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TOPOLOGICAL STRUCTURE OF SOLUTION SETS TO PARABOLIC PROBLEMS

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Dedicated to Professor V. Šeda

Abstract. In this paper we deal with the Peano phenomenon for general initial-boundary value problems of quasilinear parabolic equations with arbitrary even order space derivatives.

The nonlinearity is assumed to be a continuous or continuously Fréchet differentiable function. Using a method of transformation to an operator equation and employing the theory of proper, Fredholm (linear and nonlinear) and Nemitskiĭ operators, we study the existence of solution of the given problem and qualitative and quantitative structure of its solution and bifurcation sets. These results can be applied to the different technical and natural science models.

Introduction

The Peano phenomenon of the existence of a solution continuum of the initial value problem for ordinary differential systems is well-known. This phenomenon has been studied by many autors in [3]–[5], [8], [17], [28]. The structure of solution sets for second order partial differential problems was observed in the authors papers [12], [13].

In this paper we shall study the existence, nonuniqueness and generic properties of quasilinear parabolic initial-boundary value problems for the equation

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of an even order with the continuous and continuously differentiable nonlinearities and the general boundary value condition. In the case of continuous nonlinearities we use the Nikol'skiĭ decomposition theorem from [30, p. 233] for linear Fredholm operators, the global inversion theorem of [9], [6] and [7, pp. 42–43] and the Ambrosetti solution quantitive results from [2, p. 216]. In the consideration on surjectivity the generalized Leray–Schauder condition is employed which is similar to that one in [20]. Stronger results are attained by the main Quinn and Smale theorem from [23] and [25] for nonlinear Fredholm operators in the case of differential nonlinearities.

The present results allow us to observe different problems describing dynamics of mechanical processes (bending, vibration), physical-heating processes, reaction-diffusion processes in chemical and biological technologies or in the ecology.

1. The formulation of problem, assumptions and spaces

The set $\Omega \subset \mathbb{R}^n$ for $n \in N$ means a bounded domain with the boundary $\partial\Omega$. The real number T will be positive and $Q := (0,T] \times \Omega$, $\Gamma := (0,T] \times \partial\Omega$. If the multiindex $k = (k_1, \dots, k_n)$ is given with $|k| = \sum_{i=1}^n k_i$, then we use the notation D_x^k for the differential operator $\partial^{|k|}/(\partial x_1^{k_1} \dots \partial x_n^{k_n})$ and D_t for $\partial/\partial t$. If the modul $|k| = 0$ then D_x^k means an identity mapping. The symbol $\operatorname{cl} M$ means the closure of the set M in $\mathbb{R}^n$.

In this paper we consider the nonlinear differential equation of an arbitrary even order $2b$ (b is a positive integer)

$$A(t,x,D_t,D_x)u + f(t,x,\overline{D}_x^\gamma u) = g(t,x) \quad \text{for } (t,x) \in Q, \tag{1.1}$$

where

$$A(t,x,D_t,D_x)u := D_t u - \sum_{|k|=2b} a_k(t,x) D_x^k u - \sum_{0 \le |k| \le 2b-1} a_k(t,x) D_x^k u$$

and $\overline{D}_x^\gamma u$ is a vector function whose components are derivatives $D_x^\gamma u$ with the different multiindex $0 \le |\gamma| \le 2b-1$.

The system of boundary conditions is given by the vector equation with b components

$$B(t,x,D_x)u|_{\operatorname{cl}\Gamma} := (B_1(t,x,D_x)u, \dots, B_b(t,x,D_x)u)^T|_{\operatorname{cl}\Gamma} = 0 \tag{1.2}$$

in which

$$B_j(t,x,D_x)u := \sum_{0 \le |k| \le r_j} b_{jk}(t,x) D_x^k u$$

for an integer $0 \le r_j \le 2b-1$ and $j = 1, \dots, b$.

Further the initial value homogeneous condition

$$u(0,x) = 0 \quad \text{for } x \in \operatorname{cl} Q \tag{1.3}$$

is considered.

Here the given functions are following mappings:

$$\begin{aligned} a_k &: \operatorname{cl} Q \to \mathbb{R} \quad \text{for } 0 \le |k| \le 2b, \\ b_{jk} &: \operatorname{cl} \Gamma \to \mathbb{R} \quad \text{for } 0 \le |k| \le r_j,\ j = 1, \dots , b, \\ f &: \operatorname{cl} Q \times \mathbb{R}^\kappa \to \mathbb{R}, \end{aligned}$$

where κ is a positive integer given by the inequality

$$\kappa \le \binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \dots + \binom{n+|\gamma|-2}{|\gamma|-1} + \binom{n+|\gamma|-1}{|\gamma|}$$

and $g: \operatorname{cl} Q \to \mathbb{R}$.

We shall be employed with parabolic problem (1.1)–(1.3) in the following sence:

The hypothesis (P) **of the uniform parabolicity.** We shall say that equation (1.1) or the differential operator $A(t,x,D_t,D_x)$ is *uniformly parabolic* with parameter δ in the sense of I. G. Petrovskiĭ on $\operatorname{cl} Q$ (or shortly *parabolic*) if and only if for the main part

$$A_0(t,x,D_t,D_x)u = D_t u - \sum_{|k|=2b} a_k(t,x) D_x^k u$$

of the equation (1.1) there exists $\delta > 0$ such that the inequality

$$(-1)^{b+1} \sum_{|k|=2b} a_k(t,x) \sigma_1^{k_1} \dots \sigma_n^{k_n} \ge \delta \bigg(\sum_{i=1}^n \sigma_i^2 \bigg)^b \tag{1.4}$$

is true for all $(t,x) \in \operatorname{cl} Q$ and all $(\sigma_1, \dots , \sigma_n) \in \mathbb{R}^n$.

For the correctness of problem (1.1)–(1.3) we have to stay a complementary condition for the boundary operators B_j, $j = 1, \dots , b$ (see [19, pp. 14–16]).

DEFINITION 1.1 (The reduction polynomial). Let $(t_0,x_0) \in \operatorname{cl} \Gamma$, $(\nu_1, \dots , \nu_n)$ be a unit inner normal vector to $\partial\Omega$ in x_0 and $\xi = (\xi_1, \dots , \xi_n)$ be a vector from the tangential space $T_{\partial\Omega}(x_0)$ to $\partial\Omega$ at the point x_0 and τ be a complex parameter. Denote by

$$\begin{aligned} \Gamma_{x_0} := \{(q,\xi) \in \mathbb{C} \times T_{\partial\Omega}(x_0) : |q| + \|\xi\|_{\mathbb{R}^n}^{2b} > 0 \\ \text{and } \operatorname{Re} q \ge -\delta_1 \|\xi\|_{\mathbb{R}^n}^{2b},\ \text{where } \delta_1 \in (0,\delta)\}. \end{aligned}$$

Here $\delta > 0$ is a constant from the parabolicity condition (1.4). Now, let us take the complex roots $\tau_j^+(t_0,x_0,q,\xi) \in \mathbb{C}$ for $j = 1, \dots , b$ with the positive imaginary part of the $2b$th degree polynomial $A_0(t_0, x_0, q, i(\xi + \tau\nu))$ in τ for an arbitrary $(q, \xi) \in \Gamma_{x_0}$. Then the polynomial of the degree b in the variable τ

$$A^+(t_0, x_0, q, \xi, \tau) = \prod_{j=1}^{b} (\tau - \tau_j^+(t_0, x_0, q, \xi))$$

is called the *reduction polynomial*.

Remark 1.2. For any $(q, \xi) \in \Gamma_{x_0}$ the polynomial $A_0(t_0, x_0, q, i(\xi + \tau\nu))$ has just $2b$ conjugate complex roots and so the reduction polynomial is correctly defined. Really, if there exist a real root τ of

$$A_0(t_0, x_0, q, i(\xi + \tau\nu)) = q - (-1)^b \sum_{|k|=2b} a_k(t, x)(\xi_1 + \tau\nu_1)^{k_1} \dots (\xi_n + \tau\nu_n)^{k_n}$$

for some $(q, \xi) \in \Gamma_{x_0}$ then from condition (1.4) we get

$$\begin{aligned} 0 &= \operatorname{Re} A_0(t_0, x_0, q, i(\xi + \tau\nu)) \geq \operatorname{Re} q + \delta[(\xi_1 + \tau\nu_1)^2 + \dots + (\xi_n + \tau\nu_n)^2]^b \\ &= \operatorname{Re} q + \delta[\xi_1^2 + \dots + \xi_n^2 + \tau^2]^b > \operatorname{Re} q + \delta_1[\xi_1^2 + \dots + \xi_n^2]^b \geq 0 \end{aligned}$$

which gives a contradiction.

Using the denotations from Definition 1.1 we can pronounce:

The hypothesis (C) **of the uniform complementarity.** Define an operator

$$B_0(t, x, D_x) = (B_{10}(t, x, D_x), \dots, B_{b0}(t, x, D_x))^T$$

formed by the main parts of the operators $B_j(t, x, D_x)$ for $j = 1, \dots, b$. Namely

$$B_{j0}(t, x, D_x)u = \sum_{|k|=r_j} b_{jk}(t, x) D_x^k u, \quad j = 1, \dots, b.$$

For $(t_0, x_0) \in \operatorname{cl} \Gamma$ and $(q, \xi) \in \Gamma_{x_0}$ we put

$$C(t_0, x_0, \xi, \tau) := B_0(t_0, x_0, i(\xi + \tau\nu)),$$

the column matrix whose rows are polynomials in τ of the degree at most $2b - 1$.

Further by $C^+(t_0, x_0, q, \xi, \tau)$ we note the column matrix which elements are remainders of a division of polynomials from the matrix $C(t_0, x_0, \xi, \tau)$ by the reduction polynomial $A^+(t_0, x_0, q, \xi, \tau)$.

Let elements $c_j^+(t_0, x_0, q, \xi, \tau)$ for $j = 1, \dots, b$ of the matrix $C^+(t_0, x_0, q, \xi, \tau)$ have the polynomial form

$$c_j^+(t_0, x_0, q, \xi, \tau) = \sum_{l=1}^{b} d_{jl}(t_0, x_0, q, \xi)\tau^{l-1}, \quad j = 1, \dots, b.$$

We shall say that problem (1.1)–(1.3) satisfies the *uniform complementary condition* (C) if and only if the rang of the matrix

$$D(t_0, x_0, q, \xi) = (d_{jl}(t_0, x_0, q, \xi))_{j,l=1}^{b}$$

is b for all $(t_0, x_0) \in \operatorname{cl}\Gamma$ and all $(q, \xi) \in \Gamma_{x_0}$.

With respect to the continuity of $|\det\, D(t_0, x_0, q, \xi)|$ the complementary condition (C) and the condition

- there is a constant $\delta^+ > 0$ such that for all $(t_0, x_0) \in \operatorname{cl}\Gamma$ and all $(q, \xi) \in \Gamma_{x_0}$ satisfying the equation $|q| + \|\xi\|_{\mathbb{R}^n}^{2b} = 1$, the inequality

  $$|\det D(t_0, x_0, q, \xi)| \geq \delta^+$$

  holds,

are mutually equivalent.

Remark 1.3. If we consider a second order differential operator

$$A(t, x, D_t, D_x)u = \sum_{i,j=1}^{n} a_{ij}(t, x) D_{ij} u + \sum_{i=1}^{n} a_i(t, x) D_i u + a_0(t, x) u$$

and

$$B(t, x, D_x)u = \sum_{i=1}^{n} b_i(t, x) D_i u + b_0(t, x) u$$

then the uniform complementary condition (C) with the constant $\delta^+ > 0$ represents the inequality

$$\sum_{i=1}^{n} b_i(t, x) |\xi_i + \tau^+(t, x, q, \xi)\nu_i| > \delta^+$$

for all $(t, x) \in \operatorname{cl}\Gamma,\ (q, \xi) \in \Gamma_x$.

Now, we define for problem (1.1)–(1.3) a compatibility condition. With respect to [19, p. 21] we have

The hypothesis (Q) **of the compatibility.** Let $0 \leq r_j \leq 2b - 1$ for $j = 1, \ldots, b$ be an order of the differential operator $B_j(t, x, D_x)$ from (1.2). We shall say that problem (1.1)–(1.3) satisfies the *compatibility condition* (Q) if and only if for all indices j for which $r_j = 0$ the equality

$$b_{jk}(0, x) g(0, x) - b_{jk}(0, x) f(0, x, \emptyset)|_{x \in \partial\Omega} = 0 \tag{1.5}$$

holds for $|k| = r_j = 0$ and $\emptyset \in \mathbb{R}^\kappa$ is the zero vector.

Remark 1.4. In the case, if r_j is a positive integer for all $j = 1, \ldots, b$, then the associated compatibility condition of problem (1.1)–(1.3) is satisfied automatically by the homogenity of boundary (1.2) and initial condition (1.3).

To formulate a smoothness assumption we define Hölder spaces. The denotations

$$\langle u\rangle^{s}_{t,\mu,Q} := \sup_{\substack{(t,x),(s,x)\in \operatorname{cl} Q\\ t\neq s}} \frac{|u(t,x)-u(s,x)|}{|t-s|^{\mu}},$$

$$\langle u\rangle^{y}_{x,\nu,Q} := \sup_{\substack{(t,x),(t,y)\in \operatorname{cl} Q\\ x\neq y}} \frac{|u(t,x)-u(t,y)|}{\|x-y\|^{\nu}_{\mathbb{R}^n}},$$

will be used.

Definition 1.5. Let $\alpha\in(0,1)$ and l be a nonnegative integer.

(a) The Banach space of continuous on $\operatorname{cl} Q$ functions $u\colon \operatorname{cl} Q\to\mathbb{R}$ with the continuous derivatives $D^k_x u$ on $\operatorname{cl} Q$ for $1\le |k|\le l$ and with the norm

$$\|u\|_{l,Q} = \sum_{0\le |k|\le l} \sup_{(t,x)\in \operatorname{cl} Q} |D^k_x u(t,x)|$$

will be denoted by $C^l_x(\operatorname{cl} Q,\mathbb{R})$.

(b) The symbol $C^{l/(2b),\,l}_{t,x}(\operatorname{cl} Q,\mathbb{R})$ represents the Banach space of continuous functions $u\colon \operatorname{cl} Q\to\mathbb{R}$ with the continuous derivatives $D^{k_0}_t D^k_x u$ for $1\le 2bk_0+|k|\le l$ on $\operatorname{cl} Q$ and with the norm

$$\|u\|_{l/(2b),l,Q} = \sum_{0\le 2bk_0+|k|\le l} \sup_{(t,x)\in \operatorname{cl} Q} |D^{k_0}_t D^k_x u(t,x)|.$$

(c) The symbol $C^{l+\alpha}_x(\operatorname{cl} Q,\mathbb{R})$ means the Banach space of continuous function $u\colon \operatorname{cl} Q\to\mathbb{R}$ with the continuous derivatives $D^k_x u$ for $|k|=1,\dots,l$ on $\operatorname{cl} Q$ and with the finite norm

$$\|u\|_{l+\alpha,Q} = \|u\|_{l,Q} + \sum_{|k|=l} \langle D^k_x u\rangle^{y}_{x,\alpha,Q}.$$

(d) By the symbol $C^{(l+\alpha)/(2b),\,l+\alpha}_{t,x}(\operatorname{cl} Q,\mathbb{R})$ we shall denote the Banach space of continuous functions $u\colon \operatorname{cl} Q\to\mathbb{R}$ with the continuous derivatives $D^k_x u$ for $|k|=1,\dots,l$ and $D^{k_0}_t D^k_x u$ for $1\le 2bk_0+|k|\le l$ on $\operatorname{cl} Q$ and with the finite norm

$$\begin{aligned}\|u\|_{(l+\alpha)/(2b),l+\alpha,Q} &= \|u\|_{l/(2b),l,Q} + \sum_{2bk_0+|k|=l} \langle D^{k_0}_t D^k_x u\rangle^{y}_{x,\alpha,Q}\\ &\quad + \sum_{0<l+\alpha-2bk_0-|k|<2b} \langle D^{k_0}_t D^k_x u\rangle^{s}_{t,(l+\alpha-2bk_0-|k|)/(2b),Q}.\end{aligned}$$

(See also 11, p. 147.)

Definition 1.6. Let $r \in (1,\infty)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. We shall say that boundary $\partial\Omega$ belongs to the class C^r if and only if:

(a) There exists a tangential space $T_{\partial\Omega}(x)$ to $\partial\Omega$ in any point $x \in \partial\Omega$.
(b) Assume $y \in \partial\Omega$ and let $(y; z_1, \dots, z_n)$ be a local orthonormal coordinate system with the center y and with the axis z_n oriented like the inner normal to $\partial\Omega$ at the point y. Then there exists a number $b > 0$ such that for every $y \in \partial\Omega$ there is an neighbourhood $O(y) \subset \mathbb{R}^n$ of y and a function $F \in C^r(\operatorname{cl} B, \mathbb{R})$, where the part of boundary

$$\begin{aligned} \partial\Omega \cap O(y) &= \{(z', F(z')) \in \mathbb{R}^n : z' = (z_1, \dots, z_{n-1}) \in B\} \text{ and} \\ B &= \{z' \in \mathbb{R}^{n-1} : \|z'\|_{\mathbb{R}^{n-1}} < b\}. \end{aligned}$$

Here $C^r(\operatorname{cl} B, \mathbb{R})$ is a space of the functions $C^l(\operatorname{cl} B, \mathbb{R})$ for $l = [r]$ and with the finite norm $\|u\|_{l+\alpha,Q}$ whereby $\alpha = r - [r] \in (0,1)$ and $r = l + \alpha$.

Definition 1.7. Let $\Omega \subset \mathbb{R}^n$ be a domain and $\partial\Omega \in C^r$ for some $r > 1$. Put $S_y := \partial\Omega \cap O(y)$ and $\Gamma_y = (0,T] \times S_y$ for $y \in \partial\Omega$, where $O(y)$ is a neighbourhood of the point y (see Definition 1.6). Let again $\alpha \in (0,1)$ and l be a nonnegative integer. The symbol $C_{t,x}^{(l+\alpha)/(2b),\, l+\alpha}(\operatorname{cl}\Gamma, \mathbb{R})$ means the Banach space of continuous functions $u \colon \operatorname{cl}\Gamma \to \mathbb{R}$ with the continuous derivatives $D_x^k u$ for $|k| = 1, \dots, l$ and $D_t^{k_0} D_x^k$ for $1 \le 2bk_0 + |k| \le l$ on $\operatorname{cl}\Gamma$ and with the finite norm

$$\|u\|_{(l+\alpha)/(2b),l+\alpha,\Gamma} = \sup_{y \in \partial\Omega} \|u\|_{(l+\alpha)/(2b),l+\alpha,\Gamma_y}.$$

The norm on the right-hand side of the last equality is defined by Definition 1.5(d) such that we write in it Γ_y instead of Q.

The hypothesis ($\mathrm{S}^{l+\alpha}$) **of the smoothness.** Let $\alpha \in (0,1)$ and l be a nonnegative integer. We shall say that problem (1.1)–(1.3) satisfies the *smoothness condition* ($\mathrm{S}^{l+\alpha}$) if and only if

(a) the coefficients of the operator $A(t, x, D_t, D_x)$ from (1.1) satisfy

$$a_k \in C_{t,x}^{(l+\alpha)/(2b),(l+\alpha)}(\operatorname{cl} Q, \mathbb{R}),$$

(b) the coefficients of $B(t, x, D_x)$ from (1.2) satisfy

$$b_{ik} \in C_{t,x}^{(l+\alpha+2b-r_j)/(2b),\, l+\alpha+2b-r_j}(\operatorname{cl}\Gamma, \mathbb{R})$$

for $0 \le r_j < 2b - 1$ for $j = 1, \dots, b$,
(c) $\partial\Omega \in C^{l+\alpha+2b}$.

In the conclusion of this section we pronounce the existence and uniqueness theorem of the classical solution of problem (1.1)–(1.3) with the nonlinear member $f = 0$.

Proposition 1.8 (see [19, p. 21] and [15, pp. 182–183]). *Let the conditions* (P), (C) *and* (S^α) *be satisfied for* $\alpha \in (0,1)$. *A necessary and sufficient condition for the existence and uniqueness of the solution*

$$u \in C^{(2b+\alpha)/(2b),2b+\alpha}_{t,x}(\operatorname{cl} Q, \mathbb{R})$$

of linear problem (1.1)–(1.3) *for* $f = 0$ *is*

$$g \in C^{\alpha/(2b),\alpha}_{t,x}(\operatorname{cl} Q, \mathbb{R})$$

with the compatibility condition (Q). *Moreover, there exists a constant* $c > 0$ *"independent of* g*" such that*

$$c^{-1}\|g\|_{\alpha/(2b),\alpha,Q} \leq \|u\|_{(2b+\alpha)/(2b),2b+\alpha,Q} \leq c\|g\|_{\alpha/(2b),\alpha,Q}.$$

2. Preliminary notions and general results

In this part we remind some notions and assertions from the nonlinear functional analysis applied in the fundamental lemmas and theorems.

Throughout this paper we shall assume that X and Y are Banach spaces either both over the real or complex field.

In the Zeidler books [33, p. 667] and [32, pp. 365–366] we find the following definitions of the linear and nonlinear Fredholm operator.

Definition 2.1. The linear operator $F: X \to Y$ is called the *Fredholm mapping* if and only if

(a) F is continuous on X and
(b) $\dim N(F) < \infty$ and $\operatorname{codim} R(F) = \dim Y/R(F) < \infty$,

where the kernel $N(F)$ of F and $R(F) = F(X)$ are closed sets in X and Y, respectively. The *index* $\operatorname{ind} F$ of the operator F is defined as the difference $\dim N(F) - \operatorname{codim} R(F)$.

The following proposition gives the necessary and sufficient condition for a linear operator to be Fredholm.

Proposition 2.2 (S. M. Nikol'skiĭ, [30, p. 233]). *A linear bounded operator* $A: X \to Y$ *is Fredholm of the zero index if and only if* $A = C + T$, *where* $C: X \to Y$ *is a linear homeomorphism and* $T: X \to Y$ *is a linear completely continuous operator.*

Definition 2.3. The nonlinear operator $F: D(F) \subset X \to Y$ defined on the open set $D(F)$ is called a *Fredholm mapping* if and only if:

(a) $F \in C^1(D(F), Y)$ and
(b) the Fréchet derivative $F'(u): X \to Y$ is a linear Fredholm operator for every $u \in D(F)$.

If the index ind $F'(u)$ is constant for all $u \in D(F)$, then we call this number the index of F and write it as ind F.

Remark 2.4. According to the perturbation invariance of the index in Proposition 8.14 from [32, p. 366] that $\operatorname{ind} F'(u)$ is constant on $D(F)$ whenever $D(F)$ is connected set and

$$\operatorname{ind} F = \dim N(F'(u)) - \operatorname{codim} R(F'(u)), \quad u \in D(F).$$

For the compact perturbation of C^1-Fredholm operator we shall use the following proposition.

Proposition 2.5 (E. Zeidler [33, p. 672]). *Let $A: D(A) \subset X \to Y$ be a C^1-Fredholm operator on the open set $D(A)$ and $B: D(A) \to Y$ be a compact mapping from the class C^1. Then $A + B: D(A) \to Y$ is a Fredholm* (*possibly nonlinear*) *operator with the same index as A at each point of $D(A)$.*

Definition 2.6. Let $D \subset X$ be a nonempty open set and $F: D \to Y$.

(a) A point $u_0 \in D$ is called *a regular point of* F if and only if the Fréchet derivative $F'(u_0): X \to Y$ is a linear homeomorphism of X onto Y (i.e. bijective and both $F'(u_0)$ and $(F'(u_0))^{-1}$ are continuous mappings).
(b) If $u_1 \in D$ is not regular point of F, then it is called *a singular point of* F.
(c) The point $u_2 \in D$ be called *a critical point of* F if and only if the equation $F'(u_2)h = 0 \in Y$ has a nontrivial solution $h \in X$. The critical point of F is a singular point of F.
(d) The image $F(u_3)$ of a singular point $u_3 \in D$ is called *a singular value of* F. If $S \subset D$ is a set of all singular points of $F: D \to Y$, then $F(S)$ is called *a set of all singular values of* F and $Y \setminus F(S)$ is *a set of all regular values of* F.
(e) A subset of a topological space Z is *residual* if and only if it is a countable intersection of dense and open subset of Z.

By the Baire theorem in any complete metric space or locally compact Hausdorff topological space, a residual set is dense in this space.

Definition 2.7. Consider the operator $F: X \to Y$ (in general nonlinear).

(a) F is called *proper* (or *σ-proper*) if and only if for each compact set $K \subseteq Y$ the set $F^{-1}(K)$, is compact (or a countable union of compact sets).
(b) The mapping F is *closed* if and only if for each closed set $S \subset X$ the set of images $F(S)$ is closed in Y.
(c) F is called a *coercive* mapping if and only if for each bounded set $S \subset Y$ the set $F^{-1}(S)$ is bounded in X.

Remark 2.8.

(a) Clearly F is coercive if and only if

$$\lim_{\|u\|_X \to \infty} \|F(u)\|_Y = \infty.$$

(b) If X and Y are finite dimensional Euclidean spaces and $F: X \to Y$ is continuous on X, then F is proper if and only if F is coercive (see [24, pp. 57–58]).

The most important theorem for nonlinear Fredholm mappings is due to S. Smale [25, p. 862] and Quinn [23]. It is also in [7, pp. 11–12] and [21, p. 217].

Proposition 2.9 (A Smale–Quinn Theorem). *If $F: X \to Y$ is a Fredholm mapping* (*possible nonlinear*) *of the class C^k in the Fréchet sence and either*

(a) *X has a countable basis* (*S. Smale*), *or*

(b) *F is σ-proper* (*Quinn*),

then the set R_F of all regular values of F is residual in Y. Moreover, if F is proper, then R_F is open and dense set in Y.

Definition 2.10. The mapping $F: X \to Y$ is called *a local C^1-diffeomorphism* at $u_0 \in X$ if and only if there exists a neighbourhood $U_1(u_0) \subset X$ of u_0 and $U_2(F(u_0)) \subset Y$ of $F(u_0)$ such that

(a) F is bijective, and

(b) both F and F^{-1} are C^1 mappings.

Proposition 2.11 (A Local Inverse Mapping Theorem, [32, p. 172]). *Let $F: U(u_0) \subset X \to Y$ be a C^1-mapping in the Frèchet sense. Then F is a local C^1-diffeomorphism at u_0 if and only if u_0 is a regular point of F.*

Proposition 2.12 ([22], [24, p. 89]). *Let* $\dim Y \geq 3$ *and $F: X \to Y$ be a Fredholm mapping of the zero index. If $u_0 \in X$ is an isolated singular point of F, then F is locally invertible at u_0.*

Definition 2.13. Let M_1, M_2 be two metric spaces and $F: M_1 \to M_2$.

(a) The mapping F is called *locally injective at the point $u_0 \in M_1$* if and only if there is a neighbourhood $U(u_0)$ of u_0 such that F is injective in $U(u_0)$. F is *locally injective in M_1* if and only if it is locally injective at each point $u \in M_1$.

(b) Let the mapping F be continuous on M_1. Then F is called *locally invertible at the point $u_0 \in M_1$* if and only if there is a neighbourhood $U_1(F(u_0))$ of $F(u_0)$ such that F is homeomorphism of $U(u_0)$ onto $U_1(F(u_0))$. F is *locally invertible in M_1* if and only if it is locally invertible at each point $u \in M_1$.

(c) Let F be continuous on M_1. We denote by Σ the set of all point $u \in M_1$ at which F is not locally invertible. The set $M_1 \setminus \Sigma$ is open and Σ is closed in M_1.

The following proposition says on the number of solutions of the operator equation $F(u) = q$.

PROPOSITION 2.14 (Ambrosetti Theorem, [2, p. 216]). *Let $F \in C(X, Y)$ be a proper mapping. Then the cardinal number* $\operatorname{card} F^{-1}(q)$ *of the set $F^{-1}(q)$ is constant and finite* (*it may be zero*) *for every q taken from the same component* (*nonempty and connected subset*) *of the set $Y \setminus F(\Sigma)$.*

A relation between the local invertibility and homeomorphism of X onto Y gives the global inverse mapping theorem.

PROPOSITION 2.15 (R. Cacciopoli [9], E. Zeidler [32, p. 174]). *Let $F \in C(X, Y)$ be a locally invertible mapping in X. Then F is a homeomorphism of X onto Y if and only if F is proper.*

The following propositions give necessary and sufficient conditions for the proper mapping.

PROPOSITION 2.16 (See [32, p. 176], [24, p. 49], [28, p. 20]). *Let $F \in C(X, Y)$.*

(a) *If F is proper, then F is a nonconstant closed mapping.*
(b) *If $\dim X = \infty$ and F is a nonconstant closed mapping, then F is proper.*

PROPOSITION 2.17 (See [24, pp. 58–59], [32, p. 498] and [28, p. 20]). *Suppose that $F: X \to Y$ and $F = F_1 + F_2$, where*

(a) *$F_1: X \to Y$ is a continuous proper mapping on X, and*
(b) *$F_2: X \to Y$ is completely continuous, or*
(c) *$F: X \to X$, $F = I - F_2$, where $I: X \to X$ is the identity and $F_2: X \to X$ is a condensing map* (*for the definition see* [10, p. 69]).

Then

(i) *The restriction of the mapping F to an arbitrary bounded closed set in X is a proper mapping.*
(ii) *If moreover, F is coercive, then F is a proper mapping.*

DEFINITION 2.18. Let $F := I - f: X \to X$ be a field ($I: X \to X$ is the identity mapping).

(a) We shall say that F is *strictly surjective* if and only if it is
  - a condensing field (i.e. f is condensing), and

- for each $y \in X$ there is a sequence $\{r_k\}_{k\in N} \subset \mathbb{R}$ such that

$$\lim_{k\to\infty} r_k = \infty \quad \text{and} \quad \deg(F - y, U(0, r_k), 0) \neq 0$$

for every $k \in N$. Here $U(0, r) = \{u \in X : \|u\|_X \leq r\}$.

(b) We shall say that F is *strictly solvable* if and only if it is

- a condensing field, and
- there exists a sequence $\{r_k\}_{k\in N} \subset \mathbb{R}$ such that $\lim_{k\to\infty} r_k = \infty$ and $\deg(F, U(0, r_k), 0) \neq 0$ for every $k \in N$.

In both definitions the degree of a condensing field is understood in the sense given in [10, pp. 69, 71–72].

Remark 2.19. It is clear that if F is strictly surjective, then it is surjective and if F is strictly solvable, then it is also solvable (i.e. there is $x \in X$ with $F(x) = 0$). Moreover, if F is strictly surjective, then it is stricty solvable, too.

Now we can formulate some sufficient conditions for the surjectivity of an operator.

Proposition 2.20 (See [28, pp. 24 and 27]). *Let X be a real Banach space. Suppose*

(a) *$P = I - f: X \to X$ is a condensing field,*
(b) *P is coercive,*
(c) *there exists a strictly solvable field $G = I - g: X \to X$ and $R > 0$ such that, for all solutions $u \in X$ of the equation*

$$P(u) = kG(u)$$

and for all $k < 0$, the estimation $\|u\|_X < R$ holds.

Then the following statements are true:

(i) *P is a proper mapping,*
(ii) *P is strictly surjective,*
(iii) $\operatorname{card} P^{-1}(q)$ *is constant, finite and nonzero for every q from the same connected component of the set $Y \setminus P(\Sigma)$.*

Proposition 2.21 (Schauder invariance of domain theorem [32, p. 705]). *Let $F: (M \subseteq X) \to X$ is continuous and locally compact perturbation of identity on the open nonempty set M in the Banach space X. Then:*

(a) *If F is locally injective on M so F is an open mapping.*
(b) *If F is injective on M so F is a homeomorphism from M onto the open set $F(M)$.*

3, A nonlinear problem and Green function

Using results on the Green function for problem (1.1)–(1.3) with $f = 0$ we shall study the existence of the given nonlinear problem from Section 1.

Definition 3.1 (Green function). A function of four variables $G\colon D(G) \to \mathbb{R}$ with the values $G(t,x;\tau,\xi)$ for

$$(t,x;\tau,\xi) \in D(G) = \{(t,x;\tau,\xi) \in \operatorname{cl} Q \times \operatorname{cl} Q : 0 \le \tau < t \le T,\ x,\xi \in \overline{\Omega}\}$$

and with the following properties:

(i) G is a continuous function on $D(G)$.
(ii) G has the first derivative with respect to t and the derivatives of $|k|$th order $D_x^k G$ for $1 \le |k| \le 2b$ on $D(G)$.
(iii) G is defined by the equality

$$G(t,x;\tau,\xi) = Z(t,x;\tau,\xi) - v(t,x;\tau,\xi), \quad (t,x;\tau,\xi) \in D(G),$$

where $Z\colon D(G) \to \mathbb{R}$ is a fundamental solution of equation (1.1) with $f = 0$ (for the definition see [14, p. 63]) and the function $v\colon D(G) \to \mathbb{R}$ satisfies the initial-boundary value problem

(a) $A(t,x,D_t,D_x)v(t,x;\tau,\xi) = 0$ for $(t,x;\tau,\xi) \in D(G)$,
(b) $v(t,x;\tau,\xi)|_{t=\tau} = 0$, if at least one of points x or ξ lies inside of the domain Ω,
(c) $B_j(t,x,D_x)v(t,x;\tau,\xi) = B_j(t,x,D_x)Z(t,x;\tau,\xi)$ for $(t,x) \in \operatorname{cl}\Gamma$ and $j = 1,\ldots,b$

is called the *Green function* of linear problem (1.1)–(1.3) with $f = 0$.

The following proposition says on the existence and estimations of the Green function.

Proposition 3.2 ([15, pp. 182–183]). *Let* $\alpha \in (0,1)$ *and the assumptions* (P), (C), (S^α) *be satisfied. Then:*

(a) *there exists the Green function of linear problem* (1.1)–(1.3) *with* $f = 0$ *which has derivatives* $D_t^{k_0} D_x^k G$ *for* $0 \le 2bk_0 + |k| \le 2b$, *thereby the estimations*

$$(3.1)\qquad |D_t^{k_0} D_x^k G(t,x;\tau,\xi)| \le c_1 (t-\tau)^{-(n+2bk_0+|k|)/(2b)} \exp\bigg\{ -c_2 \frac{\|x-\xi\|_{\mathbb{R}^n}^r}{(t-\tau)^{1/(2b-1)}} \bigg\}$$

for $(t,x;\tau,\xi) \in D(G)$,

$$(3.2)\qquad |D_t^{k_0} D_x^k G(t,x;\tau,\xi) - D_t^{k_0} D_x^k G(t,y;\tau,\xi)| \le c_1 \|x-y\|_{\mathbb{R}^n}^\alpha (t-\tau)^{-(n+2bk_0+|k|+\alpha)/(2b)} \exp\bigg\{ -c_2 \frac{\|x^*-\xi\|_{\mathbb{R}^n}^r}{(t-\tau)^{1/(2b-1)}} \bigg\}$$

for $(t,x;\tau,\xi),(t,y;\tau,\xi)\in D(G)$, *if*

$$2bk_0+|k|=2b,\quad \|x^*-\xi\|_{\mathbb{R}^n}=\min\{\|x-\xi\|_{\mathbb{R}^n},\|y-\xi\|_{\mathbb{R}^n}\},$$

$$\tag{3.3} \begin{aligned} &|D_t^{k_0}D_x^kG(t,x;\tau,\xi)-D_t^{k_0}D_x^kG(s,x;\tau,\xi)| \\ &\quad\le c_1(t-s)^{(2b(1-k_0)-|k|+\alpha)/(2b)}(s-\tau)^{-(n+2b+\alpha)/(2b)} \\ &\qquad\qquad\cdot\exp\bigg\{-c_2\frac{\|x-\xi\|_{\mathbb{R}^n}^r}{(t-\tau)^{1/(2b-1)}}\bigg\}\end{aligned}$$

for $(t,x;\tau,\xi),(s,x;\tau,\xi)\in D(G)$ *such that* $\tau<s<t$ *and* $0<2bk_0+|k|\le 2b$ *hold.*

Here $r=2b/(2b-1)$ *and the constants* c_1, c_2 *depend on* δ *and* δ^+ *from hypotheses* (P) *and* (C), *respectively, on the constant which bounds the associated norms of all coefficients* a_k, b_{jk} *from* (1.1)–(1.2), *respectively, on the measure of the variety* $\partial\Omega$ *from condition* (S^α) *and on the numbers* n, b, r_j *for* $j=1,\dots,b$ *and* α, T.

(b) *If moreover to the hypotheses of* (a) *we take* $g\in C_{t,x}^{\alpha/2,\alpha}(\mathrm{cl}\,Q,\mathbb{R})$ *and hypotheses* (Q) *for* $f=0$, *then the function* $u\colon \mathrm{cl}\,Q\to\mathbb{R}$ *defined by*

$$\tag{3.4} u(t,x)=\int_0^t d\tau\int_\Omega G(t,x;\tau,\xi)g(\tau,\xi)\,d\xi$$

is a solution of linear problem (1.1)–(1.3) *for* $f=0$ *and belongs to* $C_{t,x}^{(2b+\alpha)/(2b),2b+\alpha}(\mathrm{cl}\,Q,\mathbb{R})$. *Hence, the operator* $L\colon D(L)\xrightarrow{\text{onto}}R(L)$ *where* $Lu=A(t,x,D_x,D_t)u$ *and*

$$\begin{aligned} D(L)&=\{u\in C_{t,x}^{(2b+\alpha)/(2b),2b+\alpha}(\mathrm{cl}\,Q,\mathbb{R}):B(t,x,D_x)u|_\Gamma=0,\ u_{t=0}=0\},\\ R(L)&=\{g\in C_{t,x}^{\alpha/2,\alpha}(\mathrm{cl}\,Q,\mathbb{R}):g(t,x)|_{t=0,\,x\in\partial\Omega}=0\}\end{aligned}$$

has the inverse

$$L^{-1}\colon R(L)\xrightarrow{\text{onto}}D(L)$$

defined by (3.4).

(c) *There exists an extension*

$$\overline{L^{-1}}\colon L_2(\mathrm{cl}\,Q,\mathbb{R})\xrightarrow{\text{onto}}R(\overline{L^{-1}})\subset L_2(\mathrm{cl}\,Q,\mathbb{R})$$

of the operator L^{-1} (*see* (b)) *and*

$$(\overline{L^{-1}}g)(t,x)=\int_0^t d\tau\int_\Omega G(t,x;\tau,\xi)g(\tau,\xi)\,d\xi$$

for $g\in L_2(\mathrm{cl}\,Q,\mathbb{R})$ (*see* [1], [27], [26] *and* [15, pp. 183, 212]).

Remark 3.3.

(a) Pay attention to Proposition 3.2. The exponent k_0 takes only values 0, 1. Estimation (3.2) holds either for the pair $(k_0,|k|)=(1,0)$ or $(0,2b)$. The estimate does not hold for $k_0=|k|=0$.

(b) Statement (c) of Proposition 3.2 says that for $g \in L_2(\mathrm{cl}\, Q, \mathbb{R})$ the integral
$$\int_0^t d\tau \int_\Omega G(t,x;\tau,\xi) g(\tau,\xi)\, d\xi$$
gives a mild solution of (1.1)–(1.3) with $f = 0$. (It is a classical solution of this problem for a sufficiently smooth right-hand side g.)

Lemma 3.4. *Let assumptions* (P), (C), (S^α) *be satisfied for some* $\alpha \in (0,1)$. *Then*

$$|D_t^{k_0} D_x^k G(t,|;\tau,\xi) \le c(t-\tau)^{-\mu} \|x-\xi\|_{\mathbb{R}^n}^{2b\mu-(n+2bk_0+|k|)} \tag{3.5}$$

for $0 \le 2bk_0 + |k| \le 2b$ *and* $\mu \le (n + 2bk_0 + |k|)/(2b)$, *thereby* $0 \le \tau < t \le T$ *and* $x, \xi \in \mathrm{cl}\,\Omega$, $x \ne \xi$. *The positive constant* c *does not depend on* t, x, τ, ξ.

Proof. From (3.1)
$$\begin{aligned}|D_t^{k_0} D_x^k G(t,|;\tau,\xi) \le{}& c_1(t-\tau)^{-\mu} \|x-\xi\|_{\mathbb{R}^n}^{2b\mu-(n+2bk_0+|k|)}\\ &\cdot [\|x-\xi\|_{\mathbb{R}^n}^{2b}/(t-\tau)]^{(n+2bk_0+|k|-2b\mu)/(2b)}\\ &\cdot \exp\{-c_2[\|x-\xi\|_{\mathbb{R}^n}^{2b}/(t-\tau)]^{1/(2b-1)}\}.\end{aligned}$$
Since $n + 2bk_0 + |k| - 2b\mu \ge 0$ and $\|x-\xi\|_{\mathbb{R}^n} < \mathrm{diam}\,\Omega$ so for $0 < \delta \le t - \tau \le T$ the estimation (3.5) is true. If $0 < t - \tau < \delta$, then with respect to
$$\lim_{y\to\infty} y^u \exp\{-cy^v\} = 0$$
for every $u, v \in \mathbb{R}$ and $c > 0$, we get estimation (3.5). □

Remark 3.5. For any $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the inequalities

$$c_n \sum_{i=1}^n |x_i| \le \|x\|_{\mathbb{R}^n} \le \sum_{i=1}^n |x_i| \tag{3.6}$$

holds for $c_n \in (0, 1/(\sqrt{2})^{n-1})$, $n \in N$ independent of x.

The aim of this part is to show that nonlinear problem (1.1)–(1.3) has at least one mild solution $u \in C_x^{2b-1}(\mathrm{cl}\, Q, \mathbb{R})$ for continuous functions f and g. Then we formulate examples of nonuniquely solvable problems.

Theorem 3.6 (The existence theorem). *Let hypotheses* (P), (C), (Q), (S^α) *for* $\alpha \in (0,1)$ *be satisfied and* $g\colon \mathrm{cl}\, Q \to \mathbb{R}$ *be a continuous function at* $\mathrm{cl}\, Q$. *Let* $f\colon \mathrm{cl}\, Q \times \mathbb{R}^\kappa \to \mathbb{R}$ *be continuous and bounded function at* $\mathrm{cl}\, Q \times \mathbb{R}^\kappa$, *where* κ *is the positive integer given in the formulation of problem* (1.1)–(1.3). *Then there is at least one mild solution* $u \in C_x^{|\gamma|}(\mathrm{cl}\, Q, \mathbb{R})$ *for* $0 \le |\gamma| \le 2b - 1$ *of* (1.1)–(1.3).

Proof. We use the Leray–Schauder fixed point theorem from [32, p. 56]. First, from Proposition 3.2(c) we can see that the mild solution $u \in C_x^{|\gamma|}(\mathrm{cl}\, Q, \mathbb{R})$

of problem (1.1)–(1.3) satisfies the equation

$$
\begin{aligned}
(3.7)\qquad u(t,x) &= \int_0^t d\tau \int_\Omega G(t,[;\tau,\xi)g(\tau,\xi) - f(\tau,\xi,\overline{D}^{\,\gamma}u(\tau,\xi))]\,d\xi \\
&=: (Su)(t,x) \quad \text{for } (t,x) \in \operatorname{cl} Q
\end{aligned}
$$

and on the contrary the solution $v \in C_x^{|\gamma|}(\operatorname{cl} Q, \mathbb{R})$ satisfying (3.7) is a mild solution of (1.1)–(1.3).

Let us take an arbitrary $u \in C_x^{|\gamma|}(\operatorname{cl} Q, \mathbb{R})$ where $0 \le |\gamma| \le 2b-1$. Then there is a constant $M > 0$ such that

$$
|g(t,x) - f(t,x,\overline{D}_x^{\,\gamma}u(t,x))| \le M
$$

for all $(t,x) \in \operatorname{cl} Q$. Put estimation (3.5) into (3.7) and embed $\operatorname{cl}\Omega$ into the ball

$$
B(x,R) := \{\xi \in \mathbb{R}^n : \|x-\xi\|_{\mathbb{R}^n} \le R,\ R > 0\}
$$

for every $x \in \operatorname{cl}\Omega$. Then

$$
\begin{aligned}
|(D_x^k Su)(t,x)| &\le \frac{Mc}{1-\mu} T^{1-\mu} \int_\Omega \|x-\xi\|_{\mathbb{R}^n}^{2b\mu-(n+|k|)}\,d\xi \\
&\le \frac{Mc}{1-\mu} T^{1-\mu} \int_{B(x,R)} \|x-\xi\|_{\mathbb{R}^n}^{2b\mu-(n+|k|)}\,d\xi.
\end{aligned}
$$

Hence, putting $x = (x_1,\dots,x_n)$, $\xi = (\xi_1,\dots,\xi_n)$ and using the spherical transformation

$$
\begin{aligned}
\xi_1 &= x_1 + r\cos\varphi_1, \\
\xi_2 &= x_2 + r\sin\varphi_1\cos\varphi_2, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\xi_{n-1} &= x_{n-1} + r\sin\varphi_1\sin\varphi_2\dots\sin\varphi_{n-2}\cos\varphi_{n-1}, \\
\xi_n &= x_n + r\sin\varphi_1\sin\varphi_2\dots\sin\varphi_{n-2}\sin\varphi_{n-1},
\end{aligned}
$$

for $r \in (0,R\rangle$, $\varphi_i \in (0,\pi)$, $i = 1,\dots,n-2$ and $\varphi_{n-1} \in (0,2\pi)$ in the last integral, we get the estimation (the Jacobi determinant of this transformation is $r^{n-1}\sin^{n-2}\varphi_1\sin^{n-3}\varphi_2\dots\sin\varphi_{n-2} \ne 0$):

$$
|(D_x^k\,Su)(t,x)| \le 2\pi^{n-1}T^{1-\mu}R^{2b\mu-|k|}Mc/(2b\mu-|k|)(1-\mu) := d_k
$$

for $(t,x) \in \operatorname{cl} Q$ and $|k|/(2b) < \mu < 1$, where $|k| = 0,\dots,2b-1$. This consideration implies the inclusion

$$
(3.8)\qquad S(G(0,d)) \subset G(0,d), \quad d \le \sum_{0\le|k|\le 2b-1} d_k,
$$

where

$$
G(0,d) := \{v \in C_x^{|\gamma|}(\operatorname{cl} Q,\mathbb{R}) : \|v\|_{C_x^{|\gamma|}(\operatorname{cl} Q,\mathbb{R})} \le d,\ 0 \le \gamma \le 2b-1\}.
$$

To prove the relative compactnes of the set $S(G(0,d))$ we apply Ascoli–Arzelà theorem [31, p. 85]. The equi-boundedness of $S(G(0,d))$ follows from (3.8). For the equi-continuity of $S(G(0,d))$, observe the difference $((t,x),(s,y)\in \operatorname{cl} Q$, $t<s$, $y=(y_1,\dots,y_n))$

$$\begin{aligned}(3.9)\qquad |(D_x^\gamma Su)(t,x) &- (D_x^\gamma Su)(s,y)| \\ &\le M\int_0^t d\tau \int_\Omega |D_x^\gamma G(t,-;\tau,\xi)D_x^\gamma G(t,[;\tau,\xi)y]|\,d\xi \\ &+ M\int_0^t d\tau \int_\Omega |D_x^\gamma G(t,[;\tau,\xi)y] - D_x^\gamma G(s,y;\tau,\xi)|\,d\xi \\ &+ M\int_t^s d\tau \int_\Omega |D_x^\gamma G(s,y;\tau,\xi)|\,d\xi.\end{aligned}$$

To estimate the first integral of (3.9) we use the mean value theorem, estimation (3.5) from Lemma 3.4 and inequalities (3.6) for the difference

$$\begin{aligned}(3.10)\quad |D_x^\gamma G(t,x;\tau,\xi) - D_x^\gamma G(t,y;\tau,\xi)| &\le \sum_{i=1}^n |x_i-y_i||D_x^{\gamma(i)}G(t,x_i^*,\tau,\xi)| \\ &\le \frac{c}{c_n}\|x-y\|_{\mathbb{R}^n}(t-\tau)^{-\mu}\sum_{i=1}^n \|x_i^*-\xi\|_{\mathbb{R}^n}^{2b\mu-(n+|\gamma(i)|)}.\end{aligned}$$

Here the multiindex $\gamma(i)=(\gamma_1,\dots,\gamma_{i-1},\gamma_i+1,\gamma_{i+1},\dots,\gamma_n)\in\mathbb{R}^n$ and $x_i^*=(y_1,\dots,y_{i-1},z_i,x_{i+1},\dots,x_n)\in\mathbb{R}^n$. The point z_i lies between the numbers x_i and y_i, $|\gamma(i)|/(2b)\le\mu<1$ and $\|x-y\|_{\mathbb{R}^n}>\|x-x_i^*\|_{\mathbb{R}^n}$. By the last inequality we obtain for $|\gamma|=0,\dots,2b-2$

$$(3.11)\quad J_{1,|\gamma|} := \int_0^t d\tau\int_\Omega |D_x^\gamma G(t,-;\tau,\xi)D_x^\gamma G(t,[;\tau,\xi)y]|\;d\xi \le C_0\|x-y\|_{\mathbb{R}^n},$$

where the constant $C_0>0$ does not depend on t, x, y.

In the case $|\gamma|=2b-1$, we take the points $x,y,\xi\in\operatorname{cl}\Omega$ satisfying the inequality $2\|x-y\|_{\mathbb{R}_n}<\|\xi-x\|_{\mathbb{R}_n}$. Then, by the triangle inequalities, it is obvious that $\|x-y\|_{\mathbb{R}_n}<\|x_i^*-\xi\|_{\mathbb{R}_n}$. Hence $\|x-\xi\|_{\mathbb{R}_n}\le\|x-x_i^*\|_{\mathbb{R}_n}+\|x_i^*-\xi\|_{\mathbb{R}_n}<\|x-y\|_{\mathbb{R}_n}+\|x_i^*-\xi\|_{\mathbb{R}_n}<2\|x_i^*-\xi\|_{\mathbb{R}_n}$. From estimation (3.10) we obtain the inequality

$$\begin{aligned}|D_x^\gamma G(t,-;\tau,\xi)&D_x^\gamma G(t,[;\tau,\xi)y]| \\ &\le (c/c_n)\cdot\|x-y\|_{\mathbb{R}_n}(t-\tau)^{-\mu}n\,(2^{-1}\|x-\xi\|_{\mathbb{R}_n})^{2b\mu-(n+2b)}.\end{aligned}$$

If we put $B_1=\{\xi\in\mathbb{R}^n:\|\xi-x\|_{\mathbb{R}_n}>2\|x-y\|_{\mathbb{R}_n}\}$, $B_2=\mathbb{R}^n-B_1$ and for $m\in N$, $m>2$ $B_3=\{\xi\in\mathbb{R}^n:\|\xi-x\|_{\mathbb{R}^n}\le m\ \|x-y\|_{\mathbb{R}^n}\}$ such that $\Omega\subset B_3$, then we have for $(2b-1+\alpha)/(2b) \le \mu < 1$, $\alpha \in (0,1)$

$$\begin{aligned} J_{1,2b-1} \le{}& (n\,c/c_n)2^{n+2b-2b\mu} \\ &\cdot \bigg[\int_0^t d\tau \int_{B_1 \cap B_3} (t-\tau)^{-\mu} \Big(\|x-y\|_{\mathbb{R}^n} \|x-\xi\|_{\mathbb{R}_n}^{2b\mu-(n+2b)} \Big)\, d\xi \\ &+ \int_0^t d\tau \int_{B_2} (t-\tau)^{-\mu} \|x-y\|_{\mathbb{R}_n} \|x-\xi\|_{\mathbb{R}_n}^{2b\mu-(n+2b)}\, d\xi \bigg] \\ \le{}& C_1 \|x-y\|_{\mathbb{R}_n}^{2b\mu-(2b-1)}, \quad C_1 > 0. \end{aligned}$$

Again employing the mean value theorem and (3.5) we find $t^* \in (t,s)$ such that

$$\begin{aligned} |D_x^\gamma G(t,y;\tau,\xi) - D_x^\gamma G(s,y;\tau,\xi)| &= |D_t D_x^\gamma G(t^*,y;\tau,\xi)|(s-t) \\ &\le c(s-t)(t-\tau)^{-\mu} \|y-\xi\|_{\mathbb{R}^n}^{2b\mu-(n+2b+|\gamma|)} \end{aligned}$$

for $\mu \le (n+2b+|\gamma|)/(2b)$ $(0 < t-\tau < t^*-\tau)$, $0 \le |\gamma| \le 2b-1$. Hence, if we put $S_1 = \{\xi \in \operatorname{cl}\Omega : \|y-\xi\|_{\mathbb{R}^n} < (s-t)^{1/(2b)}\}$ and $S_2 = \operatorname{cl}\Omega - S_1$, then by estimate (3.5) we get for the two last integral members of (3.9) $(0 \le |\gamma| \le 2b-1)$

$$\begin{aligned} J_{2,|\gamma|} :={}& \int_0^t d\tau \int_\Omega |D_x^\gamma G(t,[;\tau,\xi)y] - D_x^\gamma G(s,y;\tau,\xi)| d\xi \\ &+ \int_t^s d\tau \int_\Omega |D_x^\gamma G(s,y;\tau,\xi)| d\xi \\ \le{}& \int_0^t d\tau \int_{S_1} |D_x^\gamma G(t,[;\tau,\xi)y]|\, d\xi + \int_0^s d\tau \int_{S_1} |D_x^\gamma G(s,y;\tau,\xi)|\, d\xi \\ &+ \int_0^t d\tau \int_{S_2} |D_x^\gamma G(t,[;\tau,\xi)y] - D_x^\gamma G(s,y;\tau,\xi)|\, d\xi \\ &+ \int_t^s d\tau \int_{S_2} |D_x^\gamma G(s,y;\tau,\xi)|\, d\xi \\ \le{}& c \int_0^t d\tau \int_{S_1} (t-\tau)^{-\lambda} \|y-\xi\|_{\mathbb{R}^n}^{2b\lambda-(n+|\gamma|)}\, d\xi \\ &+ c \int_0^s d\tau \int_{S_1} (s-\tau)^{-\nu} \|y-\xi\|_{\mathbb{R}^n}^{2b\nu-(n+|\gamma|)}\, d\xi \\ &+ c \int_0^t d\tau \int_{S_2} (s-t)(t-\tau)^{-\mu} \|y-\xi\|_{\mathbb{R}^n}^{2b\mu-(n+2b+|\gamma|)}\, d\xi \\ &+ c \int_t^s d\tau \int_{S_2} (s-\tau)^{-\sigma} \|y-\xi\|_{\mathbb{R}^n}^{2b\sigma-(n+|\gamma|)}\, d\xi \end{aligned} \tag{3.12}$$

for $0 < \lambda \le (n+|\gamma|)/(2b)$, $0 < \nu \le (n+|\gamma|)/(2b)$, $0 < \mu \le (n+2b+|\gamma|)/(2b)$ and $0 < \sigma \le (n+|\gamma|)/(2b)$. If apply the spherical transformation for ξ with the center y and radius $r \in (0,(s-t)^{1/(2b)})$ in the two integrals over S_1, such for

$|\gamma|/(2b) < \lambda < 1$ and $|\gamma|/(2b) < \nu < 1$

$$(2.13)\quad \int_0^t d\tau \int_{S_1} (t-\tau)^{-\lambda} \|y-\xi\|_{\mathbb{R}^n}^{2b\lambda-(n+|\gamma|)}\, d\xi \\ \leq 2\pi^{n-1} T^{1-\lambda} (s-t)^{(2b\lambda-|\gamma|)/(2b)}/(2b\lambda-|\gamma|)(1-\lambda)$$

and

$$(3.14)\quad \int_0^t d\tau \int_{S_1} (s-\tau)^{-\nu}\, \|y-\xi\|^{2b\nu-(n+|\gamma|)}\, d\xi \\ \leq 2\pi^{n-1} T^{1-\nu} (s-t)^{(2b\nu-|\gamma|)/(2b)}/(2b\nu-|\gamma|)(1-\nu).$$

If we embed the set S_2 into the set

$$B(y,(s-t)^{1/(2b)},R) := \{\xi \in \mathbb{R}^n : (s-t)^{1/(2b)} \leq \|y-\xi\|_{\mathbb{R}^n} \leq R,\ R>0\} \supset S_2$$

and we use the spherical substitution for ξ with the center y and radius $r \in ((s-t)^{1/(2b)}, R)$ in the two integrals over S_2, then we get for $|\gamma|/(2b) < \mu < 1$ and $|\gamma|/(2b) < \sigma < 1$

$$(3.15)\quad (s-t)\int_0^t d\tau \int_{S_2} (t-\tau)^{-\mu}\, \|y-\xi\|_{\mathbb{R}^n}^{2b\mu-(n+2b+|\gamma|)}\, d\xi \\ \leq 2\pi^{n-1} T^{1-\mu} (s-t)^{(2b\mu-|\gamma|)/(2b)}/(2b+|\gamma|-2b\mu)(1-\mu)$$

and

$$(3.16)\quad \int_t^s d\tau \int_{S_2} (s-\tau)^{-\sigma}\, \|y-\xi\|_{\mathbb{R}^n}^{2b\sigma-(n+|\gamma|)} d\xi \\ \leq 2\pi^{n-1} R^{2b\sigma-|\gamma|} (s-t)^{1-\sigma}/(2b\sigma-|\gamma|)(1-\sigma).$$

From inequality (3.9) and estimations (3.11)–(3.16) we can conclude that the operator S is compact. $\square$

The following examples ilustrate a non-uniqueness of classical solution of (1.1)–(1.3) type initial-boundary value problems.

Example 3.7. Consider the two Neumann type initial-boundary value problems (parabolic and non-parabolic)

$$(3.1^*)\quad \frac{\partial u}{\partial t} = \pm\frac{\partial^2 u}{\partial x^2} + f(t,x,u), \quad (t,x) \in (0,T\rangle \times \Omega = Q,$$

$$(3.2^*)\quad \frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,1) = 0, \quad t \in \langle 0,T\rangle,$$

$$(3.3^*)\quad u(0,x) = 0, \quad x \in \overline{\Omega}.$$

(a) If $f(t,x,u) = |u|^\alpha$, $\alpha \in (0,1)$, the given problem has a continuum of the solutions $u_r \in C^{1,2}_{t,x}(\operatorname{cl} Q, \mathbb{R})$ for $r \in (0,T)$

$$u_r(t,x) = \begin{cases} 0 & \text{if } (t,x) \in \langle 0,r\rangle \times \overline{\Omega}, \\ (1-\alpha)^{1/(1-\alpha)}(t-r)^{1/(1-\alpha)} & \text{if } (t,x) \in (r,T\rangle \times \overline{\Omega}, \end{cases}$$

$u_0(t,x) = (1-\alpha)^{1/(1-\alpha)} t^{1/(1-\alpha)}$ and $u_T(t,x) = 0$ are solutions of (3.1*)–(3.3*), too.

(b) Similarly, if $f(t,x,u) = |u|^{1/2} - au$, $a > 0$, we have a continuum of solutions of (3.1*)–(3.3*) for $r \in (0,T)$

$$u_r(t,x) = \begin{cases} 0 & \text{if } (t,x) \in \langle 0,r\rangle \times \overline{\Omega}, \\ \dfrac{1}{a^2}\left(1 - \exp\left\{-\dfrac{a}{2}(t-r)\right\}\right)^2 & \text{if } (t,x) \in (r,T\rangle \times \overline{\Omega}. \end{cases}$$

The functions $u_0(t,x) = (1/a^2)(1-\exp\{-at/2\})^2$, $u_T(t,x) = 0$ are solutions of the given problem, too.

(iii) We obtain an analogical situation for $f(t,x,u) = t^\beta |u|^\alpha$ with $\alpha \in (0,1)$ and $\beta > 0$. Other nonlinearities f can be taken, too.

Example 3.8 (see [18, p. 48]). (i) Consider the initial-boundary value problem for the nonlinear equation

$$\text{(3.1**)} \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \sqrt{\frac{2}{\pi}}\left|\int_0^\pi u(t,y)\sin y\,dy\right|^{1/2} \sin x + \sqrt{\frac{2}{\pi}}\left|\int_0^\pi u(t,y)\sin 2y\,dy\right|^{1/2} \sin 2x$$

for $(t,x) \in (0,T\rangle \times (0,\pi)$, with the Dirichlet type boundary value condition

$$\text{(3.2**)} \qquad u(t,0) = u(t,\pi) = 0, \quad t \in \langle 0,T\rangle$$

and the initial value condition

$$\text{(3.3**)} \qquad u(0,x) = 0, \quad x \in \langle 0,\pi\rangle.$$

A continuum of solutions belonging to $C^{1,2}_{t,x}(\operatorname{cl} Q, \mathbb{R})$ of this problem represents the set of functions

$$u_r(t,x) = a_r(t)\sin x + b_r(t)\sin 2x, \quad (t,x) \in \operatorname{cl} Q$$

for $r \in \langle 0,T\rangle$. Here for $r \in (0,T)$

$$a_r(t) = \begin{cases} 0 & \text{if } t \in \langle 0,r\rangle, \\ (1-\exp\{-(t-r)/2\})^2 & \text{if } t \in (r,T\rangle, \end{cases}$$

and

$$b_r(t) = \begin{cases} 0 & \text{if } t \in \langle 0,r\rangle, \\ \dfrac{1}{16}(1-\exp\{-2(t-r)\})^2 & \text{if } t \in (r,T\rangle. \end{cases}$$

Further, $a_0(t) = (1-\exp\{-t/2\})^2$, $a_T(t) = 0 = b_T(t)$, $b_0(t) = (1/16)(1-\exp\{-2t\})^2$.

The functions a_r and $b_r\colon \langle 0,T\rangle \to \mathbb{R}$ are the solutions of the initial value problems

$$\begin{aligned} \frac{du}{dt} + a &= |a|^{1/2}, \quad t\in(0,T\rangle,\ a(0)=0,\\ \frac{db}{dt} + 4b &= |b|^{1/2}, \quad t\in(0,T\rangle,\ b(0)=0, \end{aligned}$$

respectively.

4. Operator formulation and fundamental lemmas

Consider the following operators:

(a)

$$A\colon X \to Y, \tag{4.1}$$

where

$$(Au)(t,x) = A(t,x,D_t,D_x)u(t,x) = D_t u(t,x) - \sum_{0\le|k|\le 2b} a_k(t,x) D_x^k u(t,x),$$

for $(t,x)\in \operatorname{cl} Q$, $u\in X$,

$$X = \{u \in C^{1,2b}_{t,x}(\operatorname{cl} Q,\mathbb{R})\colon B_j(t,x,D_x)u|_\Gamma = 0,\ j=1,\dots,b,\\ u(0,x)=0 \text{ for } x\in\operatorname{cl} Q\}$$

and $Y = C(\operatorname{cl} Q,\mathbb{R})$.

(b) The *Nemitskiĭ operator* for the function f from (1.1)

$$N\colon X\to Y, \tag{4.2}$$

where $(Nu)(t,x) = f(t,x,\overline{D}_x^\gamma u(t,x))$ for $(t,x)\in\operatorname{cl} Q$, $u\in X$.

(c) The operator

$$F\colon X\to Y, \tag{4.3}$$

where $(Fu)(t,x) = (Au)(t,x) + (Nu)(t,x)$ for $(t,x)\in\operatorname{cl} Q$, $u\in X$.

Together with the solution sets of given problem (1.1)–(1.3) we shall search the bifurcation points sets.

Definition 4.1.

(a) A couple $(u,g)\in X\times Y$ will be called the *bifurcation point* of (1.1)–(1.3) if and only if u is a solution of this problem and there exists a sequence $\{g_k\}_{k\in N}\subset Y$ such that $\lim_{k\to\infty} g_k = g$ in Y and initial-boundary value problem (1.1)–(1.3) with $g=g_k$ has at least two different solutions u_k, v_k for each $k\in N$ and $\lim_{k\to\infty} u_k = \lim_{k\to\infty} v_k = u$ in X.

(b) The set of all solutions $u \in X$ of (1.1)–(1.3) (or the set of all functions $g \in Y$) such that (u, g) is a bifurcation point of (1.1)–(1.3) will be called the *domain of bifurcation* (resp. the *bifurcation range*) of (1.1)–(1.3).

EXAMPLE 4.2. The point $(u_r, 0) \in X \times Y$ for $r \in \langle 0, T\rangle$ is a bifurcation point of the Neumann problem from Example 3.7(a) and (b). Really, there is the zero sequence $\{g_k\}_{k\in N}$, where $g_k = 0$ for $k \in N$, of the right-hand side of (1.1) for which there exist two different sequences of solutions

$$\{u_k\}_{k\in N} = \{u_{r(k+1)/(k+2)}\}_{k\in N} \quad \text{and} \quad \{v_k\}_{k\in N} = \{u_{(rk)/(k+1)}\}_{k\in N}$$

with the same limit $u_r \in X$. □

The following equivalence result is true.

LEMMA 4.3.

(a) *The function $u \in X$ is a solution of initial-boundary value problem* (1.1)–(1.3) *for $g \in Y$ if and only if $Fu = g$.*

(b) *The couple $(u, g) \in X \times Y$ is a bifurcation point of* (1.1)–(1.3) *if and only if $Fu = g$ and u is a point at which F is not locally invertible, i.e. $u \in \Sigma$ (see Definition* 2.13*).*

PROOF. The first assertion is clear.

(b) If (u, g) is a bifurcation point of (1.1)–(1.3), then with respect to Definition 4.1 we get $Fu = g$, $Fu_k = g_k = Fv_k$, $u_k \neq v_k$. Thus F is not locally injective at u. Hence, F is not locally invertible at u, i.e. $u \in \Sigma$. Conversely, if F is not locally invertible at u and $Fu = g$, then F is not locally injective at u. Hence, it follows that the couple $(u, g) \in X \times Y$ is a bifurcation point of (1.1)–(1.3). □

The following lemma gives sufficient conditions for the operator A to be of Fredholm type.

LEMMA 4.4. *Let the operator A from* (4.1) *satisfy smoothness hypothesis* (S^α)*, $\alpha \in (0, 1)$ and*

(A.1) *There exists a linear homeomorphism $H: X \to Y$ with*

$$Hu = D_t u - H(t, x, D_x)u, \quad u \in X,$$

where

$$H(t, x, D_x)u = \sum_{|k|=2b} a_k(t, x) D_x^k u + \sum_{0\le|k|\le 2b-1} h_k(t, x) D_x^k u$$

satisfying (S^α)*, $\alpha \in (0, 1)$.*

Then:

(a) $\dim X = \infty$.
(b) *The operator* $A: X \to Y$ *is a linear bounded Fredholm operator of the zero-index.*

Proof. (a) We use the decomposition theorem from [29, p. 139]: Let Z be a linear space and $0 \neq x^*: Z \to \mathbb{R}$ be a linear mapping on Z and $x_0 \in Z \setminus M$, where $M = \{x \in Z : x^*(x) = 0\}$. Then every element $x \in Z$ can be expressed by the formula

$$x = \frac{x^*(x)}{x^*(x_0)} x_0 + m,$$

where $m \in M$, i.e. there is a one-dimensional subspace L_1 of Z such that $Z = L_1 \oplus M$.

Now, we put

$$M_l = \{u \in C^{1,2b}_{t,x}(\mathrm{cl}\, Q, \mathbb{R}) : B_j(t, x, D_x)u|_\Gamma = 0 \text{ for } j = 1, \dots, l\}$$

for $l = 1, \dots, b$. We see that $C^{1,2b}_{t,x}(\mathrm{cl}\, Q, \mathbb{R}) \supset M_1 \supset \dots \supset M_b$. There exist one-dimensional spaces L_l for $l = 1, \dots, b$ such that

$$C^{1,2b}_{t,x}(\mathrm{cl}\, Q, \mathbb{R}) = L_1 \oplus M_1, \quad M_1 = L_2 \oplus M_2, \dots, M_{b-1} = L_b \oplus M_b.$$

If we put

$$\begin{aligned} M_{b+1} = \{u \in C^{1,2b}_{t,x}(\mathrm{cl}\, Q, \mathbb{R}) : B_j(t, x, D_x)u|_\Gamma = 0 \text{ for } j = 1, \dots, b, \\ u(0, x) = 0 \text{ on } \mathrm{cl}\, \Omega\} = D(A) = X \subset M_b, \end{aligned}$$

then we can write

$$\begin{aligned} C^{1,2b}_{t,x}(\mathrm{cl}\, Q, \mathbb{R}) &= L_1 \oplus \dots \oplus L_b \oplus M_b \\ &= L_1 \oplus \dots \oplus L_b \oplus L_{b+1} \oplus M_{b+1} \\ &= L_1 \oplus \dots \oplus L_b \oplus L_{b+1} \oplus X, \end{aligned} \tag{4.4}$$

where L_{b+1} is a one-dimensional subspace of M_b. Since

$$\dim C^{1,2b}_{t,x}(\mathrm{cl}\, Q, \mathbb{R}) = \infty,$$

from (4.4) we get $\dim X = \infty$.

(b) Since the coeficients a_k for $0 \leq |k| \leq 2b$ are continuous on the compact set $\mathrm{cl}\, Q$ there is a positive constant $K > 0$

$$\|Au\|_Y \leq K(\|D_t u\|_Y + \sum_{0 \leq |k| \leq 2b} \|D^k_x u\|_Y) = K\|u\|_X$$

for all $u \in X$, whence the operator A is bounded on X.

By Proposition 2.2 [30, p. 233], it is sufficient to show that

$$Au = Hu + (H(t, x, D_x) - A(t, x, D_x))u := Hu + Tu,$$

thereby the mapping $T: X \to Y$ is the linear completely continuous operator. It will be proved by the Ascoli–Arzelà theorem from [31, p. 85].

From the hypothesis (S^α), the equi-boundedness of

$$Tu = \sum_{0 \le |k| \le 2b-1} (h_k(t,x) - a_k(t,x)) D_x^k u$$

holds at the bounded set $S \subset X$, i.e. there is a constant $K_1(n, \alpha, T, \Omega) > 0$ such that $\|Tu\|_Y \le K_1 \|u\|_X$ for all $u \in S$.

With respect to (S^α) we obtain for all $u \in S$ and $(t,x), (s,y) \in \operatorname{cl} Q$

$$\begin{aligned} |Tu(t,x) - Tu(s,y)| & \\ \le & \sum_{0 \le |k| \le 2b-1} |(h_k - a_k)(t,x) - (h_k - a_k)(s,y)| \colon |D_x^k u(t,x)| \\ & + \sum_{0 \le |k| \le 2b-1} |h_k(s,y) - a_k(s,y)| \colon |D_x^k u(t,x) - D_x^k u(s,y)| \\ \le & K_2 \sum_{0 \le |k| \le 2b-1} |(h_k - a_k)(t,x) - (h_k - a_k)(s,y)| \\ & + K_3 \sum_{0 \le |k| \le 2b-1} |D_x^k u(t,x) - D_x^k u(s,y)|, \end{aligned}$$

where K_2, K_3 are positive constants only dependent of n, α, T, Ω. Using assumption (S^α) for the first member and the mean value theorem for the second member in the previous estimation, we obtain

$$\begin{aligned} |Tu(t,x) - Tu(s,y)| & \\ \le & K_2 K_4 \operatorname{card}\{k : 0 \le |k| \le 2b-1\}[|t-s|^{\alpha/(2b)} + \|x-y\|^\alpha_{\mathbb{R}^n}] \\ & + K_3 \sum_{0 \le |k| \le 2b-1} \bigg[|D_t D_x^k u(t^*, x)||t-s| + \sum_{i=1}^n |D_x^{k(i)} u(t, x_i^*)||x_i - y_i| \bigg]. \end{aligned}$$

Here t^* lays between t and s, $x_i^* = (y_1, \ldots, y_{i-1}, \xi_i, x_{i+1}, \ldots, x_n)$ with ξ_i laying between x_i and y_i. The modul of multiindex $k(i) = (k_1, \ldots, k_{i-1}, k_i + 1, \ldots, k_n)$ is $|k(i)| = |k| + 1 \le 2b$ for $i = 1, \ldots, n$.

For $|t-s| < \delta$, $\|x-y\|_{\mathbb{R}^n} < \delta$ with a sufficiently small $\delta > 0$ the every member of the last inequality is smaller than a fixed arbitrary $\varepsilon > 0$. This proves the equi-continuity of the set $T(S)$. □

Corollary 4.5. *Let $\mathcal{L}$ mean the set of all linear differential operators $A = D_t - A(t,x,D_x): X \to Y$ satisfying the hypothesis* (S^α), *$\alpha \in (0,1)$. Then, for each $A \in \mathcal{L}$, the initial boundary value homogeneous problem: $Au = 0$,* (1.2), (1.3) *has a nontrivial solution or any $A \in \mathcal{L}$ is a linear bounded Fredholm operator of the zero index.*

Proof. Really, if there exists an operator $A \in \mathcal{L}$ such that the problem $Au = 0$, (1.2), (1.3) has only trivial solution, then A is a homeomorphism of X onto Y. Then, by Lemma 4.4, all operators of $\mathcal{L}$ are Fredholm of the zero index. □

Lemma 4.6. *Suppose*

(N.1) $f \in C(\operatorname{cl} Q \times \mathbb{R}^\kappa, \mathbb{R})$.

Then the Nemitskiĭ operator $N\colon X \to Y$ *from* (4.2) *is completely continuous on* X.

Proof. For any bounded set $S \subset X$ the N is equi-bounded in Y. Also, for $|t-s|^2 + \|x-y\|^2_{\mathbb{R}^n} < \delta^2$ with a sufficiently small $\delta > 0$ we get the equi-continuity of N. □

Lemma 4.7. *Let* (S^α), (A.1), (N.1) *and an almost coercivity condition*

(F.1) *Let r be an integer $0 \le r \le 2b-1$. Suppose that coefficients a_k and h_k of operators A and H from* (4.1) *and* (A.1), *respectively are equal for $|k| = r+1, \dots, 2b$ at $\operatorname{cl} Q$ and there is a multiindex k with $|k| = r$ for which $a_k \ne h_k$ at $\operatorname{cl} Q$. Put $a = \max\{|\gamma|, r\}$. Moreover, we assume, there exists a constant $K_a > 0$ such that the inequality*

$$\|u\|_{a,\,Q} = \sum_{0 \le |k| \le a} \sup_{(t,x) \in \operatorname{cl} Q} |D^k_x u(t,x)| \le K_a \tag{4.5}$$

holds for all solutions $u \in X$ of problem (1.1)–(1.3) *with the right–hand sides g from bounded set $S \subset Y$.*

be satisfied. Then:

(a) *F from* (4.3) *is coercive on X.*
(b) *F is proper and continuous.*

Proof. (a) We need to prove that if the set $S \subset Y$ is bounded in Y, then the set of arguments $F^{-1}(S) \subset X$ is bounded in X.

By (4.5) and assumption (F.1) it follows that the set $F^{-1}(S)$ is bounded in the norm $\|\cdot\|_{a,Q}$. Hence and by (N.1) one obtains the estimation $\|Nu\|_Y \le K_4$ for all $u \in F^{-1}(S)$. From Lemma 4.4(b) also $\|Au\|_Y \le \|Fu\|_Y + \|Nu\|_Y \le K_5$ for any $u \in F^{-1}(S)$, where K_4, K_5 are positive constants.

On the other hand, condition (A.1) ensures the existence and uniqueness of the solution $u \in X$ of the linear equation $Hu = y$ for any $y \in Y$ and (see the Green representation of solution from Proposition 3.2 and the estimation (3.5)) the estimation

$$\|u\|_X \le K_6 \|y\|_Y, \quad K_6 > 0,\ u \in F^{-1}(S) \tag{4.6}$$

is true. Then for $u \in F^{-1}(S)$ we have

$$Hu = Au + \sum_{0 \le |k| \le 2b} (a_k(t,x) - h_k(t,x)) D_x^k u.$$

With respect to (S^α) and (F.1)

$$\begin{aligned}\|y\|_Y = \|Hu\|_Y \le \|Au\|_Y + \sum_{0 \le |k| \le r} \|a_k - h_k\|_Y \, \|D_x^k u\|_Y \\ \le K_5 + K_7 \|u\|_{r,\,Q} \le K_5 + K_7 \|u\|_{a,\,Q} \le K_5 + K_7 K_a, \quad K_7 > 0.\end{aligned}$$

Hence and by (4.6)

$$\|u\|_X \le K_6(K_5 + K_7 K_a), \quad u \in F^{-1}(S).$$

(b) Since $\dim X = \infty$ and A is a nonconstant and closed mapping on X, then by Proposition 2.16(b) it is proper on X. From Lemma 4.6 the operator N is completely continuous on X. From (b) of this lemma F is coercive on X. Proposition 2.17(b) concludes the proof of (b) and the proof of Lemma 4.7. □

The following lemma gives conditions for the continuous F-differentiability of the Nemitskiĭ operator N.

Lemma 4.8. *Let the Nemitskiĭ operator $N: X \to Y$ satisfy the condition* (N.1) *and*

(N.2) $\partial f / \partial v_\beta \in C(\mathrm{cl}\, Q \times \mathbb{R}^\kappa, \mathbb{R})$ *for the multiindices β with the modul $0 \le |\beta| \le 2b - 1$, where κ represents the number of the components in the vector function $\overline{v}_\beta = \overline{D}_x^\beta u$ from* (1.1).

Then

(a) *the operator N is continuously Fréchet differentiable on X, i.e. $N \in C^1(X, Y)$.*
(b) *If moreover* (S^α) *for $\alpha \in (0,1)$ holds, then $F \in C^1(X,Y)$.*

Proof. (a) We need prove that the Fréchet derivative $N': X \to L(X,Y)$ defined by the equation

$$N'(u)h(t,x) = \sum_{\substack{0 \le |\beta| \le 2b-1 \\ \mathrm{card}\{\beta\} = \kappa}} \frac{\partial f}{\partial v_\beta}[t, x, \overline{D}_x^\gamma u(t,x)] D_x^\beta h(t,x) \tag{4.7}$$

is continuous on X for every $u, h \in X$. Here $\beta = (\beta_1, \dots, \beta_n)$ represents every multiindex $\gamma = (\gamma_1, \dots, \gamma_n)$ appearing in the nonlinearity f. It is sufficient to show for every fixed $v \in X$ the condition:

$$\forall \varepsilon > 0 \ \exists \delta(\varepsilon, v) > 0 \ \forall u \in X, \ \|u - v\|_X < \delta \Rightarrow \|N'u - N'v\|_{L(X,Y)} < \varepsilon,$$

i.e.

$$\sup_{h\in X,\ \|h\|_X\le 1} \|N'(u)h - N'(v)h\|_Y < \varepsilon. \tag{4.8}$$

Let us take an arbitrary $\varepsilon > 0$ and $u \in X$ such that $\|u - v\|_X < \delta$, i.e. $|D_t u(t,x) - D_t v(t,x)| < \delta$ and $|D_x^k u(t,x) - D_x^k v(t,x)| < \delta$ for all multiindices $0 \le |k| \le 2b$ on $\operatorname{cl} Q$. Hence with the respect to the uniform continuity of $\partial f/\partial v_\beta$ for $0 \le |\beta| \le 2b-1$ on every compact subset of $\operatorname{cl} Q \times \mathbb{R}^\kappa$ we get

$$\begin{aligned} &|N'(u)h(t,x) - N'(v)h(t,x)| \\ &\quad\le \sum_{\substack{0\le|\beta|\le 2b-1\\ \operatorname{card}\{\beta\}=\kappa}} \left|\frac{\partial f}{\partial v_\beta}[t,x,\overline{D}_x^\gamma u(t,x)] - \frac{\partial f}{\partial v_\beta}[t,x,\overline{D}_x^\gamma v(t,x)]\right| |D_x^\beta h(t,x)| < \varepsilon \end{aligned}$$

for $\|h\|_X \le 1$ and all $(t,x) \in \operatorname{cl} Q$. It finishes the proof of (4.8).

(b) We easily see that Fréchet derivative $F' \colon X \to L(X,Y)$ is defined by the equation

$$F'(u)h(t,x) = D_t h(t,x) - \sum_{0\le|k|\le 2b} a_k(t,x) D_x^k h(t,x) + N'(u)h(t,x)$$

for $u, h \in X$. Hence and by (c) of Theorem 5.2 we get $F \in C^1(X,Y)$. $\square$

Lemma 4.9. *Let the hypotheses* (S^α), $\alpha \in (0,1)$, (A.1), (N.1) *and* (N.2) *be satisfied. Then $F = A + N \colon X \to Y$ is a nonlinear Fredholm operator of the zero index on X.*

Proof. According to Lemma 4.4(b) the operator $A \colon X \to Y$ is a linear continuous and C^1-Fredholm mapping of the zero index. By Lemma 4.6 the operator $N \colon X \to Y$ is compact. By Lemma 4.8 it belongs to the classs C^1. Then Proposition 2.5 implies that F is a nonlinear Fredholm operator with the zero index. $\square$

5. The structure of solution sets for continuous nonlinearities

The first result for that proper mapping F is given by the following theorem.

Theorem 5.1. *Let hypotheses* (S^α) *for* $\alpha \in (0,1)$, (A.1), (N.1) *hold. Then:*

- (a) *For any compact set of the right-hand sides $g \in Y$ of* (1.1) *the corresponding set of all solutions of* (1.1)–(1.3) *is a countable union of compact sets.*
- (b) *For $u_0 \in X$ there exists a neighbourhood $U(u_0)$ of u_0 and $U(F(u_0))$ of $F(u_0) \in Y$ such that for each $g \in U(F(u_0))$ there is a unique solution of* (1.1)–(1.3) *if and only if the operator F is locally injective at u_0.*

(c) *Let moreover* (F.1) *hold. Then for any compact set of the right–hand sides* $g \in Y$ *from* (1.1), *the set of all solutions of* (1.1)–(1.3) *is compact* (*possibly empty*).

Proof. (a) Since $F = A + N$ (see (4.3)) by the decomposition of $A = C + T$ (Proposition 2.2) we have $F = C + (T + N)$ where C is a continuous and proper mapping from X onto Y (see Proposition 2.16), A is a Fredholm operator of the zero index, T and N are completly continuous mappings. Since X is a countable union of closed balls in X, so with respect to Proposition 2.17(a) the operator F is σ-proper (continuous). Lemma 4.3(a) implies the assertion (a).

(b) Suppose that F is injective in a neighbourhood $U(u_0)$ of $u_0 \in X$. From the decomposition (for H see Lemma 4.4)

$$F = H + (T + N)$$

we obtain $H^{-1}F = I + H^{-1}(T + N)$ which is a completely continuous and injective perturbation of the identity $I: X \to Y$ in $U(u_0)$. According to Proposition 2.21(a) the set $H^{-1}F(U(u_0))$ is open in X and the restriction $H^{-1}F|_{U(u_0)}$ is a homeomorphism of $U(u_0)$ onto $H^{-1}F(U(u_0))$. Therefore F is locally invertible at u_0. Again by Lemma 4.3() we obtain (b).

(c) By Lemma 4.7(b) the operator $F: X \to Y$ is proper which implies the given assertion and includes the proof of Theorem 5.1. □

On futher qualitative and quantitative properties of solutions of (1.1)–(1.3) the following theorem says.

Theorem 5.2. *Let the hypotheses* (S^α), $\alpha \in (0,1)$, (A.1), (N.1), (F.1) *be satisfied. For solutions of* (1.1)–(1.3) *the following statements are true:*

(a) *The set of solutions for each* $g \in Y$ *is compact* (*possibly empty*).
(b) *The set* $R(F) = \{g \in Y$: *there exists at least one solution* $u \in X$ *of* (1.1)–(1.3)$\}$ *is closed and connected in* Y.
(c) *The domain of bifurcation* D_b *is closed in* X *and the bifurcation range* R_b *is closed in* Y. *The set* $F(X \setminus D_b)$ *is open in* Y.
(d) *If* $Y \setminus R_b \neq \emptyset$, *then each component of* $Y \setminus R_b$ *is a nonempty open set* (*i.e. domain*).
(e) *If* $Y \setminus R_b \neq \emptyset$, *the number* n_g *of solutions is finite and constant* (*it may be zero*) *on each component of* $Y \setminus R_b$, *i.e.* n_g *is the same nonnegative integer for each* g *belonging to the same component of* $Y \setminus R_b$.
(f) *If* $R_b = \emptyset$, *then the given problem has a unique solution* $u \in X$ *for each* $g \in Y$ *and this solution continuously depends on* g *as a mapping from* Y *onto* X.
(g) *If* $R_b \neq \emptyset$, *then the boundary* $\partial F(X \setminus D_b)$ *is a subset of* $F(D_b) = R_b$ ($\partial F(X \setminus D_b) \subset F(D_b)$).

Proof. Assertion (a) follows directly from Theorem 5.1(c).

(b) Take the sequence $\{g_n\}_{n\in N} \subset R(F) \subset Y$ converging to $g \in Y$ as $n \to \infty$. Since F is proper, the set $F^{-1}(\{g_1, g_2, \dots\} \cup \{g\}) \subset X$ is compact. Thus there exists a subsequence $\{u_{n_k}\}_{k\in N} \subset F^{-1}(\{g_1, g_2, \dots\} \cup \{g\})$ converging to $u \in X$ and $F(u_{n_k}) = g_{n_k} \to g$ in Y as $n \to \infty$. Since the mapping F is proper (Lemma 4.7(b)) by Proposition 2.16(a) it is closed, whence $F(u) = g$, i.e. $g \in R(F)$. The set $R(F)$ is closed. $R(F) = F(X)$ is connected as a continuous image of the connected set X.

(c) According to Lemma 4.3(b) $D_b = \Sigma$ and $R_b = F(D_b) = F(\Sigma)$. Since $X \setminus \Sigma$ is an open set then D_b is closed in X and its continuous image R_b is a closed set in Y.

The difference, $X \setminus D_b = X \setminus \Sigma$ represents the set of all points at which the mapping F is locally invertible. Then for each $u_0 \in X \setminus D_b$ there is a neighbourhood $U_1(F(u_0)) \subset F(X \setminus D_b)$. It means that the set $F(X \setminus D_b)$ is open.

(d) The set $Y \setminus R_b = Y \setminus F(D_b) \neq \emptyset$ is open in Y. Then each its component is nonempty and open, too.

(e) This directly follows from Proposition 2.14.

(f) By $R_b = \emptyset$ we have $D_b = \emptyset$ and the mapping F is locally invertible in X. Proposition 2.17(b) asserts that F is a proper mapping. Then the global inverse mapping theorem (Proposition 2.15) implies that F is homeomorphism from X onto Y.

(g) From Lemma 4.3(b) $D_b = \Sigma$ and by (c) of Theorem 5.2 D_b and $F(D_b)$ are closed. Then $\partial F(X \setminus D_b) = \partial F(D_b) \subset F(D_b)$.

This finishes the proof of the theorem. □

The following two theorems concern the surjectivity corresponding to problem (1.1)–(1.3).

Theorem 5.3. *Under the assumptions* (S^α), $\alpha \in (0,1)$, (A.1), (N.1), (F.1) *each of the following conditions is sufficient to the solvability of problem* (1.1)–(1.3) *for each* $g \in Y$*:*

(a) *For each* $g \in R_b$ *there is a solution* $u \in X \setminus D_b$ *of* (1.1)–(1.3).
(b) *The set* $Y \setminus R_b$ *is connected and there is* $g \in R(F) \setminus R_b$ *(for* $R(F)$ *see Theorem* 5.2(b)*).*

Proof. First of all we can see that conditions (a) and (b) are mutually equivalent to the conditions:

(a') $F(D_b) \subset F(X \setminus D_b)$,
(b') $Y \setminus R_b$ is a connected set and $F(X \setminus D_b) \setminus R_b \neq \emptyset$,

respectively.

From the proof of Theorem 5.2(c) we have $D_b = \Sigma$.

(a) From (a') we have $F(X) = F(D_b) \cup F(X \setminus D_b) = F(X \setminus D_b)$. So $R(F) = F(X)$ is closed and connected in Y (Theorem 5.2(b)) as well as open set in Y (see Theorem 5.2(c)). Thus $R(F) = Y$ which implies the surjectivity of F.

(b) By (e) of Theorem 5.2, $\operatorname{card} F^{-1}(\{g\})$ is a constant $k \geq 0$ for every g from the same component of $Y \setminus R_b$.

If $k = 0$ for all $g \in Y \setminus R_b$, then $F(X) = R_b$. Hence $F(X \setminus D_b) \subset R_b$. However, it is a contradiction with (b'). □

THEOREM 5.4. *Let* (S^α), $\alpha \in (0,1)$, (A.1), (N.1), (F.1) *hold together with hypothesis*

(S.1) *All solutions $u \in X$ of the initial-boundary value problem for the equation*
$$Hu + \mu(Au - Hu + Nu) = 0, \quad \mu \in (0,1)$$
with data (1.2), (1.3) *fulfil inequality* (4.5) *from Lemma* 4.7. *H is the linear homeomorphism from hypothesis* (A.1).

Then:

(a) *problem* (1.1)–(1.3) *has at least one solution for each $g \in Y$,*
(b) *the number n_g of solutions* (1.1)–(1.3) *is finite, constant and different from zero on each component of the set $Y \setminus R_b$* (for all g belonging to the same component of $Y \setminus R_b$).

PROOF. (a) It is sufficient to prove the surjectivity of $F: X \to Y$. By Lemma 4.4 (see proof of (b)) we can write
$$F = A + N = H + (T + N).$$
The mapping
$$H^{-1}F = I + H^{-1}(T + N): X \to X$$
is a completely continuous and condensing field (see [32, p. 496]).

Let $S \subset X$ be a bounded set. Then $H(S)$ is a bounded set in Y. From the coercivity of F (see Lemma 4.7(a)) the set $F^{-1}[H(S)] = (H^{-1}F)^{-1}(S)$ is bounded at X. Hence $H^{-1}F$ is coercive.

Now we show that condition (c) from Proposition 2.20 is satisfied for the condensing and coercive field $P = H^{-1}F$. Take the strictly solvable field $G(u) = u$. The equation $P(u) = kG(u)$ is equivalent to
$$(H^{-1}F)(u) = ku.$$
Hence we get, for $u \in X$ and $k < 0$,
$$Hu + (1 - k)^{-1}[Au - Hu + Nu] = 0,$$

where $(1-k)^{-1} \in (0,1)$. With respect to condition (S.1)

$$\|u\|_{a,Q} \le K_a$$

for $a = \max\{|\gamma|, r\}$, where $|\gamma| = 0, 1, \dots, 2b-1$ and $0 \le r \le 2b-1$ are fixed. Using the same method as in Lemma 4.7(a) we obtain for all solutions of

$$(H^{-1}F)u = ku$$

the estimation $\|u\|_X \le K_8$, $K_8 > 0$. By Proposition 2.20 we have the strict surjectivity of $H^{-1}F$ and so F. This proves (a).

(b) From the surjectivity of F on X it follows that $n_g \neq 0$. The other assertions of (b) follow from Theorem 5.2(e). $\square$

6. The solution set of C^1 nonlinearities

With respect to the C^1-differentiability of the operator N from (4.2) we prove here several stronger results than in Section 5 for the solutions of (1.1)–(1.3).

Theorem 6.1. *Suppose that* (S^α), $\alpha \in (0,1)$, (A.1), (N.1) *and* (N.2) *be satisfied and R_b means the bifurcation range of* (1.1)–(1.3). *Then the set $Y \setminus R_b$ is open and dence in Y and thus the bifurcation range R_b of initial-boundary value problem* (1.1)–(1.3) *is nowhere dense in Y.*

Proof. The openess of $Y \setminus R_b$ follows from the statement (c) of Theorem 5.2.

From Lemmas 4.8 and 4.9 the operator $A: X \to Y$ is a linear continuous Fredholm mapping of the zero index and the Nemitskiĭ operator $N: X \to Y$ is compact and $N \in C^1(X,Y)$.

For every $u \in X$ the linear operator $N'(u): X \to Y$ from (4.7) is completely continuous on X. By the Nikol'skiĭ decomposition theorem (see Proposition 2.2) the operator $F'(u) = A + N'(u): X \to Y$ is a linear Fredholm mapping of the zero index for each $u \in X$. By Lemma 4.8(b) we know that $F \in C^1(X,Y)$ and by Lemma 4.9 the F is a nonlinear Fredholm operator of the zero index.

According to the Banach open mapping theorem (see [31, p. 77]) the mutual equivalence is true: $F'(u)$ is a linear homeomorphism if and only if it is a bijective mapping. Since $F'(u)$ for every $u \in X$ is a linear Fredholm mapping of the zero index so $F'(u)$ is bijective if and only if it is injective (in this case the the injectivity implies surjectivity, see Proposition 8.14 (1) from [32, p. 366]). Then by Definition 2.6 we see that $u \in X$ is a singular point of the Fredholm operator F if and only if u is a critical point of F.

From Proposition 2.11 we obtain that set Σ (of all points $u \in X$ for which F is not locally invertible) is contained in the subset of all critical points of F.

Then, evidently Σ is a subset of the set S of all singular points of F, i.e. $\Sigma \subset S$. Hence we get for the set of regular values R_F of the operator F the relations

$$R_F = Y \setminus F(S) \subset Y \setminus F(\Sigma) \subset Y \setminus R_b \subset Y,$$

where $R_b \subset F(\Sigma)$ is a bifurcation range of F.

Since $F: X \to Y$ is a nonconstant closed mapping with $\dim X = \infty$, by Proposition 2.12 we obtain that F is a proper mapping. By Proposition 2.9 (the Quinn version) the set R_F is residual, open and dense in Y. Hence $Y \setminus R_b$ is dense in Y, too. With respect to Lemma 4.3(b) we can conclude the proof. $\square$

In the following results we shall deal with the linear problem in $h \in X$

$$Ah(t,x) + \sum_{\substack{0 \le |\beta| \le 2b-1 \\ \mathrm{card}\{|\beta|\} = \kappa}} \frac{\partial f}{\partial v_\beta}[t, x, D_x^\gamma u(t,x)] D_x^\beta h(t,x) = g(t,x) \tag{6.1}$$

for $(t,x) \in Q$ and some fixed $u \in X$ with the conditions (1.2), (1.3). The left-hand side of equation (6.1) represents the Fréchet derivative $F'(u)h$ of the operator $F = A + N: X \to Y$.

Theorem 6.2. *Let the hypotheses* (S)$^\alpha$, $\alpha \in (0,1)$, (A.1), (N.1), (N.2) *and* (F.1) *be satisfied. Then*

(a) *For any compact set of Y* (*of the right-hand sides $g \in Y$ of equation* (1.1)) *the set of all corresponding solutions of* (1.1)–(1.3) *is compact* (*possibly empty*).

(b) *The number solutions of* (1.1)–(1.3) *is constant and finite* (*it may be zero*) *on each connected component of the open set $Y \setminus F(S)$, i.e. for any g belonging to the same connected component of $Y \setminus F(S)$. Here S means the set of all critical points of the operator $F = A + N: X \to Y$.*

(c) *Let $u_0 \in X$ be a regular solution of* (1.1)–(1.3) *with the right-hand side $g_0 \in Y$. Then there exists a neighbourhood $U(g_0) \subset Y$ of g_0 such that for any $g \in U(g_0)$ initial-boundary value problem* (1.1)–(1.3) *has one and only one solution $u \in X$. This solution continuously depends on g. Associated linear problem* (6.1), (1.2), (1.3) *for $u = u_0$ has a unique solution $h \in X$ for any g from a neighbourhood $U(g_0)$ of $g_0 = F(u_0)$. This solution continuously depends on g.*

(d) *Denote by G the set of all right-hand side $g \in Y$ of equation* (1.1) *for which all corresponding solutions $u \in X$ of problem* (1.1)–(1.3) *are its critical points. Then G is closed nowhere dense in Y.*

(e) *If the singular points set of* (1.1)–(1.3) *is empty, then this problem has unique solution $u \in X$ for each $g \in Y$. It continuously depends on the right-hand side g.*

Proof. (a) Since the operator F is proper (see Lemma 4.7) we have the assertion (a).

(b) In the proof of Theorem 6.1 we have showed that set of all singular points of F is equal to the set of all critical points of F. Then the Ambrosetti theorem (see Proposition 2.14) implies the statement (b).

(c) Since $u_0 \in X \setminus S$, where S is a set of all singular (in our case all critical) points (see Definition 2.6(b) and (c)), then by Proposition 2.11 the mapping F is a local C^1-diffeomorphism at u_0. This proves the first part of (c) for (1.1)–(1.3).

Since F is a C^1-diffeomorphism, it follows that $F' \in C(X,Y)$, $(F^{-1})' \in C(X,Y)$, where $F'(u)h$ is the left-hand side of (6.1) nd $(F^{-1})'(Fu) = (F'(u))^{-1}$ for every $u \in X$. Hence linear problem (6.1), (1.2), (1.3) for $u = u_0$ has a unique solution $h \in X$ for any $g \in U(g_0)$ with $g_0 = F(u_0)$. This solution continuously depends on a right-hand side g. The proof of (c) is completed.

(d) In our case the equality $G = F(S)$ holds. By the Smale–Quinn theorem (Proposition 2.9) we obtain the expected results.

(e) By Proposition 2.11, the operator $F: X \to Y$ is a local C^1-diffeomorphism at any point $u \in X$. Hence, the last assertion follows. $\square$

By the point (c) of Theorem 6.2 we obtain the following.

Corollary 6.3. *Let the hypotheses of Theorem* 6.2 *hold and moreover:*

(H.1) *The linear homogeneous problem* (6.1), (1.2), (1.3) (*for* $g = 0$) *has only the zero solution* $h = 0 \in X$ *for any* $u \in X$.

Then initial-boundary value problem (1.1)–(1.3) *has a unique solution* $u \in X$ *for any* $g \in Y$. *Moreover, linear problem* (6.1), (1.2), (1.3) *has a unique solution* $h \in X$ *for any* $u \in X$ *and the right-hand side* $g \in Y$ *of* (6.1). *This solution continuously depends on* g.

Corollary 6.4. *Let the assumptions of Theorem* 6.2 *be satisfied. Then we have:*

(a) *If the set* S *of all singular* (*in our case all critical*) *points of* F *is nonempty, then* $\partial F(X \setminus S) \subset F(S)$.
(b) *If* $F(S) \subset F(X \setminus S)$, *then problem* (1.1)–(1.3) *has the solution* $u \in X$ *for any* $g \in Y$, *i.e.* $R(F) = Y$ (F *is a surjection of* X *onto* Y).
(c) *If* $Y \setminus F(S)$ *is connected and* $X \setminus S \neq \emptyset$, *then* $R(F) = Y$ (*the solvability of* (1.1)–(1.3) *for any* $g \in Y$).

Proof. (a) By Theorem 6.2(d) the set $F(S)$ is closed in Y and by Proposition 2.9 $F(X \setminus S)$ is open in Y. Also the set $F(X)$ is closed by Lemma 4.7. Hence we obtain the equations

$$F(S) \cup F(X \setminus S) = F(X) = \overline{F(X)} = F(S) \cup \overline{F(X \setminus S)}. \tag{6.2}$$

The inclusion in (a) follows directly from (6.2).

(b) From the first equation of (6.2) we have $F(X) = F(X \setminus S)$ and so $R(F)$ is an open as well as a closed subset of the connected space Y. Thus $R(F) = Y$.

(c) Since $Y \setminus F(S)$ is connected, by the Ambrosetti theorem (see Proposition 2.14) we obtain the card $F^{-1}(\{g\}) = \text{const} =: k \geq 0$ for each $g \in Y \setminus F(S)$.

If it was $k = 0$, then there would be $F(X) = F(S)$ and $F(X \setminus S) \subset F(S)$ and this is a contradiction with $X \setminus S \neq \emptyset$. Then $k > 0$. □

Theorem 6.5. *Suppose that hypotheses* (S^α), $\alpha \in (0,1)$, (A.1), (N.1), (N.2) *and* (F.1) *hold together with the condition*

(H.2) *Each point $u \in X$ is either a regular point or an isolated critical point of problem* (1.1)–(1.3)

Then for every $g \in Y$ there exists exactly one solution $u \in X$ of (1.1)–(1.3). *It continuously depends on g.*

Proof. The associated operator $F: X \to Y$ is a proper C^1-Fredholm mapping of the zero index. By Proposition 2.11 the F is a local C^1-diffeomorphism at a regular point of F. In the isolated singular point, by Proposition 2.12, F is locally invertible. Since F is proper, the global inverse mapping theorem (see Proposition 2.15) implies the statement of this problem. □

In the conclusion of this paper, let us notice that the previous results can be proved without parabolic (P), complementary (C) and compatibility (Q) conditions of initial-boundary value problem (1.1)–(1.3). Thus all previous generic properties keep for the general evolution problems of type (1.1)–(1.3). Such models describe different natural science phenomena (a reaction-diffusion and environment models, a diffusive waves in fluid dynamics — the Burges equation, the wave propagation in a large number of biological and chemical systems — the Fisher equation, a nerve pulse propagation in nerve fibers and wall motion in liquid crystals).

The results of the present paper can be generalized also to the quasilinear parabolic and general evolution systems of type (1.1)–(1.3). It enables to apply the Fredholm theory to hyperbolic equations modeling different nonlinear vibration problems, to a nonlinear dispersion (the nonlinear Klein–Gordan equation), a propagation of magnetic flux and the stability of fluid notions (the nonlinear Sine–Gordan equation) and so on.

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