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ON STABILITY OF FIXED POINTS OF MULTIVALUED MAPS

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Abstract. The criterion for the stability of a fixed point of a compact or condensing multimap in a Banach space with respect to a small perturbation is expressed in terms of its topological index.

1. Introduction

In this paper we consider necessary and sufficient conditions for the stability of an isolated fixed point of a convex-valued multivalued map in a Banach space. The coresponding results of B. O'Nill (see [4]) and G. Gabor (see [3]) are generalized. Section 2 contains definitions used in the paper. Section 3 is devoted to the research of the criterion for the stability of an isolated singular point for a completely continuous multivalued vector field in a Banach space. In Section 4 we extend the result to the case of a multivalued vector field condensing with respect to the Hausdorff measure of noncompactness.

2. Definitions

Let E be a Banach space; $Kv(E)$ denote a collection of all nonempty convex compact subsets of E; $G \subset E$ be an open set.

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We will consider an upper semicontinuous (u.s.c.) multimap $F : G \to Kv(E)$ and a corresponding multifield $\Phi = i - F : G \to Kv(E)$, $\Phi(x) = x - F(x)$.

Let χ be a Hausdorff measure of noncompactness (MNC) in E:

$$\chi(\Omega) = \inf\{\varepsilon > 0 : \Omega \text{ has a finite } \varepsilon\text{-net}\}.$$

Definition 1. An u.s.c. multimap $F : G \to Kv(E)$ is said to be (k,χ)-*condensing*, $0 \le k < 1$, provided $\chi(F(\Omega)) \le k\chi(\Omega)$ for every bounded $\Omega \subset G$ (see, e.g. [1], [2]).

Definition 2. For $\varepsilon > 0$, a (k',χ)-*condensing* multifield $\Phi_\varepsilon = i - F_\varepsilon : G \to Kv(E)$, $\Phi_\varepsilon(x) = x - F_\varepsilon(x)$ is said to be ε-close to a (k,χ)-*condensing* multifield $\Phi : G \to Kv(E)$ provided

$$\rho(\Phi_\varepsilon(x), \Phi(x)) = \sup_{y \in \Phi_\varepsilon(x)} \operatorname{dist}(y, \Phi(x)) < \varepsilon$$

for all $x \in G$.

Definition 3. (cf. [3], [4]). An isolated singular point $x_* \in G$, $0 \in \Phi(x_*)$ of a (k,χ)-*condensing* multifield Φ is said to be stable provided for every sufficiently small neighbourhood $U = U(x)$ there exists such $\varepsilon > 0$ that every ε-*close* to Φ on $\overline{U}$ (k',χ)-*condensing* multifield Φ_ε has a singular point in U.

In particular, an isolated singular point $x_* \in G, 0 \in \Phi(x_*)$ of a completely continuous (i.e. corresponding to an u.s.c. and compact multimap) multifield Φ is said to be stable provided for every sufficiently small neighbourhood $U = U(x)$ there exists such $\varepsilon > 0$ that every ε-*close* to Φ on $\bar{U}$ completely continuous multifield Φ_ε has a singular point in U.

3. Criterion of stability of an isolated singular point of a completely continuous multifield in a Banach space

Theorem 1. *Let $x_* \in G$ be an isolated singular point of a completely continuous multifield $\Phi = i - F : G \to Kv(E)$. The condition of nontriviality of its topological index*

$$\operatorname{ind}(x_*, \Phi) \ne 0$$

is sufficient, and in case $\Phi(x_*) = \{0\}$, also necessary condition for the stability of x_*.

Proof. (1) *Sufficiency.* A sufficiency follows from properties of a topological degree for a completely continuous multifields. (see [1], [2])

(2) *Necessity.* Let $x_* \in G$ be an isolated singular point of a multifield Φ such that $\Phi(x_*) = \{0\}$ and $\operatorname{ind}(x_*, \Phi) = 0$. Let us show that the point x_* is unstable. For simplicity we are considering the case when $x_* = 0$. Let us fix arbitrary $\varepsilon > 0$ and take any $\varepsilon_1 > 0$ such that $0 \le \varepsilon_1 \le \varepsilon/3$.

We will choose a ball $\overline{B}_R \subset E$ with the center at zero and the radius $R > 0$ such that

1) $R \le (\varepsilon - 3\varepsilon_1)/2$;
2) x_* is the only singular point of Φ in $\overline{B}_R$;
3) for all $x \in \overline{B}_R$ we have that

$$\|\Phi(x)\| := \sup\{\|\phi\| : \phi \in \Phi(x)\} \le \varepsilon_1.$$

We will construct multifield Φ_ε on a $\overline{B}_R$ *ε-close* to Φ and such that Φ_ε is fixed point free.

Since multifield Φ is fixed point free on the boundary S_R there exists $\nu > 0$ such that $\|\Phi(x)\| \ge \nu$ for all $x \in S_R$.

Now we will choose a δ-approximation f_δ of a multimap $F(x)$, where $0 < \delta < \min\{\nu/2, \varepsilon_1\}$ such that coresponding field $\phi_\delta = i - f_\delta$ is fixed point free on sphere S_R and its degree $\deg(\phi_\delta, S_R)$ is equal to zero (see [1], [2]).

According to the definition of a δ-approximation for any $x \in S_R$ there exists $x' \in S_R : \|x - x'\| < \delta$ such that

$$f_\delta(x) \cup F(x) \subset W_\delta(F(x')). \tag{1}$$

It means that

$$\operatorname{dist}(x, \overline{\operatorname{co}}\,(f_\delta(x) \cup F(x)) \ge \operatorname{dist}(x', F(x')) - 2\delta \ge \nu - 2\delta = \xi > 0.$$

Lemma 1. *There exist $\beta > 0$ and $\alpha > 0$ such that for all $\lambda \in [1-\alpha; 1]$ and $x \in S_R$ $\|x - \lambda f_\delta(x)\| < \beta$.*

Proof. Let us assume a contrary. Then we can choose sequences $\beta_i \to 0$, $\alpha_i \to 0$ and $\lambda_i \in [1-\alpha_i; 1]$, $x_i \in S_R$ such that $\|x_i - \lambda_i f_\delta(x_i)\| < \beta_i$.

Since map f_δ is completely continuous we my assume w.l.o.g. that $f_\delta(x_i) \to y$ and since $\lambda_i \to 1$ then $\lambda_i f_\delta(x_i) \to y$. Then $x_i = \lambda_i f_\delta(x_i) + h_i$, where $\|h_i\| < \beta_i$.

Therefore $h_i \to 0$, $x_i \to y \in S_R$ and $f_\delta(y) = y$ since f_δ is continuous. But this contraclicts to the fact that f_δ is fixed point free. □

Now we will choose $R_1 > 0$ such that:

1) $R - R_1 < \min\{\xi, \beta\}$,
2) $R_1/R > 1 - \alpha$.

Let us define the extension of $F : S_R \to Kv(E)$ on a ball layer $P = \{x \in E : R_1 \le \|x\| \le R\}$ as a complitely continuous multimap $\widetilde{F}_\varepsilon : P \to Kv(E)$,

$$\widetilde{F}_\varepsilon(x) = \frac{1}{R - R_1}\left[(\|x\| - R_1)F\left(\frac{Rx}{\|x\|}\right) + (R - \|x\|)f_\delta\left(\frac{Rx}{\|x\|}\right)\right].$$

Lemma 2. *Multifield $\widetilde{\Phi}_\varepsilon = i - \widetilde{F}_\varepsilon$ is fixed point free on P.*

Proof. Notice that, for all $x \in P$,

$$\widetilde{F}_\varepsilon(x) \subset \overline{\text{co}}\left(f_\delta\left(\frac{Rx}{\|x\|}\right) \cup F\left(\frac{Rx}{\|x\|}\right)\right) \Rightarrow \text{dist}\left(\left(\frac{Rx}{\|x\|}\right), \widetilde{F}_\varepsilon(x)\right) \geq \xi.$$

Then for all $x \in P$ and $y \in \widetilde{F}_\varepsilon(x)$ we have that

$$\begin{aligned}\|x-y\| &= \left\|\frac{Rx}{\|x\|} - y - \left(\frac{Rx}{\|x\|} - x\right)\right\| \geq \left\|\left(\frac{Rx}{\|x\|} - y\right)\right\| - \left\|\left(\frac{Rx}{\|x\|} - x\right)\right\| \\ &\geq \xi - (R - \|x\|) \geq \xi - (R - R_1) > 0.\end{aligned}$$

Therefore $\widetilde{\Phi}_\varepsilon$ is fixed point free on P. □

Notice that $\|F(x)\| = \|x - \Phi(x)\| \leq \|x\| + \|\Phi(x)\| \leq R + \varepsilon_1 = M_1$. Since $\widetilde{F}_\varepsilon(x) \subset \overline{\text{co}}\,(f_\delta(Rx/\|x\|) \cup F(Rx/\|x\|)$ we obtain from the inclusion (1) that

$$\|\widetilde{F}_\varepsilon(x)\| \leq M_1 + \delta = M_2 \quad \text{for all } x \in P.$$

Notice that multimap $\widetilde{F}_\varepsilon(x)$ on S_{R_1} is a completely continuous map $f_\delta(R/R_1 x)$, and from Lemma 2 it follows that it is fixed point free.

We will show that coresponding field

$$x - f_\delta\left(\frac{R}{R_1}x\right)$$

has a zero degree on a sphere S_{R_1}. For this we will show that fields

$$x - f_\delta\left(\frac{R}{R_1}x\right) \tag{2}$$

and

$$x - \frac{R_1}{R} f_\delta\left(\frac{R}{R_1}x\right) \tag{3}$$

are homotopic on S_{R_1}.

In fact, we will consider a map $h : S_{R_1} \times [0,1] \to E$

$$h(x,\mu) = (1-\mu)\frac{R_1}{R} f_\delta\left(\frac{R}{R_1}x\right) + \mu f_\delta\left(\frac{R}{R_1}x\right) = f_\delta\left(\frac{R}{R_1}x\right)\left[(1-\mu)\frac{R_1}{R} + \mu\right].$$

Since $1 - \alpha < R_1/R < 1$ we obtain that $\|x - h(x,\mu)\| \geq \beta$. Therefore $h(x,\mu)$ realizes a homotopy of fields (2) and (3) on S_{R_1} and therefore these fields have the same degree. It is easy to see that field (3) on S_{R_1} is obtained from field $x - f_\delta(x)$ on S_R by homotetic transformation and then

$$\deg\left(x - \frac{R_1}{R} f_\delta\left(\frac{R}{R_1}x\right), S_{R_1}\right) = 0,$$

and therefore

$$\deg\left(x - f_\delta\left(\frac{R}{R_1}x\right), S_{R_1}\right) = 0.$$

From Theorem 20.9 [5] it follows that the map $f_\delta(Rx/R_1)$ can be extended from S_{R_1} to a completely continuous map $f_1 : \overline{B}_{R_1} \to E$ whithout fixed points. Let us take a retraction $\rho : E \to \overline{B}_{M_2}$ and consider a continuous map $\widetilde{f}(x)$ on a ball $\overline{B}_{R_1}$ defined as

$$\widetilde{f}(x) = \rho \circ f_1(x).$$

Since $\|f_1(x)\| \leq M_2$ for $x \in S_{R_1}$ the map $\widetilde{f}(x)$ coincides with $f_1(x) = f_\delta(Rx/R_1)$ on S_{R_1}. So $\widetilde{f}(x)$ is the extension of a multimap $\widetilde{F}_\varepsilon(x)$ on $\overline{B}_{R_1}$ and

$$\|\widetilde{f}(x)\| \leq M_2, \quad \text{for all } x \in \overline{B}_{R_1}.$$

Lemma 3. *The map $\widetilde{f}(x)$ is fixed point free on $\overline{B}_{R_1}$.*

Proof. Let us assume the contrary. Then there exists a point $x \in \overline{B}_{R_1}$ such that $x = \widetilde{f}(x)$. Then $x = \rho \circ f_1(x)$. Let us consider following cases:

(i) $\|f_1(x)\| \leq M_2$, then $\rho \circ f_1(x) = f_1(x)$ and hence $x = f_1(x)$. But f_1 is fixed point free.

(ii) $\|f_1(x)\| > M_2$, then $\|x\| = \|\rho \circ f_1(x)\| = M_2 > R$. But $\|x\| \leq R_1 < R$.

□

So we have an extension $\Phi_\varepsilon = i - F_\varepsilon$ of a multifield $\Phi = i - F$ from S_R on all ball $\overline{B}_R$, where

$$F_\varepsilon(x) = \begin{cases} \widetilde{F}_\varepsilon(x) & \text{for } x \in P, \\ \widetilde{f}(x) & \text{for } x \in \overline{B}_{R_1}. \end{cases}$$

This extension is fixed point free and satisfy the following estimate

$$\|\Phi_\varepsilon(x)\| \leq \|x\| + \|F_\varepsilon(x)\| \leq R + M_2 = 2R + \varepsilon_1 + \delta < 2R + 2\varepsilon_1 \leq \varepsilon - \varepsilon_1 = \varepsilon.$$

So we have

$$\rho(\Phi_\varepsilon(x), \Phi(x)) \leq \sup\{\|\Phi_\varepsilon(x)\| + \|\Phi(x)\| \leq \varepsilon - \varepsilon_1 + \varepsilon_1 = \varepsilon,$$

proving the theorem. □

4. Stability of singular points of condensing multifields

Now we can extend the above result to the case of χ-condensing multifield in a Banach space.

Theorem 2. *Let $x_* \in G$ be an isolated singular point of a (k, χ)-condensing multifield Φ. The condition of nontriviality of its topological index*

$$\operatorname{ind}(x_*, \Phi) \neq 0$$

is sufficient, and in case $\Phi(x_*) = \{0\}$, also necessary condition for the stability of x_*.

Proof. (1) *Sufficiency.* Let U is a neighbourhood of x_* such that Φ has no other singular point in $\overline{U}$. Then $\deg(\Phi, \partial U) \neq 0$. Let us show that we can find $\varepsilon > 0$ such that every ε-*close* (k', χ)-*condensing* multifield Φ_ε is homotopic on ∂U to a multifield Φ and moreover, this homotopy $G_\varepsilon : [0,1] \times \partial U \to Kv(E)$ the form

$$G_\varepsilon(\lambda, x) = \lambda F_\varepsilon(x) + (1-\lambda)F(x).$$

Indeed, it is easy to see that the family G_ε is (k'', χ)-*condensing*, where $k'' = \max\{k, k'\}$. Moreover, it is fixed point free if $\varepsilon > 0$ is sufficently small. If we assume the contrary then we will have sequences $\{\varepsilon_n\}_{n=1}^\infty$, $\varepsilon_n > 0$, $\varepsilon_n \to 0$; $\{\lambda_n\}_{n=1}^\infty \subset [0,1]$ and $\{x_n\}_{n=1}^\infty \subset \partial U$ such that $x_n \in G_{\varepsilon_n}(\lambda_n, x_n)$, $n \geq 1$. But then

$$x_n \in \lambda_n F_{\varepsilon_n}(x_n) + (1-\lambda_n)F(x_n) \in F(x_n) + B_{\varepsilon_n} \tag{4}$$

where B_{ε_n} is a ball with a center at the origin and radius ε_n. Therefore for any $m > 1$ we have that

$$\{x_n\}_{n=m}^\infty \subset F(\{x_n\}_{n=m}^\infty) + B_{\varepsilon_m}$$

and by virtue of properties of nonsingularity, monotonicity and algebraical semi-additivity of the Hausdorff MNC (see, for example, [1], [2]) we have

$$\begin{aligned}\chi(\{x_n\}_{n=m}^\infty) &= \chi(\{x_n\}_{n=m}^\infty) \leq \chi(F(\{x_n\}_{n=m}^\infty) + B_{\varepsilon_m}) \\ &\leq \chi(F(\{x_n\}_{n=m}^\infty)) + \chi(B_{\varepsilon_m}) \leq k\chi(\{x_n\}_{n=m}^\infty) + \varepsilon_m \\ &= k\chi(\{x_n\}_{n=m}^\infty) + \varepsilon_m.\end{aligned}$$

Since $k < 1$, m is arbitrary we obtain $\chi(\{x_n\}_{n=1}^\infty) = 0$, that is the sequence $\{x_n\}_{n=1}^\infty$ is relatively compact and so we may consider w.l.o.g. that $x_n \to x_0 \in \partial U$. Since a multimap F is closed ([1], [2]) from the inclusion (4) it follows that $x_0 \in F(x_0)$ contrary to the fact that F is a fixed point free on ∂U.

From the homotopy of Φ and Φ_ε we have $\deg(\Phi_\varepsilon, \partial U) = \deg(\Phi, \partial U) \neq 0$. Hence Φ_ε has at least one singular point in U.

(2) *Necessity.* Let $x_* \in G$ be an isolated singular point of a multifield Φ such that $\Phi(x_*) = \{0\}$. We will show that if

$$\operatorname{ind}(x_*, \Phi) = 0$$

then x_* is unstable. For simplicity we are considering the case when $x_* = 0$.

Let us fix arbitrary $\varepsilon > 0$ and take any $\varepsilon_* > 0$ such that $0 < \varepsilon_* \leq \varepsilon/7$.

We will choose a ball $\overline{B}_R \subset E$ with the center at zero and sufficiently small radius $R > 0$ such that

(1) $R \leq \varepsilon/112$,

(2) x_* is the only singular point of Φ in $\overline{B}_R$,
(3) for all $x \in \overline{B}_R$ we have that $\|\Phi(x)\| \leq \varepsilon_*$.

We will construct a χ-*condensing* multifield Φ_ε ε-*close* to Φ on $\overline{B}_R$ and such that Φ_ε is fixed point free.

It is known that we may take an essential fundamental set $T \subset E$ of a multimap F on $S = \partial B_R$ (see [1], [2]), i.e. T is a convex closed set satisfying the following conditions:

(a) $S \cap T \neq \emptyset$,
(b) set $F(S \cap T) \subseteq T$ is relatively compact,
(c) if $x \in S$, $x \in \overline{\mathrm{co}}\,(F(x) \cup T))$ then $x \in T$.

It is known that there exists a completely continuous multimap $\widetilde{F} : S \to Kv(E)$ such that $\widetilde{F}(S) \subseteq T$ and $\widetilde{F}|_{S\cap T} = F|_{S\cap T}$. In fact, if $\rho : E \to \overline{\mathrm{co}}\, F(S\cap T)$ is an arbitrary retraction then F my be defined as $\widetilde{F}(x) = \overline{\mathrm{co}}\,\rho(F(x))$. Furthermore

$$\deg(\widetilde{\Phi}, S) = \deg(\Phi, S) = 0.$$

Let R_1, $0 < R_1 < R$ be such that $l = R/R_1 < 1/k$. Consider the retraction $r : \overline{B}_{RR_1} \to S$ of a ball layer $\overline{B}_{RR_1} = \{x : R_1 \leq x \leq R\}$ on sphere S, $r(x) = Rx/\|x\|$.

Let us define the extension $G : B_{R_1,R} \to Kv(E)$ of F from S to $\overline{B}_{RR_1}$ by the following formula:

$$G(x) = \frac{1}{R - R_1}[(\|x\| - R_1)F(r(x)) + (R - \|x\|)\widetilde{F}(r(x))].$$

Lemma 4. *Multimap G is (kl, χ)-condensing.*

Proof. It is easy to see that the retraction r is a l-Lipschitz map: $\|r(x) - r(y)\| \leq l\|x - y\|$. But then a multimap $F \circ r$ is a (kl, χ)-*condensing*. Now for any set $\Omega \subseteq \overline{B}_{R_1R}$ we have that

$$\chi(G(\Omega) \leq \chi(\overline{\mathrm{co}}\, F(r(\Omega)) \cup \widetilde{F}(r(\Omega))) = \chi(F(r(\Omega))) \leq kl\chi(\Omega). \qquad \square$$

Lemma 5. *If $R - R_1 > 0$ is sufficiently small then a multimap G is fixed point free.*

Proof. Let us assume the contrary. Then we will have a sequence $\{x_n\} \subset \overline{B}_{R_1R}$, $\|x_n\| \to R$ such that

$$x_n \in \lambda_n F\left(\frac{Rx_n}{\|x\|}\right) + (1 - \lambda_n)\widetilde{F}\left(\frac{Rx_n}{\|x\|}\right),$$

where $0 \leq \lambda_n \leq 1$. Then we obtain

$$\chi(\{x_n\}) \leq \chi(\overline{\mathrm{co}}\,(F(r(\{x_n\})) \cup \widetilde{F}(r(\{x_n\})))) = \chi(F(r(\{x_n\}))) < kl\chi(\{x_n\}).$$

Therefore $\chi(\{x_n\} = 0$ and so the sequence $\{x_n\}$ is relatively compact and we may assume w.l.o.g. that $x_n \to x_0 \in S$. Since we may suppose also that $\lambda_n \to \lambda_0$, we obtain:

$$x_0 \in \lambda_0 F(x_0) + (1-\lambda_0)\widetilde{F}(x_0) \subset \overline{\mathrm{co}}\,(F(x_0) \cup T).$$

Hence $x_0 \in S \cap T$ and therefore $x_0 \in F(x_0)$, contrary to the assumption that F is fixed point free on S. □

Notice now that the restriction $G|_{S_1}$ on $S_1 = \partial B_{R_1}$ is a comletely continuous multimap $\widetilde{F}'(x) = \widetilde{F}(r(x))$. Let us show that the topological degree $\deg(\widetilde{\Phi}', S_1)$ is equal to zero for R_1 sufficiently close to R.

Indeed, a completely continuous multifield $\widetilde{\Phi}''$, given on S_1 as $\widetilde{\Phi}''(x) = x - (R_1/R)\widetilde{F}'(x)$ can be obtained from the multifield $\widetilde{\Phi}$ on S by the "homotetic" transformation, and therefore $\deg(\widetilde{\Phi}'', S_1) = 0$.

But if R_1 is sufficiently close to R then multifields $\widetilde{\Phi}'$ and $\widetilde{\Phi}''$ have no oppositely directed vectors on S_1. In fact, if we assume the contrary then we will have sequences

$$\{x_n\}, \|x_n\| = R_n \to R, \quad \{y_n\}, \{z_n\} \subset \widetilde{F}\bigg(\frac{Rx_n}{R_n}\bigg)$$

and $\mu_n > 0$ such that

$$x_n - y_n = -\mu_n\bigg(x - \frac{R_n}{R} z_n\bigg).$$

Then

$$x_n = \frac{1}{1+\mu_n} y_n + \frac{\mu_n}{1+\mu_n}\frac{R_n}{R} z_n. \tag{5}$$

Since a multimap $\widetilde{F}$ is completely continuous then we may consider sequences $\{y_n\}$, $\{z_n\}$ and $\{x_n\}$ as tending to points y_0, z_0 and $x_0 \in S$ respectively and moreover $y_0, z_0 \in \widetilde{F}(x_0)$, but from (5) we obtain that $x_0 \in \widetilde{F}(x_0)$, contrary to the fact that $\widetilde{F}$ is fixed point free on S.

Since multifields $\widetilde{\Phi}'$ and $\widetilde{\Phi}''$ are not oppositely directed on S_1 then they are homotopic and therefore $\deg(\widetilde{\Phi}', S_1) = \deg(\widetilde{\Phi}'', S_1) = 0$.

For a multimap F on $\overline{B}_R$ we have the following estimate:

$$\|F(x)\| \le \|x\| + \|\Phi(x)\| \le R + \varepsilon_* \le \frac{\varepsilon}{112} + \frac{\varepsilon}{7} = \frac{17\varepsilon}{112}.$$

Since for a multimap G we have that $G(B_{R_1,R}) \subset \overline{\mathrm{co}}\,F(S)$ then for G we have the same estimate $\|G(x)\| \le 17\varepsilon/112$ for all $x \in B_{R_1,R}$. Hence

$$\|\widetilde{\Phi}'(x)\| \le \|x\| + \|\widetilde{F}'(x)\| \le R + \|\widetilde{F}'(x)\| \le \frac{\varepsilon}{112} + \frac{17\varepsilon}{112} = \frac{9\varepsilon}{56} = \varepsilon_1$$

for all $x \in S_1 = \partial B_{R_1}$.

Now, applying the results of the Section 3 we may extend a completely continuous multifield $\widetilde{\Phi}'$ from S_1 on $\overline{B}_{R_1}$ as completely cintinuous multifield $\widetilde{\Phi}_\varepsilon : \overline{B}_{R_1} \to Kv(E)$ without singular points satisfying the estimate: $\|\widetilde{\Phi}_\varepsilon(x)\| \leq \varepsilon - \varepsilon_1 = 47/56\varepsilon$.

Now define a multifield $\Phi_\varepsilon : \overline{B}_R \to Kv(E)$ as

$$\Phi_\varepsilon(x) = \begin{cases} x - G(x) & \text{for } x \in \overline{B}_{R_1,R}, \\ \widetilde{\Phi}_\varepsilon(x) & \text{for } x \in \overline{B}_{R_1}. \end{cases}$$

The multifield Φ_ε is a desirable one. Indeed, it is easy to see that Φ_ε is a χ-*condensing*. Further, let us evaluate the deviation $\rho(\Phi_\varepsilon(x), \Phi(x))$.

If $x \in \overline{B}_{R_1,R}$ then

$$\begin{aligned}\rho(\Phi_\varepsilon(x)), \Phi(x)) &\leq \|\Phi_\varepsilon(x)\| + \|\Phi(x)\| \leq \|x\| + \|G(x)\| + \|\Phi(x)\| \\ &\leq \frac{\varepsilon}{112} + \frac{17\varepsilon}{112} + \frac{\varepsilon}{7} = \frac{17}{56}\varepsilon < \varepsilon.\end{aligned}$$

If $x \in \overline{B}_{R_1}$ then

$$\rho(\Phi_\varepsilon(x), \Phi(x)) \leq \|\widetilde{\Phi}_\varepsilon(x)\| + \|\Phi(x)\| \leq \frac{47\varepsilon}{56} + \frac{\varepsilon}{7} = \frac{55}{56}\varepsilon < \varepsilon.$$

Since Φ_ε has no singular point we proved that x_* is unstable. □

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