from the method of lines.

2. Some basic assumptions and preliminary results

From here on, k denotes a generic constant, which can vary with different situations.

We make the following assumptions.

(2.1) β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$, such that the range of β contains the origin $(0, 0)$.

(2.2) α is a continuously differentiable function on $[0,1]\times\mathbb{R}$, such that $\alpha_\xi(x,\xi) \geq k > 0$ and $\alpha(x,0) \equiv 0$ for all x and ξ.

(2.3) α_x/α_ξ has at most linear growth in ξ, so that there is a continuous function $M(x) \geq k > 0$, for which

$$\left|\frac{\alpha_x}{\alpha_\xi}\right| \leq M(x)(1 + |\xi|).$$

(2.4) g is a continuous function on $[0,1] \times \mathbb{R}$, such that $g(x,\xi)$ is monotone non-increasing in ξ and $g(x,0) \equiv 0$ for all x.

Define a nonlinear operator $A : D(A) \subset L^2(0,1) \to L^2(0,1)$ as follows

$$D(A) = \{u \in W^{2,2}(0,1) : (\alpha(0,u'(0)), -\alpha(1,u'(1))) \in \beta(u(0),u(1))\}$$

and

$$Au = \frac{d\alpha(x,u')}{dx} + g(x,u) \quad \text{for } u \in D(A).$$

Proposition 1. *For each $h \in C[0,1], \lambda > 0$, and $a, b \in \mathbb{R}$, there is a unique solution to the equation*

$$\begin{cases} u - \lambda\dfrac{d\alpha(x,u')}{dx} - \lambda g(x,u) = h, \\ u(0) = a, \quad u(1) = b. \end{cases} \tag{3}$$

Proof. Since the properties of α and g are not affected when multiplied by λ, it suffices to consider only the case of $\lambda = 1$.

Let $w \in C^1[0,1]$ and let Tw be the unique solution to

$$\begin{cases} u - \alpha_x(x,w') - \alpha_\xi(x,w')u'' - g(x,w) = h, \\ u(0) = a, \quad u(1) = b, \end{cases} \tag{4}$$

by linear ordinary differential equation theory [10], for all u.

We show that the nonlinear operator $T: C^1[0,1] \to C^1[0,1]$ satisfies $\|u\|_{C^1} \leq k$ for which, $\sigma Tu = u$, $\sigma \in [0,1]$, and that T is compact and continuous.

Let $\sigma Tu = u$. Then (4) gives that

$$\begin{cases} u - \sigma\alpha_x(x,u') - \alpha_\xi(x,u')u'' - \sigma g(x,u) = \sigma h, \\ u(0) = \sigma a, \quad u(1) = \sigma b. \end{cases} \tag{5}$$

If the maximum of u occurs at end points, then $\|u\|_\infty$ is uniformly bounded from (5); if instead, it occurs at some interior point x_0 in $(0,1)$, then we have that $u'(x_0) = 0$ and $u(x_0)u''(x_0) \leq 0$ by the first and second derivative tests. With those plugged into (5), we have that, by the monotonicity assumption of g,

$$u^2(x_0) \leq \sigma[u(x_0)\alpha_x(x_0,0) + h(x_0)u(x_0)]$$

and so again, $\|u\|_\infty$ is uniformly bounded.

We continue to estimate u'. Equation (5) gives that

$$u'' + \sigma\frac{\alpha_x(x,u')}{\alpha_\xi(x,u')} = \frac{(u - \sigma g(x,u) - \sigma h)}{\alpha_\xi(x,u')}. \tag{6}$$

The assumptions (2.2) and (2.3) imply that (6) is a uniformly elliptic equation with bounded coefficients and bounded right side, and so, $\|u'\|_\infty$ and $\|u''\|_\infty$ are all uniformly bounded by linear ordinary differential equations theory [10]. Thus $\|u\|_{C^2} \leq k$.

Next, let w_n be a bounded sequence in $C^1[0,1]$. By the definition of T, we have that

$$\begin{cases} u_n - \alpha_x(x,w_n'') - \alpha_\xi(x,w_n')u'' - g(x,w_n) = h, \\ u_n(0) = a, \quad u_n(1) = b, \end{cases} \tag{7}$$

if $u_n = Tw_n$. By the above arguments, we have that $\|u_n\|_{C^2} \leq k$, and so, u_n has a convergent subsequence in $C^1[0,1]$ by the Ascoli–Arzela theorem. Therefore, T is compact.

Next, let w_n converge to w in $C^1[0,1]$ (and so, w^n is uniformly bounded in $C^1[0,1]$). Then $u_n \equiv Tw_n$ has a convergent subsequence u_{n_k}, converging to some u in $C^1[0,1]$ since T is compact. It follows that (7) converges to (3) with $\lambda = 1$ through the subsequences u_{n_k} and w_{n_k}, and so, $Tw_{n_k} = u_{n_k}$ converges to $u = Tw$. Here we have used the fact that the first differential operator d/dx with $C^1[0,1]$ as its domain is closed in $C[0,1]$. This arguments, when repeated, shows that every subsequence of Tw_n has, in turn, a convergent subsequence conveging to Tw, and so, T is continuous.

With the above properties, T has a fixed point by the Schauder fixed point theorem [7], which is a solution to (3) with $\lambda = 1$.

We continue to prove uniqueness. Let u_1 and u_2 satisfy (3) with $\lambda = 1$. Then

$$(8)\qquad \begin{gathered}(u_1 - u_2) - \frac{\alpha(x, u_1') - \alpha(x, u_2')}{dx} - [g(x, u_1) - g(x, u_2)] = 0,\\ (u_1 - u_2)(0) = (u_1 - u_2)(1) = 0.\end{gathered}$$

Integrating (8) gives that

$$0 \le \int_0^1 (u_1 - u_2)^2\,dx = \sum_{i=1}^{3} I_i,$$

where

$$I_1 = \int_0^1 (u_1 - u_2)[g(x, u_1) - g(x, u_2)\,dx, \le 0$$

since $g(x, \eta)$ is monotone non-increasing in η,

$$I_2 = (u_1 - u_2)[\alpha(x, u_1') - \alpha(x, u_2')]\big|_0^1 = 0,$$

by the boundary condition in (3),

$$I_3 = -\int_0^1 (u_1' - u_2')[\alpha(x, u_1') - \alpha(x, u_2')]\,dx \le 0,$$

by the assumption (2.2).

Thus, $\int_0^1 (u_1 - u_2)^2\,dx = 0$, and so, $u_1 \equiv u_2$ since $u_1, u_2 \in C^1[0, 1]$. □

3. The evolution equation approach

We rewrite (1) as

$$\begin{cases} \dfrac{du}{dt} = Au & \text{for } t > 0,\\ u(0) = u_0, \end{cases}$$

in the Hilbert space $(L^2(0, 1), \|\cdot\|)$, where the nonlinear operator A is defined Section 2.

Lemma 1. *The nonlinear operator A has the dissipativity condition* (i) *on $L^2(0, 1)$.*

Proof. Let $u_i \in D(A), \lambda > 0$, and $h_i = u_i - \lambda A u_i$, where $i = 1, 2$. Using integration by parts, we have that

$$\int_0^1 (u_1 - u_2)((h_1 - h_2)\,dx = \int_0^1 (u_1 - u_2)^2\,dx + \lambda \sum_{i=1}^{3} J_i,$$

where

$$J_1 = -\int_0^1 (u_1 - u_2)[g(x, u_1) - g(x, u_2)\,dx, \ge 0$$

since $g(x, \eta)$ is monotone non-increasing in η,

$$J_2 = \int_0^1 (u_1' - u_2')[\alpha(x, u_1') - \alpha(x, u_2')]\,dx \ge 0,$$

by the uniformly elliptic assumption of (2.2),

$$J_3 = -(u_1 - u_2)[\alpha(x, u_1') - \alpha(x, u_2')]\Big|_0^1 \geq 0,$$

using the monotonicity assumption (2.1) of β and the boundary condition in $D(A)$. Thus,

$$\|u_1 - u_2\|^2 \leq \int_0^1 (u_1 - u_2)(h_1 - h_2)\, dx \leq \|u_1 - u_2\| \|h_1 - h_2\|$$

by the Hölder inequality, and so, $\|u_1 - u_2\| \leq \|h_1 - h_2\|$. This proves the dissipativity of A. □

PROPOSITION 2. *For $\lambda > 0$, the range of $(I - \lambda A)$ contains $C[0,1]$ and so, is dense in $L^2(0,1)$.*

PROOF. It suffices to consider only the case of $\lambda = 1$. Let $h \in C[0,1]$ and $a, b \in \mathbb{R}$. Consider the equation

$$\begin{cases} u - \dfrac{d\alpha(x, u')}{dx} - g(x, u) = h, \\ u(0) = a, \quad u(1) = b. \end{cases} \tag{9}$$

Proposition 1 implies that (9) has a unique solution u. Define the nonlinear operator $S : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by

$$S(a, b) = \beta(a, b) + B(a, b),$$

where

$$B(a, b) = -(\alpha(0, u'(0)), -\alpha(1, u'(1))).$$

We show that B is monotone and hemicontinuous, and that S is coercive.

Let u_1 be the solution to (9), corresponding to the pair (a_1, b_1). Similarly, let u_2 correspond to the pair (a_2, b_2) through (9). Here, $a_i, b_i \in \mathbb{R}$, $i = 1, 2$. Then

$$\begin{cases} u_i - \dfrac{d\alpha(x, u_i')}{dx} - g(x, u_i) = h, \\ (u_i(0), u_i(1)) = (a_i, b_i), \quad i = 1, 2. \end{cases} \tag{10}$$

Integration by parts applied to (10) gives that

$$\begin{aligned} C &\equiv (u_1 - u_2)[\alpha(x, u_1') - \alpha(x, u_2')]\Big|_0^1 \\ &= \int_0^1 (u_1 - u_2)^2\, dx + \int (u_1' - u_2')[\phi(x, u_1') - \phi(x, u_2')]\, dx \\ &\quad - \int_0^1 (u_1 - u_2)[g(x, u_1 - g(x, u_2)]\, dx \geq 0, \end{aligned}$$

by the arguments as in proving Lemma 1. Let $\langle \cdot, \cdot \rangle$ be the inner product in $\mathbb{R} \times \mathbb{R}$. Then

$$\langle (a_1 - b_1) - (a_2 - b_2), B(a_1 - b_1) - B(a_2 - b_2) \rangle = C \geq 0,$$

and so, B is monotone.

Next, let $t \in [0,1]$ and u_t be the unique solution to (9), corresponding to the pair $(a+tc, b+td) = (a,b) + t(c,d)$, that is, let u_t satisfy

$$\begin{cases} u_t - \dfrac{d\alpha(x,u_t')}{dx} - g(x,u_t) = h, \\ u_t(0) = a+tc, \quad u_t(1) = b+td. \end{cases} \tag{11}$$

Similarly, let u correspond to the pair (a,b) through (9). Then, it follows from as in proving Proposition 1 that $\|u_t\|_{C^2[0,1]} \le k$ for $t \in [0,1]$. Therefore, we can use the Ascoli–Arzela theorem to derive that (11) converges to (9) through some subsequence of u_t as $t \to 0$ and then, through the very sequence u_t as in proving Proposition 1. Consequently, we have that

$$-(\alpha(0,u_t'(0)), -\alpha(1,u_t'(1))) \to -(\alpha(0,u'(0)), -\alpha(1,u'(1))),$$

that is, $B((a,b)+t(c,d))$ converges to $B(a,b)$, and so, B is hemicontinuous.

Next, let $x = (u(0),u(1)) = (a,b)$. Then $\langle Sx, x\rangle = J_1 + J_2$, where

$$J_1 = \langle \beta(u(0),u(1)), (u(0),u(1))\rangle \ge 0,$$

by the monotonicity assumption (2.1) of β,

$$\begin{aligned} J_2 &= \langle -(\alpha(0,u'(0)), -\alpha(1,u'(1))), (u(0),u(1))\rangle \\ &= u\alpha(x,u')\big|_0^1 = \int_0^1 (u^2 + u'\alpha(x,u') - ug(x,u) - uh)\, dx, \end{aligned}$$

by integrating (9), which we denote as $\sum_{i=1}^4 I_i$. Here,

$$\begin{aligned} I_1 &= \int_0^1 u^2\, dx \ge 0, \\ I_2 &= \int_0^2 u'\alpha(x,u')\, dx \ge k \int_0^1 (u')^2\, dx, \end{aligned}$$

by the uniform elliptic assumption (2.2) of α,

$$I_3 = -\int_0^1 ug(x,u)\, dx \ge 0,$$

by the monotone non-increasing assumption (2.4) of g together with $g(x,0) = 0$ and by the Hölder inequality

$$\begin{aligned} I_4 &= -\int_0^1 uh\, dx \ge -\int_0^1 |uh|\, dx \\ &\ge -\Big(\int_0^1 |u|^2\, dx\Big)^{1/2}\Big(\int_0^1 |h|^2\, dx\Big)^{1/2} \ge \frac{\|u\| + \|h\|}{-2}. \end{aligned}$$

So, if we let $M = \|u\|^2$ and $N = \|u'\|^2$, then we have that

$$\langle Sx, x\rangle \ge k(M+N) - \|h\|^2/2.$$

We estimate further. By the fundamental theorem of calculus, for $0 \le x \le 1$, we have that

$$|b| = |u(1)| = \left| u(x) + \int_x^1 u'(t)\,dt \right|, \le |u| + \int_0^1 |u'|\,dx,$$

and so, by the Hölder inequality,

$$\begin{aligned} |b|^2 &\le |u|^2 + \left(\int_0^1 |u'|\,dx \right)^2 + 2|u| \int_0^1 |u'|\,dx \\ &\le |u|^2 + \left(\int_0^1 |u'|\,dx \right)^2 + \left[|u|^2 + \left(\int_0^1 |u'|\,dx \right)^2 \right] \le 2|u|^2 + 2\|u'\|^2. \end{aligned}$$

Integrating both sides gives that $|b|^2 \le 2(M+N)$. Similarly, we have that $a^2 = |u(0)|^2 \le 2(M+N)$. So, we obtain that

$$\frac{\langle Sx, x\rangle}{|x|} = \frac{\langle Sx, x\rangle}{\sqrt{a^2+b^2}} \ge \frac{2k(a^2+b^2) - \|h\|^2}{2\sqrt{a^2+b^2}},$$

which converges to ∞ as $|x| = |(a,b)| \to \infty$. So, S is concercive.

Now, we have shown that B is monotone and hemicontinuous and that S is coercive and so, S is onto [1]; in particular, we have that $(0,0) \in S(a,b)$ for some $(a,b) \in \mathbb{R} \times \mathbb{R}$. Thus, given $h \in C[0,1]$, there exists a solution u to

$$\begin{cases} u - \dfrac{d\alpha(x,u')}{dx} - g(x,u) = h, \\ (\alpha(0,u'(0)), -\alpha(1,u'(1))) \in \beta(u(0),u(1)), \end{cases} \tag{12}$$

which implies that the range of $(I - A)$ contains $C[0,1]$. □

Since A satisfies the dissipativity condition (i) and the range of $(I - \lambda A) \supset C[0,1] \supset D(A)$ for $\lambda > 0$, we have by the Crandall–Liggett theorem or the Komura theorem in the Hilbert space case that

THEOREM 1. *Problem* (1) (*written as* (2) *on* $L^2(0,1)$) *has a unique strong solution for every* $u_0 \in D(A)$.

REMARK. In fact, A is m-dissipative on $L^2(0,1)$. For this, it suffices to show that A is closed in $L^2(0,1)$ since $C[0,1]$ is dense in $L^2(0,1)$.

Let $w_n \in D(A) \to w$ and $Aw_n \to v$. We need to show that $w \in D(A)$ and $Aw = v$. Let

$$v_n = Aw_n = \frac{d}{dx}\alpha(x, w_n') + g(x, w_n). \tag{13}$$

Since $Aw_n \to v$ in $L^2(0,1)$, we have $\|v_n\| \le k$. Multiplying (13) by w_n and using integration by parts, we have

$$\int_0^1 w_n' \alpha(x, w_n')\,dx - \int_0^1 w_n g(x, w_n)\,dx + w_n \alpha(x, w_n')|_1^0 = -\int_0^1 w_n v_n\,dx,$$

which gives that

$$k\|w_n'\| \leq \int_0^1 w_n'\alpha(x, w_n')\,dx \leq \|w_n\|\|v_n\|,$$

by (2.2), (2.4), and the boundary condition in $D(A)$. So we have $\|w_n'\| \leq k$.

Now, as in proving the coerciveness of S, we have that

$$(w_n(1))^2 \leq 2(\|w_n\|^2 + \|w_n'\|^2)$$

and so, $|w_n(1)| \leq k$. By the fundamental theorem of calculus, we have

$$|w_n(x)| \leq |w_n(1)| + \int_0^1 |w_n'|\,dx \leq k + \|w_n'\|$$

and so, $\|w_n\|_\infty \leq k$. Next, (13) gives that

$$\|w_n''\| \leq \frac{\|v_n\| + \|g(x, w_n)\|}{k} + k\|1 + w_n'\|,$$

by using (2.2) and (2.3) and so, $\|w_n''\| \leq k$. Now as in proving the coerciveness of S, we have

$$(w_n'(1))^2 \leq 2(\|w_n'\|^2 + \|w_n''\|^2),$$

and so $|w_n'(1)| \leq k$. Then as above, $\|w_n'\|_\infty \leq k$. It follows from (13) that $\|w_n''\|_\infty \leq k$. Thus by the Ascoli–Arzela theorem, we have $w_n \to w$ in $C^{1+\nu}[0,1]$ for $0 < \nu < 1$ and so, w satisfies the boundary condition in $D(A)$ since $(I-\beta)^{-1} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}\times\mathbb{R}$ is nonexpansive (and so continuous) and w_n satisfies the boundary condition in $D(A)$.

Next, for each $\phi \in L^2(0,1)$, (13) gives formally that

$$\begin{aligned}\int v_n\phi\,dx &= \int(\alpha_x(x, w_n') + \alpha_\xi(x, w_n')w_n'' + g(x, w_n))\phi\,dx\\ &= \int(\alpha_x(x, w_n') - \alpha_x(x, w'))\phi\,dx\\ &\quad + \int(\alpha_\xi(x, w_n')w_n'' - \alpha_\xi(x, w')w'')\phi\,dx\\ &\quad + \int g(x, w_n) - g(x, w))\phi\,dx\\ &\quad + \int\left(\frac{d}{dx}\alpha(x, w') + g(x, w)\right)\phi\,dx,\end{aligned}$$

which we denote as $\sum_{i=1}^4 I_i$. Here the integration range $[0,1]$ is omitted.

Since w_n converges to w in $C^{1+\nu}[0,1]$ and $\alpha_x(x,\xi)$ is continuous in ξ, we have $|I_1| \to 0$.

Next, rewrite I_2 as

$$\int \alpha_\xi(x, w)(w_n'' - w'')\phi\,dx + \int(\alpha_\xi(x, w_n') - \alpha_\xi(x, w'))w_n''\phi\,dx,$$

which we denote as $J_1 + J_2$. We have $|J_2| \to 0$ since

$$|J_2| \le \|\alpha_\xi(x, w_n') - \alpha_\xi(x, w')\|_\infty \|w_n''\| \|\phi\|$$

and $\|w_n\|_{C^2[0,1]} \le k$.

On the other hand, we have $|J_1| \to 0$ since w_n converges weakly in $W^{2,2}(0,1)$ by the Alaoglu theorem and since $\alpha_\xi(x, w')\phi \in L^2(0,1)$.

Next, to see $|I_3| \to 0$, we note that w_n converges in $C^{1+\nu}[0,1]$ and g is continuous and the Lebesgue convergence theorem applies.

Thus, we have shown

$$\int v_n \phi \, dx \ \to \ I_4 = \int \left(\frac{d}{dx} \alpha(x, w') + g(x, w) \right) \phi \, dx$$

for each $\phi \in L^2$ and so, $w \in D(A)$ and $Aw = v$. This shows that A is closed in $L^2(0,1)$.

4. The difference scheme from the method of lines

Let $T > 0$ and $n \in \mathbb{N}$ large. Time-discretize (2) to have

$$u_i - \varepsilon A u_i = u_{i-1}, \quad u_i \in D(A), \tag{14}$$

where $\varepsilon = T/n$ and $i = 1$ to n.

We assume that $u_0 \in D(A)$. Proposition 2 applied to (14) gives the existence of a u_1. The dissipativity proof for Lemma 1 shows immediately that u_1 exists uniquely. By induction, u_i exists uniquely for $i = 1$ to n. For convenience, we define

$$u_{-1} = u_0 - \varepsilon A u_0.$$

Next, we estimate u_i. From (14), we have that

$$\frac{u_i - u_{i-1}}{\varepsilon} - (Au_i - Au_{i-1}) = \frac{u_{i-1} - u_{i-2}}{\varepsilon}. \tag{15}$$

Multiplying (15) by $(u_i - u_{i-1})/\varepsilon$ and using integration by parts, we have, as in proving dissipativity of A, that $\|v_{i,\varepsilon}\| \le \|v_{i-1,\varepsilon}\|$, if we let $v_{i,\varepsilon} = (u_i - u_{i-1})/\varepsilon$, and so, $\|v_{i,\varepsilon}\|$ is uniformly bounded since $\|v_{0,\varepsilon}\| = \|Au_0\| \le k$. Here, $\|\cdot\|$ is the norm in $L^2(0,1)$. The same arguments also show that $\|u_i\| \le \|u_0\| \le k$.

Now, rewrite (14) as

$$\frac{d\alpha(x, u_i')}{dx} + g(x, u_i) = v_{i,\varepsilon}, \quad u_i \in D(A). \tag{16}$$

Multiplying (16) by u_i and using integration by parts, we have that

$$\int_0^1 u_i' \alpha(x, u_i') \, dx + \int_0^1 (-u_i') g(x, u_i) \, dx + u_i \alpha(x, u_i') \Big|_1^0 = -\int_0^1 u_i v_{i,\varepsilon} \, dx,$$

which gives that

$$k\|u_i'\|^2 \le \int_0^1 u_i'\alpha(x,u_i')\,dx \le \|u_i\|\|v_{i,\varepsilon}\|$$

by the uniformly elliptic assumption (2.2) of α, the monotone non-increasing assumption (2.4) of g, and the boundary condition in $D(A)$. Therefore, we have that $\|u_i'\| \le k$.

Now, as in proving the coerciveness of S in Section 3, we have that

$$(u_i(1))^2 \le 2(\|u_i\|^2 + \|u_i'\|^2)$$

and so, $|u_i(1)| \le k$. By the fundamental theorem of calculus formula

$$u_i(x) = u_i(1) + \int_1^x u_i'(t)\,dt,$$

we have that

$$|u_i(x)| \le |u_i(1)| + \int_0^1 |u_i'|\,dx \le k + \|u_i'\|,$$

by the Hölder inequality, and so $\|u_i\|_\infty$ is uniformly bounded.

Next, rewrite (16) as

$$u_i'' = \frac{v_{i,\varepsilon} - g(x,u_i)}{\alpha_\xi(x,u_i')} - \frac{\alpha_x(x,u_i')}{\alpha_\xi(x,u_i')}, \tag{17}$$

which implies that

$$\|u_i''\| \le \frac{\|v_{i,\varepsilon}\| + \|g(x,u_i)\|}{k} + k\|1 + u_i'\|,$$

by the uniformly elliptic assumption (2.2) of α and the most possible linear growth assumption (2.3) of $\alpha(x,\xi)$ in ξ. So, $\|u_i''\|$ is uniformly bounded.

Next, again as in proving the coerciveness of S in Section 3, we have that

$$(u_i'(1))^2 \le 2(\|u_i'\|^2 + \|u_i''\|^2),$$

and so, $|u_i'(1)|$ is uniformly bounded. Thus, by the fundamental theorem of calculus, we have that

$$|u_i'(x)| \le |u_i'(1)| + \int_0^1 |u_i''|\,dx,$$

which is less than or equal to $(k + \|u_i''\|)$ by the Hölder inequality. Thus, $\|u_i'\|_\infty$ is uniformly bounded. With this, (17) implies that $\|u_i''\|_\infty$ is uniformly bounded. Therefore, we have shown that $\|u_i\|_{C^2}$ is uniformly bounded.

Next, we construct the Rothe's functions [11], [20]. Let

$$\chi^n(0) = u_0, \quad \chi^n(t) = u_i$$

for $t \in (t_{i-1}, t_i]$, and let

$$u^n(t) = u_{i-1} + \frac{u_i - u_{i-1}}{\varepsilon}(t - t_{i-1}) \quad \text{for } t \in [t_{i-1}, t_i], \tag{18}$$

where, as before, $n \in \mathbb{N}$ is large, $\varepsilon = T/n$, and $i = 1$ to n. By the definition of $\chi^n(t)$ and $u^n(t)$, and by $\|v_{i,\varepsilon}\| \le k$, we have that

$$\sup_{t\in[0,1]} \|u^n(t) - \chi^n(t)\|_\infty \to 0,$$

$$\|u^n(t) - u^n(\tau)\| \le k|t - \tau| \quad \text{for } t, \tau \in [t_{i-1}, t_i], \tag{19}$$

and

$$\frac{du^n(t)}{dt} = A\chi^n(t), \quad u^n(0) = u_0, \tag{20}$$

where the last equation has values in $B([0,1]; L^2(0,1))$, the real Banach space of all bounded functions from $[0,1]$ to $L^2(0,1)$ since $\|u_i\|_{C^2}$ is uniformly bounded.

Next, we show convergence of $u^n(t)$. Since $\|u_i\|_{C^2} \le k$, we have that

$$\sup_{t\in[0,T]} \|u^n(t)\|_{C2} \le k,$$

and so, $u^n(t)$ has a t-uniformly convergent subsequence in $C^{1+\nu}[0,1]$ (and so in $L^2(0,1)$) by using the Ascoli–Arzela theorem. Here, $0 < \nu < 1$. Thus, for each t, $u^n(t)$ is relatively compact in $L^2(0,1)$. Since $u^n(t)$ is also equi-continuous in $C([0,1]; L^2(0,1))$ by (19), we have that $u^n(t)$ (actually, its some subsequence) converges to, say $u(t) \in C([0,1]; L^2(0,1))$ by using the Ascoli-Arzela theorem [19] again.

Since $(I + \beta)^{-1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is nonexpansive (and so continuous), $u^n(t)$ converges t-uniformly in $C^{1+\nu}[0,1]$ to $u(t)$, and u_i satisfies the boundary condition in (1), we see easily that $u(t)$ also satisfies the boundary condition in (1). Here we notice, from the above, that $\sup_{t\in[0,T]} \|u(t)\|_{C^{1+\nu}[0,1]} \le k$.

Next, from (20), we have formally that for each $\phi \in L^2(0,1)$,

$$\begin{aligned}
\int \frac{du^n}{dt}\phi\,dx &= \int \left[\alpha_x\left(x, \frac{d\chi^n}{dx}\right) + \alpha_\xi\left(x, \frac{d\chi^n}{dx}\right)\frac{d^2\chi^n}{dx^2} + g(x,\chi^n)\right]\phi\,dx \\
&= \int \left[\alpha_x\left(x, \frac{d\chi^n}{dx}\right) - \alpha_x\left(x, \frac{du}{dx}\right)\right]\phi\,dx \\
&\quad + \int \left[\alpha_\xi\left(x, \frac{d\chi^n}{dx}\right)\frac{d^2\chi^n}{dx^2} - \alpha_\xi\left(x, \frac{du}{dx}\right)\frac{d^2u}{dx^2}\right]\phi\,dx \\
&\quad + \int [g(x,\chi^n) - g(x,u)]\phi\,dx + \int \left[\frac{d\alpha(x, du/dx)}{dx} + g(x,u)\right]\phi\,dx,
\end{aligned}$$

which we denote as $\sum_{i=1}^4 I_i$. Here, we omit the integration range $[0,1]$.

Now, we estimate I_i. Since u^n converges t-uniformly to u in $C^{1+\nu}[0,1]$ and $\alpha_x(x,\xi)$ is continuous in ξ, we have that $|I_1| \to 0$ t-uniformly.

Next, rewrite I_2 as

$$\int \alpha_\xi\left(x, \frac{du}{dx}\right)\left(\frac{d^2\chi^n}{dx^2} - \frac{d^2u}{dx^2}\right)\phi\,dx + \int \left[\alpha_\xi\left(x, \frac{d\chi^n}{dx}\right) - \alpha_\xi\left(x, \frac{du}{dx}\right)\right]\frac{d^2\chi^n}{dx^2}\phi\,dx,$$

which we denote as $J_1 + J_2$. We have that $|J_2| \to 0$ since

$$|J_2| \le \left\|\alpha_\xi\left(x, \frac{d\chi^n}{dx}\right) - \alpha_\xi\left(x, \frac{du}{dx}\right)\right\|_\infty \left\|\frac{d^2\chi^n}{dx^2}\right\| \|\phi\|$$

and $\|u^n\|_{C^2} \le k$. On the other hand, we have that $|J_1| \to 0$ since $u^n(t)$ converges weakly in $W^{2,2}(0,1)$ by the Alaoglu's theorem and since $\alpha_\xi(x, du/dx)\phi \in L^2(0,1)$.

Next, to see that $|I_3| \to 0$, we note that $u^n(t)$ converges to $u(t)$ t-uniformly in $C^{1+\nu}[0,1]$ and g is continuous and the Lebesgue dominated convergence theorem applies. Thus, we have shown that

$$\int \frac{du^n}{dt}\phi\,dx \to I_4 = \int \left[\frac{d}{dx}\alpha(x, \frac{du}{dx}) + g(x,u)\right]\phi\,dx,$$

for each $\phi \in L^2(0,1)$, which we rewrite as

$$\left(\frac{du^n(t)}{dt}, \phi\right) \to (Bu(t), \phi)$$

t-uniformly, where $(\,\cdot\,,\,\cdot\,)$ is the inner product in $L^2(0,1)$. So, by the Fubini theorem, we have that

$$(u^n(t) - u^n(0), \phi) = \left(\int_0^t \frac{du^n}{dt}\,dt, \phi\right) = \int_0^t \left(\frac{du^n}{dt}, \phi\right) dt,$$

which converges to

$$(u(t) - u_0, \phi) = \int_0^t (Bu(\tau), \phi)\,d\tau,$$

by the Lebesgue dominated convergence theorem since

$$\left|\left(\frac{du^n(t)}{dt}, \phi\right)\right| \le \left\|\frac{du^n(t)}{dt}\right\| \|\phi\| \le k.$$

Now, by the Fubini theorem again, we have that

$$(u(t) - u_0, \phi) = \left(\int_0^t Bu(\tau)\,d\tau, \phi\right)$$

for each $\phi \in L^2(0,1)$, and so,

$$u(t) - u_0 = \int_0^t Bu(\tau)\,d\tau.$$

Hence, by the fundamental theorem of calculus, we have that

$$\begin{cases} \dfrac{du}{dt} = Bu(t) \quad \text{almost everywhere in } t, \\ u(0) = u_0. \end{cases} \tag{21}$$

To prove uniqueness of solution, let u_1 and u_2 be two solutions of (21). By integration by parts, we have that

$$\frac{1}{2}\frac{d\|u_1(t)-u_2(t)\|^2}{dt} = \frac{1}{2}\frac{d\int_0^1 (u_1(t)-u_2(t))^2\,dx}{dt} = \int_0^1 (Bu_1(t)-Bu_2(t))(u_1(t)-u_2(t))\,dx \le 0,$$

and so,

$$0 \le \|u_1(t)-u_2(t)\|^2 \le \|u_1(0)-u_2(0)\|^2 = 0$$

and so, $u_1 \equiv u_2$ in $L^2(0,1)$ for almost every t. Thus, we have proved that

THEOREM 2. *If* $u_0 \in D(A)$, *then there is a unique solution* u *satisfying* (1) *on* $(0,T)$ ($T \in \mathbb{R}$ *is given*) *almost everywhere in* t, *with the properties that*

$$\left\|\frac{du}{dt}\right\| \le k \quad \text{for almost every } t$$

and

$$\sup_{t\in[0,T]} \|u(t)\|_{C^{1+\nu}[0,1]} \le k.$$

Here $0 < \nu < 1$.

REMARK. Since $u_i = (I-\varepsilon A)^{-[t/\varepsilon]}u_0$ for each $t \in [t_i, t_{i+1})$, we have the solution u from the difference scheme coincides with the solution from the Crandall–Liggett theorem or the Komura theorem in the Hilbert space case.

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