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A CLASS OF REAL COCYCLES OVER AN IRRATIONAL ROTATION FOR WHICH ROKHLIN COCYCLE EXTENSIONS HAVE LEBESGUE COMPONENT IN THE SPECTRUM

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Abstract. We describe a class of functions $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that for each irrational rotation $Tx = x + \alpha$, where α has the property that the sequence of aritmethical means of its partial quotients is bounded, the corresponding weighted unitary operators $L^2(\mathbb{R}/\mathbb{Z}) \ni g \mapsto e^{2\pi i c f} \cdot g \circ T$ have a Lebesgue spectrum for each $c \in \mathbb{R}\setminus\{0\}$. We show that for such f and T and for an arbitrary ergodic $\mathbb{R}$-action $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ on $(Y, \mathcal{C}, \nu)$ the corresponding Rokhlin cocycle extension $T_{f,\mathcal{S}}(x, y) = (Tx, S_{f(x)}y)$ acting on $(\mathbb{R}/\mathbb{Z} \times Y, \mu \otimes \nu)$ has also a Lebesgue spectrum in the orthogonal complement of $L^2(\mathbb{R}/\mathbb{Z}, \mu)$ and moreover the weak closure of powers of $T_{f,\mathcal{S}}$ in the space of self-joinings consists of ergodic elements.

1. Introduction

We will study spectral properties of Anzai skew products on $\mathbb{R}/\mathbb{Z} \times \mathbb{T}$ of the form

$$(x, z) \mapsto (Tx, e^{2\pi i f(x)} z),$$

where $Tx = x + \alpha$ is an irrational rotation and $f\colon \mathbb{R}/\mathbb{Z} \mapsto \mathbb{R}$ is a measurable function which will satisfy certain additional assumptions. The problem can be

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reduced to the spectral analysis of weighted operators $V^{e^{2\pi i f},T}$ on $L^2(\mathbb{R}/\mathbb{Z})$ given by

$$g \mapsto e^{2\pi i f} \cdot g \circ T.$$

It is known from Helson's analysis (see [10], [14]) that the spectrum of $V^{e^{2\pi i f},T}$ satisfies the so called "purity low", that is it is pure i.e. either discrete or continuous and purely singular, or equivalent to Lebesgue. Now we recall the classic results concerning the spectrum of $V^{e^{2\pi i f},T}$ first in case it is Lebesgue and at the end when it is singular.

In the results cited below concerning Lebesgue spectrum the function f has non-zero topological degree (i.e. the lift $\widetilde{f}: \mathbb{R} \to \mathbb{R}$ of f satisfies $\widetilde{f}(1) - \widetilde{f}(0) \in \mathbb{Z} \setminus \{0\}$). Namely, it has been proved by Anzai (see [1]) that if $f(x) = nx$, $n \in \mathbb{Z}\setminus\{0\}$ then $V^{e^{2\pi i f},T}$ has a Lebesgue spectrum, therefore $V^{e^{2\pi i c f},T}$ has a Lebesgue spectrum for each $c \in \mathbb{Z} \setminus \{0\}$. On the other hand if $f(x) = nx + g(x)$, where $n \in \mathbb{Z}\setminus\{0\}$, $g \in C^2(\mathbb{R}/\mathbb{Z})$ and $g' + 1 > 0$, then $V^{e^{2\pi i f},T}$ has a Lebesgue spectrum, which was proved by Kushnirenko (see [3, Chapter 13, Theorem 2]). Choe in [2] proved the same assuming only $g \in C^2(\mathbb{R}/\mathbb{Z})$. Moreover, it has been shown in [14] that for $g: \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ absolutely continuous with g' of bounded variation, we obtain also a Lebesgue spectrum of $V^{e^{2\pi i f},T}$. One more sufficient condition for a Lebesgue spectrum, given in terms of the Fourier coefficients of g, is presented in [12]. More precisely the function $g \in C^1$ need to satisfy $\sum |n|^3 |\widehat{g}(n)|^2 < \infty$. The following result was shown in [11] (see also [8]): If $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is absolutely continuous then $V^{e^{2\pi i f},T}$ has a singular spectrum. It follows that also for each $c \in \mathbb{R} \setminus \{0\}$ the operator $V^{e^{2\pi i c f},T}$ has a singular spectrum. It has been proved in [13] that if $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ is piecewise absolutely continuous with a single non-integer jump discontinuity, then $V^{e^{2\pi i f},T}$ has a continuous singular spectrum over any irrational rotation.

M. Guenais in [9] obtained some interesting results concerning the multiplicity of the spectrum of $V^{e^{2\pi i f},T}$, namely it is bounded by $\max(2, 2\pi \mathrm{Var}(f)/3)$ if f is of bounded variation, by $|\beta| + 1$ if $f(x) = \beta x$, $0 \neq \beta \in \mathbb{R}$ and is equal to 1 in case f is absolutely continuous and homotopically trivial.

Consider now the Rokhlin cocycle extension given by

$$(x, y) \xmapsto{T_{f,\mathcal{S}}} (Tx, S_{f(x)} y),$$

where $\mathcal{S} = (S_t)_{t \in \mathbb{R}}$ is an ergodic $\mathbb{R}$-action. There is strict relationship between the maximal spectral type of $T_{f,\mathcal{S}}$ and the maximal spectral type of $V^{e^{2\pi i c f},T}$, $c \neq 0$ (see Lemma 2.4 below). Hence a natural question arises whether it is possible to find $f: \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ such that for each $c \in \mathbb{R} \setminus \{0\}$, $V^{e^{2\pi i c f},T}$ has a Lebesgue spectrum. Then we could conclude that the spectrum of $T_{f,\mathcal{S}}$ is also Lebesgue, what would enlarge the list of properties of the Rokhlin cocycle extensions recently published in [15].

In this paper we will consider measurable functions $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$, with piecewise continuous second derivative such that f' has finitely many discontinuity points for which the one-sided limits exist with at least one equal to infinity and $f'(x) > 0$ for all $x \in \mathbb{R}/\mathbb{Z}$. With these assumptions $V^{e^{2\pi i f,T}}$ has a Lebesgue spectrum whenever α has the property that the sequence of aritmethical means of its partial quotients is bounded (see Theorem 3.1 below). Moreover, we will show that for such a function f we have that $V^{e^{2\pi i c f,T}}$ has a Lebesgue spectrum for all $c \in \mathbb{R}\setminus\{0\}$. The methods we use are widely inspired by [14]. We will carry out one more observation. For $f\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ satisfying the above assumptions, the Rokhlin cocycle extensions given by $T_{f,\mathcal{S}}$ have also a Lebesgue spectrum in orthogonal complement of $L^2(\mathbb{R}/\mathbb{Z}, \mu)$.

As an application we will show that if the flow $\mathcal{S}$ is additionaly weakly mixing, then the weak closure of powers of $T_{f,\mathcal{S}}$ in the space of self-joinings consists of ergodic elements that is the Rokhlin cocycle extension lies in the class of ELF automorphisms recently introduced in [6].

Throughout the paper we will identify $\mathbb{R}/\mathbb{Z}$ with $[0, 1)$ (with addition mod 1). Each function defined on $[0, 1)$ we will treat as a 1-periodic function on $\mathbb{R}$. By $\mathbb{T}$ we will denote the set $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Considering the space (X, μ), if $X = [0, 1)$ then μ will be understood as Lebesgue measure, while on $\mathbb{T}$ the Lebesgue measure will be denoted by λ.

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2. Preliminaries

In this section we shall establish notation and recall some definitions and some results needed in the rest of the paper.

2.1. Spectral theory. Let $(X, \mathcal{B}, \mu)$ be a standard probability Borel space. Let $V\colon L^2(X, \mathcal{B}, \mu) \to L^2(X, \mathcal{B}, \mu)$ be a unitary operator. By $\sigma_{f,V}$ we denote the spectral measure of f, it means the only finite positive Borel measure on $\mathbb{T}$ such that

$$\widehat{\sigma}_{f,V}(n) = \int_{\mathbb{T}} z^n \, d\sigma_{f,V}(z) = \int_X V^n(f) \cdot \overline{f} \, d\mu, \quad n \in \mathbb{Z}.$$

Given $f, g \in L^2(X, \mathcal{B}, \mu)$ by $\sigma_{f,g,V}$ we will denote the complex measure determined by

$$\widehat{\sigma}_{f,g,V}(n) = \int_{\mathbb{T}} z^n \, d\sigma_{f,g,V}(z) = \int_X V^n(f) \cdot \overline{g} \, d\mu, \quad n \in \mathbb{Z}.$$

Recall that $\sigma_{f,g,V} \ll \sigma_{f,V}$ and $\sigma_{f,g,V} \ll \sigma_{g,V}$ (see e.g. [19, p. 18]). By σ_V we will denote the maximal spectral type of V.

Denote by $L_0^2(X,\mathcal{B},\mu)$ the subspace of $L^2(X,\mathcal{B},\mu)$ of the zero mean functions. Given an automorphism $T\colon (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ by U_T we will denote a unitary operator on $L^2(X,\mathcal{B},\mu)$ acting by the formula $U_T f = f\circ T$. Writing about spectrum (or the maximal spectral type) of an automorphism T we will always understand the spectrum (or the maximal spectral type) of U_T. The maximal spectral type of U_T on $L_0^2(X,\mathcal{B},\mu)$ will be denoted by σ_T.

Let us now consider a measurable flow $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$, that is for each $t \in \mathbb{R}$, S_t is an automorphism on a probability standard Borel space $(Y,\mathcal{C},\nu)$ and the corresponding unitary representation of $\mathbb{R}$ on $L^2(Y,\mathcal{C},\nu)$ given by the formula $U_{\mathcal{S}}(t)(f) = f\circ S_t$ is measurable.

For $f \in L^2(Y,\mathcal{C},\nu)$, by $\sigma_{f,\mathcal{S}}$ we will denote the spectral measure of f, that is the measure on $\mathbb{R}$ (equal to the character group $\widehat{\mathbb{R}}$) such that

$$\widehat{\sigma}_{f,\mathcal{S}}(t) = \int_{\mathbb{R}} e^{2\pi itc}\, d\sigma_{f,\mathcal{S}}(c) = \int_Y f\circ S_t \cdot \overline{f}\, d\nu, \quad t\in\mathbb{R}.$$

Similarly as before $\sigma_{f,g,\mathcal{S}}$ will mean the measure on $\mathbb{R}$ determined by

$$\widehat{\sigma}_{f,g,\mathcal{S}}(t) = \int_{\mathbb{R}} e^{2\pi itc}\, d\sigma_{f,g,\mathcal{S}}(c) = \int_Y U_{\mathcal{S}}(t)f\ \overline{g}\, d\nu, \quad t\in\mathbb{R}.$$

We easily verify that whenever $\mathcal{S}$ is ergodic we have

$$\sigma_{f,g,\mathcal{S}}(\{0\}) = \int_Y f\, d\nu \cdot \int_Y \overline{g}\, d\nu \tag{2.1}$$

for each $f,g \in L^2(Y,\mathcal{C},\nu)$.

Let $\sigma_{\mathcal{S}}$ denote the maximal spectral type of $\mathcal{S}$ (meaning $U_{\mathcal{S}}$) on $L_0^2(Y,\mathcal{C},\nu)$.

2.2. Weighted operators. Let $T\colon (X,\mathcal{B},\mu) \to (X,\mathcal{B},\mu)$ be an ergodic automorphism and $\varphi\colon X \to \mathbb{T}$ be measurable. Define a unitary operator on $L^2(X,\mathcal{B},\mu)$ by the formula

$$(V^{\varphi,T}h)(x) = \varphi(x)h(Tx).$$

Then we have

$$(V^{\varphi,T})^n h(x) = \varphi^{(n)}(x)h(T^n x),$$

where $\varphi^{(\,\cdot\,)}(\,\cdot\,)\colon \mathbb{Z}\times X \to \mathbb{T}$ is given by

$$\varphi^{(n)}(x) = \begin{cases} \varphi(x)\varphi(Tx)\ldots\varphi(T^{n-1}x) & \text{for } n>0,\\ 1 & \text{for } n=0,\\ (\varphi(T^n x)\ldots\varphi(T^{-1}x))^{-1} & \text{for } n<0, \end{cases} \tag{2.2}$$

that is $\varphi^{(\,\cdot\,)}(\,\cdot\,)$ is the cocycle.

Consider now the operator $V^{\overline{\varphi},T}$. We have

$$\langle (V^{\overline{\varphi},T})^n h, h\rangle = \overline{\langle (V^{\varphi,T})^n \overline{h}, \overline{h}\rangle} = \langle (V^{\varphi,T})^{-n}\overline{h}, \overline{h}\rangle.$$

Notice that

$$\text{(2.3)} \qquad \begin{cases} \text{if the operator } V^{\varphi,T} \text{ has a Lebesgue spectrum,} \\ \text{then the operator } V^{\overline{\varphi},T} \text{ has also a Lebesgue spectrum.} \end{cases}$$

Indeed, by putting $s\colon \mathbb{T}\to\mathbb{T}$, $s(z)=z^{-1}=\overline{z}$, for each $n\neq 0$ and $h\in L^2(X,\mathcal{B},\mu)$ we obtain

$$\begin{aligned}\int_{\mathbb{T}} z^{-n}\, d\sigma_{\overline{h},V^{\overline{\varphi},T}} &= \langle (V^{\overline{\varphi},T})^{-n}\overline{h},\overline{h}\rangle = \langle (V^{\varphi,T})^n h, h\rangle \\ &= \int_{\mathbb{T}} z^n\, d\sigma_{h,V^{\varphi,T}} = \int_{\mathbb{T}} z^{-n}\, ds_*(\sigma_{h,V^{\varphi,T}}),\end{aligned}$$

where by $s_*(\sigma_{h,V^{\varphi,T}})$ we denote the image of $\sigma_{h,V^{\varphi,T}}$ via s. Thus

$$\sigma_{\overline{h},V^{\overline{\varphi},T}} = s_*(\sigma_{h,V^{\varphi,T}}).$$

Therefore if $\sigma_{h,V^{\varphi,T}}$ is equivalent to Lebesgue measure (or absolutely continuous), then $\sigma_{\overline{h},V^{\overline{\varphi},T}}$ so is. We obtain that $V^{\overline{\varphi},T}$ has a Lebesgue spectrum.

2.3. Anzai skew products and weighted operators. Let T be an automorphism on a standard probability Borel space $(X,\mathcal{B},\mu)$. Given a cocycle $\varphi\colon X\to\mathbb{T}$ we can define a map

$$T_\varphi\colon (X\times\mathbb{T},\mathcal{B}\otimes\mathcal{B}(\mathbb{T}),\mu\otimes\lambda)\to(X\times\mathbb{T},\mathcal{B}\otimes\mathcal{B}(\mathbb{T}),\mu\otimes\lambda)$$

(λ stands for Lebesgue measure) acting by a formula

$$T_\varphi(x,z)=(Tx,\varphi(x)z).$$

Notice that T_φ preserves measure $\mu\otimes\lambda$. Recall that T_φ is called an *Anzai skew product*.

Put $H_n = L^2(X,\mu)\otimes z^n$, $n\in\mathbb{Z}$. Notice, that H_n is a closed subspace of $L^2(X\times\mathbb{T},\mu\otimes\lambda)$ and from Fubini's Theorem we get

$$L^2(X\times\mathbb{T},\mu\otimes\lambda)=\bigoplus_{n=-\infty}^{\infty} H_n.$$

Moreover, H_n are U_{T_φ}-invariant (i.e. $U_{T_\varphi}H_n = H_n$) since

$$U_{T_\varphi}(h(\,\cdot\,)\otimes\,\cdot\,^n)(x,z)=(\varphi(x))^n h(Tx)z^n.$$

Observe that $U_{T_\varphi}|H_0 = U_T$ and $U_{T_\varphi}|H_1$ is isomorphic to $V^{\varphi,T}$ (the isomorphism is given by $h(x)z\mapsto h(x)$). Similarly the map $h(x)z^n\mapsto h(x)$ defines an isomorphism between $U_{T_\varphi}|H_n$ and $V^{\varphi^n,T}$ (here $\varphi^n(x)=(\varphi(x))^n$).

Let us fix an irrational rotation $Tx=x+\alpha$ acting on $(X,\mathcal{B},\mu)$, where $X=[0,1)$ and μ denotes Lebesgue measure. Take a cocycle $\varphi\colon[0,1)\to\mathbb{T}$.

Recall now Helson's analysis of spectral properties of $V^{\varphi,T}$ (see [10], also [14]). Put $M\colon L^2(X,\mu)\to L^2(X,\mu)$, $(Mh)(x)=e^{2\pi ix}h(x)$. We have $(V^s\circ M)(h)=e^{2\pi is\alpha}(M\circ V^s)(h)$ for all $s\in\mathbb{Z}$. Thus $\langle V^s(Mh),Mh\rangle=e^{2\pi is\alpha}\langle V^sh,h\rangle$ and therefore $\sigma_{Mh,V}=\sigma_{h,V}*\delta_{e^{2\pi i\alpha}}$. By a Wiener's Lemma (see [18], Appendix) combined with the ergodicity of translation T on $([0,1),\mathcal{B},\mu)$ it follows that if $H\subset L^2(X,\mu)$ is a closed subspace $V^{\varphi,T}$- and M-invariant, then either $H=\{0\}$ or $H=L^2(X,\mu)$.

Since the subspaces

$$\begin{aligned} H_{ac} &= \{h\in L^2(X,\mu)\colon \sigma_{h,V^{\varphi,T}}\ll\lambda\},\\ H_s &= \{h\in L^2(X,\mu)\colon \sigma_{h,V^{\varphi,T}}\perp\lambda\},\\ H_d &= \{h\in L^2(X,\mu)\colon \sigma_{h,V^{\varphi,T}} \text{ is discrete}\}. \end{aligned}$$

are closed M- and $V^{\varphi,T}$-invariant and $L^2(X,\mu)=H_{ac}\oplus H_s\oplus H_d$, only one of these subspaces is equal to $L^2(X,\mu)$. Moreover, if $H_{ac}=L^2(X,\mu)$ then the maximal spectral type $\sigma_{V^{\varphi,T}}$ of $V^{\varphi,T}$ on $L^2(X,\mu)$ is the Lebesgue type (see [10]).

Recall that if σ is a positive finite measure on $\mathbb{T}$ and $(\widehat{\sigma}(n))_{n\in\mathbb{Z}}\in l_2(\mathbb{Z})$, then $\sigma\ll\lambda$. Notice, that

$$\widehat{\sigma}_{1,V^{\varphi,T}}(n)=\langle (V^{\varphi,T})^n1,1\rangle=\int_X \varphi^{(n)}(x)\,d\mu(x).$$

Since

$$\int_X \varphi^{(-n)}\,d\mu=\overline{\int_X \varphi^{(n)}\circ T^{-n}\,d\mu}=\overline{\int_X \varphi^{(n)}\,d\mu}, \tag{2.4}$$

therefore

$$\text{if } \left(\int_X \varphi^{(n)}(x)d\mu(x)\right)_n\in l_2=l_2(\mathbb{N}),\ \text{then } \sigma_{1,V^{\varphi,T}}\ll\lambda \tag{2.5}$$

and we obtain that the operator $V^{\varphi,T}$ has a Lebesgue spectrum (therefore U_{T_φ} has a Lebesgue spectrum on H_1). Analogously, if $(\int_X((\varphi(x))^k)^{(n)}d\mu(x))_n\in l_2$ then $V^{\varphi^k,T}$ has a Lebesgue spectrum (therefore U_{T_φ} has this property on H_k).

2.4. Continued fraction expansion. In this section we will recall some facts about continued fraction expansion of an irrational number needed in the sequel.

Every point $\alpha\in(0,1)$ can be represented as a continued fraction

$$\alpha=\frac{1}{a_1+1/(a_2+\ldots)}:=[a_1,a_2,\ldots]=\lim_{n\to\infty}[a_1,\ldots,a_n]\quad\text{where } a_i\in\mathbb{N}_+.$$

The above expansion is finite if and only if α is rational. For each $i\in\mathbb{N}_+$ the number a_i is called a *partial quotient* of α and the rational number $r_n=$

$[a_1, \ldots, a_n]$ is called the *nth convergent to* α. Then $\alpha = \lim_{n\to\infty} r_n = [a_1, a_2, \ldots]$. Put

$$\Lambda_{mb} = \left\{ \alpha \in [0,1) : \alpha \text{ is irrational and } \left(\frac{1}{n} \sum_{i=1}^{n} a_i \right)_n \text{ is bounded} \right\}.$$

Notice that the Lebesgue measure of the set Λ_{mb} is zero (see [3, Chapter 7, § 4, Theorem 4]).

Let $x_1, \ldots, x_N$ be a finite sequence of real numbers. The *discrepancy* of this sequence is defined as

$$D_N = D_N(x_1, \ldots, x_N) = \sup_{0 \le a < b \le 1} \left| \frac{A([a,b),N)}{N} - (b-a) \right|$$

where $A([a,b),N) = \operatorname{card}\{1 \le i \le N : a \le x_i \le b\}$. In case ω is an infinite sequence, the $D_N(\omega)$ mean the discrepancy of the first N its elements.

Consider now the sequence $(n\alpha)$, where α is an irrational number. We have the following

Theorem 2.1 ([4, p. 53]). *Let α be irrational. Then $ND_N(n\alpha) = O(\log N)$ if and only if the sequence $((1/n)\sum_{i=1}^{n} a_i)_n$ is bounded (i.e. $\alpha \in \Lambda_{mb}$).*

The following corollary is hence an immediate consequence of the definition of D_N.

Corollary 2.2. *If $((1/n)\sum_{i=1}^{n} a_i)_n$ is bounded, then there exists a positive constant M such, that for all $n \ge 2$ and for each interval $[a,b) \subset [0,1)$, for which*

$$b - a \ge M \frac{\log n}{n}$$

there exists $0 \le j \le n-1$ such that $j\alpha \in [a,b)$.

Let $v \in [0,1)$. We will denote $\|v\| = \operatorname{dist}(v, \mathbb{Z})$. Notice that the function $\|\cdot\| \colon [0,1) \to \mathbb{R}_+$ is an *F-norm*, that is it satisfies the following conditions

- $\|v\| = 0$ iff $v = 0$,
- $\|-v\| = \|v\|$,
- $\|v + w\| \le \|v\| + \|w\|$.

2.5. Rokhlin cocycle extensions. Let T be an ergodic automorphism on a standard probability Borel space $(X, \mathcal{B}, \mu)$ and $\mathcal{S} = \{S_t\}_{t\in\mathbb{R}}$ be a measurable flow on $(Y, \mathcal{C}, \nu)$. Given a cocycle $f \colon X \to \mathbb{R}$ we can define an automorphism $T_{f,\mathcal{S}}$ on $(X \times Y, \mathcal{B} \otimes \mathcal{C}, \mu \otimes \nu)$ by putting

$$T_{f,\mathcal{S}}(x,y) = (Tx, S_{f(x)}y).$$

Recall that so defined automorphism $T_{f,\mathcal{S}}$ is called a *Rokhlin cocycle extension of T* (see e.g. [15], [16]).

We will now recall some results from [16] needed in what follows.

REMARK 2.3. Although in the lemmas below we are talking about types of measures, a precise meaning of the integrals is done by a special choise of measure belonging to the relevant spectral type (see [16] for details).

LEMMA 2.4 ([16]). *The maximal spectral type of* $U_{T_{f,\mathcal{S}}}$ *on* $L^2(X\times Y,\mu\otimes\nu)\ominus(L^2(X,\mu)\otimes 1_Y)$ *is equal to*

$$\sigma_{T_{f,\mathcal{S}}}=\int_{\mathbb{R}}\sigma_{V^{e^{2\pi icf},T}}\,d\sigma_{\mathcal{S}}(c).$$

PROOF (see [16]). Let $\{f_n\}_{n\geq 0}$, $\{g_n\}_{n\geq 0}$ ($f_0=g_0=1$) be orthogonal, linearly dense families of functions in $L^2(X,\mathcal{B},\mu)$, $L^2(Y,\mathcal{C},\nu)$ respectively. Then the following equalities are true (understood as the equivalence of measures)

$$\begin{aligned}\sigma_{T_{f,\mathcal{S}}}&=\sum_{(m,n)\neq(0,0)}2^{-(m+n)}\sigma_{f_n\otimes g_m,T_{f,\mathcal{S}}},\\ \sigma_{V^{e^{2\pi icf},T}}&=\sum_{n\geq 0}2^{-n}\sigma_{f_n,V^{e^{2\pi icf},T}},\\ \sigma_{\mathcal{S}}&=\sum_{m\geq 1}2^{-m}\sigma_{g_m,\mathcal{S}},\quad \sigma_T=\sum_{n\geq 1}2^{-n}\sigma_{f_n,T}.\end{aligned}$$

For each $k\in\mathbb{Z}$ we get

$$\begin{aligned}\widehat{\sigma}_{f_n\otimes g_m,T_{f,\mathcal{S}}}(k)&=\int_{X\times Y}(f_n\otimes g_m)\circ(T_{f,\mathcal{S}})^k\cdot\overline{f_n\otimes g_m}\,d(\mu\otimes\nu)\\ &=\int_X f_n(T^kx)\overline{f_n(x)}\int_{\mathbb{R}}e^{2\pi icf^{(k)}(x)}\,d\sigma_{g_m,\mathcal{S}}(c)\,d\mu(x)\\ &=\int_{\mathbb{R}}\widehat{\sigma}_{f_n,V^{e^{2\pi icf},T}}(k)\,d\sigma_{g_m,\mathcal{S}}(c).\end{aligned}$$

Hence

$$\begin{aligned}\widehat{\sigma}_{T_{f,\mathcal{S}}}(k)&=\sum_{(m,n)\neq(0,0)}2^{-(m+n)}\widehat{\sigma}_{f_n\otimes g_m,T_{f,\mathcal{S}}}(k)\\ &=\sum_{m\geq 1}2^{-m}\int_{\mathbb{R}}\Big(\sum_{n\geq 0}2^{-n}\widehat{\sigma}_{f_n,V^{e^{2\pi icf},T}}(k)\Big)\,d\sigma_{g_m,\mathcal{S}}(c)\\ &\quad+\sum_{n\geq 1}2^{-n}\int_{\mathbb{R}}\widehat{\sigma}_{f_n,V^{e^{2\pi icf},T}}(k)\,d\sigma_{g_0,\mathcal{S}}(c)\\ &=\int_{\mathbb{R}}\widehat{\sigma}_{f_n,V^{e^{2\pi icf},T}}(k)\,d\sigma_{\mathcal{S}}(c)+\widehat{\sigma}_T(k)\end{aligned}$$

and the result follows. □

Assume now that $g\in L^2_0(Y,\nu)$ and let $Z(g)$ denote the cyclic space generated by g, i.e. $Z(g)=\overline{\operatorname{span}}\{g\circ S_t:t\in\mathbb{R}\}$. In the similar way as above we prove

Lemma 2.5 ([16]). *The maximal spectral type of* $U_{T_{f,\mathcal{S}}}$ *on* $L^2(X,\mu)\otimes Z(g)$ *is equal to*

$$\sigma_{T_{f,\mathcal{S}}} = \int_{\mathbb{R}} \sigma_{V^{e^{2\pi icf}},T}\, d\sigma_{g,\mathcal{S}}(c).$$

2.6. Automorphisms whose weak limits of powers consists of ergodic joinings. Let $T_i\colon (X_i,\mathcal{B}_i,\mu_i)\to(X_i,\mathcal{B}_i,\mu_i)$, $i=1,2$ be an automorphism. The probability measure ϱ on $(X_1\times X_2,\mathcal{B}_1\otimes\mathcal{B}_2)$ will be called a *joining* of T_1 and T_2 if ϱ is $T_1\times T_2$-invariant and

(a) $\varrho(A_1\times X_2)=\mu_1(A_1)$ for each $A_1\in\mathcal{B}_1$,
(b) $\varrho(X_1\times A_2)=\mu_2(A_2)$ for each $A_2\in\mathcal{B}_2$.

The set of all joinings between T_1 and T_2 will be denoted by $J(T_1,T_2)$. In case $T_1=T_2=T$ we write $J(T)$ instead of $J(T,T)$ and call elements of $J(T)$ *self-joinings.*

Having a joining $\varrho\in J(T_1,T_2)$ we can define a map $\Phi_\varrho\colon L^2(X_1,\mathcal{B}_1,\mu_1)\to L^2(X_2,\mathcal{B}_2,\mu_2)$ such that

$$\int_{X_2}\Phi_\varrho(f)g\,d\mu_2=\int_{X_1\times X_2} f\otimes g\,d\varrho.$$

Notice that

$$1\otimes\Phi_\varrho(f)=E(f\otimes 1|\{\emptyset,X_1\}\otimes\mathcal{B}_2). \tag{2.6}$$

Thus we get a *Markov operator* $\Phi_\varrho\colon L^2(X_1,\mathcal{B}_1,\mu_1)\to L^2(X_2,\mathcal{B}_2,\mu_2)$, i.e. a linear, bounded operator satisfying:

- $\Phi_\varrho 1=\Phi_\varrho^* 1=1$,
- $\Phi_\varrho f\geq 0$ whenever $f\geq 0$.

Conversely having a Markov operator $\Phi\colon L^2(X_1,\mathcal{B}_1,\mu_1)\to L^2(X_2,\mathcal{B}_2,\mu_2)$ we can define a measure on $(X_1\times X_2,\mathcal{B}_1\otimes\mathcal{B}_2)$ by putting

$$\varrho(A_1\times A_2)=\int_{A_2}\Phi(\chi_{A_1})\,d\mu_2$$

for all $A_i\in\mathcal{B}_i,\ i=1,2$.

We have that projections of ϱ on $\mathcal{B}_1\otimes\{\emptyset,X_2\}$ and $\{\emptyset,X_1\}\otimes\mathcal{B}_2$ are equal to μ_1 and μ_2, respectively. Moreover, the fact that ϱ is $T_1\times T_2$-invariant is read as $\Phi\circ U_{T_1}=U_{T_2}\circ\Phi$ (see e.g. [17], [20]). Thus we can identify the set $J(T_1,T_2)$ with the set of all Markov operators satisfying $\Phi\circ U_{T_1}=U_{T_2}\circ\Phi$ (which will be denoted by $\mathcal{J}(T_1,T_2)$). The set $\mathcal{J}(T_1,T_2)$ is a closed subset (in weak operator topology) of the unit ball in the Banach space of all linear, bounded operators from $L^2(X_1,\mathcal{B}_1,\mu_1)$ to $L^2(X_2,\mathcal{B}_2,\mu_2)$, therefore $\mathcal{J}(T_1,T_2)$ is compact in the weak operator topology. Having identified sets $J(T_1,T_2)$ and $\mathcal{J}(T_1,T_2)$ we can define the weak topology on the set of joinings transfering the weak operator topology from $\mathcal{J}(T_1,T_2)$ and hence obtaining:

$$\varrho_n \to \varrho \text{ if and only if } \varrho_n(A_1\times A_2)\to\varrho(A_1\times A_2) \text{ for all } A_i\in\mathcal{B}_i\ (i=1,2).$$

For each $\varrho\in J(T_1,T_2)$ we have an automorphism $T_1\times T_2\colon (X_1\times X_2,\mathcal{B}_1\otimes\mathcal{B}_2,\varrho)\to(X_1\times X_2,\mathcal{B}_1\otimes\mathcal{B}_2,\varrho)$.

Assume that T_1 and T_2 are ergodic and let $\varrho\in J(T_1,T_2)$. In case $T_1\times T_2$ acting on $(X_1\times X_2,\varrho)$ is ergodic we say that ϱ is ergodic. The subset of such joinings will be denoted by $J^e(T_1,T_2)$ and the set of corresponding operators by $\mathcal{J}^e(T_1,T_2)$. The above considerations are in particular true for self-joinings and we obtain sets $\mathcal{J}(T)$, $J^e(T)$, $\mathcal{J}^e(T)$, respectively.

Recall, that if T_i is an automorphism on $(X_i,\mathcal{B}_i,\mu_i)$, $i=1,2$ and S establishes an isomorphism between T_1 and T_2 then we can define a joining of T_1 and T_2 by putting

$$\Delta_S(A_1\times A_2)=\mu_1(A_1\cap S^{-1}A_2)\quad\text{for all } A_i\in\mathcal{B}_i,\ i=1,2.$$

Notice that $\Phi_{\Delta_S}=U_S$. Such a joining is called a *graph-joining*.

Given a *factor* $\mathcal{A}\subset\mathcal{B}$ of an automorphism T, i.e. the T-invariant sub-σ-algebra, by $J(T,\mathcal{A})$ we will denote the set of self-joinings of the quotient map $\widetilde{T}$ on $(X/\mathcal{A},\mathcal{A},\mu)$. Having $\varrho\in J(T,\mathcal{A})$ by putting

$$\widehat{\varrho}(A_1\times A_2)=\int_{X/\mathcal{A}\times X/\mathcal{A}} E(\chi_{A_1}|\mathcal{A})(\overline{x})E(\chi_{A_2}|\mathcal{A})(\overline{y})\,d\varrho(\overline{x},\overline{y}),$$

for $A_i\in\mathcal{B}_i$, $i=1,2$, we obtain an element $\widehat{\varrho}\in J(T)$. Such a self-joining of T is called the *relatively independent extension of* ϱ.

Assume that T is ergodic. Having its factor $\mathcal{A}$ we can define the *relative product* over $\mathcal{A}$ as the self-joining $\mu\otimes_{\mathcal{A}}\mu$ in $J(T)$ determined by

$$\mu\otimes_{\mathcal{A}}\mu=\int_{X/\mathcal{A}}\mu_{\overline{x}}\otimes\mu_{\overline{x}}\,d\mu(\overline{x}),$$

where $\mu=\int_{X/\mathcal{A}}\mu_{\overline{x}}\,d\mu(\overline{x})$ is the disintegration of μ over a factor $\mathcal{A}$. Then $\mu\otimes_{\mathcal{A}}\mu(A\times B)=\int_{X/\mathcal{A}}E(\chi_A|\mathcal{A})E(\chi_B|\mathcal{A})\,d\mu$ (see [7]).

Following [7] we say that T is *relatively weakly mixing over* $\mathcal{A}$ if $\mu\otimes_{\mathcal{A}}\mu\in J^e(T)$.

PROPOSITION 2.6 ([16]). *$T_{f,\mathcal{S}}$ is relatively weakly mixing over T if and only if $T_{f,\mathcal{S}}$ is ergodic and $\mathcal{S}$ is weakly mixing.*

Following [6] an automorphism $T\colon(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ is called ELF if $\overline{\{U_{T^n}\}}\subset\mathcal{J}^e(T)$.

For example the ergodic rotation on the circle is ELF.

3. A class of weighted operators with a Lebesgue spectrum over irrational rotations

Throughout this section only irrational rotations $Tx = x+\alpha$ on $([0,1),\mathcal{B},\mu)$, where $\alpha \in \Lambda_{mb}$ are considered.

We will consider the set Γ of functions $f\colon [0,1) \to \mathbb{R}$ satisfying the following properties:

(a) f' has only finitely many discontinuity points:
- $s \geq 0$ of them, say $p_0,\dots,p_{s-1}$ being jumps,
- $m \geq 1$ of them, say $y_0,\dots,y_{m-1}$ being such that the one-side limits of f at y_i exists and $\lim_{x\to y_i^+} f'(x) = \infty$ or $\lim_{x\to y_i^-} f'(x) = \infty$,

(b) $f'(x) > 0$ for all $x \in [0,1)$,

(c) let $\delta > 0$ be such that the intervals of length 2δ centered at all discontinuity points are pairwise disjoint and f' is decreasing in $(y_i, y_i+\delta)$ in case $\lim_{x\to y_i^+} f'(x) = \infty$ or f' is increasing in $(y_i-\delta, y_i)$ in case $\lim_{x\to y_i^-} f'(x) = \infty$,

(d) if we denote by $z_0,\dots,z_{m+s-1}$ all discontinuity points of f' in increasing ordering, then $f \in C^2(z_i, z_{i+1})$ for each $i = 0,\dots,m+s-1$.

Theorem 3.1. *Let $f \in \Gamma$. Assume that there exists a sequence $(\varepsilon_n)_{n\geq 1}$ of positive reals such that*

(a) $(n\varepsilon_n)_n \in l_2$,

(b) *for all $M > 0$*

$$\left(n\Big/\min\left\{\min_{\substack{0\leq i\leq m-1\\ \lim_{x\to y_i^+} f'(x)=\infty}} f'\left(y_i + M\frac{\log n}{n}\right),\ \min_{\substack{0\leq i\leq m-1\\ \lim_{x\to y_i^-} f'(x)=\infty}} f'\left(y_i - M\frac{\log n}{n}\right)\right\}\right)_n \in l_2$$

(c) *for all $M > 0$*

$$\left(n\left(\sum_{i=0}^{m+s-1} \mathrm{Var}_{z_i+\varepsilon_n}^{z_{i+1}-\varepsilon_n} f'\right)\Big/\left[\min\left\{\min_{\substack{0\leq i\leq m-1\\ \lim_{x\to y_i^+} f'(x)=\infty}} f'\left(y_i + M\frac{\log n}{n}\right),\ \min_{\substack{0\leq i\leq m-1\\ \lim_{x\to y_i^-} f'(x)=\infty}} f'\left(y_i - M\frac{\log n}{n}\right)\right\}\right]^2\right)_n \in l_2.$$

Then for each $\alpha \in \Lambda_{mb}$, the automorphism $T_{e^{2\pi i f}}$ has a Lebesgue spectrum on $L^2([0,1)\times\mathbb{T},\mu\otimes\lambda) \ominus (L^2([0,1),\mu)\otimes 1_{\mathbb{T}})$.

Proof. In view of (2.5) it is enough to show that $(\int_{[0,1)} e^{2\pi i f^{(n)}(x)}dx)_n \in l_2$, where $f^{(n)}(x) = f(x) + f(Tx) + \ldots + f(T^{n-1}x)$. Indeed, for each $k \in \mathbb{N}_+$,

$\widetilde{f} = kf$ also satisfies (b), (c). Using (2.3), $\widetilde{f} = kf$ satisfies (b), (c) also for each $k \in \mathbb{Z} \setminus \{0\}$.

Fix $n \geq 1$. Consider the sequence $\{z_i - j\alpha\}_{j=0}^{n-1}$, $i = 0, \dots, m+s-1$. Put its terms in an increasing sequence $x_0 \leq \dots \leq x_{n(m+s)-1}$.

Let $A_j = (x_j - \varepsilon_n, x_j + \varepsilon_n)$, $j = 0, \dots, (m+s)n - 1$. Notice that each discontinuity point of $f^{(n)}$ must belong to $\{x_0, x_1, \dots, x_{n(m+s)-1}\}$.

Consider the set $[0,1) \setminus \bigcup_{j=0}^{n(m+s)-1} A_j = B_0 \cup \dots \cup B_{(m+s)n-1} = B$ (notice that some of B_i may be empty).

Notice that if $x \in B$, then no point of the form $x, x+\alpha, \dots, x+(n-1)\alpha$ belongs to $\bigcup_{i=0}^{m+s-1} (z_i - \varepsilon_n, z_i + \varepsilon_n)$. Indeed, if $x + r\alpha \in (z_i - \varepsilon_n, z_i + \varepsilon_n)$ for some r and i, then $\|x - (z_i - r\alpha)\| = \|(x + r\alpha) - z_i\| < \varepsilon_n$, which means that $x \in (z_i - r\alpha - \varepsilon_n, z_i - r\alpha + \varepsilon_n) = A_j$ for some j and we get a contradiction. We have

$$\mu\Big(\bigcup_{j=0}^{n(m+s)-1} A_j\Big) \leq \sum_{j=0}^{n(m+s)-1} \mu(A_j) = 2(m+s)n\varepsilon_n.$$

Further

$$\begin{aligned}\Big|\int_{[0,1)} e^{2\pi i f^{(n)}(x)}\,dx\Big| &= \Big|\int_B e^{2\pi i f^{(n)}(x)}\,dx + \int_{[0,1)\setminus B} e^{2\pi i f^{(n)}(x)}\,dx\Big| \\ \leq \Big|\int_B e^{2\pi i f^{(n)}(x)} dx\Big| + \mu\Big(\bigcup_{j=0}^{n(m+s)-1} A_j\Big) &\leq \Big|\int_B e^{2\pi i f^{(n)}(x)}\,dx\Big| + 2(m+s)n\varepsilon_n.\end{aligned}$$

Notice that in $[x_j + \varepsilon_n, x_{j+1} - \varepsilon_n]$ the function $x \mapsto e^{2\pi i f^{(n)}(x)}$ is continuous and function $x \mapsto 1/2\pi i (f^{(n)})'(x)$ is of bounded variation, thus there exists the Stieltjes integral

$$\int_{x_j+\varepsilon_n}^{x_{j+1}-\varepsilon_n} e^{2\pi i f^{(n)}(x)}\,d\frac{1}{2\pi i (f^{(n)})'(x)}.$$

Moreover,

$$\begin{aligned}\Big|\int_B e^{2\pi i f^{(n)}(x)}\,dx\Big| &= \Big|\sum_{j=0}^{n(m+s)-1} \int_{x_j+\varepsilon_n}^{x_{j+1}-\varepsilon_n} e^{2\pi i f^{(n)}(x)}\,dx\Big| \\ &= \Big|\sum_{j=0}^{n(m+s)-1} \int_{x_j+\varepsilon_n}^{x_{j+1}-\varepsilon_n} \frac{1}{2\pi i (f^{(n)})'(x)}\,de^{2\pi i f^{(n)}(x)}\Big| \\ &= \Big|\sum_{j=0}^{n(m+s)-1} \frac{e^{2\pi i f^{(n)}(x)}}{2\pi i f'^{(n)}(x)}\Big|_{x_j+\varepsilon_n}^{x_{j+1}-\varepsilon_n} \\ &\quad - \int_{x_j+\varepsilon_n}^{x_{j+1}-\varepsilon_n} e^{2\pi i f^{(n)}(x)}\,d\frac{1}{2\pi i f'^{(n)}(x)}\Big| \\ &\leq \Big|\sum_{j=0}^{n(m+s)-1} \frac{e^{2\pi i f^{(n)}(x_{j+1}-\varepsilon_n)}}{2\pi i f'^{(n)}(x_{j+1}-\varepsilon_n)} - \frac{e^{2\pi i f^{(n)}(x_j+\varepsilon_n)}}{2\pi i f'^{(n)}(x_j+\varepsilon_n)}\Big|\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=0}^{n(m+s)-1} \left| \int_{x_j+\varepsilon_n}^{x_{j+1}-\varepsilon_n} e^{2\pi i f^{(n)}(x)}\, d\frac{1}{2\pi i f'^{(n)}(x)} \right| \\
&\le \frac{1}{2\pi} \sum_{j=0}^{n(m+s)-1} \left(\frac{1}{|f'^{(n)}(x_{j+1}-\varepsilon_n)|} + \frac{1}{|f'^{(n)}(x_j+\varepsilon_n)|} \right) \\
&+ \frac{1}{2\pi} \sum_{j=0}^{n(m+s)-1} \mathrm{Var}_{x_j+\varepsilon_n}^{x_{j+1}-\varepsilon_n} \frac{1}{f'^{(n)}}.
\end{aligned}
$$

Since $D_n(x, x+\alpha, \dots, x+(n-1)\alpha) = O((\log n)/n)$, there exist $M > 0$, N such that for all $n > N$, for each interval I of length $M(\log n)/n$ and for an arbitrary x there exists $r \in \{0, \dots, n-1\}$ such that $x + r\alpha \in I$. In particular for each $i = 0, \dots, m-1$ such that $\lim_{x\to y_i^+} f'(x) = \infty$ there exists $r_i^+ \in \{0, \dots, n-1\}$ such, that $x + r_i^+\alpha \in (y_i, y_i + M(\log n)/n]$ (analogously if $\lim_{x\to y_i^-} f'(x) = \infty$ we can find r_i^- such that $x + r_i^-\alpha \in [y_i - M(\log n)/n, y_i))$. We consider only n so that $M(\log n)/n < \delta$. Then for all $x \in B$

$$
\begin{aligned}
f'^{(n)}(x) &= f'(x) + f'(x+\alpha) + \dots + f'(x+(n-1)\alpha) \\
&\ge \sup_r f'(x+r\alpha) \ge \min\{f'(x+r_i^+\alpha), f'(x+r_i^-\alpha)\} \\
&\ge \min\Bigg\{ \min_{\substack{0\le i\le m-1 \\ \lim_{x\to y_i^+} f'(x)=\infty}} f'\left(y_i + M\frac{\log n}{n}\right), \\
&\qquad \min_{\substack{0\le i\le m-1 \\ \lim_{x\to y_i^-} f'(x)=\infty}} f'\left(y_i - M\frac{\log n}{n}\right)\Bigg\} =: \eta(n, M).
\end{aligned}
$$

Let us continue our estimation

$$
\begin{aligned}
\left| \int_B e^{2\pi i f^{(n)}(x)}\, dx \right| &\le \frac{1}{2\pi} n(m+s) \frac{2}{\eta(n,M)} + \frac{1}{2\pi} \sum_{j=0}^{n(m+s)-1} \mathrm{Var}_{x_j+\varepsilon_n}^{x_{j+1}-\varepsilon_n} \frac{1}{f'^{(n)}} \\
&\le \frac{m+s}{\pi} \frac{n}{\eta(n,M)} \\
&\quad + \frac{1}{2\pi} \sum_{j=0}^{n(m+s)-1} \frac{\mathrm{Var}_{x_j+\varepsilon_n}^{x_{j+1}-\varepsilon_n} f'^{(n)}}{(\min_{x_j+\varepsilon_n \le x \le x_{j+1}-\varepsilon_n} f'^{(n)}(x))^2} \\
&\le \frac{m+s}{\pi} \frac{n}{\eta(n,M)} + \frac{1}{2\pi} \frac{1}{(\eta(n,M))^2} \sum_{j=0}^{n(m+s)-1} \mathrm{Var}_{x_j+\varepsilon_n}^{x_{j+1}-\varepsilon_n} f'^{(n)} \\
&= \frac{m+s}{\pi} \frac{n}{\eta(n,M)} + \frac{1}{2\pi} \frac{1}{(\eta(n,M))^2} \\
&\quad \cdot \sum_{j=0}^{n(m+s)-1} \mathrm{Var}_{B_j}(f' + f' \circ T + \ldots + f' \circ T^{n-1}).
\end{aligned}
$$

But for a fixed $r \in \{0, \ldots, n-1\}$

$$\sum_{j=0}^{n(m+s)-1} \mathrm{Var}_{B_j}(f' \circ T^r) = \sum_{j=0}^{n(m+s)-1} \mathrm{Var}_{T^r B_j} f' \le \sum_{i=0}^{m+s-1} \mathrm{Var}_{z_i+\varepsilon_n}^{z_{i+1}-\varepsilon_n} f'.$$

The last inequality results from following reasoning: given $B_j = (x_j + \varepsilon_n, x_{j+1} - \varepsilon_n)$ we have $B_j \cap B_i = \emptyset$, $i \neq j$ what implies $T^r B_j \cap T^r B_i = \emptyset$, $i \neq j$. Moreover, if $x \in T^r B_j$ then $x - r\alpha \in B_j \subset B$, hence for all $i = 0, \ldots, m+s-1$, $j = 0, \ldots, n-1$ $x - r\alpha + j\alpha \notin (z_i - \varepsilon_n, z_i + \varepsilon_n)$, therefore for all $i = 0, \ldots, m+s-1$ we have that $x \notin (z_i - \varepsilon_n, z_i + \varepsilon_n)$. Hence

$$T^r B_j \subset \bigcup_{i=0}^{m+s-1} (z_i + \varepsilon_n, z_{i+1} - \varepsilon_n).$$

Continuing

$$\left| \int_B e^{2\pi i f^{(n)}(x)}\, dx \right| \le \frac{m+s}{\pi} \frac{n}{\eta(n,M)} + \frac{1}{2\pi} \frac{n \sum_{i=0}^{m+s-1} \mathrm{Var}_{z_i+\varepsilon_n}^{z_{i+1}-\varepsilon_n} f'}{(\eta(n,M))^2}.$$

Finally we obtain

$$\left| \int_{[0,1)} e^{2\pi i f^{(n)}(x)}\, dx \right| \le \frac{m+s}{\pi} \frac{n}{\eta(n,M)} + \frac{1}{2\pi} \frac{n \sum_{i=0}^{m+s-1} \mathrm{Var}_{z_i+\varepsilon_n}^{z_{i+1}-\varepsilon_n} f'}{(\eta(n,M))^2} + 2(m+s)n\varepsilon_n.$$

The result follows from this estimation and from (a)–(c). $\square$

Immediately from the above proof we obtain the following

COROLLARY 3.2. *Let $f \in \Gamma$. We assume that there exists a sequence (ε_n) of positive reals such that*

(a) $n\varepsilon_n \to 0$, $n \to \infty$,
(b) *for all* $M > 0$

$$n \Big/ \min\Big\{ \min_{\substack{0 \le i \le m-1 \\ \lim_{x \to y_i^+} f'(x) = \infty}} f'\Big(y_i + M \frac{\log n}{n}\Big), \min_{\substack{0 \le i \le m-1 \\ \lim_{x \to y_i^-} f'(x) = \infty}} f'\Big(y_i - M \frac{\log n}{n}\Big) \Big\} \to 0, \quad \textit{as } n \to \infty,$$

(c) *for all* $M > 0$

$$n\Big(\sum_{i=0}^{m+s-1} \mathrm{Var}_{z_i+\varepsilon_n}^{z_{i+1}-\varepsilon_n} f'\Big) \Big/ \Big[\min\Big\{ \min_{\substack{0\le i\le m-1\\ \lim_{x\to y_i^+} f'(x)=\infty}} f'\Big(y_i + M\frac{\log n}{n}\Big),$$

$$\min_{\substack{0\le i\le m-1\\ \lim_{x\to y_i^-} f'(x)=\infty}} f'\Big(y_i - M\frac{\log n}{n}\Big)\Big\}\Big]^2 \to 0, \quad \textit{as } n\to\infty.$$

Then for each $\alpha \in \Lambda_{mb}$ *the automorphism* $T_{e^{2\pi i f}}$ *is mixing on*

$$L^2([0,1)\times\mathbb{T}, \mu\otimes\lambda) \ominus (L^2([0,1),\mu)\otimes 1_{\mathbb{T}}).$$

Let us see some examples of functions satisfying the assumptions of Theorem 3.1.

Example 3.3. Let $f(x) = -1/x^{2+\delta}$, $\delta > 0$ and $\varepsilon_n = 1/n^{3/2+\varepsilon}$, where $\varepsilon < \delta/(2(3+\delta))$.

One can see that the condition (a) holds. In this case (b) is reduced to

$$\sum_{n\ge1}\Big(\frac{n}{f'(M(\log n)/n)}\Big)^2 = (2+\delta)^{-2} M^{6+2\delta} \sum_{n\ge 1} \frac{(\log n)^{6+2\delta}}{n^{4+2\delta}} < \infty,$$

while for (c) let us see that

$$\begin{aligned}\Big(\frac{n\mathrm{Var}_{\varepsilon_n}^1 f'}{(f'(M(\log n)/n))^2}\Big)^2 &\le \Big(\frac{nf'(\varepsilon_n)}{(f'(M(\log n)/n))^2}\Big)^2\\ &= (2+\delta)^{-2}M^{12+4\delta}\frac{(\log n)^{12+4\delta}}{n^{1+\delta-2\varepsilon(3+\delta)}}\end{aligned}$$

and whenever $\varepsilon < \delta/(2(3+\delta))$ we have

$$\sum_{n\ge1}\frac{(\log n)^{12+4\delta}}{n^{1+\delta-2\varepsilon(3+\delta)}} < \infty.$$

Similarly we show that $f(x) = -(1/x^{2+\delta} - 1/(1-x)^{2+\delta})$ also satisfies the assumptions (a)–(c) with the same sequence (ε_n).

Remark 3.4. Notice that in general if f has only one discontinuity point at 0 (such that $\lim_{x\to0^+} f'(x) = \infty$) and it satisfies the assumptions of Theorem 3.1 then $\widetilde{f}(x) = f(x) - f(1-x)$ also satisfies these assumptions (indeed, take the same sequence (ε_n)).

Example 3.5. Let $f(x) = -1/x^\delta$, $\delta > 0$ then the assumptions of Corollary 3.2 are satisfied ($\varepsilon_n = 1/n^\mu$, for some $1 < \mu < (1+2\delta)/(1+\delta)$).

Remark 3.6. Let us observe that if a cocycle f satisfies the assumptions of Theorem 3.1 then for each $c > 0$, cf also satisfies the same assumptions and it follows from (2.3) that for all $c \neq 0$ the operator $V^{e^{2\pi icf},T}$ has also a Lebesgue spectrum.

Assume now that $\mathcal{S} = (S_t)_{t\in\mathbb{R}} : (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu)$ is an ergodic flow. Denote by $\sigma_{\mathcal{S}}$ the maximal spectral type of $U_{\mathcal{S}}$ on $L_0^2(Y, \mathcal{C}, \nu)$. Since $\mathcal{S}$ is ergodic, $\sigma_{\mathcal{S}}$ has no atom at 0.

Consider $T_{f,\mathcal{S}}(x, y) = (Tx, S_{f(x)}y)$ (acting on $(X \times Y, \mu \otimes \nu)$, $X = [0, 1)$). Take f satisfying the assumptions of Theorem 3.1. Since for all $c \neq 0$, $V^{e^{2\pi icf},T}$ has a Lebesgue spectrum, we have $\sigma_{T_{f,\mathcal{S}}} \equiv \lambda$. Indeed, we have $\sigma_{V^{e^{2\pi icf},T}} \equiv \lambda$ for all $c \neq 0$. Now we use Lemma 2.4. If $\lambda(A) = 0$, then $\sigma_{V^{e^{2\pi icf},T}}(A) = 0$ and we obtain

$$\sigma_{T_{f,\mathcal{S}}}(A) = \int_{\mathbb{R}} \sigma_{V^{e^{2\pi icf},T}}(A)\, d\sigma_{\mathcal{S}}(c) = 0.$$

Thus we have $\sigma_{T_{f,\mathcal{S}}} \ll \lambda$.

Conversely, if $\sigma_{T_{f,\mathcal{S}}}(A) = 0$ then for some $c \neq 0$, $\sigma_{V^{e^{2\pi icf},T}}(A) = 0$ (since $\sigma_{\mathcal{S}}$ has no atom at 0), what implies that $\lambda(A) = 0$ and we get $\lambda \ll \sigma_{T_{f,\mathcal{S}}}$.

COROLLARY 3.7. *If $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ is ergodic and $f: [0, 1) \to \mathbb{R}$ satisfies the assumptions of Theorem* 3.1 *then $U_{T_{f,\mathcal{S}}}$ has a Lebesgue spectrum on*

$$L^2(X \times Y, \mu \otimes \nu) \ominus (L^2(X, \mu) \otimes 1_Y).$$

Since on each subspace $L^2(X, \mu) \otimes Z(g)$ the spectrum is also Lebesgue (see Lemma 2.5), $U_{T_{f,\mathcal{S}}}$ has a Lebesgue spectrum of infinite multiplicity whenever $\dim L^2(Y, \nu) = \infty$.

Using the same argument as before we obtain

COROLLARY 3.8. *If $\mathcal{S} = (S_t)_{t\in\mathbb{R}}$ is ergodic and $f: [0, 1) \to \mathbb{R}$ satisfies the assumptions of Corollary* 3.2 *then $T_{f,\mathcal{S}}$ is mixing on*

$$L^2(X \times Y, \mu \otimes \nu) \ominus (L^2(X, \mu) \otimes 1_Y).$$

We will now show that if $\mathcal{S}$ is weakly mixing and if f satisfies the assumptions of Corollary 3.2 then the automorphism $T_{f,\mathcal{S}}$ is ELF.

We have a weakly mixing flow

$$\mathcal{S} = (S_t)_{t\in\mathbb{R}}: (Y, \mathcal{C}, \nu) \to (Y, \mathcal{C}, \nu).$$

Take an irrational rotation on $X = [0, 1)$, $Tx = x + \alpha$ and a cocycle $f: [0, 1) \to \mathbb{R}$ satisfying the assumptions of Corollary 3.2. We will describe all elements of the weak closure of $\{U_{T^n_{f,\mathcal{S}}}\}_n$. Consider a weakly convergent sequence $(U_{T^{n_i}_{f,\mathcal{S}}})_i$. Then the sequence of integrals

$$\int_{[0,1)} \int_Y (U_{T^{n_i}_{f,\mathcal{S}}} F)\overline{G}\, d\mu\, d\nu$$

converges for each $F, G \in L^2(X \times Y, \mu \otimes \nu)$. Since functions of the form $j \otimes k$, $j \in L^2(X, \mu)$, $k \in L^2(Y, \nu)$ span $L^2(X \times Y, \mu \otimes \nu)$, we may suppose that

$$F = f_1 \otimes f_2, \quad G = g_1 \otimes g_2, \quad f_1, g_1 \in L^2(X, \mu), \quad f_2, g_2 \in L^2(Y, \nu).$$

We have

$$\begin{aligned}
&\int_{[0,1)} \int_Y F \circ T^{n_i}_{f,\mathcal{S}} \overline{G} \, d\mu \, d\nu \\
&\quad = \int_{[0,1)} \int_Y f_1(T^{n_i} x) f_2(S_{f^{(n_i)}(x)} y) \overline{g_1(x) g_2(y)} \, d\mu(x) \, d\nu(y) \\
&\quad = \int_{[0,1)} f_1(T^{n_i} x) \overline{g_1(x)} \left(\int_Y f_2(S_{f^{(n_i)}(x)} y) \overline{g_2(y)} \, d\nu(y) \right) d\mu(x) \\
&\quad = \int_{[0,1)} f_1(T^{n_i} x) \overline{g_1(x)} \left(\int_{\mathbb{R}} e^{2\pi i c f^{(n_i)}(x)} \, d\sigma_{f_2, g_2, \mathcal{S}}(c) \right) d\mu(x) \\
&\quad = \int_{\mathbb{R}} \left(\int_{[0,1)} f_1(T^{n_i} x) \overline{g_1(x)} e^{2\pi i c f^{(n_i)}(x)} \, d\mu(x) \right) d\sigma_{f_2, g_2, \mathcal{S}}(c) \\
&\quad = \int_{\mathbb{R}} \langle (V^{e^{2\pi i c f}, T})^{n_i} f_1, g_1 \rangle \, d\sigma_{f_2, g_2, \mathcal{S}}(c) \\
&\quad = \int_{\mathbb{R}} \widehat{\sigma}_{f_1, g_1, V^{e^{2\pi i c f}, T}}(n_i) \, d\sigma_{f_2, g_2, \mathcal{S}}(c) \\
&\quad = \int_{\mathbb{R} \setminus \{0\}} \widehat{\sigma}_{f_1, g_1, V^{e^{2\pi i c f}, T}}(n_i) \, d\sigma_{f_2, g_2, \mathcal{S}}(c) \\
&\qquad + \sigma_{f_2, g_2, \mathcal{S}}(\{0\}) \int_{[0,1)} f_1 \circ T^{n_i} \overline{g_1} \, d\mu.
\end{aligned}$$

Since for all $c \neq 0$ the maximal spectral type of $V^{e^{2\pi i c f}, T}$ is a Rajchman measure (the measure σ on the circle is Rajchman measure if $\widehat{\sigma}(n) \to 0$, $n \to \infty$), $\widehat{\sigma}_{f_1, g_1, V^{e^{2\pi i c f}, T}}(n_i) \to 0$, $i \to \infty$. Thus from the Lebesgue Dominated Convergence Theorem we obtain that

$$\int_{\mathbb{R}} \widehat{\sigma}_{f_1, g_1, V^{e^{2\pi i c f}, T}}(n_i) d\sigma_{f_2, g_2, \mathcal{S}}(c) \to 0, \quad i \to \infty.$$

Now take functions of form

$$F(x, y) = f_1 \otimes 1_Y, \quad G(x, y) = g_1 \otimes 1_Y.$$

Then

$$\int_{[0,1)} \int_Y F \circ T^{n_i}_{f,\mathcal{S}} \overline{G} \, d\mu \, d\nu = \int_{[0,1)} f_1 \circ T^{n_i} \overline{g_1} \, d\mu.$$

Hence $U_{T^{n_i}}$ converges weakly. Since T is a rotation, we get immediately that $\lim_{i \to \infty} n_i \alpha = \beta$ and $T^{n_i} \to S$ weakly $(i \to \infty)$, where $Sx = x + \beta$. Hence

$$\int_{[0,1)} \int_Y F \circ T^{n_i}_{f,\mathcal{S}} \overline{G} \, d\mu \, d\nu \to \sigma_{f_2, g_2, \mathcal{S}}(\{0\}) \int_{[0,1)} f_1 \circ S \cdot \overline{g_1} \, d\mu.$$

On the other hand

$$\begin{aligned}
\int_{([0,1)\times Y)\times([0,1)\times Y)} & F\otimes\overline{G}\,d\widehat{\Delta}_{S^{-1}} \\
&= \int_{([0,1)\times Y)/\mathcal{B}_1\times\{Y,\emptyset\})^2} E(f_1\otimes f_2|X)\otimes E(\overline{g_1\otimes g_2}|X)\,d\Delta_{S^{-1}} \\
&= \int_{[0,1)}\Big(\int_Y f_2\,d\nu\cdot f_1\circ S\Big)\Big(\int_Y \overline{g_2}\,d\nu\cdot\overline{g_1}\Big)\,d\mu \\
&= \Big(\int_Y f_2\,d\nu\Big)\Big(\int_Y \overline{g_2}\,d\nu\Big)\int_{[0,1)} f_1\circ S\cdot\overline{g_1}\,d\mu
\end{aligned}$$

and by (2.1) we obtain that $U_{T^{n_i}_{f,\mathcal{S}}}\to\Phi_{\widehat{\Delta}_{S^{-1}}}$ weakly ($i\to\infty$).

PROPOSITION 3.9. *Under the above assumptions* $\overline{\{U^n_{T_{f,\mathcal{S}}}\}}\subset\mathcal{J}^e(T_{f,\mathcal{S}})$.

PROOF. From the calculations above it remains to show that the automorphism $T_{f,\mathcal{S}}\times T_{f,\mathcal{S}}\colon(X\times Y\times X\times Y,\widehat{\Delta}_{S^{-1}})\to(X\times Y\times X\times Y,\widehat{\Delta}_{S^{-1}})$ is ergodic. Observe first, that this automorphism is isomorphic to $T_{f\times f\circ S^{-1},\mathcal{S}'}$, where $(f\times f\circ S^{-1})(x)=(f(x),f(S^{-1}x))$ and $\mathcal{S}'=(S_t\times S_{t'})_{(t,t')\in\mathbb{R}^2}$. The latter automorphism acts on $(X\times Y\times Y,\mu\otimes\nu\otimes\nu)$. Indeed, since the measure $\widehat{\Delta}_{S^{-1}}$ is concentrated on the set $\{(x_1,y_1,S^{-1}x_1,y_2):x_1\in X,y_1,y_2\in Y\}$, the map $(x_1,y_1,S^{-1}x_1,y_2)\mapsto(x_1,y_1,y_2)$ establishes the required isomorphism.

Since $\mathcal{S}$ is weakly mixing (in particular ergodic), $T_{f,\mathcal{S}}$ is mixing on $L^2(X\times Y,\mu\otimes\nu)\ominus(L^2(X,\mu)\otimes 1_Y)$ (see Corollary 3.8) (so it is ergodic there) and on (X,μ) it is an irrational rotation, so $T_{f,\mathcal{S}}$ is ergodic. Hence from Proposition 2.6 we obtain that $T_{f,\mathcal{S}}$ is relatively weakly mixing over T. Similarly we get, that $T_{f\circ S^{-1},\mathcal{S}}$ is also relatively weakly mixing over T.

Let us observe, that relative product $T_{f,\mathcal{S}}$ and $T_{f\circ S^{-1},\mathcal{S}}$ over T is isomorphic to $T_{f\times f\circ S^{-1},\mathcal{S}'}$. Since $T_{f,\mathcal{S}}$ and $T_{f\circ S^{-1},\mathcal{S}}$ are relatively weakly mixing over T, so is their relative product and thereby $T_{f\times f\circ S^{-1},\mathcal{S}'}$ has also the same property (see [7, Proposition 6.4]). It follows that $T_{f\times f\circ S^{-1},\mathcal{S}'}$ is ergodic. Therefore $T_{f,\mathcal{S}}\times T_{f,\mathcal{S}}$ (acting on $(X\times Y\times X\times Y,\widehat{\Delta}_{S^{-1}})$) is also ergodic. □

4. Final remarks

We have been unable to decide whether it is possible to find an integrable function f for which a sequence (ε_n) exists so that Theorem 3.1 holds (compare Example 3.5 with Example 3.3). It should be noted, however, that in [5] appeared an example of skew product on $(\mathbb{R}^4/\mathbb{Z}^4)\times\mathbb{T}$ having a countable Lebesgue spectrum in the orthogonal complement of $L^2(\mathbb{R}^4/\mathbb{Z}^4)$. More precisely, this example is derived from a skew product $\widetilde{T}_{e^{2\pi i\phi}}$ which is constructed from the minimal translation on $\mathbb{R}^2/\mathbb{Z}^2$ (defined by $\widetilde{T}(x,y)=(x+\alpha,y+\alpha'))$ and the real analytic function on $\mathbb{R}^2$ ($\mathbb{Z}^2$-periodic) given by

$$\phi(x,y) = 1 + \mathrm{Re}\bigg(\sum_{j=0}^{\infty} \frac{e^{2\pi i q_j x}}{e^{q_j}}\bigg) + \mathrm{Re}\bigg(\sum_{j=0}^{\infty} \frac{e^{2\pi i q'_j y}}{e^{q'_j}}\bigg), \tag{4.1}$$

where α and α' are rationally independent such that the denominators of their convergents, q_n and q'_n, satisfy for n sufficiently large

$$q_n \geq e^{3nq'_{n-1}}, \quad q'_n \geq e^{3nq_n}. \tag{4.2}$$

Then the following holds

Proposition 4.1 ([5]). *For $l \in \mathbb{Z} \setminus \{0\}$ and for any $\varepsilon > 0$ we have for $\psi(x,y,z) = z^l$:*

$$\int_{(\mathbb{R}^2/\mathbb{Z}^2)\times\mathbb{T}} (\psi \circ \widetilde{T}^n_{e^{2\pi i\phi}})(x,y,z)\overline{\psi(x,y,z)}\, dx\, dy\, dz = O\bigg(\frac{1}{n^{1/3-\varepsilon}}\bigg),$$

when n goes to infinity.

Now, constructing from $\widetilde{T}_{e^{2\pi i\phi}}$ a skew product on $(\mathbb{R}^4/\mathbb{Z}^4) \times \mathbb{T}$ by putting

$$\widetilde{S}(x_1,x'_1,x_2,x'_2,z) = (x_1+\alpha_1, x'_1+\alpha'_1, x_2+\alpha_2, x'_2+\alpha'_2, ze^{2\pi i(\phi_1(x_1,x'_1)+\phi_2(x_2,x'_2))}),$$

where the couples (α_i, α'_i), $i = 1,2$ satisfy (4.2) and ϕ_1, ϕ_2 are as in (4.1), we get for $\psi(x_1,x'_1,x_2,x'_2,z) = z^l$ ($l \in \mathbb{Z}$)

$$\langle U^n_{\widetilde{S}}\psi, \psi\rangle = \int_{\mathbb{R}^2/\mathbb{Z}^2} e^{2\pi i l\phi_1^{(n)}(x_1,x'_1)}\, dx_1\, dx'_1 \int_{\mathbb{R}^2/\mathbb{Z}^2} e^{2\pi i l\phi_2^{(n)}(x_2,x'_2)}\, dx_2\, dx'_2$$

and using Proposition 4.1 the following has been proved

Theorem 4.2 ([5]). *$U_{\widetilde{S}}$ has a Lebesgue spectrum on $L^2((\mathbb{R}^4/\mathbb{Z}^4) \times \mathbb{T}) \ominus (L^2(\mathbb{R}^4/\mathbb{Z}^4) \otimes 1_{\mathbb{T}})$.*

Remark 4.3. We remark that for all $c \in \mathbb{R}\setminus\{0\}$ the function $c\phi$ has the same required properties as ϕ, so we get the same result for $\widetilde{S}$ constructed from the minimal translation on $\mathbb{R}^4/\mathbb{Z}^4$ and $\widehat{\phi}(x_1,x'_1,x_2,x'_2) = c(\phi_1(x_1,x'_1) + \phi_2(x_2,x'_2))$ for all $c \neq 0$.

We will now use a similar argument as above to obtain a Lebesgue component in the spectrum of a skew product over higher dimensional torus with $F(x_1,\ldots,x_\tau) = f(x_1) + \ldots + f(x_\tau)$ for some τ, where f (and ε_n) are from Example 3.5.

For $f(x) = -1/x^\delta$, $\delta > 0$ ($\varepsilon_n = 1/n^\mu$, $\mu > 1$) we have the following estimate in Corollary 3.2

$$\bigg|\int_{[0,1)} e^{2\pi i f^{(n)}(x)}\, dx\bigg| \leq 2(m+s)\frac{1}{n^{\mu-1}} + \frac{m+s}{\pi}M^{1+\delta}\frac{(\log n)^{1+\delta}}{n^\delta} + \frac{1}{2\pi}M^{2+2\delta}\frac{(\log n)^{2+2\delta}}{n^{1-\mu(1+\delta)+2\delta}},$$

hence $\widehat{\sigma}_{1,V^{e^{2\pi i f,T}}}(n)=O(1/n^{\kappa})$, $0<\kappa<\min\{\mu-1,\delta,1-\mu(1+\delta)+2\delta\}$. Consider now the skew product on $X\times\mathbb{T}$, where $X=\mathbb{R}^{\tau}/\mathbb{Z}^{\tau}$ $(\tau=[1/(2\kappa)]+1)$ given by

$$\widetilde{T}_{e^{2\pi iF}}(x_1,\ldots,x_\tau,z)=(x_1+\alpha_1,\ldots,x_\tau+\alpha_\tau,ze^{2\pi iF(x_1,\ldots,x_\tau)}),$$

where $F(x_1,\ldots,x_\tau)=f(x_1)+\ldots+f(x_\tau)$ and $\alpha_1,\ldots,\alpha_\tau\in\Lambda_{mb}$ are rationally independent. Then

$$\begin{aligned}\Big|\int_{[0,1)^\tau}&e^{2\pi iF^{(n)}(x_1,\ldots,x_\tau)}\,dx_1\ldots dx_\tau\Big|\\&=\Big|\int_{[0,1)^\tau}e^{2\pi if^{(n)}(x_1)}\ldots e^{2\pi if^{(n)}(x_\tau)}\,dx_1\ldots dx_\tau\Big|\\&=\Big|\int_{[0,1)}e^{2\pi if^{(n)}(x_1)}\,dx_1\ldots\int_{[0,1)}e^{2\pi if^{(n)}(x_\tau)}\,dx_\tau\Big|=O\Big(\frac{1}{n^{\kappa\tau}}\Big).\end{aligned}$$

Since $\kappa\tau>1/2$, we have that $\widehat{\sigma}_{1,V^{e^{2\pi iF,\widetilde{T}}}}(n)\in l_2$ (where $V^{e^{2\pi iF,\widetilde{T}}}$ is the corresponding weighted operator acting on $L^2(\mathbb{R}^\tau/\mathbb{Z}^\tau)$) and we obtain a Lebesgue spectrum of $\widetilde{T}_{e^{2\pi iF}}$ in the orthogonal complement of $L^2(\mathbb{R}^\tau/\mathbb{Z}^\tau)$.

REMARK 4.4. We state as an open question whether given an irrational rotation T we can find $f\colon[0,1)\to\mathbb{R}$ so that the conclusion of Remark 3.6 holds and at the same time the skew product T_f on $([0,1)\times\mathbb{R},\mu\otimes\lambda)$ (λ stands for Lebesgue measure on $\mathbb{R}$) given by $T_f(x,y)=(x+\alpha,f(x)+y)$ is ergodic?

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