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A RESULT ON THE SINGULAR PERTURBATION THEORY FOR DIFFERENTIAL INCLUSIONS IN BANACH SPACES

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Dedicated to the memory of Juliusz P. Schauder

Abstract. We provide conditions which ensure that the solution set of the Cauchy problem for a singularly perturbed system of differential inclusions in infinite dimensional Banach spaces is upper semicontinuous with respect to the parameter $\varepsilon \geq 0$ of the perturbation. The main tools are represented by suitable introduced measures of noncompactness and the topological degree theory in locally convex spaces.

Introduction

The aim of this paper is to provide conditions under which we have the upper semicontinuity at $\varepsilon = 0$, in a convenient topology, which will be precised later, of the solution map $\varepsilon \to \Sigma_\varepsilon$ of the following Cauchy problem for a singularly perturbed system of differential inclusions in infinite dimensional spaces:

$$\begin{cases} x'(t) \in A\,x(t) + f_1(t, x(t), y(t)), \\ \varepsilon y'(t) \in B\,y(t) + f_2(t, x(t), y(t)), \quad t \in [0, d], \end{cases} \tag{1}$$

$$x(0) = x_0, \quad y(0) = y_0 \tag{2}$$

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where A and B are infinitesimal generators of C^0-semigroups of linear operators e^{At} and e^{Bt}, $t \geq 0$, $x \in E_1$ and $y \in E_2$ with E_1, E_2 infinite dimensional Banach spaces. The nonlinear multivalued operators $f_i,\ i = 1, 2$, have nonempty, convex and compact values and satisfy suitable conditions expressed in terms of the Hausdorff measure of noncompactness. All the assumptions will be precised in the next section.

In finite dimensional spaces for singularly perturbed differential inclusions several attempts to obtain a version of the classical Tikhonov's theorem have been made. For this the main difficult is the choice of the topology for the convergence as $\varepsilon \to 0$ of the solutions $(x_\varepsilon, y_\varepsilon)$, $\varepsilon > 0$, to a solution (x_0, y_0) at $\varepsilon = 0$. In [13], [20] in the case when one considers the uniform convergence with respect to the x-variable and the weak topology of L^2 with respect to the y-variable then the solution map of the Cauchy problem associated to the system $\varepsilon \to S(\varepsilon)$ turns out to be upper semicontinuous at $\varepsilon = 0$. When the uniform topology is also considered for the y-variable then we have to restrict the attention to a suitable subset $\widehat{S}(\varepsilon)$ of $S(\varepsilon)$ in order to obtain the upper semicontinuity at $\varepsilon = 0$ (see [11], [23] and [24]). In fact, in general, even in the linear case the map $\varepsilon \to S(\varepsilon)$ is not upper semicontinuous at $\varepsilon = 0$ as shown in [12].

In [14] and [15], (see also the references therein), an approach in order to approximate the slow motions of a singularly perturbed control system in finite dimension by a limit differential inclusion was proposed. This approach is based on the averaging method applied to the fast dynamics, as result the uniform convergence of the slow motions to a solution of the limit differential inclusion is obtained. Furthermore, any such solution is the uniform limit of slow motions. Finally, singular perturbation methods for partial differential equations are intensively studied (see e.g. [21] and [22]).

This paper can be considered in itself an attempt of developing a singular perturbation theory for differential inclusions in infinite dimensional spaces by means of topological methods. For our purposes it turns out convenient to consider the uniform topology for the x-variable and the weak topology for the y-variable. In fact, with this choice of the topology we will be able to show the upper semicontinuity at $\varepsilon = 0$ of a suitably defined condensing operator F_ε, whose fixed points represent the set Σ_ε of the solutions of our problem (Theorem 1). Furthermore, we will show that $\operatorname{ind}(\Sigma_0, F_0) = 1$ (Lemma 1). Therefore, from these results we obtain that the solution map $\varepsilon \to \Sigma_\varepsilon$ has nonempty values and it is upper semicontinuous at $\varepsilon = 0$ (Theorem 2). Here the topological index is that for condensing operators in locally convex spaces.

Observe that, for singularly perturbed differential inclusions in finite dimensional spaces, a variant of the necessary Tikhonov's stability conditions is given in an explicit form, for instance, in [11] and [24]. In this paper these conditions are represented by the assumptions (S_0), (A_0) and (A_3).

We have an other reason for considering singularly perturbed systems of differential inclusions in infinite dimensional Banach spaces. Indeed, in [4], combining the classical singular perturbation theory in finite dimensional spaces with the control technique based on the sliding manifolds, several tracking control problems were solved. In fact, in [4] the control function is designed by means of a differential equation depending on a small parameter $\varepsilon > 0$, on the dynamics to be controlled and on the assigned sliding manifold, which is given as zeros of a differentiable function. Applications of this control technique to concrete problems, treated as tracking problems, can be found in [5], [6] and [17]. To this regard the results presented in this paper can be also considered as a first contribution to the development of the control technique proposed in [4] to the infinite dimensional case when the dynamics is modelled by a differential inclusion as in the case of uncertain systems.

We would like to point out that systems of differential inclusions involving noncompact operators have been also considered in [7] and [18] to solve nonlinear boundary value control problems.

The paper is organized as follows. In Section 1 we state the problem and we formulate the assumptions which permit to solve it. Then we introduce convenient operators in order to rewrite our problem in terms of a multivalued fixed point problem. In Section 2 we prove in Theorem 1 the relevant properties of the resulting fixed point operator F_ε, in particular the condensivity with respect to a suitably introduced measure of noncompactness and the upper semicontinuity in the considered topology. In Lemma 1 we will show that the topological index of the reduced problem at $\varepsilon = 0$ is one. Then, by using the previous results, in Theorem 2 we state the desired properties for the solution map $\varepsilon \to \Sigma_\varepsilon$. Finally, in Section 3, we provide an example illustrating how the conditions of Section 2 can be verified.

1. Statement of the problem, definitions and assumptions

Through this paper we consider the Cauchy problem for a system of differential inclusions of the following form

$$\begin{cases} x'(t) \in A\,x(t) + f_1(t, x(t), y(t)), \\ \varepsilon y'(t) \in B\,y(t) + f_2(t, x(t), y(t)), \quad t \in [0, d], \end{cases} \tag{1}$$

$$x(0) = x_0, \quad y(0) = y_0, \tag{2}$$

where A and B are infinitesimal generators of C^0-semigroups of linear operators e^{At} and e^{Bt}, $t \geq 0$, respectively, acting in the separable Banach spaces E_1 and E_2 with E_2 satisfying the Radon–Nikodym condition (see [10]), $\varepsilon > 0$ is a small parameter and $f_i : \mathbb{R}_+ \times E_1 \times E_2 \to Kv(E_i)$, $i = 1, 2$, are multivalued operators. Here $Kv(E)$ denotes the set of all the nonempty, convex, compact subsets of the Banach space E.

Statement of the problem. We want to provide conditions under which the solution set Σ_ε, $\varepsilon \geq 0$ of system (1)–(2) is upper semicontinuous at $\varepsilon = 0$ in a suitable functional space $\mathcal{F}$, equipped with the uniform topology with respect to the x-variable and with the weak topology with respect to the y-variable. For this, first we show that Σ_ε can be represented as a fixed points set of a condensing operator F_ε, $\varepsilon \geq 0$ defined in $\mathcal{F}$. Moreover, we will show that, under these conditions, the related topological index $\mathrm{ind}(\Sigma_0, F_0)$ is different from zero, (see e.g. [3]).

To make precise the setting in which we will solve the above problem we first choose for the functions $t \to x(t)$ and $t \to y(t)$, $t \in [0, d]$ the functional spaces $C(E_1)$ and $L^1(E_2)$ respectively, and so $\mathcal{F} = C(E_1) \times L^1(E_2)$. We recall that $C(E)$ denotes the space of continuous functions $x : [0, d] \to E$ equipped with the uniform norm: $\max_{t\in[0,d]} \|x(t)\|_E$ and $L^1(E)$ is the space of strongly measurable functions $x : [0, d] \to E$ having finite norm $\|x\|_{L^1} := \int_0^d \|x(t)\|_E \, dt$. In the sequel by ${}^wE$ we will denote the space E equipped with the weak topology of E, while $Kv - w(E)$ will be the set of all the nonempty, convex, weakly compact subsets of E. Furthermore, by $C_\tau(E_1)$ and $L^1_\tau(E_2)$ will denote the Banach spaces of the functions as defined before restricted to the interval $[0, \tau] \subset [0, d]$.

We assume that

- (S_0) there exists a positive constants $\gamma > 0$ such that $\|e^{Bt}\|_{E_2} \leq e^{-\gamma t}$ for any $t \geq 0$. Moreover, $D(B^*)$, the domain of the adjoint operator B^*, is dense in E_2^* (see [19]).
- (A_0) The Nemytskiĭ operators $\Phi_i : C(E_1) \times L^1(E_2) \to Kv\text{-}w(L^1(E_i))$ generated by $f_i : \mathbb{R}_+ \times E_1 \times E_2 \to Kv(E_i)$, $i = 1, 2$, as follows

$$\Phi_i(x, y) = \{g \in L^1(E_i) : g(t) \in f_i(t, x(t), y(t)) \text{ for almost all } t \in [0, d]\}$$

are well defined. Furthermore, for any $\tau \in (0, d]$, we put

$$\Phi_i^\tau(x, y) = \{g \in L^1_\tau(E_i) : g(t) \in f_i(t, x(t), y(t)) \text{ for almost all } t \in [0, \tau]\}.$$

The following assumptions are formulated in terms of the Nemytskiĭ operators Φ_i, $i = 1, 2$.

- (A_1) For any pair of bounded sets $\Omega_1 \subset C(E_1)$, $\Omega_2 \subseteq Q \subset L^1(E_2)$, where Q is the convex, bounded closed set of the asumption (A_3) in the sequel, there exists a function $\varphi \in L^1(\mathbb{R})$ such that $\|g_i(t)\|_{E_i} \leq \varphi(t)$ for almost all $t \in \mathbb{R}$ and any $g_i \in \Phi_i(x, y)$, $i = 1, 2$, whenever $(x, y) \in \Omega_1 \times \Omega_2$.
- (A_2) Φ_i are upper semicontinuous multivalued operators from $C(E_1) \times {}^wL^1(E_2)$ to ${}^wL^1(E_i)$.

REMARK 1. Explicit conditions on f_i, $i = 1, 2$, which ensure that the related Nemytskiĭ operators are well defined will be given in Section 3. For the finite dimensional case (see [2]).

We also need suitable compactness conditions on Φ_i, $i = 1, 2$, expressed in terms of the Hausdorff measure of noncompactness. To this aim we give the following definitions.

DEFINITION 1. Let E be a Banach space. Let $\Omega \subset E$ be a bounded set. The Hausdorff measure of noncompactness $\chi_E(\Omega)$ of the set Ω is the infimum of the numbers $\alpha > 0$ such that Ω has a finite α-net in E. For the relevant properties of χ_E we refer to [1].

DEFINITION 2. Let E be a Banach space. Let $\Omega \subset E$ be a bounded set of E. The measure of weak noncompactness $\chi_w(\Omega)$ of the set Ω is the infimum of the number $\alpha > 0$ such that Ω has a weakly compact α-net in E. This measure of weak compactness and its properties have been studied by De Blasi in [9].

DEFINITION 3. Let Ω be a bounded set of $L^1(E)$. A function $b \in L^1(\mathbb{R})$ is called a bound for the weak measure of noncompactness χ_w of the set $\Omega \subset L^1(E)$ if for every $\delta > 0$ there exist a measurable set $e_\delta \subset [0, d]$ and a compact set $K_\delta \subset E$ such that $\operatorname{meas} e_\delta < \delta$ and for every $f \in \Omega$ there exists $g \in L^1(E)$ satisfying $g(t) \in K_\delta$ for almost all $t \in [0, d]$ and

$$\|f(t) - g(t)\|_E \leq b(t) + \delta \qquad \text{for almost all } t \in [0, d] \setminus e_\delta.$$

In the sequel the set of all the functions $b \in L^1(\mathbb{R})$ with the previous properties will be denoted by $WB(\Omega)$. Observe that we can always assume that $g(t) = 0$ for $t \in e_\delta$. Furthermore, for any $\tau \in (0, d]$, we put $\chi_w(\Omega)(\tau) = \chi_w(\Omega|_{[0,\tau]})$.

We introduce now the operators F_ε, $\varepsilon \in [0, 1]$, in order to represent the solutions of (1)–(2) in a convenient way. For this, we need first to define the linear operators $\Lambda_\varepsilon : L^1(E_1) \times L^1(E_2) \to C(E_1) \times L^1(E_2)$ as follows

$$\Lambda_\varepsilon \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(t) = \begin{pmatrix} \Lambda_1 g_1 \\ \Lambda_2(\varepsilon) g_2 \end{pmatrix}(t) = \begin{pmatrix} \int_0^t e^{A(t-s)} g_1(s)\, ds \\ (1/\varepsilon) \int_0^t e^{(1/\varepsilon)B(t-s)} g_2(s)\, ds \end{pmatrix}, \quad \varepsilon > 0,$$

$$F_\varepsilon(x, y)(t) = \Lambda_\varepsilon \begin{pmatrix} \Phi_1(x, y) \\ \Phi_2(x, y) \end{pmatrix}(t) + \begin{pmatrix} e^{At} x_0 \\ e^{(1/\varepsilon)Bt} y_0 \end{pmatrix}.$$

While, for $\varepsilon = 0$ we set $\Lambda_2(0) = -B^{-1}$ and 0 instead of $e^{(1/\varepsilon)Bt} y_0$.

We formulate now the assumptions under which F_ε, $\varepsilon \in [0, 1]$, is a well defined condensing operator.

(A$_3$) There exists a convex, bounded and closed set $Q \subset L^1(E_2)$ such that

$$\Lambda_2(\varepsilon)\Phi_2^\tau : C([0, \tau], E_1) \times Q|_{[0,\tau]} \multimap Q|_{[0,\tau]}, \quad \tau \in [0, d].$$

(A_4) There exist positive constants k_{11} and k_{12} such that for any pair of bounded sets $\Omega_1 \subset C(E_1)$ and $\Omega_2 \subset Q$ we have that

$$k_{11}\chi_{E_1}(\Omega_1(\,\cdot\,)) + k_{12}\chi_w(\Omega_2)(\,\cdot\,) \in WB(\Phi_1(\Omega_1 \times \Omega_2)).$$

(A_5) There exist positive constants k_{21} and k_{22} such that for any pair of bounded sets $\Omega_1 \subset C(E_1)$ and $\Omega_2 \subset Q$ one has

$$\chi_w(\Phi_2(\Omega_1 \times \Omega_2))(\tau) \le k_{21} \sup_{t\in[0,\tau]} \chi_{E_1}(\Omega_1(t)) + k_{22}\chi_w(\Omega_2)(\tau), \quad \tau \in [0,d].$$

Finally, we now formulate the last assumption

(A_6) $k_{22}/\gamma < 1$.

Remark 2. The assumption (A_3) is verified if, for instance, there exist positive constants M and l such that

$$\|f_2(t,x,y)\|_{E_2} \le M + l\|y\|,$$

with $l/\gamma < 1$. In this case, we have $Q = Q_R$, where

$$Q_R := \{g \in L^1(E_2) : \|g(t)\|_{E_2} \le R, \text{ for almost all } t \in \mathbb{R}\}.$$

and $R > 0$ is sufficiently large.

2. Results

We introduce now suitable measures of noncompactness with respect to which we will show that F_ε, $\varepsilon \in [0,1]$, is condensing. For this, given a bounded set $\Omega \subset C(E_1) \times L^1(E_2)$ we put

$$\mu(\Omega)(\tau) = \begin{pmatrix} \chi_{E_1}(P_1(\Omega)(\tau)) \\ \chi_w(P_2(\Omega))(\tau) \end{pmatrix},$$

where P_1 is the projector on the first coordinate of the Cartesian product $C(E_1)\times L^1(E_2)$, while P_2 is the projector on the second coordinate of the same space. Note that $\chi_w(P_2(\Omega))(\tau)$ is a nondecreasing function with respect to τ and so measurable. Furthermore, we also define a measure of noncompactness as follows

$$\nu(\Omega) = (\mu(\Omega), \mathrm{mod}_c(P_1\Omega)),$$

where $\mathrm{mod}_c(P_1\Omega)$ denotes the modulus of equicontinuity of the functions in $P_1\Omega$. The measure of noncompactness ν takes values in the cone K defined as follows

$$K = \{(a,b,c) \in L^1(\mathbb{R}) \times L^1(\mathbb{R}) \times \mathbb{R} : a(t) \ge 0, b(t) \ge 0 \text{ for almost all } t \in [0,d], \text{ and } c \ge 0\}.$$

We can now prove the following.

THEOREM 1. *Assume that the conditions* (S_0), (A_0)$-$(A_6) *are satisfied, then the operator* $F_\varepsilon : C(E_1) \times Q \multimap C(E_1) \times Q$ *is upper semicontinuous at any* $\varepsilon \in [0,1]$ *and* $F_\varepsilon(x,y)$ *is* ν*-condensing with respect to the variables* x, y, ε.

PROOF. We prove first that from $g_n \to g_0$ weakly in $L^1(E_2)$ and $\varepsilon_n \to 0$ it follows that $\Lambda_2(\varepsilon_n)g_n \to \Lambda_2(0)g_0$ weakly in $L^1(E_2)$. For this, let v^* be the functional generated by the function

$$y^*(t) = \sum_{i=0}^{m-1} y_i^* \psi_{[t_i,t_{i+1})}(t), \tag{3}$$

where $y_i^* \in E_2^*$, $0 = t_0 < \ldots < t_m = d$, and $\psi_{[t_i,t_{i+1})}$ is characteristic function of the interval $[t_i, t_{i+1})$ then

$$\begin{aligned}
&\int_0^d \Big\langle y^*(t), \frac{1}{\varepsilon_n} \int_0^t e^{(1/\varepsilon_n)B(t-s)} g_n(s)\,ds \Big\rangle\,dt \\
&\quad = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \Big\langle y_i^*, \frac{1}{\varepsilon_n} \int_0^t e^{(1/\varepsilon_n)B(t-s)} g_n(s)\,ds \Big\rangle\,dt \\
&\quad = \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \Big\langle y_i^*, \frac{1}{\varepsilon_n} \int_0^{t_i} e^{(1/\varepsilon_n)B(t-s)} g_n(s)\,ds \Big\rangle\,dt \\
&\qquad + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \Big\langle y_i^*, \frac{1}{\varepsilon_n} \int_{t_i}^t e^{(1/\varepsilon_n)B(t-s)} g_n(s)\,ds \Big\rangle\,dt \\
&\quad = \sum_{i=0}^{m-1} \int_0^{t_i} \Big\langle \frac{1}{\varepsilon_n} \int_{t_i}^{t_{i+1}} e^{(1/\varepsilon_n)B^*(t-s)} y_i^*\,dt, g_n(s) \Big\rangle\,ds \\
&\qquad + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \Big\langle \frac{1}{\varepsilon_n} \int_s^{t_{i+1}} e^{(1/\varepsilon_n)B^*(t-s)} y_i^*\,dt, g_n(s) \Big\rangle\,ds \\
&\quad = \sum_{i=0}^{m-1} \int_0^{t_i} \langle (B^*)^{-1}(e^{(1/\varepsilon_n)B^*(t_{i+1}-s)} - e^{(1/\varepsilon_n)B^*(t_i-s)}) y_i^*, g_n(s) \rangle\,ds \\
&\qquad + \sum_{i=0}^{m-1} \int_{t_i}^{t_{i+1}} \langle (B^*)^{-1}(e^{(1/\varepsilon_n)B^*(t_{i+1}-s)} - I) y_i^*, g_n(s) \rangle\,ds \\
&\quad = \sum_{i=0}^{m-1} \int_0^{t_{i+1}} \langle (B^*)^{-1} e^{(1/\varepsilon_n)B^*(t_{i+1}-s)} y_i^*), g_n(s) \rangle\,ds
\end{aligned}$$

$$- \sum_{i=0}^{m-1} \int_0^{t_i} \langle (B^*)^{-1} e^{(1/\varepsilon_n)B^*(t_i-s)} y_i^*, g_n(s) \rangle \, ds$$
$$+ \int_0^d \langle (-B^*)^{-1} y^*(s), g_n(s) \rangle \, ds.$$

We now prove that the first two terms tend to zero as $n \to \infty$. For this, note that since $g_n \to g_0$ weakly in $L^1(E_2)$, we have that for any $\delta > 0$ there exists $\mu > 0$ such that for any set $e \subset [0,d]$ with meas $(e) < \mu$ it follows $\int_e \|g_n(s)\|_{E_2} \, ds < \delta$ for any $n \in \mathbb{N}$, (see [10]). Let $t \in [0,d]$, first write $\int_0^t = \int_0^{t-\mu} + \int_{t-\mu}^t$. Then we can estimate by means of (S_0) these integrals and taking into account the observation above we can conclude that

$$\int_0^t \langle (B^*)^{-1} e^{(1/\varepsilon_n)B^*(t-s)} y_i^*, g_n(s) \rangle \, ds \to 0$$

as $n \to \infty$. We leave the details to the reader. Finally, the last term tends to

$$- \int_0^d \langle y^*(s), B^{-1} g_0(s) \rangle \, ds.$$

In conclusion, we have weak convergence of $(1/\varepsilon_n) \int_0^t e^{(1/\varepsilon_n)B(t-s)} g_n(s) \, ds$ to $-B^{-1} g_0$ with respect to the functionals v^*. On the other hand, E_2 has the Radon–Nikodym property, then by [10, Theorem 1] we have that $(L^1(E_2))^* = L^\infty(E_2^*)$. Now we can approximate in the "almost every" convergence any function $y \in L^\infty(E_2^*)$ by a function of the form (3). For this it is possible to consider the continuous function

$$z_h(s) = \frac{1}{h} \int_s^{s+h} y(t) \, dt, \quad s \in [0,d]$$

which tends to $y(s)$ as $h \to 0$ for almost all $s \in [0,d]$ (see [10, Theorem 9]) and then approximate $z_h(\,\cdot\,)$ by step functions. Applying now Egorov's theorem we finally obtain that, for any $y \in L^\infty(E_2^*)$,

$$\int_0^d \left\langle y(t), \frac{1}{\varepsilon_n} \int_0^t e^{(1/\varepsilon_n)B(t-s)} g_n(s) \, ds \right\rangle dt$$

tends to

$$- \int_0^d \langle y(s), B^{-1} g_0(s) \rangle \, ds$$

as $n \to \infty$. Therefore, the operator $\Lambda_2(\varepsilon)\Phi_2(x,y)$ is upper semicontinuous from $[0,1] \times C(E_1) \times Q$ to $Kv\text{-}w(Q)$.

Observe that $\|\Lambda_2(\varepsilon)\|_{L^1(E_2)\to L^1(E_2)} \le 1/\gamma$. In fact, let $\varepsilon > 0$, and consider

$$\int_0^d \frac{1}{\varepsilon}\Big\| \int_0^t e^{B(t-s)} g(s)\, ds \Big\|_{E_2} dt \le \int_0^d \frac{1}{\varepsilon} \int_0^t e^{-(1/\varepsilon)\gamma(t-s)} \|g(s)\|_{E_2}\, ds\, dt$$
$$= \int_0^d \frac{1}{\varepsilon} \int_s^d e^{-(1/\varepsilon)\gamma(t-s)} \|g(s)\|_{E_2}\, ds\, dt + \frac{1}{\gamma} \int_0^d (1 - e^{-(1/\varepsilon)\gamma(d-s)}) \|g(s)\|_{E_2}\, ds$$
$$= \frac{1}{\gamma} \int_0^d \|g(s)\|_{E_2}\, ds.$$

Finally, for $\varepsilon = 0$, since $B^{-1} = \int_0^\infty e^{Bt}\, dt$, we have $\|B^{-1}\| \le 1/\gamma$.

We prove now that F is ν-condensing. Let $\Omega \subset C(E_1) \times Q$, one has

$$\nu(\Omega) \le \nu\Big(\bigcup_{\varepsilon\in[0,1]} F_\varepsilon(\Omega) \Big).$$

Then

$$\mu(\Omega)(t) \le \mu\Big(\bigcup_{\varepsilon\in[0,1]} F_\varepsilon(\Omega) \Big)(t) \quad \text{for almost all } t \in [0,d] \tag{4}$$

and

$$\mathrm{mod}_c(P_1\Omega) \le \mathrm{mod}_c\Big(P_1 \bigcup_{\varepsilon\in[0,1]} F_\varepsilon(\Omega) \Big). \tag{5}$$

Therefore

$$\mu(\Omega)(t) \le \mu\Big(\bigcup_{\varepsilon\in[0,1]} F_\varepsilon(P_1\Omega \times P_2\Omega) \Big)(t) \quad \text{for almost all } t \in [0,d]. \tag{6}$$

From (A_4), see [8], we obtain for all $t \in [0,d]$

$$v(t) \le \int_0^t C(k_{11} v(s) + k_{12} \chi_w(P_2\Omega)(s))\, ds, \tag{7}$$

where $C = \sup_{t\in[0,d]} \|e^{At}\|$, $v(t) = \chi_{E_1}(P_1\Omega(t))$. Since, if $C \subset L^1(E_2)$ is any weakly compact set then $\bigcup_{\varepsilon\in[0,1]} \Lambda_2(\varepsilon) C$ is weakly compact, from (A_5), for almost all $t \in [0,d]$, we have

$$\chi_w(P_2\Omega)(t) \le \frac{k_{21}}{\gamma} \sup_{s\in[0,t]} v(s) + \frac{k_{22}}{\gamma} \chi_w(P_2\Omega)(t). \tag{8}$$

From (8), using (A_6), we obtain

$$\chi_w(P_2\Omega)(s) \le \Big(1 - \frac{k_{22}}{\gamma} \Big) \frac{k_{21}}{\gamma} \sup_{\xi\in[0,s]} v(\xi), \quad s \in [0,\tau]. \tag{9}$$

Consider $\tau \le h$, where h is chosen in such a way that

$$C\,h \max\left\{k_{11},\, k_{12}\left(1-\frac{k_{22}}{\gamma}\right)\frac{k_{21}}{\gamma}\right\} < 1$$

then from (7) we have $v(t) = 0$ for $t \in [0,h]$ and from (9) $\chi_w(P_2\Omega)(s) = 0$ for almost all $s \in [0,h]$. Therefore, the inequality (7) has the form

$$v(t) \le \int_h^t C(k_{11}v(s) + k_{12}\chi_w(P_2\Omega)(s))\,ds \tag{10}$$

and the inequality (9) takes the form

$$\chi_w(P_2\Omega)(s) \le \left(1-\frac{k_{22}}{\gamma}\right)\frac{k_{21}}{\gamma}\sup_{\xi\in[h,s]} v(\xi), \qquad s \in [0,\tau]. \tag{11}$$

Let now $t \in [h,2h]$, then from (10) and (11) we have $v(s) = 0$, $\chi_w(P_2\Omega)(s) = 0$ for almost all $s \in [h,2h]$. Continuing with this process we obtain

$$v(s) = 0, \qquad s \in [0,d], \tag{12}$$

$$\chi_w(P_2\Omega) = 0. \tag{13}$$

Therefore, from (A_4),

$$0 \in WB(\Phi_1(P_1\Omega \times P_2\Omega))$$

and consequently $\bigcup_{\varepsilon\in[0,1]} P_1F_\varepsilon(P_1\Omega \times P_2\Omega)$ is relatively compact, see [8]. Thus

$$\operatorname{mod}_c\left(P_1 \bigcup_{\varepsilon\in[0,1]} F_\varepsilon(P_1\Omega \times P_2\Omega)\right) = 0$$

and from (5) we have that

$$\operatorname{mod}_c(P_1\Omega) = 0 \tag{14}$$

In conclusion from (12), (13) and (14) we have the relative compactness of Ω. $\square$

Consider now the reduced problem

$$\begin{cases} x'(t) \in A\,x(t) + f_1(t, x(t), y(t)), \\ y(t) \in B^{-1}f_2(t, x(t), y(t)), \quad t \in [0,d], \end{cases} \tag{15}$$

$$x(0) = x_0. \tag{16}$$

Let Σ_0^τ the solution set of (15) and (16) defined in the interval $[0,\tau]$. Observe that if Σ_0^τ is bounded then, by Theorem 1, we obtain that Σ_0^τ is a compact subset of $C_\tau(E_1) \times {}^wQ|_{[0,\tau]}$. We have the following.

LEMMA 1. *Assume that the conditions* (S_0), (A_0)$-$(A_6) *are satisfied. Furthermore, assume that* Σ_0 *is nonempty, bounded and* $\Sigma_0^\tau = \Sigma_0|_{[0,\tau]}$ *for all* $\tau \in [0,d]$. *Then the topological index* $\operatorname{ind}(\Sigma_0, F_0) = 1$.

PROOF. Let $\Gamma = \{(t,x) : (t,x) \in (t, P_1\Sigma_0(t)\}$. Since $P_1\Sigma_0$ is compact in $C(E_1)$ then Γ is compact in $\mathbb{R} \times E_1$. Consider now the continuous Uryson function

$$H(t,x) = \begin{cases} 1 & \text{if } (t,x) \in B(\Gamma, r), \\ 0 & \text{if } (t,x) \notin B(\Gamma, 2r), \end{cases}$$

where $B(\Gamma, r)$ is a r-ball neighbourhood of Γ. Let us consider the operator

$$\widetilde{f_1}(t,x,y) = H(t,x) f_1(t,x,y).$$

Then the superposition operator

$$\widetilde{\Phi}_1(x,y)(t) = \widetilde{f_1}(t, x(t), y(t))$$

is well defined and upper semicontinuous, since Φ_1 is upper semicontinuous and $\mathcal{H}(x)(t) = H(t, x(t))$ is continuous.

We now prove that the operator $\widetilde{\Phi}_1$ satisfies assumption (A_4). In fact, since Φ_1 satisfies (A_4), given $\delta > 0$ we can find a set $e_{\delta/2}$ such that $\operatorname{meas}(e_{\delta/2}) < \delta/2$ and

$$\chi_{E_1}(\Phi_1(\Omega_1 \times \Omega_2)(t)) \le k_{11}\chi_{E_1}(\Omega_1(t)) + k_{12}\chi_w(\Omega_2)(t) + \delta/2,$$

for $t \in [0,d] \setminus e_{\delta/2}$. On the other hand

$$\begin{aligned} \chi_{E_1}(\widetilde{\Phi}_1(\Omega_1 \times \Omega_2)(t)) &\le \chi_{E_1}(\mathcal{H}(\Omega_1)(t)\Phi_1(\Omega_1 \times \Omega_2)(t)) \\ &\le \chi_{E_1}(\overline{co}(\Phi_1(\Omega_1 \times \Omega_2)(t))) = \chi_{E_1}(\Phi_1(\Omega_1 \times \Omega_2)(t)). \end{aligned}$$

Therefore, if we define as zero on the set $e_{\delta/2}$ all the functions in $\widetilde{\Phi}_1(\Omega_1 \times \Omega_2)$, then we can apply the result of [8] to derive the existence of a set $\widehat{e}_{\delta/2}$ with $\operatorname{meas}(\widehat{e}_{\delta/2}) < \delta/2$, of a compact $K_{\delta/2}$ and of a set of functions $G_{\delta/2} \subset L^1(E_1)$ such that $g(t) \in K_{\delta/2}$ for all $g \in G_{\delta/2}$ and for almost all $t \in [0,d]$. Moreover, for every f belonging to $\widetilde{\Phi}_1(\Omega_1 \times \Omega_2)$, there exists $g \in G_{\delta/2}$ such that

$$\|f(t) - g(t)\| \le k_{11}\chi_{E_1}(\Omega(t)) + k_{12}\chi_w(\Omega_2)(t) + \delta, \ \text{ for } t \in [0,d] \setminus (e_{\delta/2} \cup \widehat{e}_{\delta/2}).$$

Therefore $k_{11}\chi_{E_1}(\Omega(\,\cdot\,)) + k_{12}\chi_w(\Omega_2)(\,\cdot\,) \in WB(\widetilde{\Phi}_1(\Omega_1 \times \Omega_2))$, and so the operator

$$\widetilde{F}(x,y) = \Lambda_0 \begin{pmatrix} \widetilde{\Phi}_1(x,y) \\ \Phi_2(x,y) \end{pmatrix}$$

is ν-condensing. Observe that it is bounded on $C(E_1) \times Q$.

Denote by $\widetilde{\Sigma}_0$ the solution set of $\widetilde{F}$ then $\operatorname{ind}(\widetilde{\Sigma}_0, \widetilde{F_0}) = 1$.

Finally, we prove that $\widetilde{\Sigma}_0 = \Sigma_0$. Assume the contrary, it means that there exists

$$\begin{pmatrix} \widetilde{x} \\ \widetilde{y} \end{pmatrix} \in \widetilde{\Sigma}_0 \setminus \Sigma_0.$$

In this case, we have that $\widetilde{x} \notin P_1\Sigma_0$. Take now $0 < \beta < r$. Since $P_1\Sigma_0$ is a compact set, we can find t^* such that

$$d(\widetilde{x}(t), P_1\Sigma_0(t)) < \beta \quad \text{for } t \in [0, t^*), \tag{17}$$

$$d(\widetilde{x}(t^*), P_1\Sigma_0(t^*)) = \beta. \tag{18}$$

On the other hand, from (17), (18), we have that $\left.\binom{\widetilde{x}}{\widetilde{y}}\right|_{[0,t^*]} \in \widetilde{\Sigma}_0^{t^*}$. But, by assumption $\Sigma_0^\tau = \Sigma_0|_{[0,\tau]}$, we have that $\Sigma_0^{t^*} = \Sigma_0|_{[0,t^*]}$ and then (18) is impossible. $\square$

Then by using standard methods of the topological degree theory for multivalued condensing operators in locally convex spaces (see [3]) we can derive the following existence result for system (1).

Theorem 2. *Assume the conditions of Lemma* 1. *Then there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in [0, \varepsilon_0]$ *the set* Σ_ε *of the solutions of system* (1) *belonging to* $C(E_1) \times Q$ *is nonempty and upper semicontinuous with respect to* ε *in the* $C(E_1) \times {}^wL^1(E_2)$ *topology.*

3. Example

In what follows we provide an example illustrating how the assumptions on the Nemytskiĭ operators Φ_i, $i = 1, 2$, presented in the previous section can be verified. This will be done by specifying a possible choice and the properties of the nonlinear operators f_i, which generate Φ_i, $i = 1, 2$. This example has been formulated having in mind a concrete application of our abstract results to a control problem in infinite dimensional spaces of the type of those considered in [4], as aldready pointed out in the Introduction. Specifically, we consider the following form for f_i, $i = 1, 2$.

$$f_1(t, x, y) = \psi_1(t, x) + b_{11}(x)y, \tag{19}$$

$$f_2(t, x, y) = \psi_2(t, x) + b_{21}(x)y + b_{22}y. \tag{20}$$

We assume the following conditions.

- (a_0) The multivalued operators $\psi_i : \mathbb{R} \times E_1 \to Kv(E_i)$ satisfy the condition that, for any $x \in E_1$ there exists a selection $g(t) \in \psi_i(t, x)$, for almost all $t \in \mathbb{R}$, belonging to $L^1(\mathbb{R})$, $i = 1, 2$.
- (a_1) For almost all $t \in \mathbb{R}$ the operators $\psi_i(t, \cdot)$, $i = 1, 2$, are upper semicontinuous.
- (a_2) There exist positive constants l_{i1} such that $\chi_{E_i}(\psi_i(t, \Omega)) \le l_{i1}\chi_{E_i}(\Omega)$, $i = 1, 2$.
- (a_3) There exist positive constants M_i such that $\|\psi_i(t, x)\|_{E_i} \le M_i$, $i = 1, 2$.

We now formulate the assumptions on the operators

$$b_{i1} : E_1 \to LK(E_2, E_i), \quad i = 1, 2, \tag{21}$$

where $LK(E_2, E_i)$ denotes the space of linear compact operators acting from E_2 to E_i.

(a_4) There exist positive constants m_{i1}, $i = 1, 2$, such that

$$\|b_{i1}(x)\|_i \le m_{i1}$$

for any $x \in E_1$. Here $\|\cdot\|_i$ denotes the operator norm in $LK(E_2, E_i)$.

(a_5) The maps $x \to b_{i1}(x)$, $i = 1, 2$, are continuous.

(a_6) There exists a positive constant l_{22} such that the bounded linear operator $b_{22} : E_2 \to E_2$ satisfies

$$\|b_{22}\| \le l_{22}. \tag{22}$$

(a_7) Finally, we assume the following

$$\frac{m_{22} + l_{22}}{\gamma} < 1. \tag{23}$$

Let $R > 0$ be sufficiently large and let

$$Q_R := \{y \in L^1(E_2) : \|y(t)\|_{E_2} \le R, \text{ for almost all } t \in \mathbb{R}\}.$$

Now we prove that by (23) we get: $\Lambda_2(\varepsilon)\Phi_2(C(E_1) \times Q_R) \subset Q_R$. This was already noticed in Remark 1 (with $Q = Q_R$). For this, let $\varepsilon > 0$ and for almost all $t \in \mathbb{R}$ we have

$$\begin{aligned}\|\Lambda_2(\varepsilon)\Phi_2(x, y)(t)\|_{E_2} &\le \frac{1}{\varepsilon}\int_0^t e^{-(1/\varepsilon)\gamma(t-s)}[M_2 + (m_{22} + l_{22})R]\, ds \\ &\le \frac{M_2 + (m_{22} + l_{22})R}{\gamma}.\end{aligned}$$

For $\varepsilon = 0$, for almost all $t \in \mathbb{R}$ we have

$$\|\Lambda_2(0)\Phi_2(x, y)(t)\|_{E_2} \le \frac{M_2 + (m_{22} + l_{22})R}{\gamma}.$$

By (23), if $R > M_2/(\gamma - m_{22} - l_{22})$ then we get the conclusion.

Let us prove now that the multivalued operators Φ_i are upper semicontinuous from $C(E_1) \times {}^wL^1(E_2)$ to $Kv\text{-}w(L^1(E_i))$, $i = 1, 2$. First observe that under our assumptions on ψ_i we have that the associated Nemytskiĭ operator are upper semicontinuous (see for instance [16]). Therefore, it is sufficient to verify that the operators $(x, y) \to b_{i1}(x(\cdot))y(\cdot)$ are continuous in the topologies which we have introduced in the previous section. From (a_5) we have

$$\langle y^*, b_{i1}(x_n)y_n\rangle \to \langle y^*, b_{i1}(x_0)y_0\rangle. \tag{24}$$

Let us now verify conditions (A_4) and (A_5). From (a_3) we have that, for any $\Omega \subset C(E_1)$,

$$\chi_{E_i}(\psi_i(t,\Omega(t))) \le l_{i1} \sup_t \chi_{E_i}(\Omega(t)), \quad i=1,2.$$

Therefore, if

$$\Gamma_i(\Omega) := \{g : g \in L^1(E_i),\ g(t) \in \psi_i(t,x(t)),\ \text{almost all } t \in [0,d] \text{ and } x \in \Omega\},$$

then

$$\chi_{E_i}(\Gamma_i(\Omega)(t)) \le l_{i1}\chi_{E_i}(\Omega(t)). \tag{25}$$

From (25), (a_5) and [8] we have for $\Omega_1 \subset C(E_1)$ and $\Omega_2 \subset Q_R$ that

$$l_{i1} \sup_t \chi_{E_1}(\Omega_1(t)) \in WB(\Gamma_i(\Omega_1)).$$

Finally, observe that if $\|\Omega_2(t)\| \le p(t)$, $p \in L^1(\mathbb{R})$ and $b \in WB(\Omega_2)$ then

$$\chi_w(\Omega_2) \le \int_0^d b(t)\,dt \quad \text{and} \quad \chi_w(b_{22}\Omega_2) \le l_{22}\chi_w(\Omega_2)$$

we obtain (A_4) and (A_5) with $k_{11}=l_{11}$, $k_{12}=0$, $k_{21}=dl_{21}$ and $k_{22}=l_{22}$. This concludes the example.

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