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CRITICAL POINTS FOR SOME FUNCTIONALS OF THE CALCULUS OF VARIATIONS

Benedetta Pellacci

Abstract. In this paper we prove the existence of critical points of non differentiable functionals of the kind

$$J(v) = \frac{1}{2}\int_\Omega A(x,v)\nabla v \cdot \nabla v - \frac{1}{p+1}\int_\Omega (v^+)^{p+1},$$

where $1 < p < (N+2)/(N-2)$ if $N > 2$, $p > 1$ if $N \leq 2$ and v^+ stands for the positive part of the function v. The coefficient $A(x,s) = (a_{ij}(x,s))$ is a Carathéodory matrix derivable with respect to the variable s. Even if both $A(x,s)$ and $A'_s(x,s)$ are uniformly bounded by positive constants, the functional J fails to be differentiable on $H^1_0(\Omega)$. Indeed, J is only derivable along directions of $H^1_0(\Omega)\cap L^\infty(\Omega)$ so that the classical critical point theory cannot be applied.

We will prove the existence of a critical point of J by assuming that there exist positive continuous functions $\alpha(s)$, $\beta(s)$ and a positive constants α_0 and M satisfying $\alpha_0|\xi|^2 \leq \alpha(s)|\xi|^2 \leq A(x,s)\xi\cdot\xi$, $A(x,0) \leq M$, $|A'_s(x,s)| \leq \beta(s)$, with $\beta(s)$ in $L^1(\mathbb{R})$.

1. Introduction

The classical theory concerning the existence of critical points for functionals of the Calculus of Variations is applicable for C^1-functionals defined on Banach spaces. However, simple examples show that this differentiability condition may fail. In order to give a model example, consider Ω a bounded open set of $\mathbb{R}^N$

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and a symmetric matrix $A(x,s) = (a_{ij}(x,s))$, where $a_{ij}(x,s)$ are Carathéodory functions, i.e. mesaurable with respect to x and continuous with respect to s. We assume that there exist a positive continuous function $\alpha(s)$ and positive constants α_0 and M such that the following conditions are satisfied for almost every x in Ω and for every s in $\mathbb{R}$

$$\begin{cases} A(x,s)\xi\cdot\xi \geq \alpha(s)|\xi|^2 & \text{for all } \xi\in\mathbb{R}^N, \\ \alpha(s)\geq\alpha_0>0, \end{cases} \tag{1.1}$$

$$A(x,0)\leq M. \tag{1.2}$$

Moreover, we suppose that $A(x,s)$ is derivable with respect to s and we denote with $A'_s(x,s)$ its derivative. Regarding the matrix $A'_s(x,s)$ we assume that there exists a positive continuous function $\beta(s)$ and positive constants R_0, β_1 and β_2 such that for almost every x in Ω and for every s in $\mathbb{R}$, the following conditions are satisfied

$$|A'_s(x,s)|\leq\beta(s), \tag{1.3}$$

$$\beta(s)\in L^1(\mathbb{R}), \tag{1.4}$$

$$\begin{cases} \dfrac{\beta(s)}{\alpha(s)}s\leq\beta_1 & \text{for all } s \text{ with } s>R_0, \\ \beta(s)s\geq-\beta_2 & \text{for all } s \text{ with } s<-R_0. \end{cases} \tag{1.5}$$

Notice that conditions (1.2) and (1.4) imply that there exists a positive constant β_0 such that

$$A(x,s)\leq\beta_0. \tag{1.6}$$

Indeed we have

$$\lg\left[\frac{A(x,s)}{A(x,0)}\right] = \int_0^s \frac{A'_s(x,t)}{A(x,t)}\,dt \leq \frac{1}{\alpha_0}\int_0^s \beta(t)\,dt < \infty.$$

So that, taking into account (1.6) we notice that condition (1.5) implies that $\beta(s)|s|\leq C$ for every $s\in\mathbb{R}$ as $\alpha(s)$ is bounded from above by β_0. Then, there exists a positive constant Λ such that

$$|A'_s(x,s)|\leq\Lambda, \qquad |A'_s(x,s)s|\leq\Lambda. \tag{1.7}$$

Let us consider the functional $I: H_0^1(\Omega)\to\mathbb{R}$ defined by

$$I(v) = \frac{1}{2}\int_\Omega A(x,v)\nabla v\cdot\nabla v - \int_\Omega F(v),$$

where $F(s) = 1/(p+1)(s^+)^{p+1}$, $1<p<(N+2)/(N-2)$ and s^+ stands for the positive part of s, i.e. $s^+=\max\{s,0\}$. From hypothesis (1.1) and by applying De Giorgi Theorem ([9]), it follows that I is weakly lower semicontinuous on $H_0^1(\Omega)$.

Condition (1.6) and the definition of F imply that I is not bounded from below on $H_0^1(\Omega)$ so that it has not a global minimum. In order to look for critical points different from minima we consider the Gateaux derivative of I. From condition (1.7) we can compute $\langle I'(u), v\rangle$ for every $u \in H_0^1(\Omega)$ and along any direction $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. We obtain

$$\langle I'(u), v\rangle = \int_\Omega A(x,u)\nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega A'_s(x,u)\nabla u \cdot \nabla u\, v - \int_\Omega (u^+)^p v.$$

Since I is not Frechét differentiable on $H_0^1(\Omega)$, we cannot apply the classical critical point theory suitable for C^1-functionals. We say that a function u in $H_0^1(\Omega)$ is a critical point of I if it satisfies $\langle I'(u), v\rangle = 0$ for every v in $H_0^1(\Omega) \cap L^\infty(\Omega)$. We will prove the following results.

THEOREM 1. *Assume* (1.1)–(1.5). *Let* $1 < p < (N+2)/(N-2)$ *and suppose that the matrix* $A'_s(x,s)$ *is negative semidefinite for* $s > R_0$. *Then there exists* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $u \geq 0$, $u \not\equiv 0$, *critical point of the functional* I.

Since (1.6) and (1.7) hold, we are considering bounded matrices with bounded derivative. Functionals with bounded coefficients have also been treated in [2], [8], [13]. In [8], [13] it is proved a multiplicity result using a "weak" notion of derivative for continuous functions defined on complete metric spaces. In [2], [3] and [6] it is proved an existence result for bounded ([2]) and unbounded [3], [6]) coefficients by means of a suitable version of the Mountain Pass Theorem for nondifferentiable functionals, which is proved using the Ekeland variational principle ([10]). In all these works, as usual when dealing with elliptic problems with quadratic gradient terms (see [5] and the references therein), it is assumed a sign condition on $A'_s(x,s)$. Namely, it is supposed that

$$A'_s(x,s)s \geq 0 \quad \text{for every } s. \tag{S}$$

Thus, Theorem 1 concerns the case in which the opposite sign in (S) is assumed. If the matrix $A'_s(x,s)$ does not satisfy any sign condition, we will prove the following Theorem.

THEOREM 2. *Assume* (1.1)–(1.5). *Moreover, suppose that the exponent* p *satisfies the condition*

$$\min\{1+\beta_1, 2\,e^{\psi_M} - 1\} < p < \frac{N+2}{N-2}, \tag{1.8}$$

where $\psi_M = \|\beta/\alpha\|_{L^1(\mathbb{R}^+)}/2$. *Then, there exists* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $u \geq 0$, $u \not\equiv 0$, *critical point of the functional* I.

We point out that even in the case in which (S) is satisfied Theorem 2 is new (see Remark 4.4).

In order to prove Theorems 1 and 2 we will use critical points theory. As I is nondifferentiable we will apply an abstract result in [2] in order to construct a Palais–Smale sequence $\{u_n\}$ for I. Then, we will demonstrate that $\{u_n\}$ is compact in $H_0^1(\Omega)$. In order to prove that $\{u_n\}$ is bounded in $H_0^1(\Omega)$, we will first take advantage of the fact that $s^+ \equiv 0$ for every $s \leq 0$ and we will prove that the negative part u_n^- ($v^- = \min\{v, 0\}$) of u_n strongly converges to zero in $H_0^1(\Omega)$. Then, we will use condition (1.8) (in the proof of Theorem 2) in order to prove that u_n^+ is bounded in $H_0^1(\Omega)$. Condition (1.8) shows an interaction between the nonlinearities $F(s)$ and $A(x,s)$. Indeed, if $A_s'(x,s) \leq 0$ for every $s \geq R_0$, then, "roughly speaking", the term involving $A_s'(x,u_n^+)$ can be ignored as n tends to infinity. Thus, the boundedness of $\{u_n^+\}$ in $H_0^1(\Omega)$ can be proved in the same way as for C^1-functionals in $H_0^1(\Omega)$ (see Lemma 3.1). On the other hand, if $A_s'(x,s) \geq 0$ or $A_s'(x,s)$ is indefinite, it seems natural to expect a competition between the terms $F(s)$ and $A(x,s)$ so that condition (1.8) raises (see Lemma 3.2).

In order to prove that u_n, bounded in $H_0^1(\Omega)$, is compact in $H_0^1(\Omega)$ we will demostrate that, given a Palais–Smale sequence u_n bounded in $H_0^1(\Omega)$, then, every u, weak cluster point of u_n, belongs to $L^\infty(\Omega)$. The key point is that this result is proved before proving that u is a critical point, so that no bootstrap arguments will be used to prove that a critical point belongs to $L^\infty(\Omega)$.
The paper is organized as follows.

Theorem 1 and 2 will be proved as a consequence of Theorems 2.1 and 2.2 respectively, that will be stated in Section 2. In Section 3 we prove some technical results that will be useful in order to prove the Palais–Smale condition. Finally in Section 4 we will prove Theorems 2.1 and 2.2.

2. Setting of the problem and statements of the results

Let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that

$$f(x,s) = 0 \quad \text{for all } s \leq 0. \tag{2.1}$$

Assume that positive constants a and b exist such that for almost every x in Ω

$$|f(x,s)| \leq a|s|^p + b \quad \text{for every } s \text{ in } \mathbb{R}^+, \tag{2.2}$$

where $1 < p < (N+2)/(N-2)$ if $N > 2$, $p > 1$ if $N = 2$, and

$$\lim_{s \to 0^+} \frac{f(x,s)}{s} = 0. \tag{2.3}$$

Consider the functional $J : H_0^1(\Omega) \to \mathbb{R}$ defined by

$$J(v) = \frac{1}{2} \int_\Omega A(x,v) \nabla v \cdot \nabla v - \int_\Omega F(x,v),$$

where $F(x,s) = \int_0^s f(x,t)\,dt$ is the primitive of f and $A(x,s)$ satisfies (1.1)–(1.5). For every v in $L^q(\Omega)$, (with $1 \le q \le \infty$) we denote with $\|v\|_q$ its norm in $L^q(\Omega)$, and for every $v \in H_0^1(\Omega)$ we denote its norm with $\|v\|_{1,2}$. We set

$$Y = H_0^1(\Omega) \cap L^\infty(\Omega), \qquad \|v\|_Y = \|v\|_{1,2} + \|v\|_\infty. \tag{2.4}$$

Notice that (1.1) and (2.2) together with the Sobolev embedding Theorem imply that J is weakly lower semicontinuous in $H_0^1(\Omega)$. In addition, from (1.6) and (1.7) we get that there exists $\langle J'(u), v\rangle$ for every $u \in H_0^1(\Omega)$, and for every v in Y. We have

$$\langle J'(u), v\rangle = \int_\Omega A(x,u)\nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega A_s'(x,u)\nabla u \cdot \nabla u\, v - \int_\Omega f(x,u)v.$$

Moreover, $\langle J'(u), v\rangle$ is continuous with respect to v for every u fixed and the map $J_v' : H_0^1(\Omega) \to \mathbb{R}$, $J_v'(u) = \langle J'(u), v\rangle$ is continuous in $u \in H_0^1(\Omega)$, for every fixed v. We take into account condition (1.7) and we say that a function $u \in H_0^1(\Omega)$ is a critical point of J if it satisfies $\langle J'(u), v\rangle = 0$ for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. Therefore, every critical point $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is a distributional solution of the following nonlinear Dirichlet problem

$$\begin{cases} -\mathrm{div}(A(x,u)\nabla u) + \dfrac{1}{2}A_s'(x,u)\nabla u \cdot \nabla u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{P}$$

Theorem 1 will be deduced from the following result.

Theorem 2.1. *Assume* (1.1)–(1.5), (2.1)–(2.3). *In addition, suppose that the following condition is satisfied*

$$A_s'(x,s) \ \textit{is negative semidefinite for every } s > R_0. \tag{2.5}$$

$$\exists\, m > 2,\ R_1 > 0,\ \textit{such that } 0 < mF(x,s) \le f(x,s)s, \tag{2.6}$$

for every $s > R_1$. Then there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $u \ge 0$, $u \not\equiv 0$, critical point of the functional J.

Define the function

$$\psi(t) = \frac{1}{2}\int_0^t \frac{\beta(s)}{\alpha(s)}\,ds.$$

From (1.4) we deduce that $\psi(t)$ is bounded. Moreover, as $\beta(s)$ is continuous, from (1.1) and (1.5) we get that there exists a positive $\overline{\beta}$ such that

$$\psi'(s)|s| \le \overline{\beta} \quad \text{for every } s \in \mathbb{R}. \tag{2.7}$$

Define the quantity

$$\psi_M = \sup_{\mathbb{R}^+} \psi(s) = \frac{1}{2}\int_0^\infty \frac{\beta(s)}{\alpha(s)}\,ds. \tag{2.8}$$

Theorem 2 will be deduced from the following result.

Theorem 2.2. *Assume* (1.1)–(1.5), (2.1)–(2.3). *Suppose that there exist* m *and* $R_1 > 0$ *such that for almost every* x *in* Ω *we have*

$$\left\{\begin{array}{ll} m > \min\{2+\beta_1, 2\,e^{\psi_M}\}, & \\ 0 < mF(x,s) \le sf(x,s) & \text{for every } s \text{ with } s \ge R_1. \end{array}\right. \tag{2.9}$$

Then, there exists $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $u \ge 0$, $u \not\equiv 0$ *critical point of the functional* J.

Note that, when we consider the model example $f(s) = (s^+)^p$ we get that condition (2.9) is reduced to hypothesis (1.8).

Remark 2.3. Notice that in the semi-linear case, i.e. $A(x,s) \equiv A(x)$, the function $\beta(s)$ (so that also the constant β_1) can be chosen equal to 0. As a consequence, assumption (2.9) is supposed for $m > 2$ and it is reduced to the classical Ambrosetti–Rabinowitz condition (see [1]).

3. Some technical results

In order to prove Theorems 2.1 and 2.2 we will use conditions (1.1), (1.6), (2.3) and either (2.6) or (2.9) to show that J has a geometrical behavior of Mountain Pass type. Then, we will apply an abstract result proved in [2] in order to contruct a Palais–Smale sequence. In this section we will prove some techincal results that will be useful when proving that J satisfies the Palais–Smale condition.

Suppose that there exists a sequence $\{u_n\} \subset Y$, where Y is defined in (2.4), such that

$$\left\{\begin{array}{l} J(u_n) \to c, \\ \|u_n\|_\infty \le 2M_n, \\ |\langle J'(u_n), v\rangle| \le \varepsilon_n \left[\dfrac{\|v\|_Y}{M_n} + \|v\|_{1,2}\right] \quad \text{for all } v \in Y, \end{array}\right. \tag{3.1}$$

where c is a positive constant, $\{M_n\} \subset \mathbb{R}^+ \setminus \{0\}$ is any sequence, and $\{\varepsilon_n\} \subset \mathbb{R}^+$ is a sequence converging to zero.

In the following Lemmas we will prove that for every $\{u_n\} \subset Y$ (where Y is defined in (2.4)) that satisfies (3.1) there exists $u \in Y$, $u \ge 0$, such that, up to a subsequence, u_n strongly converges to u in $H_0^1(\Omega)$.

In the following Lemma we prove that u_n is bounded in $H_0^1(\Omega)$ under the assumptions of Theorem 2.2.

Lemma 3.1. *Assume* (1.1)–(1.5), (2.1), (2.2), (2.9). *Let* $\{u_n\} \subset Y$ *be a sequence that satisfies* (3.1). *Then* u_n *is bounded in* $H_0^1(\Omega)$.

Proof. We will first show that $u_n^- = \min\{u_n, 0\}$ strongly converges to zero in $H_0^1(\Omega)$. We take $v = (u_n^-)e^{-\psi(u_n)}$ as test function in (3.1). Note that this choice is admissible since ψ and ψ' are bounded. Moreover, from (2.7) we have that

$$\|v\|_{1,2} \le c_0(1+\overline{\beta})\|u_n^-\|_{1,2} = C\|u_n^-\|_{1,2}, \qquad \|v\|_\infty \le c'\|u_n\|_\infty.$$

We get

$$\int_\Omega e^{-\psi(u_n)} A(x,u_n)\nabla u_n \cdot \nabla u_n^- + \frac{1}{2}\int_\Omega (u_n^-) e^{-\psi(u_n)}\left[A'_s(x,u_n) - A(x,u_n)\frac{\beta(u_n)}{\alpha(u_n)}\right]\nabla u_n \cdot \nabla u_n \le \varepsilon_n[c' + C\|u_n^-\|_{1,2}].$$

Notice that conditions (1.1) and (1.3) imply that

$$A'_s(x,u_n)\nabla u_n \cdot \nabla u_n \le \frac{\beta(u_n)}{\alpha(u_n)} A(x,u_n)\nabla u_n \cdot \nabla u_n, \tag{3.2}$$

so that, as $u_n^- \le 0$, we deduce that

$$\int_\Omega e^{-\psi(u_n)} A(x,u_n)\nabla u_n \cdot \nabla u_n^- \le \varepsilon_n[c' + C\|u_n^-\|_{1,2}]. \tag{3.3}$$

Hypothesis (1.1) yields

$$\alpha_0\|u_n^-\|_{1,2}^2 \le \varepsilon_n[c' + C\|u_n^-\|_{1,2}]. \tag{3.4}$$

Thus, u_n^- is bounded in $H_0^1(\Omega)$. Then, again from (3.4) we get that $u_n^- \to 0$ strongly in $H_0^1(\Omega)$. Moreover, from (3.3) and by applying hypothesis (1.1) we deduce that

$$\lim_{n\to\infty}\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^- = 0. \tag{3.5}$$

This, together with the fact that $F(x,s) \equiv 0$ for every $s \le 0$, implies that

$$J(u_n^+) \to c, \tag{3.6}$$

where c is defined in (3.1).

In order to prove that u_n^+ is bounded in $H_0^1(\Omega)$, consider first the case in which $\min\{2+\beta_1, 2\,e^{\psi_M}\} = 2e^{\psi_M}$. We take $v = (u_n^+)e^{-\psi(u_n)}$ as test function in (3.1). Note that this choice is admissible since ψ and ψ' are bounded and u_n belongs to Y (where Y is defined in (2.4)). Moreover, from (2.7) we have that $\|v\|_{1,2} \le C_0\|u_n^+\|_{1,2}$, for some $C_0 \in \mathbb{R}^+$ and since ψ is a bounded function we get that $\|v\|_\infty \leq \|u_n\|_\infty$. It results

$$\int_\Omega e^{-\psi(u_n)} A(x,u_n)\nabla u_n \cdot \nabla u_n^+ - \int_\Omega f(x,u_n)u_n^+ e^{-\psi(u_n)} + \frac{1}{2}\int_\Omega e^{-\psi(u_n)} u_n^+ \left[A'_s(x,u_n) - A(x,u_n)\frac{\beta(u_n)}{\alpha(u_n)}\right]\nabla u_n \cdot \nabla u_n \geq -\varepsilon_n(2 + C_0\|u_n^+\|_{1,2}).$$

From (3.2) and as $u_n^+ \geq 0$, we obtain

$$\int_\Omega e^{-\psi(u_n)} A(x,u_n)\nabla u_n \cdot \nabla u_n^+ - \int_\Omega f(x,u_n)u_n^+ e^{-\psi(u_n)} \geq -\varepsilon_n(2 + C_0\|u_n^+\|_{1,2}).$$

Thus we get the following estimate

$$\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+ - e^{-\psi_M}\int_\Omega f(x,u_n)u_n^+ \geq -\varepsilon_n(2 + C_0\|u_n^+\|_{1,2}), \tag{3.7}$$

where ψ_M is defined in (2.8). Notice that (3.6) implies that there exists a positive constant C' such that

$$me^{-\psi_M} J(u_n^+) \leq C'. \tag{3.8}$$

When we subtract (3.7) from (3.8), we obtain

$$\left[\frac{m}{2}e^{-\psi_M} - 1\right]\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+ \leq C' + \varepsilon_n(2 + C_0\|u_n^+\|_{1,2}) + e^{-\psi_M}\int_\Omega [mF(x,u_n) - f(x,u_n)u_n^+].$$

From conditions (2.2) and (2.9) we deduce that there exists a positive function $a_0(x) \in L^1(\Omega)$ such that

$$mF(x,s) \leq f(x,s)s + a_0(x) \quad \text{for every } s \text{ in } \mathbb{R}. \tag{3.9}$$

Furthermore, by (2.9), we can fix ε_0 such that $2e^{\psi_M}(1+\varepsilon_0) \leq m$. Then

$$me^{-\psi_M}/2 \geq (1+\varepsilon_0)$$

and we get

$$\varepsilon_0\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+ \leq c_1 + \varepsilon_n(2 + C_0\|u_n^+\|_{1,2}),$$

so that, from condition (1.1), we get that $\|u_n^+\|_{1,2}$ is bounded. This, together with (3.4), implies that u_n is bounded in $H_0^1(\Omega)$.

Now let us deal with the case in which $\min\{2+\beta_1, 2\,e^{\psi_M}\} = 2+\beta_1$. First, we want to prove that for every positive $\sigma > 0$ and $r > 0$ there exists a positive constant $M_{r,\sigma}$ such that for all n in $\mathbb{N}$, it results

$$\int_{\Omega^-_{n,r}} A(x,u_n)\nabla u_n^+ \cdot \nabla u_n \leq \sigma \int_{\Omega\setminus\Omega^-_{n,r}} A(x,u_n)\nabla u_n^+ \cdot \nabla u_n + M_{r,\sigma} + \varepsilon_n[C_1(1+\|u_n^+\|_{1,2}) + C_n], \tag{3.10}$$

where $\Omega^-_{n,r} = \{x \in \Omega : u_n^+(x) \leq r\}$, C_1 is a positive constant, $C_n \geq 0$, $C_n \to 0$ as n tends to infinity and ε_n is given in (3.1). In order to prove (3.10) we will use an idea of [8]. For every δ, with $0 < \delta < \min\{1, \alpha_0\}$, let us define the function $\vartheta_\delta(s) : \mathbb{R} \to \mathbb{R}^+$ by

$$\vartheta_\delta(s) = \begin{cases} s & \text{for } 0 \leq s \leq r, \\ r + \delta r - \delta s & \text{for } r \leq s \leq (r+\delta r)/\delta, \\ 0 & \text{for } s \geq (r+\delta r)/\delta \text{ or } s \leq 0. \end{cases} \tag{3.11}$$

Consider $v = e^{\psi(u_n)}\vartheta_\delta(u_n)$. From condition (2.7) and from the definition of $\vartheta_\delta(s)$ we have that

$$\|v\|_{1,2} \leq [c_0\|u_n^+\|_{1,2} + c_1\|\vartheta_\delta(u_n)\|_{1,2}], \qquad \|v\|_\infty \leq c_0\|u_n\|_\infty.$$

As $u_n^+ \in Y$ (where Y is defined in (2.4)) and $u_n^+ \geq 0$, from (3.11) we deduce that v belongs to Y and $v \geq 0$. When we take v as test function in (3.1), we get from (1.3)

$$\begin{aligned} \int_\Omega e^{\psi(u_n)} A(x,u_n)\nabla u_n \cdot \nabla\vartheta_\delta(u_n) - \int_\Omega f(x,u_n)e^{\psi(u_n)}\vartheta_\delta(u_n) \\ + \int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n e^{\psi(u_n)}\psi'(u_n)\vartheta_\delta(u_n) \\ \leq \varepsilon_n[c_0(1+\|u_n^+\|_{1,2})] + \frac{1}{2}\int_\Omega \beta(u_n)|\nabla u_n|^2 + \varepsilon_n c_1\|\vartheta_\delta(u_n)\|_{1,2}. \end{aligned}$$

From the definition of ψ and from conditions (1.1), (2.2) it follows

$$\begin{aligned} \int_\Omega e^{\psi(u_n)} A(x,u_n)\nabla u_n \cdot \nabla\vartheta_\delta(u_n) \\ \leq \left[a\left(\frac{r+\delta r}{\delta}\right)^p + b\right] e^{\psi_M}|\Omega| r + \varepsilon_n[c_0(1+\|u_n^+\|_{1,2}) + c_1\|\vartheta_\delta(u_n)\|_{1,2}], \end{aligned}$$

where $|\Omega|$ denotes the Lebesgue measure of Ω. Applying Young inequality we get

$$\int_\Omega e^{\psi(u_n)} A(x,u_n)\nabla u_n \cdot \nabla \vartheta_\delta(u_n) \le \delta\|\vartheta_\delta(u_n)\|_{1,2}^2 + C_{r,\delta} + \varepsilon_n[c_0(1+\|u_n^+\|_{1,2}) + \varepsilon_n c_1^2/4\delta], \tag{3.12}$$

where $C_{r,\delta} = [a((r+\delta r)/\delta)^p + b]e^{\psi_M}|\Omega| r$. On the other hand, since $\delta \in (0,1)$, from (1.1) it results

$$\delta\|\vartheta_\delta(u_n)\|_{1,2}^2 \le \delta\|u_n^+\|_{1,2}^2 \le \frac{\delta}{\alpha_0}\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+. \tag{3.13}$$

From (3.11)–(3.13) we get

$$\int_{\Omega_{n,r}^-} e^\psi(u_n) A(x,u_n)\nabla u_n \cdot \nabla u_n^+ \le C_{r,\delta} + \varepsilon_n[c_0(1+\|u_n^+\|_{1,2}) + \varepsilon_n c_1^2/4\delta] + \frac{\delta}{\alpha_0}\int_\Omega A(x,u_n)\nabla u_n\cdot\nabla u_n^+ + \delta\int_{\Omega_{r,\delta}} e^{\psi(u_n)}A(x,u_n)\nabla u_n\cdot\nabla u_n^+,$$

where $\Omega_{r,\delta} = \{x \in \Omega : r < u_n^+(x) \le (r+\delta r)/\delta\}$. Therefore, we obtain

$$[1-\delta/\alpha_0]\int_{\Omega_{n,r}^-} A(x,u_n)\nabla u_n\cdot\nabla u_n^+ \le C_{r,\delta} + \varepsilon_n[c_0(1+\|u_n^+\|_{1,2}) + \varepsilon_n c_1^2/4\delta] + \delta[e^{\psi_M} + 1/\alpha_0]\int_{\Omega\setminus\Omega_{n,r}^-} A(x,u_n)\nabla u_n\cdot\nabla u_n^+,$$

where ψ_M is defined in (2.8). For every $\sigma > 0$ we fix $\delta < \min\{1,\alpha_0\}$ such that

$$\delta\left(\frac{\alpha_0 e^{\psi_M}+1}{\alpha_0-\delta}\right) < \sigma.$$

Moreover, we set $C_1 = \alpha_0 c_0/(\alpha_0-\delta)$, $C_n = c_1^2\alpha_0\varepsilon_n/4\delta(\alpha_0-\delta)$ and we define $M_{r,\sigma} = \alpha_0 C_{r,\delta}/(\alpha_0-\delta)$ so that (3.10) follows.

Let us now take $v = u_n^+$ as test function in (3.1) and obtain

$$\int_\Omega A(x,u_n)\nabla u_n\cdot\nabla u_n^+ + \frac{1}{2}\int_\Omega A_s'(x,u_n)\nabla u_n\cdot\nabla u_n u_n^+ \ge \int_\Omega f(x,u_n)u_n - \varepsilon_n(1+\|u_n^+\|_{1,2}). \tag{3.14}$$

Let R_0 and R_1 be given in (1.5), (2.9) and consider $r = \max\{R_0, R_1\}$. We define $\beta_r = \sup_{[0,r]} \beta(s)$. From (1.1), (1.3) and (1.5) it follows

$$\begin{aligned}
&\frac{1}{2}\int_\Omega A'_s(x,u_n)\nabla u_n \cdot \nabla u_n u_n^+ \\
&\quad\le \frac{r\beta_r}{2}\int_{\Omega^-_{n,r}} |\nabla u_n^+|^2 + \frac{\beta_1}{2}\int_{\Omega\setminus\Omega^-_{n,r}} \alpha(u_n)|\nabla u_n^+|^2 \\
&\quad\le \frac{\beta_r r}{2\alpha_0}\int_{\Omega^-_{n,r}} A(x,u_n)\nabla u_n \cdot \nabla u_n^+ + \frac{\beta_1}{2}\int_{\Omega\setminus\Omega^-_{n,r}} \alpha(u_n)|\nabla u_n^+|^2.
\end{aligned}$$

We set $r\beta_r/\alpha_0 = C_0$. From (1.1) and (3.10) we deduce

$$\begin{aligned}
&\frac{1}{2}\int_\Omega A'_s(x,u_n)\nabla u_n \cdot \nabla u_n u_n^+ \\
&\quad\le \frac{C_0}{2}\sigma\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+ + \frac{\beta_1}{2}\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+ \\
&\qquad + \frac{C_0}{2}M_{r,\sigma} + \frac{C_0}{2}\varepsilon_n[C_1\|u_n^+\|_{1,2} + C_n].
\end{aligned} \tag{3.15}$$

From (3.14) and (3.15) we get

$$\begin{aligned}
&\frac{1}{2}[2 + C_0\sigma + \beta_1]\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+ - \int_\Omega f(x,u_n)u_n \\
&\qquad\qquad \ge -\varepsilon_n[C'_n + C_3 + C_2\|u_n^+\|_{1,2}] - \frac{C_0}{2}M_{r,\sigma},
\end{aligned} \tag{3.16}$$

where $C'_n \to 0$ as n tends to infinity. Notice that (3.6) implies that there exists a positive constant C' such that

$$mJ(u_n^+) \le C'. \tag{3.17}$$

When we subtract (3.16) from (3.17), we obtain

$$\begin{aligned}
&\frac{1}{2}[m - 2 - \beta_1 - C_0\sigma]\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+ \\
&\quad\le C' + \varepsilon_n[C'_n + C_3 + C_2\|u_n^+\|_{1,2}] + \frac{C_0}{2}M_{r,\sigma} + \int_\Omega [mF(x,u_n) - f(x,u_n)u_n].
\end{aligned}$$

From (3.9) it follows

$$\begin{aligned}
&\frac{1}{2}[m - 2 - \beta_1 - C_0\sigma]\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+ \\
&\qquad\qquad \le \|a_0\|_1 + C' + \frac{C_0}{2}M_{r,\sigma} + \varepsilon_n[C'_n + C_3 + C_2\|u_n^+\|_{1,2}].
\end{aligned}$$

Since $m > 2 + \beta_1$, there exists $\varepsilon_0 > 0$ such that $m - 2 \geq \beta_1 + \varepsilon_0$. Let σ be fixed by $\sigma = \varepsilon_0/2C_0$. Then, we get

$$\frac{\varepsilon_0}{4}\|u_n^+\|_{1,2} \leq \|a_0\|_1 + C' + \varepsilon_n[C_n' + C_3 + C_2\|u_n^+\|_{1,2}] + \frac{C_0}{2}M_{r,\sigma},$$

so that also in this case we have proved that u_n^+ is bounded in $H_0^1(\Omega)$. This, together with (3.4), implies again that u_n is bounded in $H_0^1(\Omega)$. □

In the following lemma we prove the boundedness of a sequence $\{u_n\} \subset Y$ that satisfies (3.1) under the assumptions of Theorem 2.1.

LEMMA 3.2. *Assume* (1.1)–(1.5), (2.1), (2.2), (2.5). *Let* $\{u_n\} \subset Y$ *be a sequence that satisfies* (3.1). *Then* u_n *is bounded in* $H_0^1(\Omega)$.

PROOF. We argue as in Lemma 3.1 and we still deduce that $u_n^- \to 0$ in $H_0^1(\Omega)$ and that (3.10) holds. Then, we fix $r = \max\{R_0, R_1\}$, where R_0 and R_1 are defined in (1.5) and (2.6) respectively. Hypothesis (2.5) yields

$$\frac{1}{2}\int_\Omega A_s'(x,u_n)\nabla u_n^+ \cdot \nabla u_n u_n \leq \frac{1}{2}\beta_r r \int_{\Omega_{n,r}^-} |\nabla u_n^+|^2 \leq \frac{\beta_r r}{2\alpha_0}\int_{\Omega_{n,r}^-} A(x,u_n)\nabla u_n \cdot \nabla u_n^+,$$

where $\beta_r = \max_{[0,r]}\beta(s)$, $\Omega_{n,r}^- = \{x \in \Omega,\, 0 \leq u_n^+(x) \leq r\}$. Let us set $C_0 = \beta_r r/(2\alpha_0)$ and take $v = u_n^+$ as test function in (3.1). From (3.10) we deduce that

$$\begin{aligned}[1 + C_0\sigma]\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+ - \int_\Omega f(x,u_n)u_n^+ \\ \geq -C_0 M_{r,\sigma} - \varepsilon_n[C_2 + C_1\|u_n^+\|_{1,2} + C_n'],\end{aligned} \tag{3.18}$$

where $C_n' \to 0$ as n tends to infinity. Note that (3.6) implies that there exists a positive constant C such that

$$mJ(u_n^+) \leq C. \tag{3.19}$$

When we subtract (3.18) from (3.19) we get

$$\begin{aligned}\frac{1}{2}[m - 2 - 2C_0\sigma]\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n^+ \\ \leq \|a_0\|_1 + C + C_0 M_{r,\sigma} + \varepsilon_n[C_2 + C_1\|u_n^+\|_{1,2} + C_n'].\end{aligned}$$

As $m > 2$ there exists ε_0 such that $m \geq 2 + \varepsilon_0$; we fix $\sigma = \varepsilon_0/4C_0$. From (1.1) we obtain

$$\frac{\alpha_0}{4}\varepsilon_0\|u_n^+\|_{1,2}^2 \leq \|a_0\|_1 + C + \varepsilon_n[C_2 + C_1\|u_n^+\|_{1,2} + C_n'] + C_0 M_{r,\sigma},$$

then u_n is bounded in $H_0^1(\Omega)$. □

REMARK 3.3. Exponential functions have often been used when dealing with elliptic problem with quadratic gradient terms (see [5] and the references therein).

The use of the map ψ combined with exponential functions has been introduced in [14] and in [7] in order to study problems of this kind without assuming any sign condition on the quadratic gradient term.

Remark 3.4. Suppose that, instead of (1.3), (1.4) and (1.5), for every s in $\mathbb{R}$, the following assumptions hold

$$|A'_s(x,s)| \le \beta_1, \quad |A'_s(x,s)s| \le \beta_2.$$

Moreover, instead of (2.9), we assume that there exist m and $R_1 > 0$ such that

$$\begin{cases} m > 2 + \beta_2/\alpha_0, \\ 0 < mF(x,s) \le f(x,s)s, \quad s > R_1. \end{cases}$$

Then, we can prove Lemma 3.1 in a more direct way. Indeed, we take $v = u_n$ as test function in (3.1). Since $J(u_n)$ is bounded, from (3.9) we get

$$\frac{1}{2}(m - 2 - \beta_2/\alpha_0)\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla u_n \le c_0 + c_1\varepsilon_n \|u_n\|_{1,2}.$$

Since $m > 2 + \beta_1/\alpha_0$ we deduce that u_n is bounded in $H^1_0(\Omega)$.

By the preceding results we get that there exists u in $H^1_0(\Omega)$ with $u \ge 0$ such that, up to a subsequence, u_n weakly converges to u in $H^1_0(\Omega)$.

Define the function

$$G_k(s) = (s-k)^+ \quad \text{for every } s > 0. \tag{3.20}$$

In order to prove that u_n is compact in $H^1_0(\Omega)$, we will first prove that u is in $L^\infty(\Omega)$. This will be done in the following result.

Lemma 3.5. *Assume conditions* (1.1)–(1.5), (2.1), (2.2). *Let* $\{u_n\}$ *be a sequence in* Y *that satisfies* (3.1) *and that weakly converges in* $H^1_0(\Omega)$ *to a function* $u \ge 0$. *Then* u *belongs to* $L^\infty(\Omega)$.

Proof. Since u_n^+ belongs to Y and $u_n^+ \ge 0$, we can take $v = e^{\psi(u_n)}G_k(u_n^+)$ as test function in (3.1). We use (1.1), (1.3) and (2.2) and we obtain

$$\int_{\Omega^+_{n,k}} A(x,u_n)\nabla u_n \cdot \nabla u_n^+ e^{\psi(u_n)} \le \int_\Omega [a + b|u_n|^p] G_k(u_n^+)e^{\psi(u_n)} + c\varepsilon_n, \tag{3.21}$$

where $\Omega^+_{n,k} = \{x \in \Omega : u_n^+(x) > k\}$. Since $0 \le G_k(u_n^+) \le u_n^+$, from hypothesis (1.1) and by applying Hölder inequality we get

$$\begin{aligned} \alpha_0 \int_\Omega |\nabla G_k(u_n^+)|^2 \le{}& c\varepsilon_n + c_0\bigg(\int_\Omega |G_k(u_n^+)|^{2^*}\bigg)^{1/2^*} |\Omega^+_{n,k}|^{1-1/2^*} \\ &+ c_0\bigg(\int_{\Omega^+_{n,k}} |u_n^+|^{2^*}\bigg)^{(p+1)/2^*} |\Omega^+_{n,k}|^{1-(p+1)/2^*}, \end{aligned} \tag{3.22}$$

where $c_0 = \max\{a,b\}e^{\psi_M}$ and $|\Omega^+_{n,k}|$ denotes the Lebesgue measure of $\Omega^+_{n,k}$. Notice that

$$\begin{aligned}\int\limits_{\Omega^+_{n,k}} |u_n^+|^{2^*} &= \int\limits_{\Omega^+_{n,k}} |u_n^+ - k + k|^{2^*} \\ &\le c_1 \int\limits_{\Omega^+_{n,k}} |G_k(u_n^+)|^{2^*} + c_1 k^{2^*}|\Omega^+_{n,k}| \\ &\le c_2\Big(\int\limits_\Omega |\nabla G_k(u_n^+)|^2\Big)^{2^*/2} + c_1 k^{2^*}|\Omega^+_{n,k}|.\end{aligned}$$

Thereofore, we obtain

$$\Big[\int\limits_{\Omega^+_{n,k}} |u_n^+|^{2^*}\Big]^{(p+1)/2^*} \le c_3\Big[\int\limits_\Omega |\nabla G_k(u_n^+)|^2\Big]^{(p+1)/2} + c_3 k^{p+1}|\Omega^+_{n,k}|^{(p+1)/2^*}. \tag{3.23}$$

From (3.22), (3.23) and applying Sobolev embedding Theorem, we deduce

$$\begin{aligned}\alpha_0\int\limits_\Omega |\nabla G_k(u_n^+)|^2 &\le c\varepsilon_n + c_4\Big(\int\limits_\Omega |\nabla G_k(u_n^+)|^2\Big)^{1/2}|\Omega^+_{n,k}|^{1-1/2^*} \\ &\quad + C_0\Big\{\Big[\int\limits_\Omega |\nabla G_k(u_n^+)|^2\Big]^{(p+1)/2} + k^{p+1}|\Omega^+_{n,k}|^{(p+1)/2^*}\Big\}|\Omega^+_{n,k}|^{1-(p+1)/2^*},\end{aligned}$$

where $C_0 = c_0c_3$. By Young inequality we get

$$\begin{aligned}\alpha_0\int\limits_\Omega |\nabla G_k(u_n^+)|^2 &\le c\varepsilon_n + C_0k^{p+1}|\Omega^+_{n,k}| + \frac{\alpha_0}{4}\int\limits_\Omega |\nabla G_k(u_n^+)|^2 \\ + C_0\Big[\int\limits_\Omega |\nabla G_k(u_n^+)|^2\Big]^{(p-1)/2}&\Big[\int\limits_\Omega |\nabla G_k(u_n^+)|^2\Big]|\Omega^+_{n,k}|^{1-(p+1)/2^*} + \frac{c_4^2}{\alpha_0}|\Omega^+_{n,k}|^{2(1-1/2^*)}.\end{aligned} \tag{3.24}$$

Since u_n^+ belongs to Y and it is bounded in $H^1_0(\Omega)$, there exists k_0 such that for every $k \ge k_0$ and for every n in $\mathbb{N}$ it holds

$$C_0\Big[\int\limits_\Omega |\nabla G_k(u_n^+)|^2\Big]^{(p-1)/2}|\Omega^+_{n,k}|^{1-(p+1)/2^*} \le \frac{\alpha_0}{4}. \tag{3.25}$$

When we use (3.25) in (3.24) we get

$$\frac{\alpha_0}{2}\int\limits_\Omega |\nabla G_k(u_n^+)|^2 \le c\varepsilon_n + C_0k^{p+1}|\Omega^+_{n,k}| + \frac{c_4^2}{2}|\Omega^+_{n,k}|^{2(1-1/2^*)}. \tag{3.26}$$

From (3.26), by applying Sobolev embedding theorem we get

$$\Big[\int_\Omega |G_k(u_n^+)|^{2^*}\Big]^{2/2^*} \le C_1\varepsilon_n + C_2 k^{p+1}|\Omega_{n,k}^+| + C_3|\Omega_{n,k}^+|^{2(1-1/2^*)}.$$

Applying Fatou Lemma and taking into account that $u \ge 0$ we obtain

$$\Big[\int_\Omega |G_k(u)|^{2^*}\Big]^{2/2^*} = \Big[\int_\Omega |G_k(u^+)|^{2^*}\Big]^{2/2^*} \le C_4 k^{p+1}|\Omega_k^+| + C_5|\Omega_k^+|^{2(1-1/2^*)},$$

where $\Omega_k^+ = \{x \in \Omega : u(x) > k\}$. Applying [11, (3.4), Chapter 5] we get that there exists $k_0 \ge k_1$ such that $|\Omega_{k_0}^+| = 0$, that is $u \in L^\infty(\Omega)$. □

REMARK 3.6. The boundedness of a nonnegative weak limit u of a sequence $u_n \in Y$ that satisfies (3.1) will be fundamental in what follows. An argument similar to the one used in Lemma 3.5 has been used in [4].

In the following Lemma we will take advantage of the boundedness of u in order to prove that u_n converges to u in $H_0^1(\Omega)$.

LEMMA 3.7. *Assume conditions* (1.1)–(1.5), (2.1), (2.2). *Let* $\{u_n\}$ *be a sequence in* Y *that satisfies* (3.1) *and that weakly converges in* $H_0^1(\Omega)$ *to a function* $u \ge 0$. *If* u *belongs to* $L^\infty(\Omega)$, u_n *strongly converges to* u *in* $H_0^1(\Omega)$.

PROOF. Let us consider $v = e^{\psi(u_n)}(u_n - u)^+$. From (1.3) we deduce that v belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover, from (2.7) we have that $\|v\|_{1,2} \le c_0[\|u_n\|_{1,2} + \|u_n - u\|_{1,2}] \le c$, $\|v\|_\infty \le c\|u_n\|_\infty$. Therefore, we can take v as test function in (3.1) and we get

$$\begin{aligned}\int_\Omega e^{\psi(u_n)} A(x,u_n)\nabla u_n\cdot\nabla(u_n-u)^+ + \int_\Omega A(x,u_n)\nabla u_n\cdot\nabla u_n e^{\psi(u_n)}(u_n-u)^+ \frac{\beta(u_n)}{\alpha(u_n)} \\ \le \frac{1}{2}\int_\Omega \beta(u_n)|\nabla u_n|^2 e^{\psi(u_n)}(u_n-u)^+ + c_1\varepsilon_n + \int_\Omega f(x,u_n)(u_n-u)^+ e^{\psi(u_n)}.\end{aligned}$$

From the definition of ψ and from conditions (1.1) and (1.3) we obtain

$$\int_\Omega e^{\psi(u_n)} A(x,u_n)\nabla u_n\cdot\nabla(u_n-u)^+ \le c_1\varepsilon_n + \int_\Omega |f(x,u_n)|(u_n-u)^+ e^{\psi(u_n)}. \tag{3.27}$$

From (2.2), since $p < (N+2)/(N-2)$, we deduce that the last term of (3.27) goes to zero as n tends to infinity. Therefore, it follows

$$\limsup_{n\to\infty} \int_\Omega e^{\psi(u_n)} A(x,u_n)\nabla u_n\cdot\nabla(u_n-u)^+ \le 0. \tag{3.28}$$

Now let us consider $v = e^{-\psi(u_n)}(u-u_n)^+$. Lemma 3.5 and (1.3) imply that we can take v as test function in (3.1). We obtain

$$\int_\Omega e^{-\psi(u_n)} A(x,u_n)\nabla u_n \cdot \nabla(u-u_n)^+ - \int_\Omega e^{-\psi(u_n)} f(x,u_n)(u-u_n)^+ \\ + \frac{1}{2}\int_\Omega e^{-\psi(u_n)}(u-u_n)^+ \left[A'_s(x,u_n) - A(x,u_n)\frac{\beta(u_n)}{\alpha(u_n)}\right]\nabla u_n \cdot \nabla u_n \geq -c_1\,\varepsilon_n.$$

From conditions (1.1) and (1.3) we deduce

$$\int_\Omega e^{-\psi(u_n)} A(x,u_n)\nabla u_n \cdot \nabla(u-u_n)^+ \geq -c_1\varepsilon_n + \int_\Omega e^{-\psi(u_n)} f(x,u_n)(u-u_n)^+.$$

Condition (2.2) implies that the last term of the previous inequality tends to zero as n goes to infinity. We take into account that $(u-u_n)^+ = -(u_n-u)^-$ and we obtain

$$\limsup_{n\to\infty} \int_\Omega e^{-\psi(u_n)} A(x,u_n)\nabla u_n \cdot \nabla(u_n-u)^- \leq 0. \tag{3.29}$$

From hypothesis (1.1) we get

$$\alpha_0 e^{-\psi_M}\|u_n-u\|_{1,2}^2 \leq e^{-\psi_M}\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla(u_n-u)^+ \\ + e^{-\psi_M}\int_\Omega A(x,u_n)\nabla u_n \cdot \nabla(u_n-u)^- - e^{-\psi_M}\int_\Omega A(x,u_n)\nabla u \cdot \nabla(u_n-u),$$

where ψ_M is defined in (2.8). Condition (1.6) allows us to apply Lebesgue Dominated Convergence Theorem to deduce that

$$\lim_{n\to\infty}\int_\Omega A(x,u_n)\nabla u \nabla(u_n-u) = 0. \tag{3.30}$$

Finally, (3.28)–(3.30) imply that u_n strongly converges to u in $H_0^1(\Omega)$. □

4. Proofs of Theorems 2.1 and 2.2

We are now able to prove Theorems 2.1 and 2.2.

Proof of Theorem 2.1 completed. From (1.1) and (2.3) we have that $u_0 = 0$ is a strict local minimum of J. Denote with φ the first positive eigenfunction of the Laplacian operator in Ω with homogeneus boundary conditions. Conditions (1.6) and (2.6) imply that the function $u_1 = \bar{t}\varphi_1$ is such that $J(u_1) < 0$, for $\bar{t}$ sufficiently large. Define

$$\Gamma = \{\gamma : [0,1] \to Y, \text{ continuous and such that } \gamma(0) = 0,\ \gamma(1) = u_1\}.$$

From the geometrical properties of J we deduce that

$$c = \inf_{\Gamma} \max_{[0,1]} J(\gamma(t)) > \max\{J(0), J(u_1)\} = 0.$$

We follow [2] and we take $\{\gamma_n\} \in \Gamma$ a sequence of paths such that

$$c \le J(\gamma_n(t)) \le c + \frac{1}{2n},$$

for every fixed $n \in \mathbb{N}$. Consider $M_n = \max_{[0,1]} \|\gamma(t)\|_\infty \ge \|u_1\|_\infty$. Notice that $|\,\|\cdot\|\,| = \|\cdot\|_\infty / M_n + \|\cdot\|_{1,2}$ is a norm in Y, equivalent to the norm $\|\cdot\|_\infty + \|\cdot\|_{1,2}$. We apply Theorem 2.1 of [2] and we get the existence of a sequence $u_n = \overline{\gamma}_n(t_n) \in \overline{\gamma}_n[0,1] \subset Y$, such that

$$\begin{gathered}
c \le \max J(\overline{\gamma}_n(t)) \le c + \frac{1}{2n}, \\
\max_{[0,1]} |\,\|\overline{\gamma}_n(t) - \gamma(t)\|\,|_n \le \sqrt{\frac{1}{n}}, \\
c - \frac{1}{n} \le J(u_n) \le c + \frac{1}{2n}, \\
|\langle J'(u_n), v\rangle| \le \sqrt{\frac{1}{n}} |\,\|v\|\,|_n \quad \text{for all } v \in Y.
\end{gathered}$$

Moreover, for n large enough, it results

$$\|u_n\|_\infty = \|\overline{\gamma}_n(t_n)\|_\infty \le \|\overline{\gamma}_n(t_n) - \gamma_n(t_n)\|_\infty + \|\gamma_n(t_n)\|_\infty \le 2M_n.$$

Thus, $u_n \in Y$ satisfies (3.1). Lemma 3.2 implies that u_n is bounded in $H_0^1(\Omega)$. Therefore, there exists u in $H_0^1(\Omega)$ with $u \ge 0$ such that, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Applying Lemma 3.5 we deduce that $u \in Y$. Lemma 3.7 implies that $u_n \to u$ strongly in$H_0^1(\Omega)$, so that u is a critical point of J, i.e.

$$\int_\Omega A(x,u)\nabla u \cdot \nabla v + \frac{1}{2}\int_\Omega A'_s(x,u)\nabla u \cdot \nabla u v - \int_\Omega f(x,u)v = 0,$$

for every $v \in H_0^1(\Omega) \cap L^\infty(\Omega)$. That is u is a distributional solution of problem (P). Moreover, $J(u_n)$ converges to $J(u)$, then $J(u) = c$ so that $u \not\equiv 0$. □

Proof of Theorem 2.2 completed. From (1.1), (1.6), (2.2), (2.3), and (2.9) we deduce that J has a geometrical behavior of Mountain Pass type. Theorem 2.1 of [2] implies the existence of a sequence $\{u_n\} \subset Y$ that satisfies (3.1). Lemma 3.1 implies that u_n is bounded in $H_0^1(\Omega)$. In addition, from Lemma 3.7 it follows that $u_n \to u$ strongly in $H_0^1(\Omega)$, so that u is a critical point of J. Moreover, $J(u) = c$, which yields $u \not\equiv 0$. □

REMARK 4.1. In [2] and [8] assumptions (1.6) and (1.7) are imposed and the existence of a critical point is proved under the additional hypotheses that there exist $R > 0$ and $\delta \in (0, m-2)$ such that almost everywhere in Ω

$$A'_s(x,s)s \geq 0 \qquad \text{for all } s : |s| \geq R, \tag{4.1}$$

$$[\delta A(x,s) - A'_s(x,s)s]\xi \cdot \xi \geq \alpha_2 |\xi|^2 \quad \text{for all } s : |s| \geq R. \tag{4.2}$$

Inequality (4.1) is a sign condition used in the literature in order to get the L^∞ summability of a critical point of J; hypothesis (4.2) is a sort of Ambrosetti-Rabinowitz type condition on $A(x,s)$ and it is used in order to prove that a Palais–Smale sequence is bounded in $H^1_0(\Omega)$.

Assumptions (1.3)–(1.5) permit to handle matrices $A(x,s)$ that do not satisfy condition (4.1), as Theorems 2.1 and 2.2 show.

EXAMPLE 4.2. Let us give an example of a nonmonotonous coefficient that satisfies all the assumptions of Theorem 2.2. Consider $\alpha(s)$ defined as follows

$$\alpha(s) = \begin{cases} C_0 & \text{for } s \leq 2k_0\pi, \\ C_0 + \displaystyle\int_{2k_0\pi}^{s} \frac{\sin t}{t^2}\, dt & \text{for } s > 2k_0\pi, \end{cases}$$

where k_0 is sufficiently large and $C_0 > 1/2k_0\pi$, so that hypothesis (1.1) is satisfied with $\alpha_0 = C_0 - 1/(2k_0\pi)$. Moreover, hypothesis (1.2) is satisfied with $M = C_0$. As $\alpha'(s) = \sin s/s^2$ for every $s \geq 2k_0\pi$ and $\alpha'(s) \equiv 0$ for every $s < 2k_0\pi$, we have that α also satisfies (1.3). Indeed, it is enough to take $\beta : \mathbb{R} \to \mathbb{R}^+$ defined by

$$\beta(s) = \begin{cases} 0 & \text{for } s < 0, \\ \dfrac{s}{(2k_0\pi)^3} & \text{for } 0 \leq s < 2k_0\pi, \\ \dfrac{1}{s^2} & \text{for } s > 2k_0\pi. \end{cases}$$

With this choice of β it results that condition (1.4) is satisfied. Moreover, $\beta(s)s/\alpha(s) \leq \beta_1$ with $\beta_1 \geq 1/(2C_0k_0\pi - 1)$, so that also hypothesis (1.5) holds.

We take

$$2 < N < 8C_0k_0\pi - 2 \quad \text{and} \quad \frac{2C_0k_0\pi}{2C_0k_0\pi - 1} < p < \frac{N+2}{N-2}.$$

So that the nonlinearity $f(s) = (s^+)^p$ satisfies (2.1)–(2.3) and (2.9). Then Theorem 2.2 yields the existence of a critical point of the functional $J : H^1_0(\Omega) \to \mathbb{R}$, defined by

$$J(v) = \frac{1}{2}\int_\Omega \alpha(v)|\nabla v|^2 - \frac{1}{p+1}\int_\Omega (v^+)^{p+1}.$$

Remark 4.3. As we noticed in the introduction condition (1.5) implies that also (1.7) holds, i.e. there exists a positive constant Λ such that

$$|A'_s(x,s)\xi\cdot\xi s| \le \Lambda|\xi|^2 \quad \text{for every } |s|\ge R.$$

This assumption on the behavior of $A'_s(x,s)$ for large s can be obtained as well as a consequence of the hypotheses on $A(x,s)$ imposed in [2] and [8]. Indeed, from (4.2) we deduce

$$|A'_s(x,s)\xi\cdot\xi s| \le (\delta\beta_0+\alpha_2)|\xi|^2 \quad \text{for every } |s|\ge R.$$

Remark 4.4. Assume conditions (1.1)–(1.5), (2.2) and (2.3). In addition assume (4.1) and suppose that (2.9) is satisfied for

$$m > 2+\beta_1, \tag{4.3}$$

i.e. $\min\{2e^{\gamma_M}, 2+\beta_1\} = 2+\beta_1$. Then, condition (4.2) is satisfied for every $s>R$. Indeed, there exists $\delta\in(0,m-2)$ such that $\delta>\beta_1$ and it holds

$$\delta A(x,s)\xi\cdot\xi > \beta_1\alpha(s)|\xi|^2 \ge \beta(s)s|\xi|^2 \ge A'_s(x,s)\xi\cdot\xi s \ge 0.$$

So that, if we assume (4.1) in addition to all the hypotheses of Theorem 2.2, and (2.9) is satisfied when (4.3) holds, then Theorem 2.2 can be obtained as a consequence of the results proved in [2] or [8].

On the other hand, if (2.9) is satisfied when $m>2e^{\psi_M}$, Theorem 2.2 implies the existence of a critical point of J also in the case in which (4.1) holds but (4.2) may not be satisfied. Indeed, there exist functions that satisfy (4.1) and all the hypotheses of Theorem 2.2 but do not satisfy (4.2). Let us consider the following example. We set

$$S=\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Let us define the function $\beta(s):\mathbb{R}\to\mathbb{R}^+$ by

$$\beta(t)=\begin{cases} 0 & \text{for } n-1\le t<n-\varepsilon_n \text{ or } t\le 0,\\ \dfrac{1}{n\varepsilon_n}(t-n+\varepsilon_n) & \text{for } n-\varepsilon_n\le t<n,\\ -\dfrac{1}{n\varepsilon_n}(t-n-\varepsilon_n) & \text{for } n\le t<n+\varepsilon_n,\end{cases}$$

where $\varepsilon_n = 1/2nS$. Notice that $\beta(s)\in L^1(\mathbb{R})$. Finally, set $m=5/4+\sqrt{6}/2$. We consider $N<6$ and Ω a bounded open set in $\mathbb{R}^N$. Let us consider the functional $J: H_0^1(\Omega)\to\mathbb{R}$ defined by

$$J(u)=\frac{1}{2}\int_\Omega \alpha(u)|\nabla u|^2 - \frac{1}{m}\int_\Omega |u|^m,$$

where $\alpha(s)=1+\int_0^s\beta(t)\,dt$. Notice that $2<m<(2N)/(N-2)$. We have that α satisfies conditions (1.1)–(1.4) with $\alpha_0=1$ and $M=1$. The function $\alpha(s)$

satisfies (1.5) with $\beta_1 = 2/3$ and with no smaller number than $2/3$. In addition, by easy computations it results that

$$\psi_M = \frac{1}{2}\int_0^{\infty}\frac{\beta(t)}{\alpha(t)}\,dt = \log\left(\sqrt{\frac{3}{2}}\right).$$

So that $2 + \beta_1 > 2e^{\psi_M}$. Since $m = 5/4 + \sqrt{6}/2$, we have that $m > 2e^{\psi_M}$ so that (2.9) is satisfied and Theorem 2.2 implies the existence of a critical point of J. Notice that α is increasing so that condition (4.1) is satisfied, while (4.2) does not hold. Indeed, for every $\delta \in (0, m-2)$ we have

$$\delta\alpha(s) - \beta(s)s \le (m-2)\alpha(s) - \beta(s)s.$$

We choose $s = n$ and we get

$$\left(\frac{\sqrt{6}}{2} - \frac{3}{4}\right)\alpha(s) - 1 \le 2\left(\frac{\sqrt{6}}{2} - \frac{3}{4}\right) - 1 = -\frac{5}{2} + \sqrt{6} < 0,$$

which implies that (4.2) is not satisfied.

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BENEDETTA PELLACCI
S.I.S.S.A.
v. Beirut 2-4
34013, Trieste, ITALY

E-mail address: pellacci@mat.uniroma1.it