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A SET-VALUED APPROACH TO HEMIVARIATIONAL INEQUALITIES

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Abstract. Let X be a Banach space, X^* its dual and let $T: X \to L^p(\Omega, \mathbb{R}^k)$ be a linear, continuous operator, where $p, k \geq 1$, Ω being a bounded open set in $\mathbb{R}^N$. Let K be a subset of X, $\mathcal{A}: K \rightsquigarrow X^*$, $G: K \times X \rightsquigarrow \mathbb{R}$ and $F: \Omega \times \mathbb{R}^k \times \mathbb{R}^k \rightsquigarrow \mathbb{R}$ set-valued maps with nonempty values. Using mainly set-valued analysis, under suitable conditions on the involved maps, we shall guarantee solutions to the following inclusion problem:

Find $u \in K$ such that, for every $v \in K$

$$\sigma(\mathcal{A}(u), v-u) + G(u, v-u) + \int_\Omega F(x, Tu(x), Tv(x) - Tu(x))dx \subseteq \mathbb{R}_+.$$

In particular, well-known variational and hemivariational inequalities can be derived.

1. Introduction

Let K be a nonempty subset of $H_0^1(\Omega)$, where Ω is a bounded open subset of $\mathbb{R}^N$ with C^1 boundary, $N \geq 1$. Many papers treat inclusion problems of the form:

Find $u \in K$ such that

$$-\triangle u \in G(x, u(x)) \quad \text{in } \Omega, \tag{1.1}$$

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where $G:\Omega\times\mathbb{R}\rightsquigarrow\mathbb{R}$ is a set-valued map with nonempty values, satisfying some growth and continuity conditions, see for instance [6] and [11]. In these papers critical point arguments were used.

Here, we suppose that G has the form

$$G(x,u(x)) = H(x,u(x)) - b(x)u(x),\quad x\in\Omega,\ u\in K, \tag{1.2}$$

where $b\in L^\infty(\Omega)$, and $H:\Omega\times\mathbb{R}\rightsquigarrow\mathbb{R}$ satisfies for all $x\in\Omega$ the following inclusion:

$$H(x,u(x))\cdot v(x) = \{h\cdot v(x) : h\in H(x,u(x))\}\subseteq[-g(x,u(x),v(x)),\infty), \tag{1.3}$$

where $g(\,\cdot\,,u(\,\cdot\,),v(\,\cdot\,))\in L^1(\Omega)$ for every $u\in K$, $v\in H_0^1(\Omega)$.

Multiplying (1.1) by $(v-u)$, integrating over Ω and applying the Gauss–Green formula, from (1.2) and (1.3) we obtain:

$$\int_\Omega \nabla u\cdot\nabla(v-u)\,dx + \int_\Omega b(x)u(x)(v(x)-u(x))\,dx + \int_\Omega [g(x,u(x),v(x)-u(x)),\infty)\,dx \subseteq \mathbb{R}_+ \tag{1.4}$$

for all $v\in K$, where the last term from the left hand side is the integral of a set-valued map in the sense of Aumann (see [2]).

If H has the form

$$H(x,u(x)) = -\partial j(x,u(x)),\quad x\in\Omega,$$

where $j:\Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that $j(x,\cdot)$ is locally Lipschitz continuous and ∂ denotes the generalized gradient, then (1.3) is verified if we take $g(x,y,z)=j_y^0(x,y;z)$, j_y^0 being the (partial) generalized directional derivative, supposing that j satisfies a growth condition (see Section 4). In this situation, (1.4) reduces to the following classical *hemivariational inequality*, see for instance Motreanu and Panagiotopoulos [8], Naniewicz and Panagiotopoulos (see [9]):

(HV$\geq$) Find $u\in K$ such that, for all $v\in K$

$$\int_\Omega \nabla u\cdot\nabla(v-u)\,dx + \int_\Omega b(x)u(x)(v(x)-u(x))\,dx + \int_\Omega j_y^0(x,u(x);v(x)-u(x))\,dx \geq 0.$$

So, it seems natural to study the following general problem.

Let X be a Banach space, X^* its dual, and let $T:X\to L^p(\Omega,\mathbb{R}^k)$ be a linear continuous operator, where $1\leq p<\infty$, $k\geq 1$, Ω being a bounded open set in $\mathbb{R}^N$.

Let K be a subset of X, let $\mathcal{A}: K \rightsquigarrow X^*$, $G: K \times X \rightsquigarrow \mathbb{R}$ and $F: \Omega \times \mathbb{R}^k \times \mathbb{R}^k \rightsquigarrow \mathbb{R}$ be set-valued maps with nonempty values, such that

(H_1) $x \in \Omega \rightsquigarrow F(x, Tu(x), Tv(x) - Tu(x))$ is a measurable set-valued map for all $u, v \in K$.

(H_2) There exist $h_1 \in L^{p/(p-1)}(\Omega, \mathbb{R}_+)$ and $h_2 \in L^{\infty}(\Omega, \mathbb{R}_+)$ such that

$$\operatorname{dist}(0, F(x, y, z)) \leq (h_1(x) + h_2(x)|y|^{p-1})|z| \quad \text{for a.e. } x \in \Omega,$$

for every $y, z \in \mathbb{R}^k$.

The aim of this paper is to study the following *hemivariational inclusion* problem:

(HV$\subseteq$) Find $u \in K$ such that, for all $v \in K$

$$\sigma(\mathcal{A}(u), v - u) + G(u, v - u) + \int_{\Omega} F(x, Tu(x), Tv(x) - Tu(x)) dx \subseteq \mathbb{R}_+.$$

We denoted by $\sigma(\mathcal{A}(u), \cdot\,)$ the support function of $\mathcal{A}(u)$, that is

$$\sigma(\mathcal{A}(u), h) = \sup_{x^* \in \mathcal{A}(u)} \langle x^*, h\rangle \quad \text{for all } h \in X.$$

The euclidean norm in $\mathbb{R}^k$ and the duality pairing between the Banach space and its dual is denoted by $|\cdot|$, respectively $\langle\cdot\,,\cdot\rangle$.

2. Preliminaries

We need some definitions and notions in order to state existence results concerning the problem (HV$\subseteq$).

Let $J: \Omega \rightsquigarrow \mathbb{R}$ be a measurable set-valued map with nonempty closed values, see [1, p. 307]. Define the set

$$\mathcal{J} = \{j \in L^1(\Omega, \mathbb{R}) : j(x) \in J(x) \text{ a.e. in } \Omega\}.$$

Definition 2.1 (see [2]). The *integral of J on Ω* is the set of integrals of integrable selections of J, i.e.

$$\int_{\Omega} J(x)\, dx = \bigg\{ \int_{\Omega} j(x)\, dx : j \in \mathcal{J} \bigg\}.$$

From the above definition we clearly have

Lemma 2.2. *Let $J_1, J_2: \Omega \rightsquigarrow \mathbb{R}$ be two measurable set-valued maps with closed values. Then the following assertions hold:*

(a) *If $J_1(x) \subseteq J_2(x)$ a.e. $x \in \Omega$, then $\int_{\Omega} J_1(x)\, dx \subseteq \int_{\Omega} J_2(x) dx$.*

(b) $\int_{\Omega} J_1(x)\, dx + \int_{\Omega} J_2(x)\, dx \subseteq \int_{\Omega} \overline{J_1(x) + J_2(x)}\, dx$.

(c) $\lambda \int_{\Omega} J_1(x)\, dx \subseteq \int_{\Omega} \lambda J_1(x)\, dx$ *for all $\lambda \in \mathbb{R}$.*

Definition 2.3. Let X be a Banach space, and let K be a nonempty subset of X. A set-valued map $\mathcal{A}: K \rightsquigarrow X^*$ with bounded values is said to be *upper demicontinuous at* $u_0 \in K$ (u.d.c. at $u_0 \in K$) if, for any $h \in X$, the real-valued function

$$u \in K \mapsto \sigma(\mathcal{A}(u), h) = \sup_{x^* \in \mathcal{A}(u)} \langle x^*, h \rangle$$

is upper semicontinuous at u_0. $\mathcal{A}$ is *upper demicontinuous on* K (u.d.c. on K) if it is udc at every $u \in K$.

Remark 2.4. If $\mathcal{A}(u) = \{A(u)\}$ for all $u \in K$, that is, if $\mathcal{A}$ is a single-valued map, then $\mathcal{A}$ is u.d.c. at $u_0 \in K$ if and only if the map $A: K \to X^*$ is w^*-demicontinuous at $u_0 \in K$, i.e. for each sequence $\{u_n\}$ in K converging to u_0 (in the strong topology), the image sequence $\{A(u_n)\}$ converges to $A(u_0)$ in the weak*-topology of X^*.

It is easy to verify that, for all $u \in K$, the function $h \in X \mapsto \sigma(\mathcal{A}(u), h)$ is lower semicontinuous, subadditive and positive homogeneous. Moreover, due to Banach–Steinhaus theorem, we can state the following useful result.

Proposition 2.5. *Let K be a nonempty subset of a Banach space X, and let $\mathcal{A}: K \rightsquigarrow X^*$ be an upper demicontinuous set-valued map with bounded values. Then the function $u \in K \mapsto \sigma(\mathcal{A}(u), v - u)$ is upper semicontinuous for all $v \in K$.*

Definition 2.6. Let W, Y be two metric spaces. A set-valued map (with nonempty values) $J: W \rightsquigarrow Y$ is called *lower semicontinuous at* $w \in W$ (l.s.c. at w) if and only if for any $y \in J(w)$ and for any sequence $\{w_n\}$, converging to w, there exists a sequence $\{y_n\}$, $y_n \in J(w_n)$ converging to y. J is said to be lower semicontinuous (l.s.c.) if it is lsc at every point $w \in W$.

Definition 2.7. Let $\{K_n\}$ be a sequence of subsets of a metric space Y. The set

$$\operatorname*{Liminf}_{n\to\infty} K_n = \{y \in Y : \lim_{n\to\infty} \operatorname{dist}(y, K_n) = 0\}$$

is the (*Kuratowski*) *lower limit of the sequence* K_n.

Remark 2.8. $\operatorname{Liminf}_{n\to\infty}$ is the set of limits of sequences $y_n \in K_n$ (see [1, p. 18]).

Proposition 2.9 (see [1, p. 42]). *Let X be a normed space. A set-valued map $F: X \rightsquigarrow \mathbb{R}$ is lower semicontinuous at $u \in X$ if and only if*

$$F(u) \subseteq \operatorname*{Liminf}_{n\to\infty} F(u_n)$$

for any sequence $\{u_n\}$ in X converging to u.

Lemma 2.10. *Let Y be a real normed space, and let $\{K_n\}, \{L_n\}$ be two sequences of subsets of Y. Then the following assertions hold:*

(a) $\operatorname{Liminf}_{n\to\infty} K_n + \operatorname{Liminf}_{n\to\infty} L_n \subseteq \operatorname{Liminf}_{n\to\infty}(K_n + L_n)$.
(b) *If $K_n \subseteq L_n$ for all $n \in \mathbb{N}$, then* $\operatorname{Liminf}_{n\to\infty} K_n \subseteq \operatorname{Liminf}_{n\to\infty} L_n$.

Definition 2.11. Let W, Y be real normed spaces, $K \subset W$ be a convex subset. The set-valued map $J: K \rightsquigarrow Y$ with nonempty values is *convex* if and only if

$$\forall\, w_1, w_2 \in K,\ \forall\, \lambda \in [0,1] : \lambda J(w_1) + (1-\lambda)J(w_2) \subseteq J(\lambda w_1 + (1-\lambda)w_2).$$

Remark 2.12. $J: K \rightsquigarrow Y$ is convex if and only if for all $w_i \in K$, for all $\lambda_i \geq 0$ such that $\sum_{i=1}^n \lambda_i = 1$, $n \in \mathbb{N}$, we have

$$\sum_{i=1}^n \lambda_i J(w_i) \subseteq J\biggl(\sum_{i=1}^n \lambda_i w_i\biggr).$$

Finally, we recall the well-known result of Ky Fan.

Lemma 2.13 (see [5]). *Let X be a Hausdorff topological vector space, K a subset of X and for each $x \in K$, let $S(x)$ be a closed subset of X, such that*

(a) *there exists $x_0 \in K$ such that the set $S(x_0)$ is compact,*
(b) *S is a KKM-map, i.e. for each $x_1, \ldots, x_n \in K$, $\operatorname{co}\{x_1, \ldots, x_n\} \subseteq \bigcup_{i=1}^n S(x_i)$, where* co *stands for the convex hull operator.*

Then $\bigcap_{x\in K} S(x) \neq \emptyset$.

3. Main results

We need some additional hypotheses to obtain a solution for (HV$\subseteq$).

(H_3) $w \in X \rightsquigarrow G(u,w)$ and $z \in \mathbb{R}^k \rightsquigarrow F(x,y,z)$ are convex for all $u \in K$, $x \in \Omega$, $y \in \mathbb{R}^k$.
(H_4) $G(u,0) \subseteq \mathbb{R}_+$ and $F(x,y,0) \subseteq \mathbb{R}_+$ for all $u \in K$, $x \in \Omega$, $y \in \mathbb{R}^k$.
(H_5) $(u,w) \in K \times X \rightsquigarrow G(u,w)$ is lower semicontinuous.
(H_6) $(y,z) \in \mathbb{R}^k \times \mathbb{R}^k \rightsquigarrow F(x,y,z)$ is lower semicontinuous for all $x \in \Omega$.

Remark 3.1. If $F: \Omega \times \mathbb{R}^k \times \mathbb{R}^k \rightsquigarrow \mathbb{R}$ is a closed-valued Carathéodory map (i.e. for any $(y,z) \in \mathbb{R}^k \times \mathbb{R}^k$, $x \in \Omega \rightsquigarrow F(x,y,z)$ is measurable and for any $x \in \Omega$, $(y,z) \in \mathbb{R}^k \times \mathbb{R}^k \rightsquigarrow F(x,y,z)$ is continuous), then the hypotheses (H_6) and (H_1) hold automatically (see [1, p. 314]).

Now, we establish the main result of this paper.

Theorem 3.2. *Let K be a nonempty compact convex subset of a Banach space X. Let $F: \Omega \times \mathbb{R}^k \times \mathbb{R}^k \rightsquigarrow \mathbb{R}$ and $G: K \times X \rightsquigarrow \mathbb{R}$ be two set-valued* *maps satisfying* (H_1)–(H_6), *of which* F *is closed-valued. If* $\mathcal{A}: K \rightsquigarrow X^*$ *is upper demicontinuous on* K *with bounded values, then* (HV$\subseteq$) *has at least a solution.*

Proof. For any $v \in K$ we set

$$S_v = \bigg\{ u \in K : \sigma(\mathcal{A}(u), v-u) + G(u, v-u) \\ + \int_\Omega F(x, Tu(x), Tv(x) - Tu(x))\,dx \subseteq \mathbb{R}_+ \bigg\}.$$

First, we prove that S_v is closed set for all $v \in K$. Fix a $v \in K$. Of course, $S_v \neq \emptyset$, since $v \in S_v$, due to (H_4). Now, let $\{u_n\}$ be a sequence in S_v which converges to $u \in X$. We prove that $u \in S_v$. Since $T: X \to L^p(\Omega, \mathbb{R}^k)$ is continuous, it follows that

$$Tu_n \to Tu \quad \text{in } L^p(\Omega, \mathbb{R}^k) \text{ as } n \to \infty.$$

Clearly, there exists a subsequence $\{u_m\}$ of $\{u_n\}$, see Proposition 2.5, such that

$$\limsup_{n\to\infty} \sigma(\mathcal{A}(u_n), v-u_n) = \lim_{m\to\infty} \sigma(\mathcal{A}(u_m), v-u_m). \tag{3.1}$$

Moreover, by [12, Lemma A.1, p. 133] there exists a subsequence $\{Tu_l\}$ of $\{Tu_m\}$ and $g \in L^p(\Omega, \mathbb{R}_+)$ such that

$$|Tu_l(x)| \le g(x), \quad Tu_l(x) \to Tu(x) \quad \text{for a.e. } x \in \Omega. \tag{3.2}$$

In the relation

$$\sigma(\mathcal{A}(u_l), v-u_l) + G(u_l, v-u_l) + \int_\Omega F(x, Tu_l(x), Tv(x) - Tu_l(x))\,dx \subseteq \mathbb{R}_+,$$

letting the lower limit and using Lemma 2.10 (with $Y = \mathbb{R}$) we obtain

$$\operatorname*{Liminf}_{l\to\infty} \sigma(\mathcal{A}(u_l), v-u_l) + \operatorname*{Liminf}_{l\to\infty} G(u_l, v-u_l) \\ + \operatorname*{Liminf}_{l\to\infty} \int_\Omega F(x, Tu_l(x), Tv(x) - Tu_l(x))\,dx \subseteq \operatorname*{Liminf}_{l\to\infty} \mathbb{R}_+ = \mathbb{R}_+. \tag{3.3}$$

Using Remark 2.8, relation (3.1) and Proposition 2.5, we obtain

$$\operatorname*{Liminf}_{l\to\infty} \sigma(\mathcal{A}(u_l), v-u_l) = \lim_{l\to\infty} \sigma(\mathcal{A}(u_l), v-u_l) \\ = \limsup_{n\to\infty} \sigma(\mathcal{A}(u_n), v-u_n) \le \sigma(\mathcal{A}(u), v-u). \tag{3.4}$$

From (H_5) and Proposition 2.9 we obtain

$$G(u, v-u) \subseteq \operatorname*{Liminf}_{l\to\infty} G(u_l, v-u_l). \tag{3.5}$$

Let $F_l = F(\,\cdot\,, Tu_l(\,\cdot\,), Tv(\,\cdot\,) - Tu_l(\,\cdot\,))$. From (H_1), F_l is measurable, for any l.

The function $x \in \Omega \mapsto \sup_l \operatorname{dist}(0, F_l(x))$ is integrable. Indeed, from (H$_2$) and relation (3.2) we have

$$\begin{aligned}\operatorname{dist}(0, F_l(x)) &\le (h_1(x) + h_2(x)|Tu_l(x)|^{p-1})|Tv(x) - Tu_l(x)| \\ &\le (h_1(x) + h_2(x)\cdot[g(x)]^{p-1})(|Tv(x)| + g(x)) \qquad \text{a.e. } x \in \Omega.\end{aligned}$$

Let $h(x) = (h_1(x) + h_2(x)\cdot[g(x)]^{p-1})(|Tv(x)| + g(x))$. From Hölder's inequality and from the conditions for h_1 and h_2 it follows that $h \in L^1(\Omega, \mathbb{R})$. Therefore, the function $x \in \Omega \mapsto \sup_l \operatorname{dist}(0, F_l(x))$ is integrable. Applying the Lebesque dominated convergence theorem for set-valued maps (see [1, p. 331]), one has

$$\begin{aligned}\text{(3.6)} \qquad &\int_\Omega \operatorname*{Liminf}_{l\to\infty} F(x, Tu_l(x), Tv(x) - Tu_l(x))\,dx \\ &\qquad\qquad \subseteq \operatorname*{Liminf}_{l\to\infty} \int_\Omega F(x, Tu_l(x), Tv(x) - Tu_l(x))\,dx.\end{aligned}$$

Of course, the first integrand is measurable (see [1, p. 312]). Using the hypothesis (H$_6$) (therefore Proposition 2.9) and (3.2), one has

$$F(x, Tu(x), Tv(x) - Tu(x)) \subseteq \operatorname*{Liminf}_{l\to\infty} F(x, Tu_l(x), Tv(x) - Tu_l(x))$$

a.e. $x \in \Omega$. From Lemma 2.2(a) and (3.6), we obtain

$$\begin{aligned}\text{(3.7)} \qquad &\int_\Omega F(x, Tu(x), Tv(x) - Tu(x))\,dx \\ &\qquad\qquad \subseteq \operatorname*{Liminf}_{l\to\infty} \int_\Omega F(x, Tu_l(x), Tv(x) - Tu_l(x))\,dx.\end{aligned}$$

Therefore, from (3.4), (3.5), (3.7) and (3.3) we obtain

$$\sigma(\mathcal{A}(u), v-u) + G(u, v-u) + \int_\Omega F(x, Tu(x), Tv(x) - Tu(x))\,dx \subseteq \mathbb{R}_+,$$

i.e. $u \in S_v$.

Finally, we prove that $S\colon K \rightsquigarrow K$ is a KKM-map. To this end, let $\{v_1, \dots, v_n\}$ be an arbitrary finite subset of K. We prove that $\operatorname{co}\{v_1, \dots, v_n\} \subseteq \bigcup_{i=1}^n S_{v_i}$. Supposing the contrary, there exist $\lambda_i \ge 0$ ($i \in \{1, \dots, n\}$) such that $\sum_{i=1}^n \lambda_i = 1$ and $\overline{v} = \sum_{i=1}^n \lambda_i v_i \notin S_{v_i}$ for all $i \in \{1, \dots, n\}$. The above relations mean that for all $i \in \{1, \dots, n\}$

$$\biggl[\sigma(\mathcal{A}(\overline{v}), v_i - \overline{v}) + G(\overline{v}, v_i - \overline{v}) + \int_\Omega F(x, T\overline{v}(x), Tv_i(x) - T\overline{v}(x))\,dx\biggr] \cap \mathbb{R}_-^* \ne \emptyset.$$

(Here, $\mathbb{R}_-^* = \,]-\infty, 0[$.) Let $I = \{i \in \{1, \dots, n\} : \lambda_i > 0\}$. From the above we obtain

$$\begin{aligned}\emptyset \ne \biggl\{ \sum_{i\in I} \lambda_i \biggl[&\sigma(\mathcal{A}(\overline{v}), v_i - \overline{v}) + G(\overline{v}, v_i - \overline{v}) \\ &+ \int_\Omega F(x, T\overline{v}(x), Tv_i(x) - T\overline{v}(x))\,dx\biggr]\biggr\} \cap \mathbb{R}_-^*.\end{aligned}$$

Using the sublinearity of the function $h \in X \mapsto \sigma(\mathcal{A}(\overline{v}), h)$, ($\mathrm{H}_3$), Lemma 2.2, the linearity of T and (H_4), we obtain

$$\begin{aligned}
\emptyset \neq \bigg\{ & \sigma\bigg(\mathcal{A}(\overline{v}), \sum_{i\in I}\lambda_i v_i - \sum_{i\in I}\lambda_i \overline{v}\bigg) + \sum_{i\in I}\lambda_i G(\overline{v}, v_i - \overline{v}) \\
& + \sum_{i\in I}\lambda_i \int_\Omega F(x, T\overline{v}(x), Tv_i(x) - T\overline{v}(x))\,dx\bigg\} \cap \mathbb{R}^*_- \\
\subseteq \bigg\{ & \sigma(\mathcal{A}(\overline{v}), 0) + G\bigg(\overline{v}, \sum_{i\in I}\lambda_i v_i - \sum_{i\in I}\lambda_i \overline{v}\bigg) \\
& + \int_\Omega \overline{\sum_{i\in I}\lambda_i F(x, T\overline{v}(x), Tv_i(x) - T\overline{v}(x))}\,dx\bigg\} \cap \mathbb{R}^*_- \\
\subseteq \bigg\{ & G(\overline{v}, 0) + \int_\Omega F\bigg(x, T\overline{v}(x), \sum_{i\in I}\lambda_i Tv_i(x) - \sum_{i\in I}\lambda_i T\overline{v}(x)\bigg)\,dx\bigg\} \cap \mathbb{R}^*_- \\
= \bigg\{ & G(\overline{v}, 0) + \int_\Omega F(x, T\overline{v}(x), 0)\,dx\bigg\} \cap \mathbb{R}^*_- \subseteq \bigg\{\mathbb{R}_+ + \int_\Omega \mathbb{R}_+\,dx\bigg\} \cap \mathbb{R}^*_- = \emptyset,
\end{aligned}$$

contradiction. This means that S is a KKM-map. Since K is compact, applying Lemma 2.13, we obtain $\bigcap_{v\in K} S_v \neq \emptyset$, i.e. (HV$\subseteq$) has at least a solution. □

When K is not compact, we can state the following result, using a coercivity assumption.

Theorem 3.3. *Let K be a nonempty closed, convex subset of a Banach space X. Let $\mathcal{A}$, G and F be as in Theorem* 3.2. *In addition, suppose that there exists a compact subset K_0 of K and an element $w_0 \in K_0$ such that*

$$\begin{aligned}
\text{(3.8)} \quad \bigg\{ & \sigma(\mathcal{A}(u), w_0 - u) + \int_\Omega F(x, Tu(x), Tw_0(x) - Tu(x))\,dx \\
& + G(u, w_0 - u)\bigg\} \cap \mathbb{R}^*_- \neq \emptyset
\end{aligned}$$

for all $u \in K \setminus K_0$. Then (HV$\subseteq$) *has at least a solution.*

Proof. We define the map S as in Theorem 3.2. Clearly, S is a KKM-map and S_v is closed for all $v \in K$, as seen above. Moreover, $S_{w_0} \subseteq K_0$. Indeed, supposing the contrary, there exists an element $u \in S_{w_0} \subseteq K$ such that $u \notin K_0$. But this contradicts (3.8). Since K_0 is compact, the set S_{w_0} is also compact. Applying again Lemma 2.13, we obtain a solution for (HV$\subseteq$). □

4. Consequences

First, we obtain a result of Browder concerning variational inequalities (see [3, Theorem 6]).

Corollary 4.1. *Let K be a nonempty compact convex subset of a Banach space X, and let $\mathcal{A}: K \rightsquigarrow X^*$ be an upper demicontinuous set-valued map with bounded values. Then there exists $\overline{u} \in K$ such that*

$$\sigma(\mathcal{A}(\overline{u}), v - \overline{u}) \geq 0 \text{ for all } v \in K.$$

Proof. Choose $F \equiv 0$ and $G \equiv 0$ in Theorem 3.2. □

In particular, Corollary 4.1 reduces to a classical result of Hartman and Stampacchia [7] if $\mathcal{A}$ is a single-valued continuous operator and X is of finite dimension.

Now, we give a solution for the hemivariational inequality treated by Panagiotopoulos, Fundo and Rădulescu (see [10]). Before to do this, we recall two elementary facts.

Lemma 4.2. *Let K be a nonempty subset of a normed space X, and let $j: K \to \mathbb{R}$ be a function. Define $J: K \rightsquigarrow \mathbb{R}$ by $J(u) = [j(u), \infty)$ for all $u \in K$. If j is upper semicontinuous on K, then J is lower semicontinuous on K.*

Lemma 4.3. *If $h: \Omega \to \mathbb{R}$ is a measurable function, then $H: \Omega \rightsquigarrow \mathbb{R}$ defined by $H(x) = [h(x), \infty)$ for all $x \in \Omega$, is also measurable (as set-valued map).*

Let Ω, X, K and T be as in the Introduction, let $A: K \to X^*$ be an operator, and we suppose that $j: \Omega \times \mathbb{R}^k \to \mathbb{R}$ is a Carathéodory function which is locally Lipschitz continuous with respect to the second variable and which satisfies the following assumption:

(j) there exist h_1 and h_2 as in (H_2) such that

$$|w| \leq h_1(x) + h_2(x)|y|^{p-1}$$

for a.e. $x \in \Omega$, every $y \in \mathbb{R}^k$ and $w \in \partial j(x, y)$.

Here $\partial j(x, y)$ is the *Clarke generalized gradient* of j, i.e.

$$\partial j(x, y) = \{w \in \mathbb{R}^k : \langle w, z\rangle \leq j_y^0(x, y; z) \text{ for all } z \in \mathbb{R}^k\},$$

where $j_y^0(x, y; z)$ is the (partial) *generalized directional derivative* of the locally Lipschitz continuous function $j(x, \cdot\,)$ at the point $y \in \mathbb{R}^k$ with respect to the direction $z \in \mathbb{R}^k$, where $x \in \Omega$, that is

$$j_y^0(x, y; z) = \limsup_{\substack{y' \to y \\ t \to 0^+}} \frac{j(x, y' + tz) - j(x, y')}{t}.$$

We consider the following hemivariational inequality problem:

(P) Find $\overline{u} \in K$ such that

$$\langle A\overline{u}, v - \overline{u}\rangle + \int_\Omega j_y^0(x, T\overline{u}(x); Tv(x) - T\overline{u}(x))\, dx \geq 0 \quad \text{for all } v \in K.$$

Corollary 4.4 (see [10]). *Let K be a nonempty compact convex subset of a Banach space X, and let $j:\Omega\times\mathbb{R}^k\to\mathbb{R}$ satisfying the condition* (j). *If the operator $A:K\to X^*$ is w^*-demicontinuous, then* (P) *has at least a solution.*

Proof. We choose $\mathcal{A}(u)=\{A(u)\}$ for all $u\in K$, $G\equiv 0$ and $F:\Omega\times\mathbb{R}^k\times\mathbb{R}^k\rightsquigarrow\mathbb{R}$ as $F(x,y,z)=[j_y^0(x,y;z),\infty)$ for all $(x,y,z)\in\Omega\times\mathbb{R}^k\times\mathbb{R}^k$. Due to Remark 2.4, the operator $\mathcal{A}$ is upper demicontinuous (with bounded values). We will verify the hypotheses from Theorem 3.2 for F.

(H_1) Using the linearity of T and the measurability of

$$x\in\Omega\mapsto j_y^0(x,Tu(x);Tv(x)-Tu(x))$$

for all $u,v\in K$ (see [8, p. 15]), from Lemma 4.3 we obtain that $x\in\Omega\rightsquigarrow F(x,Tu(x),Tv(x)-Tu(x))$ is measurable.

(H_2) Since $j_y^0(x,y;z)=\max\{\langle w,z\rangle : w\in\partial j(x,y)\}=\langle w_0,z\rangle$, for some $w_0\in\partial j(x,y)$ (using (j)) we have

$$|j_y^0(x,y;z)|\le|w_0|\cdot|z|\le(h_1(x)+h_2(x)|y|^{p-1})|z|.$$

Since $\operatorname{dist}(0,F(x,y,z))\le|j_y^0(x,y;z)|$, we obtain the desired relation.

(H_3) Since $z\in\mathbb{R}^k\mapsto j_y^0(x,y;z)$ is convex (see [4, p. 25]) we obtain that $z\in\mathbb{R}^k\rightsquigarrow F(x,y,z)$ is convex for all $x\in\Omega$ and all $y\in\mathbb{R}^k$.

(H_4) Since $j_y^0(x,y;0)=0$, we have $F(x,y,0)=\mathbb{R}_+$ for all $x\in\Omega$ and all $y\in\mathbb{R}^k$.

(H_6) Since $(y,z)\in\mathbb{R}^k\times\mathbb{R}^k\mapsto j_y^0(x,y;z)$ is upper semicontinuous (see [4, p. 25]), and using Lemma 4.2 we obtain that $(y,z)\in\mathbb{R}^k\times\mathbb{R}^k\rightsquigarrow F(x,y,z)$ is lower semicontinuous for all $x\in\Omega$.

Therefore, from Theorem 3.2 we have a solution $\overline{u}\in K$ such that

$$\langle A\overline{u},v-\overline{u}\rangle+\int_\Omega F(x,T\overline{u}(x),Tv(x)-T\overline{u}(x))dx\subseteq\mathbb{R}_+\quad\text{for all } v\in K.$$

In particular, for the "lower" selection of $F(\cdot,T\overline{u}(\cdot),Tv(\cdot)-T\overline{u}(\cdot))$, i.e. for $j_y^0(\cdot,T\overline{u}(\cdot);Tv(\cdot)-T\overline{u}(\cdot))$, which is integrable due to (j), we have

$$\langle A\overline{u},v-\overline{u}\rangle+\int_\Omega j_y^0(x,T\overline{u}(x);Tv(x)-T\overline{u}(x))\,dx\ge 0\quad\text{for all } v\in K,$$

i.e. $\overline{u}$ is a solution for (P). □

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