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GLOBAL SPECIAL REGULAR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN A CYLINDRICAL DOMAIN WITHOUT THE AXIS OF SYMMETRY

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Abstract. Global existence of regular solutions to the Navier–Stokes equations in a bounded cylindrical domain without the axis of symmetry and with boundary slip conditions is proved. We showed the existence of solutions without restrictions on the magnitude of the initial velocity assuming only that the L_2-norms of the angular derivative of the cylindrical components of the initial velocity and the external force are sufficiently small. To prove global existence some decay estimates on the external force are imposed.

1. Introduction

We consider a motion of a viscous incompressible fluid described by the Navier–Stokes equations in a bounded cylinder without the axis of symmetry

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under ideal boundary slip conditions (see [4]):

$$
\begin{aligned}
v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) &= f && \text{in } \Omega^T = \Omega\times(0,T),\\
\operatorname{div} v &= 0 && \text{in } \Omega^T,\\
v\cdot\overline{n} &= 0 && \text{on } S^T = S\times(0,T),\\
\overline{n}\cdot\mathbb{T}(v,p)\cdot\overline{\tau}_\alpha &= 0, && \alpha=1,2, \text{ on } S^T,\\
v|_{t=0} &= v(0) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t)) \in \mathbb{R}^3$ is the velocity vector, $p = p(x,t) \in \mathbb{R}$ the pressure, $f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ the external force field, $\overline{n}$ the unit outward vector normal to $S = \partial\Omega$, $\overline{\tau}_\alpha$, $\alpha = 1,2$, are tangent vectors to S.

By $\mathbb{T}(v,p)$ we denote the stress tensor of the form

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - pI,$$

where ν is the constant viscosity coefficient, $\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}$ the dilatation tensor and I the unit matrix.

To describe domain Ω and the considered motion we introduce the cylindrical coordinates r, φ, z by the relations $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$, where x_1, x_2, x_3 are the Cartesian coordinates.

We assume that $\Omega = \{x\in\mathbb{R}^3 : 0 < R_1 < r < R_2,\ -a < z < a,\ \varphi\in[0,2\pi]\}$. Then $S = S_1\cup S_2$, where $S_1 = \{x\in\mathbb{R}^3 : r \text{ is either } R_1 \text{ or } R_2,\ -a<z<a,\ \varphi\in[0,2\pi]\}$ and $S_2 = \{x\in\mathbb{R}^3 : z \text{ is either } -a \text{ or } a,\ R_1<r<R_2 \text{ and } \varphi\in[0,2\pi]\}$.

Let u be any vector. We introduce the cylindrical coordinates of u in the following way: $u_r = u\cdot\overline{e}_r$, $u_\varphi = u\cdot\overline{e}_\varphi$, $u_z = u\cdot\overline{e}_z$ where $\overline{e}_r = (\cos\varphi, \sin\varphi, 0)$, $\overline{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$, $\overline{e}_z = (0,0,1)$ and dot denotes the scalar product in $\mathbb{R}^3$.

The above implies that Ω is a cylinder without the axis of symmetry. We cutted of the axis of symmetry to simplify considerations. Our aim is to prove global existence of solutions which are close to the axially symmetric solutions (see the definition below). For this purpose we simplify considerations as much as possible. Otherwise we should use weighted Sobolev spaces and repeat some considerations from [4]. This needs a lot of additional considerations connected with techniques of weighted spaces.

Definition 1.1. By an *axially symmetric solution* to (1.1) we mean such solution that the cylindrical components of v, f, $v(0)$ and p do not depend on φ.

Following [4] we distinguish the quantities:

$$
\begin{aligned}
h &= h_r\overline{e}_r + h_\varphi\overline{e}_\varphi + h_z\overline{e}_z, \quad & h_r &= v_{r,\varphi}, \quad & h_\varphi &= v_{\varphi,\varphi}, \quad & h_z &= v_{z,\varphi},\\
q &= p_{,\varphi}, & w &= v_\varphi, & \alpha &= \operatorname{rot} v, & \chi &= \alpha_\varphi.
\end{aligned}
$$

To show global existence we need some additional problems which help us to obtain an global estimate. First we have the problem for h and q

$$
\begin{aligned}
h_{,t} - \operatorname{div} \mathbb{D}(h) + \nabla q &= -v \cdot \nabla h - h \cdot \nabla v + g && \text{in } \Omega^T, \\
\operatorname{div} h &= 0 && \text{in } \Omega^t, \\
h \cdot \overline{n} &= 0 && \text{on } S^T, \\
\overline{n} \cdot \mathbb{D}(h) \cdot \overline{\tau}_\alpha &= 0, && \alpha = 1,2, \text{ on } S^T, \\
h|_{t=0} &= h(0) && \text{in } \Omega,
\end{aligned}
\tag{1.2}
$$

where $g = f_{r,\varphi}\overline{e}_r + f_{\varphi,\varphi}\overline{e}_\varphi + f_{z,\varphi}\overline{e}_z$.

The cylindrical components of vorticity have the form

$$
\alpha_r = \frac{1}{r}(h_z - rw_{,z}), \quad \alpha_\varphi = v_{r,z} - v_{z,r} \equiv \chi, \quad \alpha_z = \frac{1}{r}[(rw)_{,r} - h_r].
$$

Next we get the following problem for χ

$$
\begin{aligned}
&\chi_{,t} + v \cdot \nabla \chi + (v_{r,r} + v_{z,z})\chi - \nu\Delta\chi + \frac{\nu\chi}{r^2} \\
&\quad = \frac{2\nu}{r^2}\left(- h_{\varphi,z} + \frac{1}{r}h_{z,\varphi}\right) - \frac{1}{r}\left(w_{,z}h_r - w_{,r}h_z + \frac{w}{r}h_z\right) \\
&\quad\quad + \frac{2}{r}ww_{,z} + F_\varphi && \text{in } \Omega^T, \\
&\chi = 0 && \text{on } S^T, \\
&\chi|_{t=0} = \chi(0) && \text{in } \Omega,
\end{aligned}
\tag{1.3}
$$

where $F = \operatorname{rot} f$. Moreover, we have also the problem for w

$$
\begin{aligned}
w_{,t} + v \cdot \nabla w + \frac{v_r}{r}w - \nu\Delta w + \nu\frac{w}{r^2} &= \frac{1}{r}q + \frac{2\nu}{r^2}h_r + f_\varphi && \text{in } \Omega^T, \\
w_{,r}|_{r=R_i} &= \frac{1}{R_i}w && i = 1,2, \text{ on } S_1^T, \\
w_{,z} &= 0 && \text{on } S_2^T, \\
w|_{t=0} &= w(0) && \text{in } \Omega.
\end{aligned}
\tag{1.4}
$$

Finally we need the following elliptic problem for $v' = (v_r, v_z)$

$$
\begin{aligned}
v_{r,z} - v_{z,r} &= \chi && \text{in } \Omega, \\
v_{r,r} + v_{z,z} &= -\frac{1}{r}(h_\varphi + v_r) && \text{in } \Omega, \\
v_r|_{S_1} &= 0, \quad v_z|_{S_2} = 0.
\end{aligned}
$$

The aim of this paper is to prove existence of global regular solutions to problem (1.1). Since we are looking for global regular solutions to (1.1) with large velocity we however need some smallness assumptions. In this paper we assume that $h(0)$ and g are sufficiently small in corresponding norms.

The paper is divided into the following steps.

In Section 3 there are found estimates necessary for Section 4, which base on the estimate for a weak solution (see Lemma 3.1). Therefore we examine regularity of weak solutions. In Section 4 we prove existence and uniqueness of local solutions with large existence time. The proof follows from the Leray–Schauder fixed point theorem and utilizes some ideas from [5]. The results are formulated in Theorems 1 and 2, respectively. In Section 5 we prove global existence by prolonging the local solution step by step. The result is presented in Theorem 3. To prove global existence we needed some decay estimates for the external force.

We have to underline that global existence is possible thanks to the energy estimate for χ (see Lemma 3.2), where the idea of the proof is taken from [2], [3] and assumptions on smallness of $h(0)$ and g which imply estimate (4.7) (see Lemma 4.2).

In this paper the motion with large angular component of velocity is considered. This fact implies serious difficulties to get the crucial estimate for χ in a neighbourhood of the axis of symmetry. We must underline that in a neighbourhood of the axis of symmetry the axially symmetric solution (see Definition 1.1) behaves as 3-dimensional (see [6]), so we are not able to obtain any global in time estimate for large v_φ. This implies that methods from [4] must be utilized (see also [6]). This is the main reason why in this paper a cylinder without the axis of symmetry is considered.

Now we formulate the main results of this paper. Let us introduce the quantities

$$\begin{aligned} F(T) &= \|g\|_{L_2(\Omega^T)} + \|f_\varphi\|_{L_{4/3}(0,T;L_4(\Omega))} + \|F_\varphi\|_{L_2(\Omega^T)} + \|f\|_{L_{5/2}(\Omega^T)}, \\ F_0 &= \|h(0)\|_{L_2(\Omega)} + \|w(0)\|_{H^1(\Omega)} + \|\chi(0)\|_{L_2(\Omega)} + \|v(0)\|_{W^{6/5}_{5/2}(\Omega)}, \\ \gamma(T) &= \|h\|_{W^{2+\beta,1+\beta/2}_\delta(\Omega^T)} + \|q\|_{G^{\beta,\beta/2}_\delta(\Omega^T)}, \\ k(T) &= \|g\|_{L_2(\Omega^T)} + \|h(0)\|_{L_2(\Omega)}, \end{aligned}$$

where notation is introduced in Section 2.

Theorem 1. *Assume that v is a weak solution to problem* (1.1). *Assume that $v(0) \in W^{6/5}_{5/2}(\Omega)$, $h(0) \in W^{2+\beta-2/\delta}_\delta(\Omega)$, $w(0) \in H^1(\Omega)$, $\chi(0) \in L_2(\Omega)$, $f \in L_{5/2}(\Omega^T)$, $g \in W^{\beta,\beta/2}_\delta(\Omega^T)$, $f_\varphi \in L_4(0,T;L_{4/3}(\Omega))$, $F_\varphi \in L_2(\Omega^T)$, $\beta < 1$, $2/\delta < \beta + 1/2$, $5/\delta < 3 + \beta$, $\delta \in (1,2)$. Assume that $A > 0$ is such that*

$$G(T,0,F(T),F_0)k(T) + c(\|g\|_{W^{\beta,\beta/2}_\delta(\Omega^T)} + \|h(0)\|_{W^{2+\beta-2/\delta}_\delta(\Omega)}) < A,$$

where G is a nonlinear positive increasing function of its arguments determined by (4.9). *Assume that $k(T)$ is so small that*

$$G(T,A,F(T),F_0)k(T) + c(\|g\|_{W^{\beta,\beta/2}_\delta(\Omega^T)} + \|h(0)\|_{W^{2+\beta-2/\delta}_\delta(\Omega)}) \le A. \tag{1.5}$$

Then there exists a solution to problem (1.2) *such that* $h \in W_{\delta}^{2+\beta,1+\beta/2}(\Omega^T)$, $q \in G_{\delta}^{\beta,\beta/2}(\Omega^T)$ (*defined in* (4.5)) *and* $\gamma(T) \le A$.

Hence the weak solution to problem (1.1) is such that $v \in W_{5/2}^{2,1}(\Omega^T)$, $\nabla p \in L_{5/2}(\Omega^T)$ and

$$\|v\|_{W_{5/2}^{2,1}(\Omega^T)} + \|\nabla p\|_{L_{5/2}(\Omega^T)} \le \varphi(A, F(T), F_0, T), \tag{1.6}$$

where φ is an increasing positive function of its arguments. Moreover (1.5) implies that T and $k(T)$ are inversely proportional.

Theorem 2. *Let solutions of* (1.1) *be such that* $v \in L_1(0,T; W_\infty^1(\Omega))$. *Then they are unique.*

Theorem 3. *Let the assumptions of Theorem* 1 *be satisfied. Let* $v(0) \in W_3^{4/3}(\Omega)$. *Let the decay estimates* (5.1), (5.2) *hold. Assume that* T, *the time of local existence, is sufficiently large. Assume that* $f(0) \in L_3(\Omega)$, $g(0) \in H^1(\Omega)$, $g_{,t}(0) \in L_2(\Omega)$, $F_\varphi(0) \in L_2(\Omega)$, $\|g\|_{L_\delta(\Omega; W_\delta^{\beta/2}(0,\infty))} < \infty$. *Assume that*

$$\|h(0)\|_{1,\Omega} + |g(0)|_{2,\Omega}$$

is sufficiently small. Then the local solution determined by Theorem 1 *can be prolonged infinitely.*

Hence there exists a global solution to problem (1.1).

2. Notation and auxiliary results

To simplify considerations we introduce

$$\begin{aligned}
|u|_{p,Q} &= \|u\|_{L_p(Q)}, && Q \in \{\Omega, S, \Omega^T, S^T\},\ p \in [1,\infty],\\
\|u\|_{s,Q} &= \|u\|_{H^s(Q)}, && Q \in \{\Omega, S\},\ 0 \le s \in \mathbb{R},\\
\|u\|_{s,Q} &= \|u\|_{W_2^{s,s/2}(Q)}, && Q \in \{\Omega^T, S^T\},\ 0 \le s \in \mathbb{R},\\
|u|_{p,q,Q^T} &= \|u\|_{L_q(0,T;L_p(Q))}, && Q \in \{\Omega, S\},\ 1 \le p,q \le \infty,\\
|u|_{1,0,\Omega} &= (\|u\|_{1,\Omega}^2 + |u_{,t}|_{2,\Omega}^2)^{1/2}.
\end{aligned}$$

To consider spaces with fractional derivatives it is convenient to introduce the notation

$$\begin{aligned}
\langle\!\langle u\rangle\!\rangle_{\alpha,p,\Omega} &= \bigg(\int_\Omega\int_\Omega \frac{|u(x)-u(x')|^p}{|x-x'|^{3+\alpha p}}\,dx\,dx'\bigg)^{1/p},\\
\langle\!\langle u\rangle\!\rangle_{\alpha,p,\Omega^T,x} &= \bigg(\int_0^T \langle\!\langle u\rangle\!\rangle_{\alpha,p,\Omega}^p dt\bigg)^{1/p},\\
\langle\!\langle u\rangle\!\rangle_{\alpha,p,(0,T)} &= \bigg(\int_0^T\int_0^T \frac{|u(t)-u(t')|^p}{|t-t'|^{1+\alpha p}}\,dt\,dt'\bigg)^{1/p},
\end{aligned}$$

$$\langle\!\langle u\rangle\!\rangle_{\alpha,p,\Omega^T,t} = \bigg(\int_\Omega \langle\!\langle u\rangle\!\rangle^p_{\alpha,p,(0,T)}\,dx\bigg)^{1/p},$$

where $\alpha\in(0,1)$, $p\in(1,\infty)$, $\Omega\subset\mathbb{R}^3$. Moreover,

$$\|u\|_{W_p^{\alpha,\alpha/2}(\Omega^T)} = \langle\!\langle u\rangle\!\rangle_{\alpha,p,\Omega^T,x} + \langle\!\langle u\rangle\!\rangle_{\alpha/2,p,\Omega^T,t} + |u|_{p,\Omega^T}.$$

The space $W_p^{2k+\alpha,k+\alpha/2}(\Omega^T)$, $k\in\mathbb{N}$, $\alpha\in(0,1)$ has the norm

$$\|u\|_{W_p^{2k+\alpha,k+\alpha/2}(\Omega^T)} = \sum_{\beta+2b\le 2k}\|D_x^\beta\partial_t^b u\|_{L_p(\Omega^T)} + \|\partial_x^{2k}u\|_{W_p^{\alpha,\alpha/2}(\Omega^T)} + \|\partial_t^k u\|_{W_p^{\alpha,\alpha/2}(\Omega^T)}.$$

By c we denote a generic constant which changes its magnitude from formula to formula. By $c(\sigma)$, $c_k(\sigma)$, $k\in\mathbb{N}$, $\varphi(\sigma)$, we understand generic functions which are always positive and increasing.

From [1] we recall the result

Lemma 2.1. *Assume $l,k,j\in\mathbb{N}$, $\alpha,\beta\in(0,1)$, $p,q_1,q_2\in(1,\infty)$, $\Omega\subset\mathbb{R}^3$. Then we have*

$$\nabla^k W_p^{l+\alpha,l/2+\alpha/2}(\Omega^T)\subset L_{q_2}(0,T;L_{q_1}(\Omega))$$

if $5/p-3/q_1-2/q_2+k\le l+\alpha$. Moreover, for $u\in W_p^{l+\alpha,l/2+\alpha/2}(\Omega^T)$ we have the interpolation inequality

$$|\nabla^k u|_{q_1,q_2,\Omega^T}\le\varepsilon^{1-\varkappa}\|u\|_{l+\alpha,p,\Omega^T}+c\varepsilon^{-\varkappa}|u|_{p,\Omega^T},$$

where

$$\varkappa=\frac{5/p-3/q_1-2/q_2+k}{l+\alpha}<1.$$

The following imbedding holds

$$\nabla^k W_p^{l+\alpha,l/2+\alpha/2}(\Omega^T)\subset W_q^{j+\beta,j/2+\beta/2}(\Omega^T)$$

if $5/p-5/q+k+j+\beta\le l+\alpha$.

Let $u\in W_p^{l+\alpha,l/2+\alpha/2}(\Omega^T)$. Then the interpolation inequality is valid

$$\|\nabla^k u\|_{j+\beta,q,\Omega^T}\le\varepsilon^{1-\varkappa_1}\|u\|_{l+\alpha,p,\Omega^T}+c\varepsilon^{-\varkappa_1}|u|_{p,\Omega^T},$$

where

$$\varkappa_1=\frac{5/p-5/q+k+j+\beta}{l+\alpha}<1.$$

Lemma 2.2. *Assume that*

$$f\in W_{\delta_1}^{\beta+\varepsilon/\delta,\beta/2+\varepsilon/(2\delta)}(\Omega^T)\cap L_{\delta_2'}(\Omega^T),\quad g\in W_{\delta_2}^{\beta+\varepsilon/\delta,\beta/2+\varepsilon/2\delta}(\Omega^T)\cap L_{\delta_1'}(\Omega^T),$$

where $\beta\in(0,1)$, $\delta\in(1,\infty)$, ε-arbitrary small positive number, $1/\delta_1+1/\delta_2=1/\delta$, $1/\delta_1'+1/\delta_2'=1/\delta$. Then

$$\|fg\|_{\beta,\delta,\Omega^T}\le c(\|f\|_{\beta+\varepsilon/\delta,\delta_1,\Omega^T}|g|_{\delta_2,\Omega^t}+\|g\|_{\beta+\varepsilon/\delta,\delta_1',\Omega^T}|f|_{\delta_2',\Omega^T}).\tag{2.1}$$

Proof. To prove the lemma it is enough to examine the highest seminorms. First we consider

$$\begin{aligned}\langle\!\langle fg\rangle\!\rangle_{\beta,\delta,\Omega^T,x} &= \bigg(\int_0^T\int_\Omega\int_\Omega \frac{|f(x,t)g(x,t)-f(x',t)g(x',t)|^\delta}{|x-x|^{3+\beta\delta}}\,dx\,dx'\,dt\bigg)^{1/\delta}\\ &\le \bigg(\int_0^T\int_\Omega\int_\Omega \frac{|f(x,t)-f(x',t)|^{\delta\lambda_1}}{|x-x'|^{3\mu_1\lambda_1+\delta\beta\lambda_1}}\,dx\,dx'\,dt\bigg)^{1/(\delta\lambda_1)}\\ &\quad\cdot\bigg(\int_0^T\int_\Omega\int_\Omega \frac{|g(x,t)|^{\delta\lambda_2}}{|x-x'|^{3\mu_2\lambda_2}}\,dx\,dx'\,dt\bigg)^{1/(\delta\lambda_2)}\\ &\quad+\bigg(\int_0^T\int_\Omega\int_\Omega \frac{|g(x,t)-g(x',t)|^{\delta\lambda_1'}}{|x-x'|^{3\mu_1'\lambda_1'+\delta\beta\lambda_1'}}\,dx\,dx'\,dt\bigg)^{1/(\delta\lambda_1')}\\ &\quad\cdot\bigg(\int_0^T\int_\Omega\int_\Omega \frac{|f(x',t)|^{\delta\lambda_2'}}{|x-x'|^{3\mu_2'\lambda_2'}}\,dx\,dx'\,dt\bigg)^{1/(\delta\lambda_2')}\equiv I,\end{aligned}$$

where $1/\lambda_1+1/\lambda_2=1$, $1/\lambda_1'+1/\lambda_2'=1$, $\mu_1+\mu_2=1$, $\mu_1'+\mu_2'=1$, and λ_i, λ_i', μ_i, μ_i', $i=1,2$, are positive.

Asumming that $\mu_2\lambda_2<1$, $\mu_2'\lambda_2'<1$ there exists $\varepsilon>0$ such that $\mu_2=1/\lambda_2-\varepsilon/3$, $\mu_2'=1/\lambda_2'-\varepsilon/3$, so $3\mu_1\lambda_1+\delta\beta\lambda_1=3+\delta\lambda_1(\beta+\varepsilon/\delta)$ and $3\mu_1'\lambda_1'+\delta\beta\lambda_1'=3+\delta\lambda_1'(\beta+\varepsilon/\delta)$ and

$$I\le\langle\!\langle f\rangle\!\rangle_{\beta+\varepsilon/\delta,\delta\lambda_1,\Omega^T,x}|g|_{\delta\lambda_2,\Omega^T}+\langle\!\langle g\rangle\!\rangle_{\beta+\varepsilon/\delta,\delta\lambda_1',\Omega^T,x}|f|_{\delta,\lambda_2',\Omega^T}.$$

Next we examine

$$\begin{aligned}\langle\!\langle fg\rangle\!\rangle_{\beta/2,\delta,\Omega^T,t} &= \bigg(\int_\Omega\int_0^T\int_0^T \frac{|f(x,t)g(x,t)-f(x,t')g(x,t')|^\delta}{|t-t'|^{1+\beta\delta/2}}\,dx\,dt\,dt'\bigg)^{1/\delta}\\ &\le \bigg(\int_\Omega\int_0^T\int_0^T \frac{|f(x,t)-f(x,t')|^{\delta\lambda_1}}{|t-t'|^{\mu_1\lambda_1+\beta\delta\lambda_1/2}}\,dx\,dt\,dt'\bigg)^{1/(\delta\lambda_1)}\\ &\quad\cdot\bigg(\int_\Omega\int_0^T\int_0^T \frac{|g(x,t)|^{\delta\lambda_2}}{|t-t'|^{\mu_2\lambda_2}}\,dx\,dt\,dt'\bigg)^{1/(\delta\lambda_2)}\\ &\quad+\bigg(\int_\Omega\int_0^T\int_0^T \frac{|g(x,t)-g(x,t')|^{\delta\lambda_1'}}{|t-t'|^{\mu_1'\lambda_1'+\beta\delta\lambda_1'/2}}\,dx\,dt\,dt'\bigg)^{1/(\delta\lambda_1')}\\ &\quad\cdot\bigg(\int_\Omega\int_0^T\int_0^T \frac{|f(x,t')|^{\delta\lambda_2'}}{|t-t'|^{\mu_2'\lambda_2'}}\,dx\,dt\,dt'\bigg)^{1/(\delta\lambda_2')}\equiv J,\end{aligned}$$

where μ_i, μ_i', λ_i, λ_i' for $i=1,2$, satisfy the same restrictions as before.

Now we take $\varepsilon>0$ such that $\mu_2=1/\lambda_1-\varepsilon/2$, $\mu_2'=1/\lambda_2'-\varepsilon/2$ are positive. Then

$$\mu_1\lambda_1+\frac{\beta}{2}\delta\lambda_1=1+\delta\lambda_1\bigg(\frac{\beta}{2}+\frac{\varepsilon}{2\delta}\bigg)\quad\text{and}\quad\mu_1'\lambda_1'+\frac{\beta}{2}\delta\lambda_1'=1+\delta\lambda_1'\bigg(\frac{\beta}{2}+\frac{\varepsilon}{2\delta}\bigg)$$

and

$$J\le\langle\!\langle f\rangle\!\rangle_{\beta/2+\varepsilon/(2\delta),\delta\lambda_1,\Omega^T,t}|g|_{\delta\lambda_2,\Omega^T}+\langle\!\langle g\rangle\!\rangle_{\beta/2+\varepsilon/(2\delta),\delta\lambda_1',\Omega^T,t}|f|_{\delta\lambda_2',\Omega^T}.$$

Adding the estimate for $|fg|_{\delta,\Omega^T}$ and putting $\delta_i = \delta\lambda_i$, $\delta_i' = \delta\lambda_i'$, $i = 1, 2$, we obtain (2.1). □

3. A priori estimates

We prove the existence of local solutions by the Leray–Schauder fixed point theorem. For this purpose we need some a priori estimates. From [5] we have

Lemma 3.1. *Assume that $v(0) \in L_2(\Omega)$, $f \in L_{2,1}(\Omega^T) \cap L_2(\Omega^T)$, $T < \infty$. Then there exist constants*

$$\begin{aligned} d_1(T) &= \int_0^T |f(t)|_{2,\Omega}\,dt + |v(0)|_{2,\Omega}, \\ d_2^2(T) &= \int_0^T |f(t)|_{2,\Omega}^2\,dt + \int_0^T |d_1(t)|^2\,dt \\ &\le 2(1+T)(|f|_{2,\Omega^T}^2 + |v(0)|_{2,\Omega}^2) \equiv \overline{d}_2^2(T), \end{aligned}$$

such that, for $t \le T$,

$$|v(t)|_{2,\Omega} \le d_1(T), \tag{3.1}$$

$$|v(t)|_{2,\Omega}^2 + \nu \int_0^t \|v(t')\|_{1,\Omega}^2\,dt' \le d_2^2(T). \tag{3.2}$$

Next we obtain an estimate for χ.

Lemma 3.2. *Let the assumptions of Lemma 3.1 hold. Let*

$$c_1 |h_\varphi|_{3/2,\Omega} \le \frac{1}{4}\nu, \tag{3.3}$$

where c_1 is the constant from the imbedding $H^1(\Omega) \subset L_6(\Omega)$. Let $F_\varphi \in L_2(\Omega^T)$, $\chi(0) \in L_2(\Omega)$ and

$$\nabla h \in L_2(0,T; L_{6/5}(\Omega)), \quad h \in L_\infty(0,T; L_3(\Omega)), \quad w \in L_4(\Omega^T). \tag{3.4}$$

Let us introduce the quantity

$$A_1^2(T) = \int_0^T |\nabla h(t)|_{6/5,\Omega}^2\,dt + \overline{d}_2^2(T) \sup_{t\in(0,T)} |h(t)|_{3,\Omega}^2 + \int_0^T |w(t)|_{4,\Omega}^4\,dt. \tag{3.5}$$

Then solutions of problem (1.3) *satisfy the inequality*

$$\left|\frac{\chi(t)}{r}\right|_{2,\Omega}^2 + \frac{\nu}{4}\int_0^t \left\|\frac{\chi(t')}{r}\right\|_{1,\Omega}^2 dt' \le c(A_1^2(t) + |F_\varphi|_{2,\Omega^t}^2) + \left|\frac{\chi(0)}{r}\right|_{2,\Omega}^2. \tag{3.6}$$

Proof. Multiplying $(1.3)_1$ by χ/r^2 and integrating the result over Ω yield

$$
(3.7)\quad \frac{1}{2}\frac{d}{dt}\left|\frac{\chi}{r}\right|_{2,\Omega}^2+\nu\left|\nabla\frac{\chi}{r}\right|_{2,\Omega}^2=\int_\Omega\frac{h_\varphi}{r}\frac{\chi^2}{r^2}\,dx
$$

$$
+2\nu\int_\Omega\frac{1}{r^2}\left(-h_{\varphi,z}+\frac{1}{r}h_{z,\varphi}\right)\frac{\chi}{r^2}\,dx-\int_\Omega\frac{1}{r}\left(w_{,z}h_r-w_{,r}h_z+\frac{w}{r}h_z\right)\frac{\chi}{r^2}dx
$$

$$
+2\int_\Omega\frac{1}{r}ww_{,z}\frac{\chi}{r^2}\,dx+\int_\Omega F_\varphi\frac{\chi}{r^2}\,dx.
$$

The first term on the r.h.s. we estimate by

$$
|h_\varphi|_{3/2,\Omega}\left|\frac{\chi}{r}\right|_{6,\Omega}^2\le c_1|h_\varphi|_{3/2,\Omega}\left|\nabla\frac{\chi}{r}\right|_{2,\Omega}^2,
$$

and the second by

$$
c\int_\Omega|\nabla h|\left|\frac{\chi}{r}\right|\,dx\le\varepsilon_1\left|\frac{\chi}{r}\right|_{6,\Omega}^2+c\left(\frac{1}{\varepsilon_1}\right)|\nabla h|_{6/5,\Omega}^2,
$$

where $\varepsilon_1\in(0,1)$

In view of the Hölder and Young invequalities the third term on the r.h.s. of (3.7) is bounded by

$$
\varepsilon_2\left|\frac{\chi}{r}\right|_{6,\Omega}^2+c\left(\frac{1}{\varepsilon_2}\right)\|w\|_{1,\Omega}^2|h|_{3,\Omega}^2.
$$

Finally the fourth term on the r.h.s. of (3.7) we examine in the way

$$
\int_\Omega\frac{1}{r}(w^2)_{,z}\frac{\chi}{r^2}\,dx=-\int_\Omega\frac{1}{r^2}w^2\left(\frac{\chi}{r}\right)_{,z}dx\equiv I,
$$

so

$$
|I|\le\varepsilon_3\int_\Omega\left(\frac{\chi}{r}\right)_{,z}^2dx+c\left(\frac{1}{\varepsilon_3}\right)\int_\Omega w^4\,dx.
$$

In view of the above considerations and assuming that ε_1, ε_2, ε_3 are sufficiently small (3.7) implies

$$
(3.8)\quad \frac{1}{2}\frac{d}{dt}\left|\frac{\chi}{r}\right|_{2,\Omega}^2+\frac{3}{4}\nu\left|\nabla\frac{\chi}{r}\right|_{2,\Omega}^2
$$

$$
\le c\left(|h_\varphi|_{3/2,\Omega}\left|\nabla\frac{\chi}{r}\right|_{2,\Omega}^2+|\nabla h|_{6/5,\Omega}^2+\|w\|_{1,\Omega}^2|h|_{3,\Omega}^2+|w|_{4,\Omega}^4+|F_\varphi|_{2,\Omega}^2\right).
$$

Integrating (3.8) with respect to time, using (3.3) and the Poincare inequality we obtain (3.6). This concludes the proof. □

To find estimates for $(3.4)_1$ we need

Lemma 3.3. *Let the assumptions of Lemma 3.1 hold and let $h \in L_\infty(0,T; L_3(\Omega))$, $g \in L_2(\Omega^T)$, $h(0) \in L_2(\Omega)$. Then*

$$(3.9)\quad |h(t)|_{2,\Omega}^2 + \nu\int_0^t \|h(t')\|_{1,\Omega}^2\,dt' \le c\bar{d}_2^{-2}\sup_{t'\in(0,t)}|h(t')|_{3,\Omega}^2 + c|g|_{2,\Omega^t}^2 + |h(0)|_{2,\Omega}^2,$$

where $t \in (0,T]$.

Proof. Multiplying $(1.2)_1$ by h, integrating the result over Ω and using the boundary conditions we obtain

$$(3.10)\quad \frac{1}{2}\frac{d}{dt}|h|_{2,\Omega}^2 + \nu\|h\|_{1,\Omega}^2 \le \varepsilon_1|h|_{6,\Omega}^2 + c\left(\frac{1}{\varepsilon_1}\right)|\nabla v|_{2,\Omega}^2|h|_{3,\Omega}^2 + \varepsilon_2|h|_{2,\Omega}^2 + c\left(\frac{1}{\varepsilon_2}\right)|g|_{2,\Omega}^2,$$

where $\varepsilon_1, \varepsilon_2 \in (0,1)$ and

$$(3.11)\quad \int_\Omega h_r\,dx = \int_\Omega h_\varphi\,dx = \int_\Omega h_z\,dx = 0.$$

Choosing ε_1 and ε_2 sufficiently small and integrating (3.10) with respect to time we obtain (3.9). $\square$

We also need

Lemma 3.4. *Let $h(0) \in L_2(\Omega)$, $g \in L_2(\Omega^T)$, $\nabla v \in L_2(0,T;L_3(\Omega))$. Then, for $t \in (0,T]$,*

$$(3.12)\quad |h(t)|_{2,\Omega}^2 \le e^{c|\nabla v|_{3,2,\Omega^t}^2}[|g|_{2,\Omega^t}^2 + e^{-\nu t}|h(0)|_{2,\Omega}^2],$$

and

$$(3.13)\quad |h|_{2,\Omega^t}^2 \le c(|\nabla v|_{3,2,\Omega^t}^2\exp(c|\nabla v|_{3,2,\Omega^t}^2) + 1)(|g|_{2,\Omega^t}^2 + |h(0)|_{2,\Omega}^2).$$

Proof. Multiplying $(1.2)_1$ by h, integrating the result over Ω and using the boundary conditions we obtain

$$(3.14)\quad \frac{1}{2}\frac{d}{dt}|h|_{2,\Omega}^2 + \nu\|h\|_{1,\Omega}^2 \le c|\nabla v|_{3,\Omega}^2|h|_{2,\Omega}^2 + c|g|_{2,\Omega}^2.$$

Continuing, we have

$$(3.15)\quad \frac{d}{dt}(|h|_{2,\Omega}^2 e^{\nu t - c\int_0^t|\nabla v(t')|_{3,\Omega}^2\,dt'}) \le c|g|_{2,\Omega}^2 e^{\nu t - c\int_0^t|\nabla v(t')|_{3,\Omega}^2\,dt'}.$$

Integrating (3.15) with respect to time we arrive to (3.12). Integrating (3.14) with respect to time and using (3.12) yield (3.13). $\square$

To estimate $|w|_{4,\Omega^T}$ which appears in A_1 (see (3.5)) we need

Lemma 3.5. *Let the assumptions of Lemma* 3.1 *hold. Let* $q,h,f_\varphi\in L_4(0,T;L_{4/3}(\Omega))$, $w(0)\in L_4(\Omega)$. *Then, for* $t\in(0,T]$,

$$\tag{3.16} \begin{aligned} &|w(t)|_{4,\Omega}+\bigg(\int_0^t\|w^2(t')\|_{1,\Omega}^2\,dt'\bigg)^{1/2}\\ &\qquad\le c(d_1+|q|_{4/3,4,\Omega^t}+|h|_{4/3,4,\Omega^t}+|f_\varphi|_{4/3,4,\Omega^t}+|w(0)|_{4,\Omega}). \end{aligned}$$

Proof. Multiplying $(1.4)_1$ by $w|w|^2$, integrating over Ω and using the boundary conditions we obtain

$$\tag{3.17} \begin{aligned} &\frac14\frac{d}{dt}|w|_{4,\Omega}^4+\nu\int_\Omega\nabla w\nabla(w|w|^2)\,dx+\nu\int_\Omega\frac{w^4}{r^2}\,dx+\int_\Omega\frac{v_r}{r}|w|^4\,dx\\ &=\frac{1}{R_1}\int_{S_1}|w(R_1)|^4\,dS_1-\frac{1}{R_2}\int_{S_1}|w(R_2)|^4\,dS_1+\int_\Omega\bigg(\frac{q}{r}+\frac{2\nu}{r^2}h_r+f_\varphi\bigg)w|w|^2\,dx. \end{aligned}$$

The second term on the l.h.s. equals to

$$\frac34\nu\int_\Omega|\nabla|w|^2|^2\,dx,$$

the last term on the l.h.s. we estimate by

$$\int_\Omega|v_r||w|^4\,dx\le|v_r|_{2,\Omega}|w|_{8,\Omega}^4\le\varepsilon_1|\nabla|w|^2|_{2,\Omega}^2+c\bigg(\frac{1}{\varepsilon_1},d_1\bigg)|w^2|_{1,\Omega}^2,$$

where $(3.2)_1$ was used. The boundary terms on the r.h.s. are bounded by

$$c|w^2|_{2,S_1}^2\le\varepsilon_2|\nabla w^2|_{2,\Omega}^2+c\bigg(\frac{1}{\varepsilon_2}\bigg)|w^2|_{1,\Omega}^2,$$

and finally the last term on the r.h.s. by

$$\varepsilon_3|w|_{12,\Omega}^4+c\bigg(\frac{1}{\varepsilon_3}\bigg)(|q|_{4/3,\Omega}^4+|h|_{4/3,\Omega}^4+|f_\varphi|_{4/3,\Omega}^4).$$

Using the above estimates in (3.17), assuming that ε_1, ε_2, ε_3 are sufficiently small and integrating with respect to time we arrive to (3.16). □

In view of (3.9) and (3.16) inequality (3.6) takes, for $t\le T$, the form

$$\tag{3.18} \begin{aligned} &\bigg|\frac{\chi(t)}{r}\bigg|_{2,\Omega}+\bigg(\int_0^t\bigg\|\frac{\chi(t')}{r}\bigg\|_{1,\Omega}^2dt'\bigg)^{1/2}\\ &\le c\bigg(\overline{d}_2(t)|h|_{3,\infty,\Omega^t}+|h|_{4/3,4,\Omega^t}^2+|q|_{4/3,4,\Omega^t}^2+|g|_{2,\Omega^t}+|f_\varphi|_{4/3,4,\Omega^t}^2\\ &\qquad+|F_\varphi|_{2,\Omega^t}+d_1^2+|h(0)|_{2,\Omega}+|w(0)|_{4,\Omega}^2+\bigg|\frac{\chi(0)}{r}\bigg|_{2,\Omega}\bigg). \end{aligned}$$

Now we obtain further estimates for v. Let Ω' be a domain obtained by intersection of Ω with the plane passing through the axis of symmetry determined by relation $\varphi = \text{const}$. Then we consider the problem

$$
\begin{aligned}
v_{r,z} - v_{z,r} &= \chi && \text{in } \Omega' \times (0,2\pi),\\
v_{r,r} + v_{z,z} &= -\frac{1}{r}(h_\varphi + v_r) && \text{in } \Omega' \times (0,2\pi),\\
v_r|_{S_1'} &= 0, \quad v_z|_{S_2'} = 0,
\end{aligned}
\tag{3.19}
$$

where S_1', S_2' are intersections of S_1, S_2 with the same plane, respectively.

To solve (3.19) we introduce new functions $u_r = rv_r$, $u_z = rv_z$, so problem (3.19) takes the form

$$
\begin{aligned}
u_{r,z} - u_{z,r} &= \chi - v_z && \text{in } \Omega' \times (0,2\pi),\\
u_{r,r} + u_{z,z} &= -h_\varphi && \text{in } \Omega' \times (0,2\pi),\\
u_r|_{S_1'} &= 0, \quad u_z|_{S_2'} = 0.
\end{aligned}
\tag{3.20}
$$

Introducing potentials σ and ψ such that

$$
\begin{pmatrix} u_r \\ u_z \end{pmatrix} = \begin{pmatrix} \sigma_{,r} + \psi_{,z} \\ \sigma_{,z} - \psi_{,r} \end{pmatrix},
\tag{3.21}
$$

problem (3.20) assumes the form

$$
\begin{aligned}
\Delta'\psi &= \chi - v_z && \text{in } \Omega' \times (0,2\pi),\\
\Delta'\sigma &= -h_\varphi && \text{in } \Omega' \times (0,2\pi),\\
\overline{n} \cdot \nabla'\sigma + \overline{\tau} \cdot \nabla'\psi &= 0 && \text{on } S' \times (0,2\pi),
\end{aligned}
\tag{3.22}
$$

where Δ', ∇' are operators with derivatives with respect to r and z only. Moreover, $\overline{n}$ is the unit normal outward vector to S' and $\overline{\tau}$ is tangent to S'. The vectors belong to the plane determined by Ω'.

Choosing $\psi = 0$ on S' we obtain the following problems

$$
\begin{aligned}
\Delta'\psi &= \chi - v_z && \text{in } \Omega' \times (0,2\pi),\\
\psi &= 0 && \text{on } S' \times (0,2\pi),
\end{aligned}
\tag{3.23}
$$

and

$$
\begin{aligned}
\Delta'\sigma &= -h_\varphi && \text{in } \Omega' \times (0,2\pi),\\
\overline{n} \cdot \nabla'\sigma &= 0 && \text{on } S' \times (0,2\pi).
\end{aligned}
\tag{3.24}
$$

The compatibility condition for problem (3.24) holds,

$$
\int_{\Omega' \times (0,2\pi)} h_\varphi r \, dr \, dz \, d\varphi = 0,
\tag{3.25}
$$

because $h_\varphi = v_{\varphi,\varphi}$.

Lemma 3.6. *Assume that* $h \in L_{3,\infty}(\Omega^T) \cap L_{4/3,4}(\Omega^T)$, $q \in L_{4/3,4}(\Omega^T)$, $g \in L_2(\Omega^T)$, $f_\varphi \in L_{4/3,4}(\Omega^T)$, $F_\varphi \in L_2(\Omega^T)$, $h(0), \chi(0) \in L_2(\Omega)$, $w(0) \in L_4(\Omega)$, $d_1 < \infty$, $\overline{d}_2 < \infty$. *Let* $v' = (v_r, v_z)$. *Then, for* $t \le T$,

$$
\begin{aligned}
\sup_{t'\le t} \|v'(t')\|_{1,\Omega} &+ \|v'\|_{L_{10/3}(0,t;W^1_{10/3}(\Omega))} + \bigg(\int_0^t \|v'(t')\|^2_{2,\Omega}\,dt'\bigg)^{1/2} \\
&\le c\bigg(\overline{d}_2|h|_{3,\infty,\Omega^t} + |h|^2_{4/3,4,\Omega^t} + |q|^2_{4/3,4,\Omega^t} + |g|_{2,\Omega^t} \\
&\quad + |f_\varphi|^2_{4/3,4,\Omega^t} + |F_\varphi|_{2,\Omega^t} + |h(0)|_{2,\Omega} \\
&\quad + \bigg|\frac{\chi(0)}{r}\bigg|_{2,\Omega} + |w(0)|^2_{4,\Omega} + d_1^2 + \overline{d}_2(t)\bigg).
\end{aligned}
\tag{3.26}
$$

Proof. For solutions of problems (3.23) and (3.24) we have

$$
\begin{aligned}
\sup_t \int_0^{2\pi} &(\|\nabla'\psi\|^2_{H^1(\Omega')} + \|\nabla'\sigma\|^2_{H^1(\Omega')})\,d\varphi \\
&+ \int_0^t dt' \int_0^{2\pi} (\|\nabla\nabla'\psi\|^2_{H^1(\Omega')} + \|\nabla\nabla'\sigma\|^2_{H^1(\Omega')})\,d\varphi \\
&\le c\bigg[\sup_t(|\chi|^2_{2,\Omega} + |v|^2_{2,\Omega} + |h|^2_{2,\Omega}) \\
&\quad + \int_0^t (\|\chi(t')\|^2_{1,\Omega} + \|v(t')\|^2_{1,\Omega} + \|h(t')\|^2_{1,\Omega})\,dt'\bigg] \equiv J,
\end{aligned}
\tag{3.27}
$$

where $H^k(\Omega')$ contains only derivatives with respect to r and z, $\nabla' = (\partial_r, \partial_z)$.

In view of (3.21) and the definition of u_r and u_z, (3.27) implies

$$
\sup_t \int_0^{2\pi} \|v'\|^2_{H^1(\Omega')} d\varphi + \int_0^t dt' \int_0^{2\pi} \|\nabla v'(t')\|^2_{H^1(\Omega')}\,d\varphi \le cJ,
\tag{3.28}
$$

where $v' = (v_r, v_z)$. Utilizing

$$
|\nabla v'_{,\varphi}|_{2,\Omega} \le c(|\nabla h|_{2,\Omega} + \|v'\|_{1,\Omega}), \qquad |v'_{,\varphi}|_{2,\Omega} \le c(|h|_{2,\Omega} + |v'|_{2,\Omega}),
$$

(3.28) yields

$$
\sup_t \|v'\|^2_{H^1(\Omega)} + \int_0^t \|v'(t')\|^2_{H^2(\Omega)}\,dt' \le cJ.
$$

Then in view of Lemmas 3.1–3.3 we obtain (3.26). □

To simplify considerations we introduce the notation

$$
\begin{aligned}
X_1(T) &= |h|_{4/3,4,\Omega^T} + |q|_{4/3,4,\Omega^T}, \\
d_3(T) &= |g|_{2,\Omega^T} + |f_\varphi|_{4/3,4,\Omega^T} + |F_\varphi|_{2,\Omega^T}, \\
b_1 &= |h(0)|_{2,\Omega} + |w(0)|_{4,\Omega} + \bigg|\frac{\chi(0)}{r}\bigg|_{2,\Omega}.
\end{aligned}
\tag{3.29}
$$

Next we obtain an additional estimate for w.

LEMMA 3.7. *Let the assumptions of Lemma* 3.6 *hold. Let*

$$\begin{aligned} X_2(T) &= |h|_{5/3,\Omega^T} + |q|_{5/3,\Omega^T} < \infty, \\ d_4(T) &= |f_\varphi|_{5/3,\Omega^T} < \infty, \\ b_2 &= \|w(0)\|_{4/5,5/3,\Omega} < \infty. \end{aligned} \tag{3.30}$$

Then, for $t \le T$, *we have*

$$\begin{aligned} \|w\|_{2,5/3,\Omega^t} \le{}& c(\overline{d}_2(t) + X_1(t) + d_3(t) + d_1)[\overline{d}_2(t)|h|_{3,\infty,\Omega^t} \\ &+ X_1(t) + X_1^2(t) + d_3(t) + d_3^2(t) + b_1 + b_1^2] \\ &+ ct^{3/5}d_1 + c(X_2(t) + d_4(t) + b_2) \equiv cJ_1. \end{aligned} \tag{3.31}$$

PROOF. For solutions of problem (1.4) we have

$$\begin{aligned} \|w\|_{2,\sigma,\Omega^t} \le{}& c(|v'\cdot\nabla w|_{\sigma,\Omega^t} + |wh_\varphi|_{\sigma,\Omega^t} + |v_r w|_{\sigma,\Omega^t} \\ &+ |w|_{\sigma,\Omega^t} + |q|_{\sigma,\Omega^t} + |h|_{\sigma,\Omega^t} + |f_\varphi|_{\sigma,\Omega^t} \\ &+ \|w\|_{1-1/\sigma,\sigma,S_1^t} + \|w(0)\|_{2-2/\sigma,\sigma,\Omega}), \end{aligned} \tag{3.32}$$

where $\sigma \le 2$ and will be chosen later. To estimate the first term on the r.h.s. of (3.32) we recall that (3.26) implies that $\nabla v' \in L_{r,q}(\Omega\times(0,t))$ with $3/r + 2/q = 3/2$. Hence $v' \in L_q(0,t;L_\sigma(\Omega))$, where $\sigma = 3r/(3-r)$, so $r = 3\sigma/(3+\sigma)$. Choosing $q = \sigma$ we obtain $3/(3\sigma/(3+\sigma)) + 2/\sigma = 3/2$, so $\sigma = 10$.

Hence by the Hölder inequality the first term on the r.h.s. of (3.32) we estimate by

$$|v'|_{\lambda_1\sigma,\Omega^t}|\nabla w|_{\lambda_2\sigma,\Omega^t} \equiv I_1,$$

where $1/\lambda_1 + 1/\lambda_2 = 1$, $\lambda_1\sigma \le 10$, $\lambda_2\sigma \le 2$, so $\sigma \le 5/3$. Then in view of $(3.2)_2$ and (3.26),

$$I_1 \le c\overline{d}_2(t)[X_1(t) + X_1^2(t) + \overline{d}_2(t)|h|_{3,\infty,\Omega^t} + d_3(t) + d_3^2(t) + b_1 + b_1^2 + d_1^2 + \overline{d}_2].$$

To estimate the second term on the r.h.s. of (3.32) we recall that (3.9) implies

$$|h|_{10/3,\Omega^t} \le c(\overline{d}_2(t)|h|_{3,\infty,\Omega^t} + d_3(t) + b_1) \tag{3.33}$$

and (3.16) gives

$$|w|_{20/3,\Omega^t} \le c(X_1(t) + d_3(t) + b_1). \tag{3.34}$$

Hence by the Hölder inequality the second term on the r.h.s. of (3.32) is bounded by

$$|w|_{\sigma\lambda_1,\Omega^t}|h|_{\sigma\lambda_2,\Omega^t} \equiv I_2,$$

where $1/\lambda_1 + 1/\lambda_2 = 1$, $\sigma\lambda_1 \le 20/3$, $\sigma\lambda_2 \le 10/3$, so $\sigma \le 20/9$, and

$$I_2 \le c(X_1(t) + d_3(t) + b_1)(\overline{d}_2(t)|h|_{3,\infty,\Omega^t} + d_3(t) + b_1).$$

Applying the Hölder inequality to the third term on the r.h.s. of (3.32) we bound it by

$$|v_r|_{\lambda_1\sigma,\Omega^t}|w|_{\lambda_2\sigma,\Omega^t} \equiv I_3,$$

where $1/\lambda_1 + 1/\lambda_2 = 1$. Assuming that $\lambda_1\sigma \le 10$, $\lambda_2\sigma \le 20/3$ we have that $\sigma \le 4$ and

$$I_3 \le c[X_1(t) + X_1^2(t) + \overline{d}_2(t)|h|_{3,\infty,\Omega^t} + d_3(t) + d_3^2(t) + b_1 + b_1^2][X_1(t) + d_3(t) + b_1].$$

The fourth term from the r.h.s. of (3.32) we estimate by $t^{1/\sigma}d_1$. The boundary term is restricted by

$$\varepsilon\|w\|_{2,\sigma,\Omega^t} + c\left(\frac{1}{\varepsilon}\right)t^{1/\sigma}d_1.$$

In view of the above considerations we choose $\sigma = 5/3$ and (3.31) follows. □

Looking for the proof of Lemma 3.7 we see that the power of integrability 5/3 in (3.31) is determined by the first term on the r.h.s. of (3.32). The other terms are less restrictive. To increase the power we use the estimate.

$$\begin{aligned}|\nabla w|_{5/3,\Omega^t} &\le c\|w\|_{2,5/3,\Omega^t} \le cJ_1 \\ &\le c(d_1^3 + \overline{d}_2^3 + d_3^3 + X_1^3 + b_1^3 + \overline{d}_2^2|h|_{3,\infty,\Omega^t}^2 + 1) \\ &\quad + ct^{3/5}d_1 + c(X_2 + d_4 + b_2),\end{aligned} \tag{3.35}$$

where the Young inequality was utilized to get the last inequality.

Let us introduce

$$\begin{aligned} X_3(t) &= |h|_{2,\Omega^t} + |q|_{2,\Omega^t}, \\ d_5(t) &= |f_\varphi|_{2,\Omega^t}, \\ b_3 &= \|w(0)\|_{1,\Omega}, \\ Y_1 &= d_1 + \overline{d}_2 + d_3 + b_1 + X_1 + \overline{d}_2|h|_{3,\infty,\Omega^t}.\end{aligned} \tag{3.36}$$

Then in view of (3.35) we have

Lemma 3.8. *Let the assumptions of Lemma* 3.7 *hold. Let $X_3(t)$, $d_5(t)$, b_3 be bounded for $t \le T$. Then*

$$\|w\|_{2,\Omega^T} \le c(Y_1^5 + 1) + cTd_1^{5/3} + c(X_2^2 + d_4^2 + b_2^2) + cT^{1/2}d_1 + c(X_3 + d_5 + b_3). \tag{3.37}$$

Proof. It is enough to examine the first term on the r.h.s. of (3.32). We estimate it now by

$$|v'|_{\lambda_1\sigma,\Omega^t}|\nabla w|_{\lambda_2\sigma,\Omega^t} \equiv I_1,$$

where $1/\lambda_1 + 1/\lambda_2 = 1$, $\lambda_1\sigma \le 10$, $\lambda_2\sigma \le 5/2$, so $\sigma \le 2$. Hence

$$\begin{aligned} I_1 &\le c[X_1 + X_1^2 + \overline{d}_2|h|_{3,\infty,\Omega^t} + d_3 + d_3^2 + b_1 + b_1^2]J_1 \\ &\le c(Y_1^2 + 1)(Y_1^3 + 1 + t^{3/5}d_1 + X_2 + d_4 + b_2) \\ &\le c[Y_1^5 + td_1^{5/3} + (X_2 + d_4 + b_2)^{5/3} + 1].\end{aligned}$$

The other terms are estimated in the same way as in the proof of Lemma 3.7. □

To simplify notation we introduce

$$
\begin{aligned}
X(T) &= \overline{d}_2|h|_{3,\infty,\Omega^T} + X_1 + X_2 + X_3,\\
d(T) &= d_1 + \overline{d}_2 + d_3 + d_4 + d_5,\\
b &= b_1 + b_2 + b_3.
\end{aligned}
\tag{3.38}
$$

In view of Lemmas 3.6 and 3.8 we have

$$
|v|_{10,\Omega^T} + |\nabla v|_{10/3,\Omega^T} \le c(1 + X^5 + d^5 + b^5 + Td_1^2). \tag{3.39}
$$

Therefore we can prove

Lemma 3.9. *Assume that $h \in L_\infty(0,T;L_3(\Omega))$, $q \in L_4(0,T;L_{4/3}(\Omega)) \cap L_2(\Omega^T)$, $f \in L_\sigma(\Omega^T)$, $f_\varphi \in L_4(0,T;L_{4/3}(\Omega))$, $g \in L_2(\Omega^T)$, $v(0) \in W_\sigma^{2-2/\sigma}(\Omega)$, $h(0) \in L_2(\Omega)$, $w(0) \in H^1(\Omega)$, $\sigma \le 5/2$. Then, for $\sigma \le 5/2$,*

$$
\|v\|_{2,\sigma,\Omega^T} \le c(1 + X^5 + d^5 + b^5 + Td_1^2)^2 + c(|f|_{\sigma,\Omega^t} + \|v(0)\|_{2-2/\sigma,\sigma,\Omega}). \tag{3.40}
$$

Proof. For solutions of problem (1.1) we have

$$
\|v\|_{2,\sigma,\Omega^t} \le c(|v\cdot\nabla v|_{\sigma,\Omega^t} + |f|_{\sigma,\Omega^t} + \|v(0)\|_{2-2/\sigma,\sigma,\Omega}),
$$

where $|v\cdot\nabla v|_{\sigma,\Omega^t} \le |v|_{\sigma\lambda_1,\Omega^t}|\nabla v|_{\sigma\lambda_2,\Omega^t}$, $1/\lambda_1 + 1/\lambda_2 = 1$. Assuming that $\sigma\lambda_1 = 10$, $\sigma\lambda_2 = 10/3$ and using (3.39) we obtain (3.40). □

Next we show

Lemma 3.10. *Let the assumptions of Lemma 3.9 hold. Let $v \in W_{5/2}^{2,1}(\Omega^T)$, $g \in W_\delta^{\beta,\beta/2}(\Omega^T)$, $h(0) \in W_\delta^{2+\beta-2/\delta}(\Omega)$. Then solutions of (1.2) satisfy*

$$
\begin{aligned}
&\|h\|_{2+\beta,\delta,\Omega^T} + \|\nabla q\|_{\beta,\delta,\Omega^T}\\
&\qquad \le \varphi(\|v\|_{2,5/2,\Omega^T})|h|_{2,\Omega^T} + c(\|g\|_{\beta,\delta,\Omega^T} + \|h(0)\|_{2+\beta-2/\delta,\delta,\Omega}),
\end{aligned}
\tag{3.41}
$$

where

$$
\varphi(a) = \sum_{i=1}^4 a^{1/(1-\varkappa_1)},
$$

and

$$
\begin{aligned}
\varkappa_1 &= \left(\frac{5}{d_1} + \beta + \frac{\varepsilon}{\delta} + 1\right)\frac{1}{2+\beta}, &\qquad \varkappa_2 &= \left(\frac{5}{d_2} + 1\right)\frac{1}{2+\beta},\\
\varkappa_3 &= \left(\frac{5}{d_3} + \beta + \frac{\varepsilon}{\delta}\right)\frac{1}{2+\beta}, &\qquad \varkappa_4 &= \left(\frac{5}{d_4}\right)\frac{1}{2+\beta},
\end{aligned}
$$

$\varkappa_i < 1$, $i = 1, \dots, 4$, $d_1 > 5$, $\beta < 5/d_2 < 1+\beta$, $1 < d_3 < 2$, $1+\beta < 5/d_4 < 2+\beta$.

Proof. For solutions of problem (1.2) we have

$$\begin{aligned}(3.42)\qquad \|h\|_{2+\beta,\delta,\Omega^T} + \|\nabla q\|_{\beta,\delta,\Omega^T} \le{} & c(\|v\cdot\nabla h\|_{\beta,\delta,\Omega^T} + \|h\cdot\nabla v\|_{\beta,\delta,\Omega^T} \\ & + \|g\|_{\beta,\delta,\Omega^T} + \|h(0)\|_{2+\beta-2/\delta,\delta,\Omega}).\end{aligned}$$

Utilizing Lemma 2.2 to the first term on the r.h.s. of (3.42) we obtain

$$\|v\cdot\nabla h\|_{\beta,\delta,\Omega^T} \le c(\|v\|_{\beta+\varepsilon/\delta,\delta_1,\Omega^T}|\nabla h|_{\delta_2,\Omega^T} + \|\nabla h\|_{\beta+\varepsilon/\delta,\delta_1',\Omega^T}|v|_{\delta_2',\Omega^T}),$$

where $1/\delta_1 + 1/\delta_2 = 1/\delta$, $1/\delta_1' + 1/\delta_2' = 1/\delta$ and $\varepsilon > 0$ is any small number. Since $v \in W^{2,1}_{5/2}(\Omega^T)$ we apply the imbeddings

$$W^{2,1}_{5/2}(\Omega^T) \subset W^{\beta+\varepsilon/\delta,\beta/2+\varepsilon/(2\delta)}_{\delta_1}(\Omega^T) \quad\text{and}\quad W^{2,1}_{5/2}(\Omega^T) \subset L_{\delta_2'}(\Omega^T),$$

which hold for $\beta < 5/\delta_1$ and any $\delta_2' > \delta$.

In virtue of the above considerations we obtain

$$(3.43)\qquad \|v\cdot\nabla h\|_{\beta,\delta,\Omega^T} \le c(|\nabla h|_{\delta_2,\Omega^T} + \|\nabla h\|_{\beta+\varepsilon/\delta,\delta_1',\Omega^T})\|v\|_{2,5/2,\Omega^T}.$$

Using the interpolation inequalities

$$(3.44)\qquad \begin{aligned}\|\nabla h\|_{\beta+\varepsilon/\delta,\delta_1',\Omega^T} &\le \bar\varepsilon_1^{1-\varkappa_1}\|h\|_{2+\beta,\delta,\Omega^T} + c\bar\varepsilon_1^{-\varkappa_1}|h|_{2,\Omega^T},\\ \|\nabla h\|_{\delta_2,\Omega^T} &\le \bar\varepsilon_2^{1-\varkappa_2}\|h\|_{2+\beta,\delta,\Omega^T} + c\bar\varepsilon_2^{-\varkappa_2}|h|_{2,\Omega^T},\end{aligned}$$

where $\bar\varepsilon_i \in (0,1)$, $i = 1,2$,

$$\varkappa_1 = \left(\frac{5}{\delta} - \frac{5}{\delta_1'} + \beta + \frac{\varepsilon}{\delta} + 1\right)\frac{1}{2+\beta} < 1 \quad\text{and}\quad \varkappa_2 = \left(\frac{5}{\delta} - \frac{5}{\delta_2} + 1\right)\frac{1}{2+\beta} < 1,$$

inequality (3.43) assumes the form

$$\begin{aligned}(3.45)\qquad \|v\cdot\nabla h\|_{\beta,\delta,\Omega^T} \le{} & \varepsilon_1\|h\|_{2+\beta,\delta,\Omega^T} \\ & + c\big(\varepsilon_1^{-\varkappa_1/(1-\varkappa_1)}\|v\|^{1/(1-\varkappa_1)}_{2,5/2,\Omega^T} + \varepsilon_1^{-\varkappa_2/(1-\varkappa_2)}\|v\|^{1/(1-\varkappa_2)}_{2,5/2,\Omega^T}\big)|h|_{2,\Omega^T},\end{aligned}$$

which holds for $\beta < 5/\delta_1 < 1+\beta$, $\delta_2' > 5$.

Exploiting Lemma 2.2 the second term on the r.h.s. of (3.42) we estimate in the way

$$\|h\cdot\nabla v\|_{\beta,\delta,\Omega^T} \le c(\|h\|_{\beta+\varepsilon/\delta,\bar\delta_1,\Omega^T}|\nabla v|_{\bar\delta_2,\Omega^T} + |h|_{\bar\delta_1',\Omega^T}\|\nabla v\|_{\beta+\varepsilon/\delta,\bar\delta_2',\Omega^T}),$$

where $1/\bar\delta_1 + 1/\bar\delta_2 = 1/\delta$ and $1/\bar\delta_1' + 1/\bar\delta_2' = 1/\delta$.

To estimate the r.h.s. we use the imbeddings

$$\begin{aligned}\nabla W^{2,1}_{5/2}(\Omega^T) &\subset L_{\bar\delta_2}(\Omega^T) && \text{with } \bar\delta \le 5,\\ \nabla W^{2,1}_{5/2}(\Omega^T) &\subset W^{\beta+\varepsilon/\delta,\beta/2+\varepsilon/(2\delta)}_{\bar\delta_2'}(\Omega^T) && \text{with } \bar\delta_2' < 5/(1+\beta).\end{aligned}$$

In view of the above considerations we have

$$(3.46)\qquad \|h\cdot\nabla v\|_{\beta,\delta,\Omega^T} \le c(|h|_{\bar\delta_1',\Omega^T} + \|h\|_{\beta+\varepsilon/\delta,\bar\delta_1,\Omega^T})\|v\|_{2,5/2,\Omega^T}.$$

Employing the interpolation inequalities

$$\tag{3.47}\begin{aligned} \|h\|_{\beta+\frac{\varepsilon}{\delta},\bar\delta_1,\Omega^T} &\le \bar\varepsilon_3^{1-\varkappa_3}\|h\|_{2+\beta,\delta,\Omega^T}+\bar\varepsilon_3^{-\varkappa_3}|h|_{2,\Omega^T},\\ |h|_{\bar\delta_1',\Omega^T} &\le \bar\varepsilon_4^{1-\varkappa_4}\|h\|_{2+\beta,\delta,\Omega^T}+c\bar\varepsilon_4^{-\varkappa_4}|h|_{2,\Omega^T}, \end{aligned}$$

where $\bar\varepsilon_i\in(0,1)$, $i=3,4$ $\varkappa_3=(5/\delta-5/\bar\delta_1+\beta+\varepsilon/\delta)/(2+\beta)<1$, and $\varkappa_4=(5/\delta-5/\bar\delta_1')/(2+\beta)<1$, inequality (3.46) takes the form

$$\tag{3.48}\begin{aligned} \|h\cdot\nabla v\|_{\beta,\delta,\Omega^T} &\le \varepsilon_2\|h\|_{2+\beta,\delta,\Omega^T}\\ &\quad + c(\varepsilon_2^{-\varkappa_3/(1-\varkappa_3)}\|v\|_{2,5/2,\Omega^T}^{1/(1-\varkappa_3)}+\varepsilon_2^{-\varkappa_4/(1-\varkappa_4)}\|v\|_{2,5/2,\Omega^T}^{1/(1-\varkappa_4)})|h|_{2,\Omega^T}. \end{aligned}$$

Summarizing, the inequlity holds for $1\le 5/\delta_2<2$, $1+\beta<5/\bar\delta_2'<2+\beta$.

Utilizing (3.45) and (3.48) in (3.42), assuming that ε_1, ε_2 are sufficiently small and defining $d_1=\delta_2'$, $d_2=\delta_1$, $d_3=\bar\delta_2$, $d_4=\bar\delta_2'$ we obtain (3.41). □

4. Local existence and uniqueness

The aim of this paper is to prove existence of more regular solutions than weak solutions described by Lemma 3.1. In other words we increase regularity of the weak solutions. We shall do it by assuming some additional regularity properties on initial data and the external force. The regularity properties will be expressed in the form that the quantities $h(0)$ and g will be small in some norms. Therefore we shall concentrate our considerations on examining the existence and uniqueness of solutions to problem (1.2), where velocity v is treated as prescribed. However if v would have the properties of weak solutions we would be able to prove nothing. Therefore to obtain higher regularity of v we use problems (1.3), (1.4), (3.19) and (1.1) by assuming that h and q in these problems are treated as given and are appropriately regular. To precise the statement let us assume that $\widetilde h$ and $\widetilde q$ in (1.3), (1.4), (3.19) are given and $\bar v$ is the weak solution described by Lemma 3.1. We assume also that $\widetilde h$ and $\widetilde q$ are solutions of (1.2) with $v=\bar v$.

Let us introduce the space

$$V_2^1(\Omega^T)=\left\{u:\sup_{t\le T}|u(t)|_{2,\Omega}+\left(\int_0^T|\nabla u(t')|_{2,\Omega}^2\,dt'\right)^{1/2}<\infty\right\}.$$

Then Lemma 3.3 determines the transformation

$$V_2^1(\Omega^T)\times L_\infty(0,T;L_3(\Omega))\ni(\bar v,\widetilde h)\to\Phi_1(\bar v,\widetilde h)=\widetilde h\in V_2^1(\Omega^T),$$

which describes some increasing of regularity of $\widetilde h$. Next Lemmas 3.3, 3.4 and 3.5 imply

$$V_2^1(\Omega^T)\times L_\infty(0,T;L_3(\Omega))\times L_{4/3,4}(\Omega^T)\ni(\bar v,\widetilde h,\widetilde q)\to\Phi_2(\bar v,\widetilde h,\widetilde q)=\chi\in V_2^1(\Omega^T).$$

Continuing Lemma 3.6 implies the transformation

$$\begin{aligned}&V_2^1(\Omega^T)\times V_2^1(\Omega^T)\times V_2^1(\Omega^T)\ni(\overline{v},\widetilde{h},\chi)\to\Phi_3(\overline{v},\widetilde{h},\chi)=v'\\&\quad=(v_r,v_z)\in L_\infty(0,T;H^1(\Omega))\cap L_{10/3}(0,T;W^1_{10/3}(\Omega))\cap L_2(0,T;H^2(\Omega)).\end{aligned}$$

Hence

$$v'=\Phi_3(\overline{v},\Phi_1(\overline{v},\widetilde{h}),\Phi_2(\overline{v},\widetilde{h},\widetilde{q}))\equiv\Phi_4(\overline{v},\widetilde{h},\widetilde{q}).$$

By imbeddings it follows that $v'\in L_{10}(\Omega^T)$, $\nabla v'\in L_{10/3}(\Omega^T)$. Finally Lemma 3.8 implies that $w=\Phi_5(\overline{v},\widetilde{h},\widetilde{q})\in W_2^{2,1}(\Omega^T)$ for

$$\overline{v}\in V_2^1(\Omega^T),\quad \widetilde{h}\in L_\infty(0,T;L_3(\Omega)),\quad \widetilde{q}\in L_{4/3,4}(\Omega^T)\cap L_2(\Omega^T). \tag{4.1}$$

Summarizing we have

$$v=\Phi_6(\overline{v},\widetilde{h},\widetilde{q})\in L_{10}(\Omega^T),\quad \nabla v=\nabla\Phi_6(\overline{v},\widetilde{h},\widetilde{q})\in L_{10/3}(\Omega^T), \tag{4.2}$$

under the assumption that $\overline{v},\widetilde{h},\widetilde{q}$ satisfy (4.1).

In view of (4.2) Lemma 3.9 implies that $v=v(\overline{v},\widetilde{h},\widetilde{q})\in W^{2,1}_{5/2}(\Omega^T)$ if (4.1) holds. Therefore to prove the existence of solutions to problem (1.1) with more regular initial data and external forces we are looking for existence of solutions to problem (1.2), where $v=v(h,q)$ is a given function. Since we are going to apply the Leray–Schauder fixed point theorem we examine the following transformation

$$\begin{aligned}h_{,t}-\operatorname{div}\mathbb{D}(h)+\nabla q&=-\lambda[v(\widetilde{h},\widetilde{q})\cdot\nabla\widetilde{h}+\widetilde{h}\cdot\nabla v(\widetilde{h},\widetilde{q})]+g,\\ \operatorname{div}h&=0,\\ h\cdot\overline{n}&=0,\\ \overline{n}\cdot\mathbb{D}(h)\cdot\overline{\tau}_\alpha&=0,\quad \alpha=1,2,\\ h|_{t=0}&=h(0),\end{aligned} \tag{4.3}$$

where parameter $\lambda\in[0,1]$, $\widetilde{h}$, $\widetilde{q}$ are treated as given functions and v depends also on a given weak solution $\overline{v}$ what is described in (4.2). Hence (4.3) implies the transformation

$$(h,q)=\Phi(\widetilde{h},\widetilde{q},\lambda). \tag{4.4}$$

The main problem of this section is to find a fixed point of transformation (4.4) for $\lambda=1$ and also its estimate. Hence to examine problem (4.3) we introduce the space

$$\begin{aligned}\mathfrak{M}(\Omega^T)=\{(h,q):h\in L_\infty(0,T;L_3(\Omega))\cap W^{\beta+\varepsilon/\delta,\beta/2+\varepsilon/(2\delta)}_{d_3'}(\Omega^T)\cap L_{d_4'}(\Omega^T),\\ \nabla h\in W^{\beta+\varepsilon/\delta,\beta/2+\varepsilon/(2\delta)}_{d_1'}(\Omega^T)\cap L_{d_2'}(\Omega^T),\ q\in L_4(0,T;L_{4/3}(\Omega))\cap L_2(\Omega^T)\},\end{aligned}$$

where d_i' are determined by the relations $1/d_i+1/d_i'=1/\delta$, $i=1,\dots,4$, d_i are introduced by Lemma 3.10, ε is arbitrary small positive number and β,δ will be determined later.

In view of estimates (3.43) and (3.46) we have that problem (4.3) for $\lambda \in (0,1]$ implies the transformation

$$\Phi\colon \mathfrak{M}(\Omega^T) \times (0,1] \to W_\delta^{2+\beta,1+\beta/2}(\Omega^T) \times G_\delta^{\beta,\beta/2}(\Omega^T), \tag{4.5}$$

where

$$G_\delta^{\beta,\beta/2}(\Omega^T) = \Big\{ q : \nabla q \in W_\delta^{\beta,\beta/2}(\Omega^T),\ \int_\Omega q\,dx = 0 \Big\}.$$

LEMMA 4.1. *Assume that* $\beta \in (0,1)$, $2/\delta < \beta + 1/2$, $5/\delta < 3+\beta$, $\delta > 1$. *Assume* $1/5 > 1/\delta - 1/d_1'$, $\beta < 5(1/\delta - 1/d_2') < 1+\beta$, $1/2 < 1/\delta - 1/d_3' < 1$, $1+\beta < 5(1/\delta - 1/d_4') < 2+\beta$. *Then the imbedding*

$$W_\delta^{2+\beta,1+\beta/2}(\Omega^T) \times G_\delta^{\beta,\beta/2}(\Omega^T) \subset \mathfrak{M}(\Omega^T) \tag{4.6}$$

is compact.

PROOF. The proof is done step by step.

The imbedding $W_\delta^{2+\beta,1+\beta/2}(\Omega^T) \subset L_\infty(0,T;L_3(\Omega))$ is compact if $5/\delta < 3+\beta$.

Imbeddings of other spaces defined h in $\mathfrak{M}(\Omega^T)$ into $W_\delta^{2+\beta,1+\beta/2}(\Omega^T)$ are compact in view of the interpolation inequalities (3.44) and (3.47).

Since $\nabla q \in W_\delta^{\beta,\beta/2}(\Omega^T)$ we have that $\nabla q \in L_{r,s}(\Omega^T)$ with $5/\delta - 3/r' - 2/s < \beta$, $r' = \max\{r,\delta\}$ and imbedding $W_\delta^{\beta,\beta/2}(\Omega^T) \subset L_{r,s}(\Omega^T)$ is compact.

Since $\int_\Omega q\,dx = 0$ we obtain that $q \in L_{3r/(3-r),s}(\Omega^T)$. Taking $3r/(3-r) \ge 4/3$, $s=4$, we get $r=1$ and $2/\delta < \beta + 1/2$. Putting $3r/(3-r) = 2$, $s=2$, we get $r = 6/5$ and $2/\delta < \beta + 1$. Hence the lemma is proved. □

For a fixed point of (4.4) we get

LEMMA 4.2. *Assume that* $v(0) \in W_{5/2}^{6/5}(\Omega)$, $h(0) \in W_\delta^{2+\beta-2/\delta}(\Omega)$, $w(0) \in H^1(\Omega)$, $\chi(0) \in L_2(\Omega)$, $f \in L_{5/2}(\Omega^T)$, $g \in W_\delta^{\beta,\beta/2}(\Omega^T)$, $f_\varphi \in L_4(0,T;L_{4/3}(\Omega))$, $F_\varphi \in L_2(\Omega^T)$, $\beta \in (0,1)$, $2/\delta < \beta + 1/2$, $5/\delta < 3+\beta$, $\delta > 1$. *Assume that* $|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}$ *is sufficiently small. Then for a fixed point of transformation* (4.4), (4.5) *there exists a constant A sufficiently large* (*see* (4.10)) *such that for sufficiently small* $|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}$ *the estimate holds*

$$\|h\|_{2+\beta,\delta,\Omega^T} + \|\nabla q\|_{\beta,\delta,\Omega^T} \le A. \tag{4.7}$$

Moreover, (4.11) *implies that T and* $|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}$ *are inversely proportional.*

PROOF. From (3.41) and (3.13) we have

$$\begin{aligned} \|h\|_{2+\beta,\delta,\Omega^T} &+ \|\nabla q\|_{\beta,\delta,\Omega^T} \\ &\le \varphi(\|v\|_{2,5/2,\Omega^T})[|\nabla v|_{3,2,\Omega^T} \exp(c|\nabla v|_{3,2,\Omega^T}^2) + 1] \\ &\quad \cdot (|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}) + c(\|g\|_{\beta,\delta,\Omega^T} + \|h(0)\|_{2+\beta-2/\delta,\delta,\Omega}), \end{aligned} \tag{4.8}$$

where

$$\varphi(\|v\|_{2,5/2,\Omega^T}) = \sum_{i=1}^{4} \|v\|_{2,5/2,\Omega^T}^{a_i},$$

$a_i = 1/(1-\varkappa_i)$, $\varkappa_i \in (0,1)$, $i = 1,\dots,4$, are defined in Lemma 3.10. Next we recall

$$\begin{aligned}
\|v\|_{2,5/2,\Omega^T} &\le c(1 + X^{10} + d^{10} + b^{10} + T^2 d_1^4) + c(|f|_{5/2,\Omega^T} + \|v(0)\|_{6/5,5/2,\Omega}),\\
X &= |h|_{4/3,4,\Omega^T} + |h|_{5/3,\Omega^T} + |h|_{2,\Omega^T} + |q|_{4/3,4,\Omega^T}\\
&\quad + |q|_{5/3,\Omega^T} + |q|_{2,\Omega^T} + \overline{d}_2 |h|_{3\infty,\Omega^T},\\
d &= d_1 + (1+T)\overline{d}_2 + |g|_{2,\Omega^T} + |f_\varphi|_{4/3,4,\Omega^T}\\
&\quad + |F_\varphi|_{2,\Omega^T} + |f_\varphi|_{5/3,\Omega^T} + |f_\varphi|_{2,\Omega^T},\\
b &= |h(0)|_{2,\Omega} + |w(0)|_{4,\Omega} + |\chi(0)|_{2,\Omega} + \|w(0)\|_{1,\Omega}.
\end{aligned}$$

We use the interpolation inequality

$$|h|_{3,\infty,\Omega^T} \le \overline{\varepsilon}_1^{1-\varkappa_0} \|h\|_{2+\beta,\delta,\Omega^T} + c\overline{\varepsilon}_1^{-\varkappa_0} |h|_{2,\Omega^T},$$

where $\varkappa_0 = (5/\delta - 1)/(2+\beta)$. From (3.39) we have that

$$|\nabla v|_{3,2,\Omega^T} \le cT^{1/5}(1 + X^5 + d^5 + b^5 + T d_1^2).$$

Let us introduce the quantities

$$\begin{aligned}
\gamma &= \|h\|_{2+\beta,\delta,\Omega^T} + \|\nabla q\|_{\beta,\delta,\Omega^T},\\
F &= |g|_{2,\Omega^T} + |f_\varphi|_{4/3,4,\Omega^T} + |F_\varphi|_{2,\Omega^T} + |f|_{5/2,\Omega^T},\\
F_0 &= |h(0)|_{2,\Omega} + \|w(0)\|_{1,\Omega} + |\chi(0)|_{2,\Omega} + \|v(0)\|_{6/5,5/2,\Omega}.
\end{aligned}$$

Then (4.8) implies

$$\gamma \le G(T,\gamma,F,F_0)(|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}) + c(\|g\|_{\beta,\delta,\Omega^T} + \|h(0)\|_{2+\beta-2/\delta,\delta,\Omega}), \tag{4.9}$$

where G is an increasing positive function of its arguments.

We recall that G is a combination of power and exponential functions. Let T be a given number. Let A be a number such that

$$\begin{aligned}
&G(T,0,F,F_0)(|g|_{2,\Omega^T} + |h(0)|_{2,\Omega})\\
&\qquad + c(\|g\|_{\beta,\delta,\Omega^T} + \|h(0)\|_{2+\beta-2/\delta,\delta,\Omega}) \le A/2.
\end{aligned} \tag{4.10}$$

Then assuming that $\gamma \le A$ we obtain from (4.9) for sufficiently small $|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}$ the inequality

$$G(T,A,F,F_0)(|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}) + c(\|g\|_{\beta,\delta,\Omega^T} + \|h(0)\|_{2+\beta-2/\delta,\delta,\Omega}) \le A. \tag{4.11}$$

Hence (4.7) holds. From (4.11) it follows that for a given A an increasing of T implies decreasing of $|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}$. □

REMARK 4.3. To satisfy (3.3) we recall that (3.12) takes the form

$$|h(t)|_{2,\Omega} \le G_2(T,\gamma,F_1,F_0)[|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}].$$

Hence (3.13) implies the restriction

$$c_1 G_3(T,A,F_1,F_0)[|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}] \le \nu/4.$$

G_2 and G_3 are increasing positive functions. Next we need

LEMMA 4.4. *Let the assumptions of Lemma* 4.2 *hold. Then the transformation* (4.4) *is uniformly continuous with respect to its arguments.*

PROOF. The uniform continuity with respect to λ is evident. Let $\lambda \in [0,1]$. To show uniform continuity with respect to (h,q) we introduce the sets of functions (h^i,q^i), $(\widetilde{h}^i,\widetilde{q}^i)$, $i=1,2$, which are connected by transformation (4.4)

$$(h^i,q^i) = \Phi(\widetilde{h}^i,\widetilde{q}^i,\lambda), \quad i=1,2.$$

Let us introduce

$$\begin{gathered} H = h^1 - h^2, \quad Q = q^1 - q^2, \quad \widetilde{H} = \widetilde{h}^1 - \widetilde{h}^2, \quad \widetilde{Q} = \widetilde{q}^1 - \widetilde{q}^2, \\ h^1(0) = h^2(0), \quad \widetilde{h}^1(0) = \widetilde{h}^2(0). \end{gathered}$$

Then problem (4.3) implies

$$\begin{aligned} H_{,t} - \operatorname{div}\mathbb{D}(H) + \nabla Q &= -\lambda[\widetilde{V}\cdot\nabla\widetilde{h}^1 + \widetilde{v}^2\cdot\nabla\widetilde{H} \\ &\qquad + \widetilde{H}\cdot\nabla\widetilde{v}^1 + \widetilde{h}^2\cdot\nabla\widetilde{V}] \equiv K, \\ \operatorname{div} H &= 0, \\ H\cdot\overline{n} &= 0, \\ \overline{n}\cdot\mathbb{D}(H)\cdot\overline{\tau}_\alpha &= 0, \quad \alpha=1,2, \\ H|_{t=0} &= 0, \end{aligned} \tag{4.12}$$

where $v^i = v(h^i,q^i)$, $\widetilde{v}^i = v(\widetilde{h}^i,\widetilde{q}^i)$, $i=1,2$, $\widetilde{V} = \widetilde{v}^1 - \widetilde{v}^2$, $V = v^1 - v^2$. We examine problem (4.12) assuming that $(\widetilde{h}^i,\widetilde{q}^i) \in W_\delta^{2+\beta,1+\beta/2}(\Omega^T) \times G_\delta^{\beta,\beta/2}(\Omega^T)$, $i=1,2$. Hence from (4.12) we have

$$\begin{aligned} \|H\|_{2+\beta,\delta,\Omega^T} + \|\nabla Q\|_{\beta,\delta,\Omega^T} &\le c\|K\|_{\beta,\delta,\Omega^T} \\ &\le c(\|\widetilde{V}\cdot\nabla\widetilde{h}^1\|_{\beta,\delta,\Omega^T} + \|\widetilde{v}^2\cdot\nabla\widetilde{H}\|_{\beta,\delta,\Omega^T} \\ &\quad + \|\widetilde{H}\cdot\nabla\widetilde{v}^1\|_{\beta,\delta,\Omega^T} + \|\widetilde{h}^2\cdot\nabla\widetilde{V}\|_{\beta,\delta,\Omega^T}) \equiv c\sum_{i=1}^4 K_i. \end{aligned} \tag{4.13}$$

In view of Lemma 2.1 and (4.7) we get

$$K_1 + K_4 \le \varphi(A)\|\widetilde{V}\|_{2,5/2,\Omega^T} \quad \text{and} \quad K_2 + K_3 \le \varphi(A)\|\widetilde{H}\|_{2+\beta,\delta,\Omega^T},$$

where φ is an increasing positive function. Hence, (4.13) implies

$$\|H\|_{2+\beta,\delta,\Omega^T} + \|\nabla Q\|_{\beta,\delta,\Omega^T} \le \varphi(A)(\|\widetilde{V}\|_{2,5/2,\Omega^T} + \|\widetilde{H}\|_{2+\beta,\delta,\Omega^T}). \tag{4.14}$$

Now we estimate the norm with $\widetilde{V}$ in the r.h.s. of (4.14). In problems (1.3) and (1.4) v is treated as a weak solution which is given and that it does not depend on h and q. Similarly w in the second term on the r.h.s. of (1.3) is treated as the weak solution. However, the third term on the r.h.s. of (1.3) must be treated as a solution of (1.4) because it can not be bounded in terms of the energy estimate. Let $\widetilde{\chi}^i = \chi(\widetilde{h}^i)$, $\widetilde{w}^i = w(\widetilde{h}^i, \widetilde{q}^i)$, $i = 1, 2$. Let

$$\widetilde{K} = \widetilde{\chi}^1 - \widetilde{\chi}^2, \qquad \widetilde{W} = \widetilde{w}^1 - \widetilde{w}^2.$$

Then problems (1.3) and (1.4) imply

$$\begin{aligned}
\widetilde{K}_{,t} + v\cdot\nabla\widetilde{K} + (v_{r,r} + v_{z,z})\widetilde{K} - \nu\Delta\widetilde{K} &= \frac{2\nu}{r^2}\bigg(-\widetilde{H}_{\varphi,z} + \frac{1}{r}\widetilde{H}_{z,\varphi}\bigg)\\
-\frac{1}{r}\bigg(w_{,z}\widetilde{H}_r - w_{,r}\widetilde{H}_z + \frac{w}{r}\widetilde{H}_z\bigg) &+ \frac{1}{r}[(w^1)^2_{,z} - (w^2)^2_{,z}],\\
\widetilde{K}|_{S^T} &= 0,\\
\widetilde{K}|_{t=0} &= 0,
\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}
\widetilde{W}_{,t} + v\cdot\nabla\widetilde{W} + \frac{v_r}{r}\widetilde{W} - \nu\Delta\widetilde{W} + \nu\frac{\widetilde{W}}{r^2} &= \frac{1}{r}\widetilde{Q} + \frac{2\nu}{r^2}\widetilde{H}_r,\\
\widetilde{W}_{,r}|_{r=R_i} &= \frac{1}{R_i}\widetilde{W}, \quad i = 1,2, \text{ on } S_1^T,\\
\widetilde{W}_{,z}|_{S_2^T} &= 0\\
\widetilde{W}|_{t=0} &= 0.
\end{aligned} \tag{4.16}$$

Multiplying $(4.15)_1$ by $\widetilde{K}/r^2$, integrating over Ω^t and using (3.3) give

$$\begin{aligned}
&\bigg|\frac{\widetilde{K}}{r}\bigg|^2_{2,\Omega} + \int_0^t \bigg\|\frac{\widetilde{K}}{r}\bigg\|^2_{1,\Omega} dt'\\
&\le c\int_0^t |\nabla\widetilde{H}|^2_{6/5,\Omega}\, dt' + d_2^2(T)|\widetilde{H}|^2_{3,\infty,\Omega^t} + \varphi(A)\int_0^t |\widetilde{W}(t')|^2_{4,\Omega}\, dt'.
\end{aligned} \tag{4.17}$$

To estimate the last term on the r.h.s. of (4.17) we multiply $(4.16)_1$ by $\widetilde{W}$ and integrate the result over Ω. Hence we get

$$\frac{d}{dt}|\widetilde{W}|^2_{2,\Omega} + \|\widetilde{W}\|^2_{1,\Omega} \le c|v_r|^2_{2,\Omega}|\widetilde{W}|^2_{2,\Omega} + c(|\widetilde{Q}|^2_{2,\Omega} + |\widetilde{H}|^2_{2,\Omega}). \tag{4.18}$$

Integrating (4.18) with respect to time and utilizing (3.2) yield

$$|\widetilde{W}|^2_{2,\Omega} + \int_0^t \|\widetilde{W}(t')\|^2_{1,\Omega}\, dt' \le c(A, d_2)(|\widetilde{H}|^2_{2,\Omega^t} + |\widetilde{Q}|^2_{2,\Omega^t}). \tag{4.19}$$

Exploiting (4.19) in (4.17) implies

$$\left|\frac{\widetilde{K}}{r}\right|_{2,\Omega}^2+\int_0^t\left\|\frac{\widetilde{K}}{r}\right\|_{1,\Omega}^2dt'\le\varphi(A,d_2)(|\nabla\widetilde{H}|^2_{6/5,2,\Omega^t}+|\widetilde{H}|^2_{3,\infty,\Omega^t}+|\widetilde{H}|^2_{2,\Omega^t}+|\widetilde{Q}|^2_{2,\Omega^t}).$$

Now we consider problem (3.19) where v on the r.h.s. of $(3.19)_2$ is treated as the weak solution. Therefore (3.19) implies

$$\widetilde{V}_{r,z}-\widetilde{V}_{z,r}=\widetilde{K},\quad \widetilde{V}_{r,r}+\widetilde{V}_{z,z}=-\frac{1}{r}\widetilde{H}_\varphi,\quad \widetilde{V}_r|_{S_1}=0,\quad \widetilde{V}_z|_{S_2}=0. \tag{4.20}$$

For solutions of (4.20) we have

$$\begin{aligned}\sup_t&\|\widetilde{V}\|_{1,\Omega}+\|\widetilde{V}\|_{L_{10/3}(0,T;W^1_{10/3}(\Omega))}+\|\widetilde{V}\|_{L_2(0,T;H^2(\Omega))}\\&\le c(|\widetilde{K}|_{2,\infty,\Omega^T}+|\widetilde{H}|_{2,\infty,\Omega^T}+\|\widetilde{K}\|_{L_2(0,T;H^1(\Omega))}+\|\widetilde{H}\|_{L_2(0,T;H^1(\Omega))})\\&\le c(|\widetilde{H}|_{3,\infty,\Omega^T}+\|\widetilde{H}\|_{L_2(0,T;H^1(\Omega))}+|\widetilde{Q}|_{2,\Omega^T})\equiv A_1.\end{aligned}$$

For solutions of (4.16) we have

$$\begin{aligned}|\widetilde{W}|_{4,\Omega}&+\bigg(\int_0^t(|\widetilde{W}(t')|^4_{4,\Omega}+|\nabla\widetilde{W}^2(t')|^2_{2,\Omega})\,dt'\bigg)^{1/4}\\&\le c(A,d_2)(|\widetilde{Q}|_{4/3,4,\Omega^t}+|\widetilde{H}|_{4/3,4,\Omega^t}+|\widetilde{Q}|_{2,\Omega^t}+|\widetilde{H}|_{2,\Omega^t})\equiv A_2,\end{aligned}$$

where

$$\int_{\Omega^t}|v_r|\,|\widetilde{W}|^4\,dx\,dt'\le\varepsilon_1\int_0^t|\nabla\widetilde{W}^2|^2_{2,\Omega}\,dt'+c\bigg(\frac{1}{\varepsilon_1},|v_r|_{2,\infty,\Omega^t}\bigg)|\widetilde{W}|^4_{2,\Omega^t}$$

and the last term we estimate in view of (4.19).

In virtue of the above considerations we have

$$|\widetilde{V}|_{10,\Omega^T}\le cA_1,\qquad |\widetilde{W}|_{20/3,\Omega^T}\le cA_2.$$

Now we consider problem (1.4) with coefficients $\widetilde{v}=v(\widetilde{h},\widetilde{q})$. Hence we obtain

$$\begin{aligned}\widetilde{w}_{,t}-\nu\Delta\widetilde{w}&=-\widetilde{v}\cdot\nabla\widetilde{w}-\frac{\widetilde{v}_r}{r}\widetilde{w}-\nu\frac{\widetilde{w}}{r^2}+\frac{1}{r}\widetilde{q}+\frac{2\nu}{r^2}\widetilde{h}_r+f_\varphi,\\ \widetilde{w}_{,r}|_{r=R_i}&=\frac{1}{R_i}\widetilde{w},\\ \widetilde{w}_{,z}|_{S_2}&=0,\\ \widetilde{w}|_{t=0}&=w(0).\end{aligned}$$

Let $\widetilde{W} = w(\widetilde{h}^1, \widetilde{q}^1) - w(\widetilde{h}^2, \widetilde{q}^2)$. Then

$$\begin{aligned}
\widetilde{W}_{,t} - \nu\Delta\widetilde{W} &= -\widetilde{V}\cdot\nabla w(\widetilde{h}^1,\widetilde{q}^1) - v(\widetilde{h}^2,\widetilde{q}^2)\cdot\nabla\widetilde{W} - \frac{1}{r}\widetilde{V}_r w(\widetilde{h}^1,\widetilde{q}^1) \\
&\quad - \frac{1}{r}\widetilde{v}_r(\widetilde{h}^2,\widetilde{q}^2)\widetilde{W} - \frac{\nu}{r^2}\widetilde{W} + \frac{1}{r}\widetilde{Q} + \frac{2\nu}{r^2}\widetilde{H}_r, \\
\widetilde{W}_{,r}|_{r=R_i} &= \frac{1}{R_i}\widetilde{W}, \quad i=1,2, \text{ on } S_1^T, \\
\widetilde{W}_{,z} &= 0 \qquad \text{on } S_2^T, \\
\widetilde{W}|_{t=0} &= 0.
\end{aligned}$$

Repeating proofs of Lemmas 3.7 and 3.8 we obtain

$$\|\widetilde{W}\|_{2,\Omega^t} \le \varphi(A)(A_1 + A_2).$$

Let us consider problem (1.1) with the nonlinear term equal to $v(\widetilde{h},\widetilde{q})\cdot\nabla v(\widetilde{h},\widetilde{q})$. Denoting a solution of such problem by $v(\widetilde{h},\widetilde{q})$ and introducing $\widetilde{V} = v(\widetilde{h}^1,\widetilde{q}^1) - v(\widetilde{h}^2,\widetilde{q}^2)$ we see that

$$\begin{aligned}
\widetilde{V}_{,t} - \nu\Delta\widetilde{V} + \nabla\widetilde{P} &= -\widetilde{V}\cdot\nabla v(\widetilde{h}^1,\widetilde{q}^1) - v(\widetilde{h}^2,\widetilde{q}^2)\cdot\nabla\widetilde{V}, \\
\operatorname{div}\widetilde{V} &= 0, \\
\overline{n}\cdot\widetilde{V} &= 0, \\
\overline{n}\cdot\mathbb{D}(\widetilde{V})\cdot\overline{\tau}_\alpha &= 0, \quad \alpha = 1,2, \\
\widetilde{V}|_{t=0} &= 0,
\end{aligned}$$

where $\widetilde{P} = p(\widetilde{h}^1,\widetilde{q}^1) - p(\widetilde{h}^2,\widetilde{q}^2)$. Repeating the proof of Lemma 3.9 we have

$$\|\widetilde{V}\|_{2,5/2,\Omega^T} \le \varphi(A)(A_1 + A_2). \tag{4.21}$$

Utilizing (4.21) in (4.14) yields

$$\|H\|_{2+\beta,\delta,\Omega^T} + \|\nabla Q\|_{\beta,\delta,\Omega^T} \le \varphi(A)(\|\widetilde{H}\|_{2+\beta,\delta,\Omega^T} + \|\nabla\widetilde{Q}\|_{\beta,\delta,\Omega^T}).$$

Hence continuity of Φ follows. □

Now we can prove the main result of this section

Proof of Theorem 1. By Lemma 3.1 we have the existence of weak solutions $\overline{v}$ to problem (1.1) with corresponding estimates. By Lemmas 3.2–3.10 we construct the transformation $v = v(h,q,\overline{v})$. Next by Lemmas 4.1–4.4 we can apply the Leray–Schauder fixed point theorem to prove the existence of a fixed point of transformation (4.4) so equivalently the existence of solutions to problem (4.3) such that $h \in W_\delta^{2+\beta,1+\beta/2}(\Omega^T)$, $q \in G_\delta^{\beta,\beta/2}(\Omega^T)$ with the corresponding estimate. Then Lemma 3.9 implies that the weak solution is such that $v \in W_{5/2}^{2,1}(\Omega^T)$, where the existence time T is determined by (4.11). From (4.11) T is large for small $|g|_{2,\Omega^T} + |h(0)|_{2,\Omega}$. □

Finally we prove uniqueness.

Proof of Theorem 2. Assume that we have two solutions of problem (1.1), v^i, p^i, $i=1,2$. Then $V=v^1-v^2$, $P=p^1-p^2$ are solutions to the problem

$$
\begin{aligned}
V_t+v^2\cdot\nabla V+V\cdot\nabla v^1-\operatorname{div}\mathbb{T}(V,P)&=0,\\
\operatorname{div}V&=0,\\
V\cdot\overline{n}&=0,\\
\overline{n}\cdot\mathbb{T}(V,P)\cdot\overline{\tau}_\alpha&=0,\quad \alpha=1,2,\\
V|_{t=0}&=0.
\end{aligned}
\tag{4.22}
$$

Multiplying $(4.22)_1$ by V and integrating over Ω yield

$$
\frac{1}{2}\frac{d}{dt}|V|_{2,\Omega}^2+\int_\Omega V\cdot\nabla v^1\cdot V\,dx+E_\Omega(V)=0.
$$

Hence

$$
\begin{aligned}
\frac{d}{dt}\bigg[|V|_{2,\Omega}^2\exp\bigg(\int_0^t|\nabla v^1(t')|_{\infty,\Omega}\,dt'\bigg)\bigg]&\\
+E_\Omega(V)\exp\bigg(\int_0^t|\nabla v^1(t')|_{\infty,\Omega}\,dt'\bigg)&=0.
\end{aligned}
\tag{4.23}
$$

Integrating (4.23) with respect to time implies uniqueness. □

5. Global existence

The aim of this section is to prove global existence of solutions to problem (1.1) by prolonging the local solution from Section 4 step by step. For this purpose we want to show that estimate (4.7) holds in any time interval $[(k-1)T, kT]$, $k\in\mathbb{N}$, utilizing that for $k=1$ it is already proved.

For this purpose we have to satisfy (4.11) for all intervals $[(k-1)T,kT]$, $k\in\mathbb{N}$, with the same T and A. Hence we must show that all quantities in (4.11) do not increase with time.

First we introduce the decay estimates,

$$
\begin{aligned}
|f(t)|_{3,\Omega}&\le|f(0)|_{3,\Omega}e^{-\delta_1t}, & |g(t)|_{2,\Omega}&\le|g(0)|_{2,\Omega}e^{-\delta_2t},\\
|F_\varphi(t)|_{2,\Omega}&\le|F_\varphi(0)|_{2,\Omega}e^{-\delta_3t}, & \|g(t)\|_{\beta,\delta,\Omega}&\le\|g(0)\|_{\beta,\delta,\Omega}e^{-\delta_4t},\\
|g(t)|_{1,0,\Omega}&\le|g(0)|_{1,0,\Omega}e^{-\delta_5t}
\end{aligned}
\tag{5.1}
$$

where δ_i, $i=1,\dots,5$, are positive constants and

$$
\|g\|_{L_\delta(\Omega;W_\delta^{\beta/2}(((k-1)T,kT))}\le a,
\tag{5.2}
$$

where a does not depend on $k\in\mathbb{N}$. In view of (5.1) and (5.2) we know that F does not increase with k.

Now we want to show that the quantity

$$F_0'(k) = |h(kT)|_{1,0,\Omega} + \|w(kT)\|_{1,\Omega} + |\chi(kT)|_{2,\Omega} + \|v(kT)\|_{6/5,5/2,\Omega} + \|h(kT)\|_{2+\beta-2/\delta,\delta,\Omega} + |w(kT)|^2_{4,\Omega} + |g(kT)|_{1,0,\Omega} \tag{5.3}$$

does not increase with k too. To show the statement we need a series of lemmas.

Lemma 5.1. *Assume that $\delta_2 > \nu$, $v \in L_2(0,T;W^1_3(\Omega))$, $g(0), h(0) \in L_2(\Omega)$. Then*

$$|h(t)|^2_{2,\Omega} \le e^{-\nu t + c|\nabla v|^2_{3,2,\Omega^t}}\left[\frac{1}{\delta_2-\nu}|g(0)|^2_{2,\Omega} + |h(0)|^2_{2,\Omega}\right]. \tag{5.4}$$

Proof. Utilizing $(5.1)_2$ in (3.15) and integrating the result with respect to time yield

$$|h(t)|^2_{2,\Omega} e^{\nu t - c\int_0^t |\nabla v(t')|^2_{3,\Omega}\,dt'} \le c|g(0)|^2_{2,\Omega}\int_0^t e^{-(\delta_2-\nu)t'}\,dt' + |h(0)|^2_{2,\Omega}. \tag{5.5}$$

Choosing $\delta_2 > \nu$ we obtain (5.4). □

Since for the local solution described by Theorem 1 we have $|\nabla v|_{3,2,\Omega^T} \le \varphi_1(A)$, where φ_1 is an increasing positive function and A is defined by (4.11), the inequality (5.4) for T sufficiently large implies $|h(T)|_{2,\Omega} \le |h(0)|_{2,\Omega}$. Next we have

Lemma 5.2. *Assume that there exists a weak solution described by Lemma 3.1. Assume that (5.1) holds, $\delta_* = \min\{\delta_1,\delta_2,\delta_3\}$ and $\nu > 4\delta_*$. Assume that $v \in L_2(0,T;W^1_3(\Omega))$, $g(0) \in L_4(\Omega)$, $f(0) \in L_{5/2}(\Omega)$, $F_\varphi(0) \in L_2(\Omega)$ and $\chi(0), h(0), w^2(0) \in L_2(\Omega)$. Assume that (5.16) with γ_1 sufficiently small holds. Then*

$$\begin{aligned}\left|\frac{\chi(t)}{r}\right|^2_{2,\Omega} &+ |h(t)|^2_{2,\Omega} + |w^2(t)|^2_{2,\Omega} \le c(d_1)d_1^2 e^{c\int_0^t \|v(t')\|^2_{1,3,\Omega}\,dt'}\\ &+ \frac{c}{\nu/2-2\delta_*}e^{c\int_0^t \|v(t')\|^2_{1,3,\Omega}\,dt' - 2\delta_* t}[\varphi(A)|g(0)|^2_{2,\Omega}\\ &+ |g(0)|^4_{4,\Omega} + |f(0)|^4_{5/2,\Omega} + |F_\varphi(0)|^2_{2,\Omega}]\\ &+ e^{-\nu t/2 + c\int_0^t \|v(t')\|^2_{1,3,\Omega}\,dt'}\left(\left|\frac{\chi(0)}{r}\right|^2_{2,\Omega} + |h(0)|^2_{2,\Omega} + |w(0)|^4_{4,\Omega}\right).\end{aligned} \tag{5.6}$$

Proof. In view of (3.3) inequality (3.8) takes the form

$$\frac{d}{dt}\left|\frac{\chi}{r}\right|^2_{2,\Omega} + \frac{\nu}{2}\left\|\frac{\chi}{r}\right\|^2_{1,\Omega} \le c(|\nabla h|^2_{6/5,\Omega} + \|w\|^2_{1,3,\Omega}|h|^2_{2,\Omega} + |w|^4_{4,\Omega} + |F_\varphi|^2_{2,\Omega}). \tag{5.7}$$

From (3.10) we have

$$\frac{d}{dt}|h|^2_{2,\Omega} + \frac{\nu}{2}\|h\|^2_{1,\Omega} \le c(|\nabla v|^2_{3,\Omega}|h|^2_{2,\Omega} + |g|^2_{2,\Omega}). \tag{5.8}$$

Adding (5.7) and (5.8) appropriately implies

$$\frac{d}{dt}\left(\left|\frac{\chi}{r}\right|^2_{2,\Omega}+|h|^2_{2,\Omega}\right)+\frac{\nu}{2}\left(\left\|\frac{\chi}{r}\right\|^2_{1,\Omega}+\|h\|^2_{1,\Omega}\right) \le c(\|v\|^2_{1,3,\Omega}|h|^2_{2,\Omega}+|w|^4_{4,\Omega}+|F_\varphi|^2_{2,\Omega}+|g|^2_{2,\Omega}). \tag{5.9}$$

From the proof of Lemma 3.5 we obtain

$$\frac{d}{dt}|w|^4_{4,\Omega}+\frac{\nu}{2}|\nabla|w|^2|^2_{2,\Omega}\le c(d_1)d_1^2+c(|q|^4_{4/3,\Omega}+|h|^4_{4/3,\Omega}+|f_\varphi|^4_{4/3,\Omega}). \tag{5.10}$$

To examine the norm $|q|_{4/3,\Omega}$ we consider the following elliptic problem for q

$$\begin{aligned} \Delta q &= -\operatorname{div}(v\cdot\nabla h+h\cdot\nabla v)+\operatorname{div} g,\\ \frac{\partial q}{\partial n}\Big|_{S_1} &= g\cdot\overline{n}-\overline{n}\cdot(v\cdot\nabla h+h\cdot\nabla v)+\frac{2\nu}{R^2}h_{\varphi,\varphi},\\ \frac{\partial q}{\partial n}\Big|_{S_2} &= g\cdot\overline{n}-\overline{n}\cdot(v\cdot\nabla h+h\cdot\nabla v). \end{aligned} \tag{5.11}$$

To obtain an estimate for $|q|_{2,\Omega}$ we introduce a function α such that

$$\Delta\alpha=q,\quad \frac{\partial\alpha}{\partial n}\Big|_S=0,\quad \int_\Omega \alpha\,dx=0. \tag{5.12}$$

Multiplying $(5.11)_1$ by α and integrating over Ω yield

$$\int_\Omega \Delta q\alpha\,dx=-\int_\Omega \operatorname{div}(v\cdot\nabla h+h\cdot\nabla v)\alpha\,dx+\frac{2\nu}{R^2}\int_{S_1}h_{\varphi,\varphi}\alpha\,dS_1+\int_\Omega \operatorname{div} g\alpha\,dx.$$

Integrating by parts and utilizing the boundary conditions $(5.11)_{2,3}$ imply

$$\int_\Omega \nabla q\nabla\alpha\,dx=-\frac{2\nu}{R^2}\int_{S_1}h_{\varphi,\varphi}\alpha\,dS_1+\int_\Omega g\cdot\nabla\alpha\,dx.$$

Integrating by parts again and using (5.12) give

$$|q|^2_{2,\Omega}=\frac{2\nu}{R^2}\int_{S_1}h_\varphi\alpha_{,\varphi}\,dS_1+\int_\Omega g\cdot\nabla\alpha\,dx. \tag{5.13}$$

Since solutions of (5.12) satisfy $\|\alpha\|_{2,\Omega}\le c|q|_{2,\Omega}$, we obtain from (5.13) the estimate

$$|q|_{2,\Omega}\le c(|h_\varphi|_{2,S_1}+|g|_{2,\Omega}). \tag{5.14}$$

Utilizing (5.14) in (5.10) and using that $\int_\Omega w^2\,dx\le d_1^2$ we obtain from (5.10) the inequality

$$\frac{d}{dt}|w^2|^2_{2,\Omega}+\frac{\nu}{2}\|w^2\|^2_{1,\Omega}\le c(d_1)d_1^2+c(\|h\|^2_{1,\Omega}+|g|^2_{2,\Omega})^2+c|f_\varphi|^4_{4/3,\Omega}. \tag{5.15}$$

Assuming that

$$\sup_t(\|h\|_{1,\Omega}+|g|_{2,\Omega})\le\gamma_1, \tag{5.16}$$

where γ_1 is sufficiently small we get from (5.9) and (5.15) the inequality

$$
\frac{d}{dt}\bigg(\bigg|\frac{\chi}{r}\bigg|^2_{2,\Omega}+|h|^2_{2,\Omega}+|w^2|^2_{2,\Omega}\bigg)+\frac{\nu}{2}\bigg(\bigg\|\frac{\chi}{r}\bigg\|^2_{1,\Omega}+\|h\|^2_{1,\Omega}+\|w^2\|^2_{1,\Omega}\bigg) \tag{5.17}
$$
$$
\le c\|v\|^2_{1,3,\Omega}|h|^2_{2,\Omega}+c(d_1)d_1^2+\varphi(A)|g|^2_{2,\Omega}+c(|g|^4_{2,\Omega}+|f_\varphi|^4_{4/3,\Omega}+|F_\varphi|^2_{2,\Omega}).
$$

To simplify notation we introduce

$$
\eta(t)=\bigg|\frac{\chi(t)}{r}\bigg|^2_{2,\Omega}+|h(t)|^2_{2,\Omega}+|w^2(t)|^2_{2,\Omega}.
$$

Then (5.17) takes the form

$$
\frac{d}{dt}\eta+\frac{\nu}{2}\eta\le c\|v\|^2_{1,3,\Omega}\eta+c(d_1)d_1^2+\varphi(A)|g|^2_{2,\Omega} \tag{5.18}
$$
$$
+c(|g|^4_{2,\Omega}+|f_\varphi|^4_{4/3,\Omega}+|F_\varphi|^2_{2,\Omega}).
$$

Continuing, (5.18) implies

$$
\frac{d}{dt}(\eta e^{\nu t/2-c\int_0^t\|v(t')\|^2_{1,3,\Omega}\,dt'})\le c(d_1)d_1^2e^{\nu t/2-c\int_0^t\|v(t')\|^2_{1,3,\Omega}\,dt'} \tag{5.19}
$$
$$
+c[\varphi(A)|g|^2_{2,\Omega}+|g|^4_{2,\Omega}+|f_\varphi|^4_{4/3,\Omega}+|F_\varphi|^2_{2,\Omega}]e^{\nu t/2-c\int_0^t\|v(t')\|^2_{1,3,\Omega}\,dt'}.
$$

Integrating (5.19) with respect to time and using (5.1) yield

$$
\eta(t)\le c(d_1)d_1^2e^{c\int_0^t\|v(t')\|^2_{1,3,\Omega}\,dt'} \tag{5.20}
$$
$$
+\frac{c}{\nu/2-2\delta_*}e^{c\int_0^t\|v(t')\|^2_{1,3,\Omega}\,dt'-2\delta_*t}[\varphi(A)|g(0)|^2_{2,\Omega}+|g(0)|^4_{4,\Omega}
$$
$$
+|f(0)|^4_{5/2,\Omega}+|F_\varphi(0)|^2_{2,\Omega}]+e^{-\nu t/2+c\int_0^t\|v(t')\|^2_{1,3,\Omega}\,dt'}\eta(0).
$$

Finally (5.20) implies (5.6). □

To satisfy (5.16) we need

Lemma 5.3. *Assume that $v\in L_2(0,T;H^2(\Omega))$, $h(0)\in H^1(\Omega)$, $g(0)\in L_2(\Omega)$. Then*

$$
\|h(t)\|^2_{1,\Omega}\le ce^{-\nu t+\int_0^t\|v(t')\|^2_{2,\Omega}\,dt'}(\|h(0)\|^2_{1,\Omega}+|g(0)|^2_{2,\Omega}). \tag{5.21}
$$

Proof. From [1, (6.3.56)] and for $2\delta_2>\nu$ we have

$$
|\mathbb{D}(h)(t)|^2_{2,\Omega}\le ce^{-\nu t+\int_0^t\|v(t')\|^2_{2,\Omega}\,dt'}(|g(0)|^2_{2,\Omega}+|\mathbb{D}(h)(0)|^2_{2,\Omega}).
$$

Emploing (5.4) and Lemma 4.2.5, Remark 4.2.6 from [1] we obtain (5.21). □

From (5.21) we obtain for a given local solution and T sufficiently large the inequality

$$
\|h(T)\|_{1,\Omega}\le\|h(0)\|_{1,\Omega}.
$$

Moreover, smallness of the r.h.s. of (5.21) implies that (5.16) might be satisfied.

LEMMA 5.4. *Assume that* $h(0) \in H^1(\Omega)$, $h_{,t}(0) \in L_2(\Omega)$, $g(0) \in L_2(\Omega)$, $g_{,t}(0) \in L_2(\Omega)$. *Assume that there exists a local solution to* (1.1) *such that* $\|v\|_{2,3,\Omega^T} \le B$ *with* T *sufficiently large. Assume* (5.1), (5.2). *Assume that* $0 < t_1 < T$. *Then*

$$\|h(T)\|_{2+\beta-2/\delta,\delta,\Omega} \le \varphi(B)e^{-\nu t_1}(|h(0)|_{1,0,\Omega} + |g(0)|_{1,0,\Omega} + \|g(0)\|_{\beta,\delta,\Omega}) + ca, \tag{5.22}$$

where φ *is an increasing positive function.*

PROOF. Let $\zeta = \zeta(t)$ be a smooth function such that $\zeta(t) = 1$ for $t \ge t_2$, $\zeta(t) = 0$ for $t \le t_1$, $0 < t_1 < t_2 < T$. Let $\widetilde{h} = h\zeta$, $\widetilde{q} = q\zeta$, $\widetilde{g} = g\zeta$. Multiplying (1.2) by ζ yields

$$\begin{aligned} \widetilde{h}_{,t} - \operatorname{div}\mathbb{D}(\widetilde{h},\widetilde{q}) &= -v\cdot\nabla\widetilde{h} - \widetilde{h}\cdot\nabla v + \widetilde{g} + h\dot{\zeta},\\ \operatorname{div}\widetilde{h} &= 0,\\ \widetilde{h}\cdot\overline{n} &= 0,\\ \overline{n}\cdot\mathbb{D}(\widetilde{h})\cdot\overline{\tau}_\alpha &= 0, \quad \alpha = 1,2,\\ \widetilde{h}|_{t=0} &= 0, \end{aligned} \tag{5.23}$$

where $\dot{\zeta} = \zeta_{,t}$. For solutions of (5.23) we have

$$\|\widetilde{h}\|_{2+\beta,\delta,\Omega^T} \le c(\|v\cdot\nabla\widetilde{h}\|_{\beta,\delta,\Omega^T} + \|\widetilde{h}\cdot\nabla v\|_{\beta,\delta,\Omega^T} + \|\widetilde{g}\|_{\beta,\delta,\Omega^T} + \|h\dot{\zeta}\|_{\beta,\delta,\Omega^T}).$$

In view of (3.45) and (3.48) we obtain

$$\|\widetilde{h}\|_{2+\beta,\delta,\Omega^T} \le \varphi(\|v\|_{2,5/2,\Omega^T})|\widetilde{h}|_{2,\Omega^T} + c\|\widetilde{g}\|_{\beta,\delta,\Omega^T} + c\|h\dot{\zeta}\|_{\beta,\delta,\Omega^T}. \tag{5.24}$$

Using that $\|v\|_{2,5/2,\Omega^T} \le \varphi(A)$ (see Theorem 1) inequality (5.24) takes the form

$$\begin{aligned} \|\widetilde{h}\|_{2+\beta,\delta,\Omega^T} \le \varphi(B)[|\widetilde{h}|_{2,\Omega^T} &+ \|h\dot{\zeta}\|_{L_\delta(0,T;W^\beta_\delta(\Omega))}\\ &+ \|h\dot{\zeta}\|_{L_\delta(\Omega;W^{\beta/2}_\delta(0,T))}] + c\|g\|_{\beta,\delta,\Omega^T}. \end{aligned} \tag{5.25}$$

The second term on the r.h.s. of (5.25) takes the form

$$\biggl[\int_0^T dt(|h\dot{\zeta}|^\delta_{\delta,\Omega} + \langle\!\langle h\dot{\zeta}\rangle\!\rangle^\delta_{\beta,\delta,x,\Omega})\biggr]^{1/\delta} \le c\biggl[\int_{t_1}^{t_2} dt(|h|^\delta_{\delta,\Omega} + \langle\!\langle h\rangle\!\rangle^\delta_{\beta,\delta,x,\Omega})\biggr]^{1/\delta} \equiv I_1,$$

because $\dot{\zeta}(t) \ne 0$ for $t \in (t_1,t_2)$. For $\delta \le 2$ and $\beta < 1$ we have by imbedding $H^1(\Omega) \subset W^\beta_\delta(\Omega)$ that

$$I_1 \le c\biggl(\int_{t_1}^{t_2} \|h(t)\|^\delta_{1,\Omega}\,dt\biggr)^{1/\delta} \le c(t_2-t_1)^{1/\delta}\sup_{t\in(t_1,t_2)}\|h(t)\|_{1,\Omega} \equiv I_2.$$

Utilizing (5.21) yields

$$I_2 \le \varphi(B)e^{-\nu t_1}(\|h(0)\|_{1,\Omega} + |g(0)|_{2,\Omega}) \equiv I_3.$$

The third term on the r.h.s. of (5.25) we estimate by

$$c(t_2-t_1)^{1/\delta}\sup_{t\in(t_1,t_2)}|h(t)|_{2,\Omega}+\bigg(\int_\Omega dx\langle\!\langle h\rangle\!\rangle^\delta_{\beta/2,\delta,t,(t_1,t_2)}\bigg)^{1/\delta}\equiv I_4.$$

The first expression in I_4 is estimated by I_3 and the second by

$$c(t_2-t_1)^{1/\delta}\|h\|_{L_2(\Omega;H^1(t_1,t_2))}\equiv I_5.$$

Continuing,

$$I_5\le\varphi(t_2-t_1)\sup_t|h(t)|_{1,0,\Omega}\equiv I_6.$$

To estimate the r.h.s. we use inequality (6.3.41) from [4],

$$|h(t)|_{1,0,\Omega}\le e^{-\nu t}\varphi(B)\bigg[|h(0)|_{1,0,\Omega}+\bigg(\int_0^t|g(t')|^2_{1,0,\Omega}e^{\nu t'}dt'\bigg)^{1/2}\bigg]. \tag{5.26}$$

Using that $|\widetilde{h}|_{2,\Omega^T}\le cI_3$, and (5.26) to estimate I_6, we obtain from (5.25) the inequality

$$\|\widetilde{h}\|_{2+\beta,\delta,\Omega^T}\le\varphi(B)e^{-\nu t_1}(|h(0)|_{1,0,\Omega}+|g(0)|_{1,0,\Omega}+\|g(0)\|_{\beta,\delta,\Omega})+ca,$$

which implies (5.22). □

To prolong the local solution we need also.

Lemma 5.5. *Assume that $h_{,t}(0)$, $g_{,t}(0)$, $F_\varphi(0),\chi(0)\in L_2(\Omega)$, $f(0)\in L_{5/2}(\Omega)$, $h(0),g(0),w(0)\in H^1(\Omega)$. Assume that there exists numbers t_1, t_2, δ_* such that $0<t_1<t_2<T$, $\delta_*=\min\{\delta_1,\delta_2,\delta,\nu/2\}$. Assume also that there exists a weak solution described by Lemma 3.1. Assume that $v\in W^{2,1}_3(\Omega^T)$ and $\|v\|_{2,3,\Omega^T}\le B$. Then*

$$\begin{aligned}\|v(T)\|_{6/5,5/2,\Omega}\le{}&\varphi(d_1,d_2,B)[e^{-\delta_*t_1}(|h(0)|_{1,0,\Omega}+|\chi(0)|_{2,\Omega}\\&+|g(0)|_{1,0,\Omega}+|f_\varphi(0)|_{2,\Omega}+|F_\varphi(0)|_{2,\Omega})+1]^{10}+\frac{c}{(t_2-t_1)^{2/5}}d_2,\end{aligned}\tag{5.27}$$

where φ is an increasing positive function.

Proof. Let ζ be the same as in Lemma 5.4. Let $\widetilde{v}=v\zeta$, $\widetilde{p}=p\zeta$, $\widetilde{f}=f\zeta$. Multiplying (1.1) by ζ we obtain

$$\begin{aligned}\widetilde{v}_{,t}+v\cdot\nabla\widetilde{v}-\operatorname{div}\mathbb{T}(\widetilde{v},\widetilde{p})&=\widetilde{f}+v\dot{\zeta},\\ \operatorname{div}\widetilde{v}&=0,\\ \widetilde{v}\cdot\overline{n}|_S&=0,\\ \overline{n}\cdot\mathbb{T}(\widetilde{v},\widetilde{p})\cdot\overline{\tau}_\alpha|_S&=0,\quad\alpha=1,2,\\ \widetilde{v}|_{t=0}&=0,\end{aligned}$$

where $\dot\zeta = \zeta_{,t}$. Repeating the proof of Lemma 3.9 applied to the above problem we obtain

$$\begin{aligned}\|\widetilde{v}\|_{2,5/2,\Omega\times(t_1,T)} \le c\Big[&\overline{d}_2|h|_{3,\infty,\Omega\times(t_1,T)} + |h|_{4/3,4,\Omega\times(t_1,T)} + |q|_{4/3,4,\Omega\times(t_1,T)}\\ &+ |h|_{5/3,\Omega\times(t_1,T)} + |q|_{5/3,\Omega\times(t_1,T)} + |h|_{2,\Omega\times(t_1,T)} + |q|_{2,\Omega\times(t_1,T)}\\ &+ d_1 + \overline{d}_2 + |g|_{2,\Omega\times(t_1,T)} + |f_\varphi|_{4/3,4,\Omega\times(t_1,T)} + |F_\varphi|_{2,\Omega\times(t_1,T)}\\ &+ |f_\varphi|_{4/3,\Omega\times(t_1,T)} + |f_\varphi|_{2,\Omega\times(t_1,T)}\\ &+ |h(0)|_{2,\Omega} + \left|\frac{\chi(0)}{r}\right|_{2,\Omega} + \|w(0)\|_{1,\Omega} + 1\Big]^{10}\\ &+ c|f|_{5/2,\Omega\times(t_1,T)} + c|\dot\zeta v|_{5/2,\Omega\times(t_1,T)}.\end{aligned}$$

In view of the decay estimates (5.1), (5.21), (5.22) we obtain

$$\begin{aligned}(5.28)\quad \|\widetilde{v}\|_{2,5/2,\Omega\times(t_1,T)} &\le \varphi(d_1,d_2,B)[e^{-\delta_* t_1}(|h(0)|_{1,0,\Omega} + |\chi(0)|_{2,\Omega}\\ &+ |g(0)|_{1,0,\Omega} + |f_\varphi(0)|_{2,\Omega} + |F_\varphi(0)|_{2,\Omega}) + 1]^{10} + \frac{c}{(t_2-t_1)^{2/5}}d_2.\end{aligned}$$

From (5.28) we get (5.27). □

LEMMA 5.6. *Let B be a positive constant. Assume that*

$$\|v\|_{2,3,\Omega\times(kT,(k+1)T)} \le B, \quad \text{for } k\in\mathbb{N}.$$

Assume (5.1) *and* $w(0), g(0), h(0) \in H^1(\Omega)$, $h_{,t}(0), g_{,t}(0), f_\varphi(0) \in L_2(\Omega)$. *Assume that T is sufficiently large* (*see* (5.33), (5.37), (5.38)). *Then*

$$\begin{aligned}(5.29)\qquad \|w(t)\|_{1,\Omega} \le\ & c(|w(0)|_{2,\Omega} + |h(0)|_{1,0,\Omega}\\ &+ |g(0)|_{1,0,\Omega}) + c|f_\varphi(0)|_{2,\Omega} + e^{-t}\|w(0)\|_{1,\Omega}.\end{aligned}$$

PROOF. By Lemma 6.3.4 from [4] have

$$\begin{aligned}(5.30)\quad &\|w(t)\|_{1,\Omega}^2\\ &\le c\exp(c\|v\|_{2,5/2,\Omega^t}^2)(1+\|v\|_{2,5/2,\Omega^t}^4)\sup_{t'\le t}|w(t')|_{2,\Omega}^2 + c\sup_{t'\le t}|w(t')|_{2,\Omega}^2\\ &\quad + \int_0^t(|q(t')|_{2,\Omega}^2 + |h(t')|_{2,\Omega}^2| + |f_\varphi(t')|_{2,\Omega}^2)\,dt' + e^{-t}\|w(0)\|_{1,\Omega}^2.\end{aligned}$$

To examine the r.h.s. of (5.30) we need the global estimate (see Lemma 6.3.5 from [4])

$$(5.31)\quad |w(t)|_{2,\Omega} \le \frac{R_2}{R_1}|w(0)|_{2,\Omega} + c\int_0^t(|q(t')|_{2,\Omega} + |h(t')|_{2,\Omega} + |f_\varphi(t')|_{2,\Omega})\,dt.$$

In view of (5.1) we obtain from (5.31) the estimate

$$(5.32)\quad |w(t)|_{2,\Omega} \le \frac{R_2}{R_1}|w(0)|_{2,\Omega} + c\int_0^t(|q(t')|_{2,\Omega} + |h(t')|_{2,\Omega})\,dt' + c|f_\varphi(0)|_{2,\Omega},$$

for $t \in \mathbb{R}_+$. Now we examine the integrals on the r.h.s. of (5.32). To estimate the integral

$$I_1(t) = \int_0^t |h(t')|_{2,\Omega}\, dt'$$

we exploit (5.4) to calculate

$$|h(T)|_{2,\Omega}^2 \le e^{-\nu T - cB^2}[|g(0)|_{2,\Omega}^2 + |h(0)|_{2,\Omega}^2] \le e^{-\nu T/2}[|g(0)|_{2,\Omega}^2 + |h(0)|_{2,\Omega}^2]$$

where used that T is so large that

$$e^{-\nu T/2 + cB^2} \le 1. \tag{5.33}$$

Next

$$\begin{aligned}|h(2T)|_{2,\Omega}^2 &\le e^{-\nu T + cB^2}[|g(T)|_{2,\Omega}^2 + |h(T)|_{2,\Omega}^2]\\ &\le e^{-\nu T/2}[e^{-\delta_2 T}|g(0)|_{2,\Omega}^2 + e^{-\nu T/2}(|g(0)|_{2,\Omega}^2 + |h(0)|_{2,\Omega}^2)]\\ &\le e^{-\nu T}[e^{-(\delta_2 - \nu/2)T}|g(0)|_{2,\Omega}^2 + |g(0)|_{2,\Omega}^2 + |h(0)|_{2,\Omega}^2].\end{aligned}$$

Continuing the considerations and using that $\delta_2 > \nu/2$ we obtain

$$|h(kT)|_{2,\Omega}^2 \le e^{-\nu kT/2}[c|g(0)|_{2,\Omega}^2 + |h(0)|_{2,\Omega}^2].$$

Now

$$\begin{aligned}I_1((k+1)T) = \sum_{s=0}^k \int_{sT}^{(s+1)T} |h(t)|_{2,\Omega}\, dt \le \sum_{s=0}^k |h(sT)|T\\ \le T\sum_{s=0}^k e^{-\nu sT/4}[c|g(0)|_{2,\Omega} + |h(0)|_{2,\Omega}] \le cT(|g(0)|_{2,\Omega} + |h(0)|_{2,\Omega}).\end{aligned}$$

Finally we estimate the expression

$$J_1 = \int_0^t |q(t')|_{2,\Omega}\, dt'.$$

From (6.3.41) in [4] we have the inequality

$$\begin{aligned}|h_{,t}(t)|_{2,\Omega}^2 + \|h(t)\|_{1,\Omega}^2 \le e^{-\nu_0 t + cB^2}\bigg[|h_{,t}(0)|_{2,\Omega}^2 + \|h(0)\|_{1,\Omega}^2\\ + c\int_0^t (|g_{,t}(t')|_{2,\Omega}^2 + \|g(t')\|_{1,\Omega}^2)e^{\nu_0 t'}\, dt'\bigg],\end{aligned} \tag{5.34}$$

where $\nu_0 < \nu$, which holds for the local solution. In view of the decay estimate (5.1) we have

$$\begin{aligned}|h_{,t}(t)|_{2,\Omega}^2 + \|h(t)\|_{1,\Omega}^2 \le e^{-\nu_0 t + cB^2}[|h_{,t}(0)|_{2,\Omega}^2 + \|h(0)\|_{1,\Omega}^2\\ + \frac{1}{\delta_5 - \nu_0}(|g_{,t}(0)|_{2,\Omega}^2 + \|g(0)\|_{1,\Omega}^2)].\end{aligned}$$

From (6.3.45) in [4] we get

$$(5.35)\qquad |q(t)|_{2,\Omega}^2 \le c(1+\sup_t \|v(t)\|_{1,\Omega}^2)(|h_{,t}(t)|_{2,\Omega}^2 + \|h(t)\|_{1,\Omega}^2) + c|g(t)|_{2,\Omega}^2.$$

Using that

$$\sup_{t'\le t}\|v(t')\|_{1,\Omega} \le c(\|v\|_{2,\Omega^t} + \|v(0)\|_{1,\Omega}) \le c(B+\|v(0)\|_{1,\Omega}),\quad t\le T,$$

and in view of the above estimates the inequality (5.35) takes the form

$$(5.36)\quad |q(t)|_{2,\Omega}^2 \le \varphi(B)e^{-\nu_0 t}[|h_{,t}(0)|_{2,\Omega}^2 + \|h(0)\|_{1,\Omega}^2 + |g_{,t}(0)|_{2,\Omega}^2 + \|g(0)\|_{1,\Omega}^2].$$

In virtue of (5.36) we have

$$\begin{aligned}
|q(kT)|_{2,\Omega}^2 &\le \varphi(B)|h(kT)|_{1,0,\Omega}^2 + c|g(kT)|_{1,0,\Omega}^2\\
&\le \varphi(B)e^{-\nu_0 T}[|h((k-1)T)|_{1,0,\Omega}^2 + |g((k-1)T)|_{1,0,\Omega}^2]\\
&\quad + ce^{-\delta_5 T}|g((k-1)T)|_{2,\Omega}^2\\
&\le e^{-\nu_0 T/2}[|h((k-1)T)|_{1,0,\Omega}^2 + |g((k-1)T)|_{1,0,\Omega}^2] \equiv I_1,
\end{aligned}$$

where we used

$$(5.37)\qquad \varphi(B)e^{-\nu_0 T/2} + ce^{-(\delta_5-\nu_0/2)T} \le 1.$$

Continuing, we have

$$\begin{aligned}
I_1 &\le e^{-\nu_0 T/2}\bigg[e^{-\nu_0 T + cB^2}\bigg(|h((k-2)T)|_{1,0,\Omega}^2 + \frac{1}{\delta_5-\nu_0}|g((k-2)T)|_{1,0,\Omega}^2\bigg)\\
&\quad + e^{-\delta_5 T}|g((k-2)T)|_{1,0,\Omega}^2\bigg]\\
&\le e^{-\nu_0 T}[|h((k-2)T)|_{1,0,\Omega}^2 + |g((k-2)T)|_{1,0,\Omega}^2],
\end{aligned}$$

where we used

$$(5.38)\qquad e^{-\nu_0 T/2 + cB^2} \le 1,\quad \frac{1}{\delta_5-\nu_0}e^{-\nu_0 T/2+cB^2} + e^{-(\delta_5-\nu_0/2)T} \le 1.$$

Summarizing, for T sufficiently large we obtain

$$|q(kT)|_{2,\Omega}^2 \le e^{-\nu_0 kT/2}[|h(0)|_{1,0,\Omega}^2 + |g(0)|_{1,0,\Omega}^2].$$

Finally we calculate

$$\begin{aligned}
J_1 &\le \sum_{k=0}^{l-1}\int_{kT}^{(k+1)T}|q(t)|_{2,\Omega}\,dt \le (|h(0)|_{1,0,\Omega}^2 + |g(0)|_{1,0,\Omega}^2)^{1/2}T\sum_{k=0}^{l-1}e^{-\nu_0 kT/2}\\
&\le (|h(0)|_{1,0,\Omega}^2 + |g(0)|_{1,0,\Omega}^2)^{1/2}\frac{T}{1-e^{-\nu_0 T/2}}.
\end{aligned}$$

In view of the above considerations (5.6) implies

$$(5.39)\quad |w(t)|_{2,\Omega} \le \frac{R_2}{R_1}|w(0)|_{2,\Omega} + T\varphi(B)(|h(0)|_{1,0,\Omega} + |g(0)|_{1,0,\Omega}) + c|f_\varphi(0)|_{2,\Omega}.$$

Utilizing (5.39) in (5.31) and additionally using the considerations leading to (5.39) yield (5.29). □

We prove global existence of solutions to problem (1.1) step by step. Therefore we introduce the quantity

$$
\begin{aligned}
F(k) = {} & |g|_{2,\Omega\times(kT,(k+1)T)} + |f_\varphi|_{4/3,4,\Omega\times(kT,(k+1)T)} \\
& + |F_\varphi|_{2,\Omega\times(kT,(k+1)T)} + |f|_{5/3,\Omega\times(kT,(k+1)T)},
\end{aligned}
\tag{5.40}
$$

for $k \in \mathbb{N}$. To prolong the local solution step by step we have to satisfy (4.11) for any k,

$$
\begin{aligned}
& G(T, A, F(k), F_0'(k))(|g|_{2,\Omega\times(kT,(k+1)T)} + |h(kT)|_{2,\Omega}) \\
& \qquad + c(\|g\|_{\beta,\delta,\Omega\times(kT,(k+1)T)} + \|h(kT)\|_{2+\beta-2/\delta,\delta,\Omega}) \le A,
\end{aligned}
\tag{5.41}
$$

where $k \in \mathbb{N}$, G does not depend explicitly on k, and $F_0'(k)$ is defined by (5.3).

Our aim is to show that T and A do not depend on k. For this purpose we have to prove that all quantities in (5.40) which depend explicitly on k do not increase with k. Therefore we need some lemmas

Lemma 5.8. *Assume that* (5.1) *holds. Then*

$$
F(k) \le F(0), \quad k \in \mathbb{N}. \tag{5.42}
$$

Proof. Let $a_1, \dots, a_4$ be positive constants. Let $|f|_{3,\Omega^T} \le a_1$, $|g|_{2,\Omega^T} \le a_2$, $|F_\varphi|_{2,\Omega^T} \le a_3$, $\|g\|_{L_\delta(0,T;W_\delta^\beta(\Omega))} \le a_4$. Then (5.1) implies

$$
\begin{aligned}
|f|_{3,\Omega\times(kT,(k+1)T)} &\le e^{-\delta_1 kT} a_1, & |g|_{2,\Omega\times(kT,(k+1)T)} &\le e^{-\delta_2 kT} a_2, \\
|F_\varphi|_{2,\Omega\times(kT,(k+1)T)} &\le e^{-\delta_3 kT} a_3, & \|g\|_{L_\delta(kT,(k+1)T;W_\delta^\beta(\Omega))} &\le e^{-\delta_4 kT} a_4.
\end{aligned}
$$

Since $F(0) = \sum_{i=1}^4 a_i$ we see that (5.42) holds. This ends the proof. □

Next we have

Lemma 5.9. *Let B, b_i, $i = 1, \dots, k_j$, $j = 1, 2, \dots$ be given constants. Let there exists a local solution to problem* (1.1) *such that $v \in W_3^{2,1}(\Omega^T)$ and*

$$
\|v\|_{2,3,\Omega^T} \le B. \tag{5.43}
$$

Let (5.1) *hold and*

$$
\begin{aligned}
|h(0)|_{2,\Omega} &\le b_1, & \left|\frac{\chi(0)}{r}\right|_{2,\Omega} + |h(0)|_{2,\Omega} + |w(0)|_{4,\Omega}^2 &\le b_2, \\
|h(0)|_{1,0,\Omega} &\le b_3, & \|w(0)\|_{1,\Omega} &\le b_4, \\
\|w(0)\|_{6/5,5/2,\Omega} &\le b_5, & \|h(0)\|_{2+\beta-2/\delta,\delta,\Omega} &\le b_6.
\end{aligned}
\tag{5.44}
$$

Let

$$
|g(0)|_{1,0,\Omega} \le k_1, \quad |f(0)|_{3,\Omega} \le k_2, \quad |F_\varphi(0)|_{2,\Omega} \le k_3, \quad \|g(0)\|_{\beta,\delta,\Omega} \le k_4. \tag{5.45}
$$

Let T and $t_1 < T$ be so large that

$$
\begin{aligned}
e^{-\nu T+cB^2}\left[\frac{1}{\delta_2-\nu}k_1^2+b_1^2\right] &\le b_1^2,\\
c(d_1)d_1^2e^{cB^2}+ce^{cB^2-c_1T}[\varphi(A)k_1^2+k_1^4+k_2^2+k_3^2]+e^{-\nu T/2+cB^2}b_2^2 &\le b_2^2,\\
ce^{-\nu T+cB^2}(b_3^2+k_1^2) &\le b_3^2,\\
\varphi(d_1,d_2)[e^{-\delta_* t_1}(b_1+b_2+b_3+k_1+k_2+k_3)+1]^{10}+\frac{cd_3}{(t_2-t_1)^{2/5}} &\le b_5,\\
\varphi(B)e^{-\nu t_1}(b_3+k_1+k_4)+ca &\le b_6.
\end{aligned}
\tag{5.46}
$$

Then there exists a constant $c_1 > 1$ such that

$$
F_0'(k)\le c_1F_0'(0),\quad k\in\mathbb{N}. \tag{5.47}
$$

Proof. In view of (5.43), $(5.44)_1$, $(5.45)_1$ it is clear that Lemma 5.1 implies

$$
|h(T)|_{2,\Omega}^2\le e^{-\nu T+cB^2}\left[\frac{1}{\delta_2-\nu}k_1^2+b_1^2\right].
$$

By $(5.46)_1$ we have that $|h(T)|_{2,\Omega}\le |h(0)|_{2,\Omega}$. By (5.43), $(5.44)_2$, $(5.45)_{2,3}$ we see that (5.6) implies

$$
\begin{aligned}
&\left|\frac{\chi(T)}{r}\right|_{2,\Omega}^2+|h(T)|_{2,\Omega}^2+|w(T)|_{4,\Omega}^2\\
&\quad\le c(d_1)d_1e^{cB^2}+ce^{cB^2-c_1T}[\varphi(A)k_1^2+k_1^4+k_2^2+k_3^2]+e^{-\nu T/2+cB^2}b_2^2.
\end{aligned}
$$

Hence $(5.46)_2$ gives

$$
\left|\frac{\chi(T)}{r}\right|_{2,\Omega}^2+|h(T)|_{2,\Omega}^2+|w(T)|_{4,\Omega}^4\le\left|\frac{\chi(0)}{r}\right|_{2,\Omega}^2+|h(0)|_{2,\Omega}^2+|w(0)|_{4,\Omega}^4.
$$

By (5.43), $(5.44)_3$, $(5.45)_1$ and (5.21) we have

$$
\|h(T)\|_{1,\Omega}^2\le ce^{-\nu T+cB^2}(b_3^2+k_1^2).
$$

Then $(5.46)_3$ gives $\|h(T)\|_{1,\Omega}\le\|h(0)\|_{1,\Omega}$. By Lemma 5.4 we have

$$
\|h(T)\|_{2+\beta-2/\delta,\delta,\Omega}\le\varphi(B)e^{-\nu t_1}(b_3+k_1+k_4)+ca
$$

To estimate the r.h.s. we use $(5.46)_5$. Emploing (5.21), $(5.44)_{1,2,3,4}$, (5.45) in (5.27) yields

$$
\|v(T)\|_{6/5,5/2,\Omega}\le\varphi(d_1,d_2,A)[e^{-\delta_* t_1}(b_1+b_2+b_3+k_1+k_2+k_3)+1]^{10}+\frac{cd_2}{(t_2-t_1)^{2/5}}.
$$

Then in view of $(5.46)_4$ we have $\|v(T)\|_{6/5,5/2,\Omega}\le b_5$.

Finally in view of the above estimate and (5.29), (5.34) we obtain (5.47). □

REMARK 5.10. Up to now we have only proved that $v \in W^{2,1}_{5/2}(\Omega^T)$ and Theorem 1 yields the estimate (1.6). Then for solutions of problem (1.1) we have the additional estimate

$$\|v\|_{2,3,\Omega^T} \le c(\|v\|^2_{2,5/2,\Omega^T} + |f|_{3,\Omega^T} + \|v(0)\|_{4/3,3,\Omega}).$$

Hence assuming that $f \in L_3(\Omega^T)$ and $v(0) \in W^{4/3}_3(\Omega)$ we have that $v \in W^{2,1}_3(\Omega^T)$ and the above construction (see Lemmas 5.1–5.9) is justified.

PROOF OF THEOREM 3. Assume that (4.11) holds with $c_1 F_0'(0)$ which replaces $F_0(0)$ and appears in Lemma 5.9. Then in view of Lemmas 5.8 and 5.9 and Remark 5.10 we see that inequality (5.41) holds for any k. Then the results of Section 4 can be prolonged step by step. □

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