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COMPACT COMPONENTS OF POSITIVE SOLUTIONS FOR SUPERLINEAR INDEFINITE ELLIPTIC PROBLEMS OF MIXED TYPE

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Abstract. In this paper we construct an example of superlinear indefinite weighted elliptic mixed boundary value problem exhibiting a mushroom shaped compact component of positive solutions emanating from the trivial solution curve at two simple eigenvalues of a related linear weighted boundary value problem. To perform such construction we have to adapt to our general setting some of the rescaling arguments of H. Amann and J. López-Gómez [2, Section 4] to get a priori bounds for the positive solutions. Then, using the theory of [1], [4] and [5], we give some sufficient conditions on the nonlinearity and the several potentials of our model setting so that the set of values of the parameter for which the problem possesses a positive solution is bounded. Finally, the existence of the component of positive solutions emanating from the trivial curve follows from the unilateral results of P. H. Rabinowitz ([18], [14]). Monotonicity methods, re-scaling arguments, Liouville type theorems, local bifurcation and global continuation are among the main technical tools used to carry out our analysis.

1. Introduction

The main goal of this paper is to give sufficient conditions on the potentials $W(x), a(x) \in L_\infty(\Omega)$ and the nonlinearity $F(x,u)$ so that the superlinear

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indefinite weighted elliptic mixed boundary value problem

$$\begin{cases} \mathcal{L}u = \lambda W(x)u - a(x)F(x,u)u & \text{in } \Omega, \\ \mathcal{B}(b)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

has a compact component of positive solutions of (1.1) bifurcating from the trivial branch $(\lambda, u) = (\lambda, 0)$ at two bifurcation values of λ, σ_1^1, σ_1^2, $\sigma_1^1 \neq \sigma_1^2$, which are simple eigenvalues of a certain linear weighted elliptic mixed boundary value problem. In order to guarantee the existence of such compact component we first adapt, to cover our more general setting here, some of the re-scaling arguments of H. Amann and J. López-Gómez in [2, Section 4] about the existence of a priori bounds for the positive solutions of (1.1), valid for the special case $W = 1$, to show the existence of a priori bounds for the positive solutions of (1.1) for general $W \in L_\infty(\Omega)$. Then, under adequate assumptions on the nonlinearity $F(x,u)$ so that the positive solutions of (1.1) have a priori bounds, and using the results in [4] and [5], we give some sufficient conditions on the weights $W(x), a(x)$ so that the range of values of the parameter λ for which (1.1) possesses a positive solution be bounded, as well as to guarantee the existence of two different bifurcation values to positive solutions from the trivial branch $(\lambda, u) = (\lambda, 0)$. These values, σ_1^1 and σ_1^2, are simple eigenvalues of a related weighted elliptic mixed boundary value problem. Finally, thanks to the Rabinowitz global bifurcation theorem (see [18]), the existence of a compact component of positive solutions of (1.1) connecting the two bifurcation values $(\sigma_1^1, 0)$ and $(\sigma_1^2, 0)$ is shown.

Throughout this paper, we make the following assumptions:

(a) Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, of class $\mathcal{C}^2$, i.e. $\overline{\Omega}$ is an N-dimensional compact connected $\mathcal{C}^2$-submanifold of $\mathbb{R}^N$ with boundary $\partial\Omega$ of class $\mathcal{C}^2$.

(b) $\lambda \in \mathbb{R}$ is regarded as the bifurcation and continuation parameter, $W \in L_\infty(\Omega)$ is a potential in front of λ, and

$$\mathcal{L} := -\sum_{i,j=1}^{N} \alpha_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} \alpha_i(x) \frac{\partial}{\partial x_i} + \alpha_0(x) \tag{1.2}$$

is an uniformly strongly elliptic differential operator in Ω with

$$\alpha_{ij} = \alpha_{ji} \in \mathcal{C}^1(\overline{\Omega}), \quad \alpha_i \in \mathcal{C}(\overline{\Omega}), \quad \alpha_0 \in L_\infty(\Omega), \quad 1 \leq i,j \leq N. \tag{1.3}$$

In the sequel we denote by $\mu > 0$ the ellipticity constant of $\mathcal{L}$ in Ω. Then, for any $\xi \in \mathbb{R}^N \setminus \{0\}$ and $x \in \overline{\Omega}$ we have that

$$\sum_{i,j=1}^{N} \alpha_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2.$$

(c) As for as the potential $a(x) \in L_\infty(\Omega)$, we assume that the weights

$$a^+(x) := \max\{a, 0\} \quad \text{and} \quad a^- := a^+ - a$$

the positive and negative part of a, respectively, are two nonnegative bounded potentials with disjoint supports, belonging to a very general class of nonnegative potentials $\mathcal{A}(\Omega) \subset L_\infty(\Omega)$, which will be introduced in Section 2. Hereafter, $\Omega^0_{a^+}$ and $\Omega^0_{a^-}$ will stand for the maximal open subsets of Ω where $a^+(x)$ and $a^-(x)$ vanish, respectively, and $\Omega^+_{a^+}$ and $\Omega^+_{a^-}$ the sets satisfying

$$\Omega^+_{a^+} := \{x \in \Omega : a^+(x) > 0\}, \quad \Omega^+_{a^-} := \{x \in \Omega : a^-(x) > 0\}.$$

In the sequel, we will set $\Omega_{a^+} := \Omega^+_{a^+}$ and $\Omega_{a^-} := \Omega^+_{a^-}$.

In Section 2 we will introduce all the details about the structure and properties of the sets $\Omega^0_{a^+}$, $\Omega^0_{a^-}$, Ω_{a^+} and Ω_{a^-}. Since we are assuming that the potentials $a^+(x)$ and $a^-(x)$ have disjoint supports, it follows that $\overline{\Omega}_{a^+} \cap \overline{\Omega}_{a^-} = \emptyset$, and hence,

$$\Omega_{a^+} \subset \Omega^0_{a^-} \quad \text{and} \quad \Omega_{a^-} \subset \Omega^0_{a^+}. \tag{1.4}$$

In the sequel we will denote by Ω_0 the maximal open subset of Ω where $a \in L_\infty(\Omega)$ vanishes and by $[\Omega^0_{a^-}]^0_{a^+}$ the maximal open subset of $\Omega^0_{a^-}$ where the positive part of a, a^+, vanishes. It should be noted that they coincide, that is to say,

$$\Omega_0 = [\Omega^0_{a^-}]^0_{a^+}. \tag{1.5}$$

We advance that with the assumptions that it will be imposed over the potential a, the open set Ω_0 will satisfy

$$\Omega_0 \in \mathcal{C}^2 \quad \text{and} \quad \operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0. \tag{1.6}$$

(d) As for as the nonlinearity $F(x,u)\colon \overline{\Omega} \times [0,\infty) \to \mathbb{R}$, satisfies the following assumptions:

($\mathcal{F}1$) $F \in \mathcal{C}^1(\overline{\Omega} \times [0,\infty); \mathbb{R})$,
($\mathcal{F}2$) $\lim_{u\nearrow\infty} F(x,u) = \infty$ uniformly in $\overline{\Omega}$,
($\mathcal{F}3$) $F(\,\cdot\,,0) = 0$ and $\partial_u F(x,u) > 0$ for each $(x,u) \in \Omega \times (0,\infty)$,

and the following growth condition at infinity

$$\lim_{u\nearrow\infty} \frac{F(x,u)}{u^{r-1}} = l(x) \quad \text{uniformly in } \Omega_{a^-}, \tag{1.7}$$

where $r > 1$ and $l \in L_\infty(\Omega_{a^-})$ is positive and bounded away from zero.

(e) $\mathcal{B}(b)$ stands for the boundary operator

$$\mathcal{B}(b)u := \begin{cases} u & \text{on } \Gamma_0, \\ \partial_\nu u + bu & \text{on } \Gamma_1, \end{cases} \tag{1.8}$$

where Γ_0 and Γ_1 are two disjoint open and closed subsets of $\partial\Omega$ with $\Gamma_0 \cup \Gamma_1 = \partial\Omega$, $b \in \mathcal{C}(\Gamma_1)$, $\nu = (\nu_1, \dots, \nu_N) \in \mathcal{C}^1(\Gamma_1, \mathbb{R}^N)$ is any outward pointing nowhere tangent vector field satisfying

$$\nu_i := \sum_{j=1}^{N} \alpha_{ij} n_j, \quad 1 \le i \le N, \tag{1.9}$$

on $\Gamma_1 \cap \partial\Omega_0$, where $n = (n_1, \dots, n_N)$ denotes the outward unit normal to Ω on Γ_1, $\partial_\nu u := \langle \nabla u, \nu \rangle$ and Ω_0 the vanishing set defined by (1.5). It should be noted as a consequence of (1.6), each component Γ^* of Γ_1 satisfies either $\Gamma^* \cap \partial\Omega_0 = \emptyset$ or $\Gamma^* \subset \partial\Omega_0$. Thus, (1.9) implies that if Γ^* is a component of Γ_1 satisfying $\Gamma^* \subset \partial\Omega_0$, then ν is the *conormal field* on Γ^* and $\partial_\nu u$ stands for the *conormal derivate* of u on Γ^*; and if Γ^* is a component of Γ_1 satisfying that $\Gamma^* \cap \partial\Omega_0 = \emptyset$, then $\nu|_{\Gamma^*} \in \mathcal{C}^1(\Gamma^*, \mathbb{R}^N)$ is any outward pointing nowhere tangent vector field to Ω on Γ^*. Moreover, Γ_0 and Γ_1 possess finitely many components. Thus, $\mathcal{B}(b)$ is the Dirichlet boundary operator on Γ_0, denoted in the sequel by $\mathcal{D}$, and the Neumann or a first order regular oblique derivative boundary operator on Γ_1. It should be pointed out that either Γ_0 or Γ_1 may be empty.

Throughout this paper, we regard to the positive solutions of (1.1) as couples (λ, u_λ) or simply by u_λ, where λ is the bifurcation parameter. Moreover, for each potential $a(x)$ with $a^+, a^- \in \mathcal{A}(\Omega)$ and each nonlinearity $F(x,u)$ satisfying the (d) assumptions, we will denote by $\Lambda(a, F)$ the set of values of λ for which (1.1) possesses a positive solution.

This paper is strongly motivated by the previous work [18], [8], [1], [12], [13], [10], [2], [5] and [4] and the main technical tools used to get our results are monotonicity methods, re-scaling arguments, Liouville type theorems, local and global bifurcation and continuation methods.

To explain the main results of this paper, we need introducing some additional concepts and notations. By a *principal eigenvalue* of the operator $\mathcal{L}$ in the domain Ω, we mean any value of $\lambda \in \mathbb{R}$ for which there exists a positive function φ satisfying

$$\begin{cases} \mathcal{L}\varphi = \lambda\varphi & \text{in } \Omega, \\ \mathcal{B}(b)\varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.10}$$

The existence and the uniqueness of the principal eigenvalue under our general assumptions goes back to H. Amann ([1]). In the sequel $\sigma_1^\Omega[\mathcal{L}, \mathcal{B}(b)]$ will denote the principal eigenvalue of (1.10). Also, given any proper subdomain $\widetilde{\Omega}$ of Ω of class $\mathcal{C}^2$ with

$$\operatorname{dist}(\Gamma_1, \partial\widetilde{\Omega} \cap \Omega) > 0, \tag{1.11}$$

we will denote by $\mathcal{B}(b,\widetilde{\Omega})$ the boundary operator build up from $\mathcal{B}(b)$ through by

$$\mathcal{B}(b,\widetilde{\Omega}) := \begin{cases} \varphi & \text{on } \partial\widetilde{\Omega}\cap\Omega, \\ \mathcal{B}(b)\varphi & \text{on } \partial\widetilde{\Omega}\cap\partial\Omega. \end{cases} \tag{1.12}$$

When $\widetilde{\Omega}=\Omega$ we set $\mathcal{B}(b,\widetilde{\Omega}) := \mathcal{B}(b)$. If $\overline{\widetilde{\Omega}}\subset\Omega$, then $\partial\widetilde{\Omega}\subset\Omega$ and $\mathcal{B}(b,\widetilde{\Omega})\varphi=\varphi$ by definition. So, $\mathcal{B}(b,\widetilde{\Omega})$ becomes into the Dirichlet boundary operator, denoted in the sequel by $\mathcal{D}$. Also, we denote by $\sigma_1^{\widetilde{\Omega}}[\mathcal{L},\mathcal{B}(b,\widetilde{\Omega})]$ the principal eigenvalue of the linear boundary value problem

$$\begin{cases} \mathcal{L}\varphi=\lambda\varphi & \text{in } \widetilde{\Omega}, \\ \mathcal{B}(b,\widetilde{\Omega})\varphi=0 & \text{on } \partial\widetilde{\Omega}. \end{cases} \tag{1.13}$$

We now recall the concept of *principal eigenvalue* for a domain with several components.

DEFINITION 1.1. If $\widetilde{\Omega}$ is an open subset of Ω with a finite number of components of class $\mathcal{C}^2$, say $\widetilde{\Omega}^j$, $1\le j\le m$ such that $\overline{\widetilde{\Omega}}^i\cap\overline{\widetilde{\Omega}}^j=\emptyset$ if $i\ne j$ and

$$\operatorname{dist}(\Gamma_1,\partial\widetilde{\Omega}\cap\Omega)>0, \tag{1.14}$$

then the principal eigenvalue of $(\mathcal{L},\mathcal{B}(b,\widetilde{\Omega}),\widetilde{\Omega})$ is defined through

$$\sigma_1^{\widetilde{\Omega}}[\mathcal{L},\mathcal{B}(b,\widetilde{\Omega})] := \min_{1\le j\le m}\sigma_1^{\widetilde{\Omega}^j}[\mathcal{L},\mathcal{B}(b,\widetilde{\Omega}^j)]. \tag{1.15}$$

REMARK 1.2. Since $\widetilde{\Omega}$ is of class $\mathcal{C}^2$, it follows from (1.14) that each of the principal eigenvalues $\sigma_1^{\widetilde{\Omega}^j}[\mathcal{L},\mathcal{B}(b,\widetilde{\Omega}^j)]$, $1\le j\le m$, is well defined. This shows the consistency of Definition 1.1.

Hereafter, for each $\lambda\in\mathbb{R}$ we will denote by $\mathcal{L}(\lambda)$ the differential operator

$$\mathcal{L}(\lambda) := \mathcal{L}-\lambda W(x). \tag{1.16}$$

Note that $\mathcal{L}(\lambda)$ is uniformly strongly elliptic in Ω, with the same ellipticity constant $\mu>0$ as $\mathcal{L}$.

We now introduce some concepts and results concerning with the existence and multiplicity of principal eigenvalues for a general class of weighted linear elliptic mixed boundary value problems.

DEFINITION 1.3. For each $W\in L_\infty(\Omega)$, any value of λ for which the problem

$$\begin{cases} \mathcal{L}\varphi=\lambda W\varphi & \text{in } \Omega, \\ \mathcal{B}(b)\varphi=0 & \text{on } \partial\Omega \end{cases} \tag{1.17}$$

admits a positive solution φ, it will be called a *principal eigenvalue* of $(\mathcal{L},W,\mathcal{B}(b),\Omega)$. If λ is a principal eigenvalue of $(\mathcal{L},W,\mathcal{B}(b),\Omega)$ with

$$N[\mathcal{L}-\lambda W]=\operatorname{span}[\varphi] \quad\text{and}\quad W\varphi\notin R[\mathcal{L}-\lambda W],$$

then it will be said that λ is a *simple eigenvalue* of $(\mathcal{L}, W, \mathcal{B}(b), \Omega)$.

The previous concept of simple eigenvalue is consistent with the concept of simple eigenvalue of $(\mathcal{L} - \lambda W, W)$ in Ω introduced in [8]. Thus, from Definition 1.3 and due the uniqueness of the principal eigenpair associated with $(\mathcal{L} - \lambda W, \mathcal{B}(b), \Omega)$, guaranteed by [1, Theorem 12.1], the principal eigenvalues of (1.17) are given by the zeroes of the map

$$\Sigma(\lambda) := \sigma_1^{\Omega}[\mathcal{L}(\lambda), \mathcal{B}(b)], \quad \lambda \in \mathbb{R}, \tag{1.18}$$

where $\mathcal{L}(\lambda)$ is the differential operator just defined by (1.16)

Hereafter, for each $a \in L_\infty(\Omega)$ with $a^+ \in \mathcal{A}(\Omega)$ we will consider the map

$$\Sigma_0(\lambda) := \sigma_1^{\Omega^0_{a^+}}[\mathcal{L}(\lambda), \mathcal{B}(b, \Omega^0_{a^+})]. \tag{1.19}$$

It should be noted, that the boundary operator $\mathcal{B}(b, \Omega^0_{a^+})$ and the principal eigenvalue (1.19) are well defined in the sense of (1.8) and (1.10), respectively, since as we will see in Section 2,

$$\Omega^0_{a^+} \in \mathcal{C}^2 \quad \text{and} \quad \operatorname{dist}(\Gamma_1, \partial\Omega^0_{a^+} \cap \Omega) > 0, \tag{1.20}$$

are followed from the fact that $a^+ \in \mathcal{A}(\Omega)$.

Throughout this paper if $\widetilde{\lambda}$ is a bifurcation value to positive solutions of (1.1) from the trivial branch $(\lambda, u) = (\lambda, 0)$, we will denote by $\mathcal{C}^+(\widetilde{\lambda})$ the global continuum of positive solutions of (1.1) emanating from the trivial branch at the bifurcation point $(\widetilde{\lambda}, 0)$, where by continuum we mean closed and connected. The main result of this paper establishes the following

Theorem 1.4. *Let $a \in L_\infty(\Omega)$ be where the positive and negative parts of a, $a^+ = \max\{a, 0\}$ and $a^- = a^+ - a$ satisfy*

$$a^+, a^- \in \mathcal{A}(\Omega), \quad \Omega^0_{a^-} \textit{ is connected}, \quad a^+ \in \mathcal{A}(\Omega^0_{a^-}), \quad \partial\Omega_{a^-} \in \mathcal{C}^1,$$

where $\mathcal{A}(\Omega)$ is a very general class of nonnegative measurable bounded potential, named Class of Admissible Potentials in Ω, that will be introduced in Section 2. Assume in addition that

$$a^-(x) = C(x)\operatorname{dist}(x, \partial\Omega_{a^-})^\gamma, \quad x \in \Omega_{a^-}, \tag{1.21}$$

with $\gamma > 0$ and $C\colon \overline{\Omega}_{a^-} \to [0, \infty)$ is a continuous function bounded away from zero near $\partial\Omega_{a^-}$.

As far as the growth condition (1.7) assume that either $N \le 2$ and $r > 1$ or

$$N \ge 3 \quad \textit{and} \quad r < \min\left\{\frac{N+2}{N-2}, \frac{N+1+\gamma}{N-1}\right\}. \tag{1.22}$$

Further, suppose that the weight W satisfy that there exist two open subsets D_+ and D_- of $\Omega^0_{a^+}$ for which $\inf_{D_+} W > 0$ and $\sup_{D_-} W < 0$ with $D_+ \cap \Omega_{a^-} \neq \emptyset$ and

$$\sup_{\lambda\in\mathbb{R}} \Sigma(\lambda) > 0, \tag{1.23}$$

where $\Sigma(\lambda)$ is the map just defined by (1.18). Finally, set $\sigma_1^1 < \sigma_1^2$ the unique roots of the map $\Sigma(\lambda)$ and $\lambda_1^0 < \lambda_2^0$ the unique roots of the map $\Sigma_0(\lambda)$, whose existence will be guaranteed. Then:

(a) *$\overline{\Lambda}(a, F) \subset (\lambda_1^0, \lambda_2^0)$ and therefore, $\Lambda(a, F)$ is bounded.*
(b) *The roots σ_1^i, $i = 1, 2$ of the map $\Sigma(\lambda)$ are simple eigenvalues of $(\mathcal{L}, W, \mathcal{B}(b), \Omega)$ in the sense of Definition 1.3 and they are bifurcation values to positive solutions of (1.1) from the trivial branch $(\lambda, u) = (\lambda, 0)$.*
(c) *The global continuum $\mathcal{C}^+(\sigma_1^1)$ of positive solutions emanating from the trivial branch at $(\lambda, 0) = (\sigma_1^1, 0)$, it is bounded in $\mathbb{R} \times L_\infty(\Omega)$ and comes back again to the trivial branch at the bifurcation point $(\lambda, 0) = (\sigma_1^2, 0)$. The same occur with the global continuum $\mathcal{C}^+(\sigma_1^2)$, which comes back to the trivial branch at the bifurcation point $(\lambda, 0) = (\sigma_1^1, 0)$. Therefore,*

$$\mathcal{C}^+(\sigma_1^1) = \mathcal{C}^+(\sigma_1^2). \tag{1.24}$$

(d) *$(\sigma_1^1, \sigma_1^2) \subset \Lambda(a, F) \subset (\lambda_1^0, \lambda_2^0)$.*

Figure 1.1 ilustrates a typical situation where (1.1) exhibits a mushroom shaped compact component of positive solutions

$$\mathcal{C}^+ := \mathcal{C}^+(\sigma_1^1) = \mathcal{C}^+(\sigma_1^2),$$

emanating from the trivial solution curve at the bifurcation values $(\sigma_1^1, 0)$ and $(\sigma_1^2, 0)$, situation whose existence is guaranteed under the assumptions of Theorem 1.4.

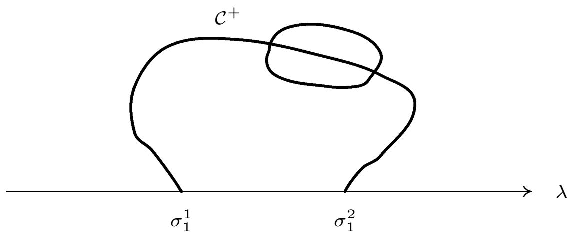

Figure 1.1. Mushroom shaped compact component of positive solutions

We now shortly describe the distribution of this paper. In Section 2 are explained all the properties of the nonnegative measurable bounded potentials belonging to the class $\mathcal{A}(\Omega)$ of *Admissible Potentials in* Ω. Section 3 contains some preliminaries, among them the existence and some properties of the principal eigenvalue $\sigma_1^\Omega[\mathcal{L},\mathcal{B}(b)]$ found in [1], the characterization of the strong maximum principle found in [2], some of the main properties of the maps $\Sigma(\lambda)$ and $\Sigma_0(\lambda)$ found in [5] and a result about the existence of positive supersolutions of a very general class of sublinear elliptic mixed boundary problem, found in [4]. Also we definitively fix the notations used throughout the paper. Section 4 contains results concerning with the existence of positive solutions of (1.1) and bifurcation values to positive solutions of (1.1) from the trivial branch $(\lambda,u)=(\lambda,0)$. Section 5 contains the results concerning with the existence of a priori bounds for the positive solutions of (1.1), where we have adapted the theory of H. Amann and J. López-Gómez ([2]) to our setting. Finally, Section 6 contains the main result of this paper, in which are given sufficient conditions to (1.1) exhibits a bounded global continuum of positive solutions connecting two different bifurcation values to positive solutions of (1.1) from the trivial branch $(\lambda,u)=(\lambda,0)$.

2. Class $\mathcal{A}(\Omega)$ of admissible potentials in Ω

In this section we introduce a very general class of nonnegative measurable bounded potentials which will be denoted by class $\mathcal{A}(\Omega)$ of *admissible potentials* in Ω. It will play a crucial role troughout this paper.

Definition 2.1. It is defined the class $\mathcal{A}(\Omega)$ of admissible potentials in Ω, as the set of nonnegative measurable bounded potentials $V>0$ for which there exist an open subset Ω_V^0 of Ω and a compact subset K_V of $\overline{\Omega}$ with Lebesgue measure zero such that

$$K_V\cap(\overline{\Omega}_V^0\cup\Gamma_1)=\emptyset, \tag{2.1}$$

$$\Omega_V^+:=\{x\in\Omega:V(x)>0\}=\Omega\setminus(\overline{\Omega}_V^0\cup K_V), \tag{2.2}$$

and each of the following conditions is satisfied:

($\mathcal{A}1$) Ω_V^0 possesses a finite number of components of class $\mathcal{C}^2$, say $\Omega_V^{0,j}$, $1\leq j\leq m$, such that $\overline{\Omega}_V^{0,i}\cap\overline{\Omega}_V^{0,j}=\emptyset$ if $i\neq j$, and

$$\operatorname{dist}(\Gamma_1,\partial\Omega_V^0\cap\Omega)>0. \tag{2.3}$$

Thus, if we denote by Γ_1^i, $1\leq i\leq n_1$, the components of Γ_1, then for each $1\leq i\leq n_1$ either $\Gamma_1^i\subset\partial\Omega_V^0$ or $\Gamma_1^i\cap\partial\Omega_V^0=\emptyset$. Moreover, if $\Gamma_1^i\subset\partial\Omega_V^0$, then Γ_1^i must be a component of $\partial\Omega_V^0$. Indeed, if $\Gamma_1^i\cap\partial\Omega_V^0\neq\emptyset$ but Γ_1^i is not a component of $\partial\Omega_V^0$, then $\operatorname{dist}(\Gamma_1^i,\partial\Omega_V^0\cap\Omega)=0$.

($\mathcal{A}2$) Let $\{i_1, \dots, i_p\}$ denote the subset of $\{1, \dots, n_1\}$ for which $\Gamma_1^j \cap \partial\Omega_V^0 = \emptyset$ if and only if $j \in \{i_1, \dots, i_p\}$. Then, V is bounded away from zero on any compact subset of

$$\Omega_V^+ \cup \bigcup_{j=1}^{p} \Gamma_1^{i_j}.$$

Note that if $\Gamma_1 \subset \partial\Omega_V^0$, then we are only imposing that V is bounded away from zero on any compact subset of Ω_V^+.

($\mathcal{A}3$) Let Γ_0^i, $1 \le i \le n_0$, denote the components of Γ_0, and let $\{i_1, \dots, i_q\}$ be the subset of $\{1, \dots, n_0\}$ for which $(\partial\Omega_V^0 \cup K_V) \cap \Gamma_0^j \neq \emptyset$ if and only if $j \in \{i_1, \dots, i_q\}$. Then, V is bounded away from zero on any compact subset of

$$\Omega_V^+ \cup \Big[\bigcup_{j=1}^{q} \Gamma_0^{i_j} \setminus (\partial\Omega_V^0 \cup K) \Big].$$

Note that if $(\partial\Omega_V^0 \cup K_V) \cap \Gamma_0 = \emptyset$, then we are only imposing that V is bounded away from zero on any compact subset of Ω_V^+.

($\mathcal{A}4$) For any $\eta > 0$ there exist a natural number $\ell(\eta) \ge 1$ and $\ell(\eta)$ open subsets of $\mathbb{R}^N$, G_j^η, $1 \le j \le \ell(\eta)$, with $|G_j^\eta| < \eta$, $1 \le j \le \ell(\eta)$, such that

$$\overline{G}_i^\eta \cap \overline{G}_j^\eta = \emptyset \quad \text{if } i \neq j, \quad K_V \subset \bigcup_{j=1}^{\ell(\eta)} G_j^\eta,$$

and for each $1 \le j \le \ell(\eta)$ the open set $G_j^\eta \cap \Omega$ is connected and of class $\mathcal{C}^2$.

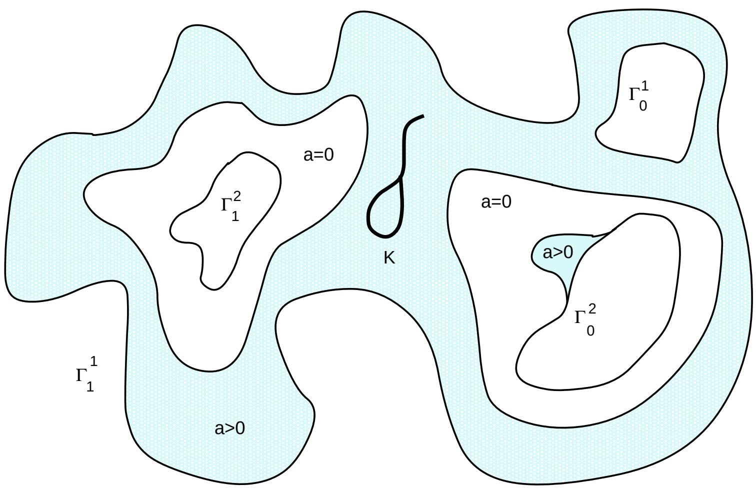

Figure 2.1. An admissible configuration

In Figure 2.1 we have represented a typical configuration for which $V \in \mathcal{A}(\Omega)$. In this case, we have $\Gamma_1 = \Gamma_1^1 \cup \Gamma_1^2$, $\Gamma_0 = \Gamma_0^1 \cup \Gamma_0^2$, and Ω_V^+ – dark area, as well as Ω_V^0 – white area, consists of two components; the compact set K_V consisting of a compact arc of curve. For such configuration, conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_4)$ are trivially satisfied. Moreover, $(\mathcal{A}_2)$ is satisfied if, and only if, V is bounded away from zero in any compact subset of $\Omega_V^+ \cup \Gamma_1^1$, and $(\mathcal{A}_3)$ holds if, and only if, V is bounded away from zero in any compact subset of $\Omega_V^+ \cup (\Gamma_0^2 \setminus \partial\Omega_V^0)$; V can vanish on the component Γ_0^1.

3. Preliminaries and notations

In this section we collect some of the main results of [1], [5] and [4] that are going to be used throughout the rest of this paper.

For each $p > 1$ we denote

$$\begin{aligned} W^2_{p,\mathcal{B}(b)}(\Omega) &:= \{u \in W^2_p(\Omega) : \mathcal{B}(b)u = 0\}, \\ W^2_{\mathcal{B}(b)}(\Omega) &:= \bigcap_{p>1} W^2_{p,\mathcal{B}(b)}(\Omega) \subset H^2(\Omega), \end{aligned}$$

and use the natural product order in $L_p(\Omega) \times L_p(\partial\Omega)$,

$$(f_1, g_1) \geq (f_2, g_2) \Leftrightarrow f_1 \geq f_2 \wedge g_1 \geq g_2.$$

It will be said that $(f_1, g_1) > (f_2, g_2)$ if $(f_1, g_1) \geq (f_2, g_2)$ and $(f_1, g_1) \neq (f_2, g_2)$.

Since $b \in \mathcal{C}(\Gamma_1)$, it follows from [17] that for each $p > 1$

$$\mathcal{B}(b) \in \mathcal{L}(W^2_p(\Omega), W_p^{2-1/p}(\Gamma_0) \times W_p^{1-1/p}(\Gamma_1)).$$

Moreover, there exists a least real eigenvalue of the problem

$$\begin{cases} \mathcal{L}\varphi = \lambda\varphi & \text{in } \Omega, \\ \mathcal{B}(b)\varphi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

denoted in the sequel by $\sigma_1^\Omega[\mathcal{L}, \mathcal{B}(b)]$ and called *principal eigenvalue of* $(\mathcal{L}, \mathcal{B}(b), \Omega)$. The principal eigenvalue is simple and associated with it there is a positive eigenfunction, unique up to multiplicative constants, denoted by $\varphi_{[\mathcal{L},\mathcal{B}(b)]}$ and called *principal eigenfunction of* $(\mathcal{L}, \mathcal{B}(b), \Omega)$. Thanks to Theorem 12.1 of [1] the principal eigenfunction satisfies

$$\varphi_{[\mathcal{L},\mathcal{B}(b)]} \in W^2_{\mathcal{B}(b)}(\Omega) \subset H^2(\Omega)$$

and it is *strongly positive in* Ω in the sense that

$$\varphi_{[\mathcal{L},\mathcal{B}(b)]}(x) > 0 \quad \text{for all } x \in \Omega \cup \Gamma_1 \quad \text{and} \quad \partial_\nu \varphi_{[\mathcal{L},\mathcal{B}(b)]}(x) < 0 \quad \text{for all } x \in \Gamma_0.$$

In addition, $\sigma_1^\Omega[\mathcal{L},\mathcal{B}(b)]$ is the only eigenvalue of (3.1) to possessing a positive eigenfunction, and it is dominant in the sense that any other eigenvalue σ of (3.1) satisfies

$$\Re\sigma > \sigma_1^\Omega[\mathcal{L},\mathcal{B}(b)].$$

Furthermore, setting $\mathcal{L}_p := \mathcal{L}|_{W^2_{p,\mathcal{B}(b)}(\Omega)}$, for each $\omega > -\sigma_1^\Omega[\mathcal{L},\mathcal{B}(b)]$ and $p > N$ we have that $(\omega+\mathcal{L}_p)^{-1} \in \mathcal{L}(L_p(\Omega))$ is a positive, compact and irreducible operator (cf. [19, V.7.7]).

Suppose $p > N$. Then, a function $\overline{u} \in W^2_p(\Omega)$ is said to be a *positive strict supersolution* of $(\mathcal{L},\mathcal{B}(b),\Omega)$ if $\overline{u} \geq 0$ and $(\mathcal{L}\overline{u},\mathcal{B}(b)\overline{u}) > 0$. A function $u \in W^2_p(\Omega)$ is said to be *strongly positive* if $u(x) > 0$ for each $x \in \Omega\cup\Gamma_1$ and $\partial_\beta u(x) < 0$ for each $x \in \Gamma_0$ with $u(x) = 0$ and any outward pointing nowhere tangent vector field $\beta \in \mathcal{C}^1(\Gamma_0,\mathbb{R}^N)$. Finally, $(\mathcal{L},\mathcal{B}(b),\Omega)$ is said to satisfy the *strong maximum principle* if $p > N$, $u \in W^2_p(\Omega)$, and $(\mathcal{L}u,\mathcal{B}(b)u) > 0$ imply that u is strongly positive. Recall that for any $p > N$

$$W^2_p(\Omega) \hookrightarrow \mathcal{C}^{2-N/p}(\overline{\Omega}) \tag{3.2}$$

and that any function $u \in W^2_p(\Omega)$ is a.e. in Ω twice differentiable (cf. [20, Theorem VIII.1]).

The following characterization of the strong maximum principle provides us with one of the main technical tools to obtain most of the results used in this paper. It comes from [15], [13] and [2].

THEOREM 3.1. *The following assertions are equivalent:*

(a) $\sigma_1^\Omega[\mathcal{L},\mathcal{B}(b)] > 0$,
(b) $(\mathcal{L},\mathcal{B}(b),\Omega)$ *possesses a positive strict supersolution,*
(c) $(\mathcal{L},\mathcal{B}(b),\Omega)$ *satisfies the strong maximum principle.*

The following results provides us with some of the main monotonicity properties of $\sigma_1^\Omega[\mathcal{L},\mathcal{B}(b)]$; they are taken from [5], (cf. there in Propositions 3.2 and 3.3).

PROPOSITION 3.2. *Let $\widetilde{\Omega}$ be a proper subdomain of Ω of class $\mathcal{C}^2$ satisfying* (1.11). *Then,*

$$\sigma_1^\Omega[\mathcal{L},\mathcal{B}(b)] < \sigma_1^{\widetilde{\Omega}}[\mathcal{L},\mathcal{B}(b,\widetilde{\Omega})],$$

where $\mathcal{B}(b,\widetilde{\Omega})$ is the boundary operator defined by (1.12).

PROPOSITION 3.3. *Let P_1, $P_2 \in L_\infty(\Omega)$ such that $P_1 < P_2$ on a set of positive measure. Then,*

$$\sigma_1^\Omega[\mathcal{L}+P_1,\mathcal{B}(b)] < \sigma_1^\Omega[\mathcal{L}+P_2,\mathcal{B}(b)].$$

As far as the maps $\Sigma(\lambda)$ and $\Sigma_0(\lambda)$ just defined by (1.18) and (1.19) it should be noted that for each $a \in L_\infty(\Omega)$ with $a^+ \in \mathcal{A}(\Omega)$, the open subset $\Omega^0_{a^+} \subset \Omega$

satisfy (1.20) and thanks to Proposition 3.2 we have that

$$\Sigma(\lambda) < \Sigma_0(\lambda). \tag{3.3}$$

Now, in order to collect some of the main properties of above maps, which will play a crucial role in the sequel, we introduce the following concept

Definition 3.4. Let $W \in L_\infty(\Omega)$ be. We will say that W changes of sign in Ω, if there exist two open subsets D_+ and D_- of Ω for which

$$\inf_{D_+} W > 0, \quad \sup_{D_-} W < 0. \tag{3.4}$$

The following properties of $\Sigma(\lambda)$ will be used along this paper and can be found in [5, Section 12].

Theorem 3.5. *Let $W \in L_\infty(\Omega)$ be. Then, the map $\Sigma(\lambda)$ defined by* (1.18) *is real holomorphic and concave. Therefore, either $\Sigma''(\lambda) = 0$ for any $\lambda \in \mathbb{R}$, or there exists a discrete set $Z \subset \mathbb{R}$ such that $\Sigma''(\lambda) < 0$ for each $\lambda \in \mathbb{R} \setminus Z$. By discrete it is meant that $Z \cap K$ is finite for any compact subset K of $\mathbb{R}$.*

Assume in addition that W changes of sign in Ω in the sense of Definition 3.4. *Then:*

(a) *The asymptotic behaviour of the map $\Sigma(\lambda)$ is*

$$\lim_{\lambda \nearrow \infty} \Sigma(\lambda) = -\infty \quad \text{and} \quad \lim_{\lambda \searrow -\infty} \Sigma(\lambda) = -\infty. \tag{3.5}$$

In particular, there exists $\lambda_0 \in \mathbb{R}$ for which

$$\Sigma(\lambda_0) = \sup_{\lambda \in \mathbb{R}} \Sigma(\lambda).$$

Moreover, $\Sigma'(\lambda_0) = 0$, $\Sigma'(\lambda) > 0$ if $\lambda < \lambda_0$, and $\Sigma'(\lambda) < 0$ if $\lambda > \lambda_0$. Therefore, λ_0 is unique.

(b) *The eigenvalue problem* (1.17) *possesses a principal eigenvalue if and only if $\Sigma(\lambda_0) \geq 0$. Moreover, λ_0 is the unique principal eigenvalue of* (1.17) *if $\Sigma(\lambda_0) = 0$, whereas* (1.17) *possesses exactly two principal eigenvalues, say $\sigma_1^1 < \sigma_1^2$, if $\Sigma(\lambda_0) > 0$. Moreover, in this case $\sigma_1^1 < \lambda_0 < \sigma_1^2$, and σ_1^i, $i = 1, 2$ is a simple eigenvalue of $(\mathcal{L}, W, \mathcal{B}(b), \Omega)$ in the sense of Definition* 1.3 *and* [8].

It should be noted that for each $a \in L_\infty(\Omega)$ with $a^+ \in \mathcal{A}(\Omega)$, the previous result can be applied to the map $\Sigma_0(\lambda)$, just defined by (1.19), substituting Ω by $\Omega^0_{a^+}$ and considering the restriction of W to $\Omega^0_{a^+}$.

Finally, we will say that a function $u \in W^2_p(\Omega)$ with $p > N$ is a *positive strict supersolution* (resp. a *positive strict subsolution*) of (1.1), if $u \geq 0$ and

$$(\mathcal{L}(\lambda)u + a(x)F(x,u)u, \mathcal{B}(b)u) > 0 \text{ (resp. } < 0).$$

The following is [4, Theorem 1.3] and it provides us a sufficient condition about the λ-parameter for the existence of positive strict supersolution of a very general sublinear elliptic weighted mixed boundary value problem.

Theorem 3.6. *Assume the general conditions of the introduction and let consider the sublineal weighted elliptic mixed boundary value problem*

$$\begin{cases} \mathcal{L}u = \lambda W u - V(x) f(x,u) u & \text{in } \Omega, \\ \mathcal{B}(b)u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.6}$$

where the potential $V \in \mathcal{A}(\Omega)$ *and the function* $f \in \mathcal{C}^1(\overline{\Omega} \times [0,\infty); \mathbb{R})$ *satisfies*

$$\lim_{u \nearrow \infty} f(x,u) = \infty \quad \text{uniformly in } \overline{\Omega}.$$

Then, (3.6) *possesses a positive strict supersolution arbitrarily large and bounded away from zero in* $\overline{\Omega}$*, provided* $\lambda \in \mathbb{R}$ *satisfies*

$$\sigma_1^{\Omega_V^0}[\mathcal{L}(\lambda), \mathcal{B}(b, \Omega_V^0)] > 0. \tag{3.7}$$

4. Existence of positive solutions and bifurcation values

In this section we give a necessary condition for the existence of positive solutions of (1.1) and we characterize the bifurcation values to positive solutions of (1.1) from the trivial branch.

Definition 4.1. A function $u_\lambda \colon \overline{\Omega} \to [0,\infty)$ is said to be a *positive solution of* (1.1) if $u_\lambda \in W_p^2(\Omega)$ with $p > N$, $u_\lambda > 0$ and u_λ satisfies $(1.1)_\lambda$ almost everywhere in Ω.

Lemma 4.2. *Let* u_0 *be a positive solution of* $(1.1)_{\lambda_0}$. *Then* u_0 *is strongly positive in* Ω *and* $u_0 \in W^2_{\mathcal{B}(b)}(\Omega)$. *In particular,* $u_0 \in \mathcal{C}^{1,\gamma}(\overline{\Omega})$ *for all* $0 < \gamma < 1$. *Moreover,* u_0 *is a.e. in* Ω *twice continuously differentiable.*

Proof. Indeed, if $(\lambda, u) = (\lambda_0, u_0)$ is a positive solution of $(1.1)_{\lambda_0}$, then $u_0 \in W_p^2(\Omega)$ for some $p > N$ and thanks to Morrey's Theorem, $u_0 \in L_\infty(\Omega)$. Thus,

$$a(\,\cdot\,)F(\,\cdot\,, u_0(\,\cdot\,)) \in L_\infty(\Omega)$$

and u_0 satisfies

$$\begin{cases} \mathcal{L}_0 u_0 = 0 & \text{in } \Omega, \\ \mathcal{B}(b)u_0 = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mathcal{L}_0 := \mathcal{L}(\lambda_0) + a(\,\cdot\,)F(\,\cdot\,, u_0(\,\cdot\,))$.

In other words, u_0 is a positive eigenfunction of $\mathcal{L}_0$ associated with the eigenvalue 0. Thus, by the uniqueness of the principal eigenpair, $(0, u_0)$ is the principal eigenpair of $\mathcal{L}_0$ in Ω and therefore $u_0 \in W^2_{\mathcal{B}(b)}(\Omega)$ and u_0 is strongly positive in Ω (cf. [1, Theorem 12.1]). The remaining assertions follow easily from the embedding $W_p^2(\Omega) \subset \mathcal{C}^{2-N/p}(\overline{\Omega})$ for each $p > N$, and [20, Theorem VIII.1]. $\square$

It should be pointed out that thanks to Lemma 4.2, (1.1) admits two kinds of non-negative solutions. Namely, $u = 0$ and the positive solutions of (1.1), which are strongly positive in Ω.

The following result gives a necessary condition for the existence of positive solutions of (1.1).

Proposition 4.3. *Under the general conditions of the introduction, if* (λ, u_λ) *is a positive solution of* (1.1)*, then*

$$\Sigma_0(\lambda) > 0, \tag{4.1}$$

where $\Sigma_0(\lambda)$ *is the map just defined by* (1.19).

Proof. Indeed, if (λ, u_λ) is a positive solution of (1.1), then u_λ is strongly positive in Ω,

$$u_\lambda \in L_\infty(\Omega), \quad a(\,\cdot\,)F(\,\cdot\,, u_\lambda(\,\cdot\,)) \in L_\infty(\Omega),$$

and $(0, u_\lambda)$ is the principal eigen-pair associated with

$$(\mathcal{L}(\lambda) + a(\,\cdot\,)F(\,\cdot\,, u_\lambda(\,\cdot\,)), \mathcal{B}(b), \Omega).$$

Moreover, since $a^+ \in \mathcal{A}(\Omega)$, we have that

$$\Omega^0_{a^+} \in \mathcal{C}^2, \quad \Omega^0_{a^+} \subset \Omega, \quad \operatorname{dist}(\Gamma_1, \partial\Omega^0_{a^+} \cap \Omega) > 0$$

and hence, Proposition 3.2 gives

$$\begin{aligned} 0 &= \sigma_1^{\Omega}[\mathcal{L}(\lambda) + (a^+ - a^-)F(\,\cdot\,, u_\lambda), \mathcal{B}(b)] \\ &< \sigma_1^{\Omega^0_{a^+}}[\mathcal{L}(\lambda) - a^- F(\,\cdot\,, u_\lambda), \mathcal{B}(b, \Omega^0_{a^+})]. \end{aligned} \tag{4.2}$$

On the other hand, owing to $(\mathcal{F}3)$ and (1.4) $a^-(\,\cdot\,)F(\,\cdot\,, u_\lambda(\,\cdot\,)) > 0$ in $\Omega^0_{a^+}$, and hence, Proposition 3.3 gives that

$$\sigma_1^{\Omega^0_{a^+}}[\mathcal{L}(\lambda) - a^- F(\,\cdot\,, u_\lambda), \mathcal{B}(b, \Omega^0_{a^+})] < \sigma_1^{\Omega^0_{a^+}}[\mathcal{L}(\lambda), \mathcal{B}(b, \Omega^0_{a^+})] = \Sigma_0(\lambda). \tag{4.3}$$

Therefore, (4.2) and (4.3) imply the result. This completes the proof. $\square$

In the sequel we will denote by

$$\Lambda_0^+ := \{\lambda \in \mathbb{R} : \Sigma_0(\lambda) > 0\}, \tag{4.4}$$

and therefore, Proposition 4.3 establishes that $\Lambda(a, F) \subset \Lambda_0^+$ for any $a \in L_\infty(\Omega)$ with $a^+ \in \mathcal{A}(\Omega)$ and $F(x, u)$ satisfying the (d) requirements of the introduction.

The following result provides us with a necessary condition for the existence of positive strict supersolution of the sublinear elliptic problem associated to (1.1) in the vanishing set of the negative part of the potential $a \in L_\infty(\Omega)$.

Proposition 4.4. *Let $a = a^+ - a^- \in L_\infty(\Omega)$ with $a^+, a^- \in \mathcal{A}(\Omega)$. Assume in addition that the $\mathcal{C}^2$-domain $\Omega^0_{a^-}$ is connected and that $a^+ \in \mathcal{A}(\Omega^0_{a^-})$. Then, the sublineal weighted elliptic mixed boundary value problem*

$$\begin{cases} \mathcal{L}u = \lambda W u - a^+(x)F(x,u)u & \text{in } \Omega^0_{a^-}, \\ \mathcal{B}(b, \Omega^0_{a^-}) = 0 & \text{on } \partial\Omega^0_{a^-}, \end{cases} \tag{4.5}$$

possesses a positive strict supersolution arbitrarily large and bounded away from zero in $\overline{\Omega}^0_{a^-}$, provided $\lambda \in \mathbb{R}$ satisfies

$$\sigma_1^{\Omega_0}[\mathcal{L}(\lambda), \mathcal{B}(b, \Omega_0)] > 0, \tag{4.6}$$

where Ω_0 stands for the vanishing set just defined by (1.5).

Proof. Indeed, since $a^- \in \mathcal{A}(\Omega)$, the vanishing set $\Omega^0_{a^-}$ satisfies ($\mathcal{A}1$) and hence

$$\Omega^0_{a^-} \in \mathcal{C}^2 \quad \text{and} \quad \operatorname{dist}(\Gamma_1, \partial\Omega^0_{a^-} \cap \Omega) > 0.$$

Therefore, the boundary operator $\mathcal{B}(b, \Omega^0_{a^-})$ is well defined in the sense of (1.12). On the other hand, since $\Omega^0_{a^-}$ is connected and $a^+ \in \mathcal{A}(\Omega^0_{a^-})$, the vanishing set of the potential a^+ in the domain $\Omega^0_{a^-}$, denoted by $[\Omega^0_{a^-}]^0_{a^+}$ which coincides with Ω_0, satisfies again the assumption ($\mathcal{A}1$) and hence,

$$\Omega_0 \in \mathcal{C}^2 \quad \text{and} \quad \operatorname{dist}(\Gamma_1, \partial\Omega_0 \cap \Omega) > 0.$$

Therefore, the boundary operator $\mathcal{B}(b, \Omega_0)$ also is well defined in the sense of (1.12). Now, since $\Omega^0_{a^-}$ is connected, $a^+ \in \mathcal{A}(\Omega^0_{a^-})$ and thanks to assumptions ($\mathcal{F}1$) and ($\mathcal{F}2$), (4.5) becomes into the abstract framework of (3.6) and therefore, Theorem 3.6 implies the result, provided $\lambda \in \mathbb{R}$ satisfies (4.6). This completes the proof. □

Remark 4.5. It should be noted that to impose that $a^+ \in \mathcal{A}(\Omega^0_{a^-})$ assuming previously that $a^+ \in \mathcal{A}(\Omega)$ in some situations does not suppose any additional restriction since under certain structural conditions about the domain $\Omega^0_{a^-}$ it is satisfied that

$$\mathcal{A}(\Omega) \subset \mathcal{A}(\Omega^0_{a^-}),$$

that is to say, it is satisfied that if the potential $a^+ \in \mathcal{A}(\Omega)$, then the restriction $a^+|_{\Omega^0_{a^-}} \in \mathcal{A}(\Omega^0_{a^-})$. In [7, Section 3] are given sufficient conditions about a subdomain $\widetilde{\Omega}$ of Ω to ensure that the nonnegative measurable potential $V|_{\widetilde{\Omega}} \in \mathcal{A}(\widetilde{\Omega})$ provided that $V \in \mathcal{A}(\Omega)$. We do not include them by do not enlarge the exposition.

Now, we characterize the set of bifurcation values to positive solutions of (1.1) from the trivial branch $(\lambda, u) = (\lambda, 0)$.

Theorem 4.6. *Let* $W \in L_\infty(\Omega)$ *be and set* $\mathcal{O} := \{\lambda \in \mathbb{R} : \Sigma(\lambda) = 0\}$ *where* $\Sigma(\lambda)$ *is the map just defined by* (1.18). *Assume that* W *changes of sign in* Ω *in the sense of Definition* 3.4 *and*

$$\sup_{\lambda\in\mathbb{R}} \Sigma(\lambda) > 0. \tag{4.7}$$

Then, $\widetilde{\lambda}_0$ *is a bifurcation value to positive solutions of* (1.1) *from the trivial branch* $(\lambda, u) = (\lambda, 0)$*, if and only if* $\widetilde{\lambda}_0 \in \mathcal{O}$.

Proof. The positive solutions of (1.1) are the zeroes of the operator $\mathcal{H}: \mathbb{R} \times \mathcal{U}^+ \to \mathcal{U}$, defined by

$$\mathcal{H}(\lambda, u) := u - (\mathcal{L} + M)^{-1}[(\lambda W(\,\cdot\,) + M)\mathcal{J}u - a(\,\cdot\,)F(\,\cdot\,, \mathcal{J}u)\mathcal{J}u],$$

where $\mathcal{U} = W^2_p(\Omega)$, $p > N$, $\mathcal{J}: \mathcal{U}^+ \hookrightarrow L_p(\Omega)$ is the inclusion operator, which is compact, and

$$M > -\sigma_1^\Omega[\mathcal{L}, \mathcal{B}(b)].$$

Moreover, $\mathcal{H}(\lambda, 0) = 0$ for each $\lambda \in \mathbb{R}$ and

$$\mathcal{D}_u\mathcal{H}(\lambda, 0) := \mathcal{I} - \mathcal{T}(\lambda)$$

where for each $\lambda \in \mathbb{R}$, $\mathcal{T}(\lambda): \mathcal{U}^+ \to \mathcal{U}$ is the operator defined by

$$\mathcal{T}(\lambda) := (\mathcal{L} + M)^{-1}[\lambda W(\,\cdot\,) + M]\mathcal{J}$$

and $\mathcal{I}$ is the identity operator in $\mathcal{U}^+$. Also,

$$\mathcal{D}_{u\lambda}\mathcal{H}(\lambda, 0) := -(\mathcal{L} + M)^{-1}W.$$

Tkanks to the compactness of $\mathcal{T}(\lambda)$ as an operator in $\mathcal{U}^+$, for each $\lambda \in \mathbb{R}$ we have that the operator $\mathcal{D}_u\mathcal{H}(\lambda, 0)$ is a Fredholm operator of index 0. Moreover,

$$N[\mathcal{D}_u\mathcal{H}(\lambda, 0)] = N[\mathcal{L}(\lambda)].$$

We now prove the necessary condition. Indeed, if $\widetilde{\lambda}_0$ is a bifurcation value to positive solutions of (1.1) from the trivial branch $(\lambda, u) = (\lambda, 0)$, necessarily $\dim[N[\mathcal{D}_u\mathcal{H}(\widetilde{\lambda}_0, 0)]] \geq 1$ and hence, $N[\mathcal{L}(\widetilde{\lambda}_0)] \neq 0$. Thus, it follows from the uniqueness of the principal eigenpair of the problem $(\mathcal{L}(\widetilde{\lambda}_0), \mathcal{B}(b), \Omega)$, that $\Sigma(\widetilde{\lambda}_0) = 0$ and therefore, $\widetilde{\lambda}_0 \in \mathcal{O}$. This proves the necessary condition.

We now prove the sufficient condition. Let $\widetilde{\lambda}_0 \in \mathcal{O}$. Since W changes of sign in Ω and (4.7) is satisfied, it follows from Theorem 3.5 that $\widetilde{\lambda}_0$ is a simple eigenvalue of the problem $(\mathcal{L}, W, \mathcal{B}(b), \Omega)$ in the sense of Definition 1.3. Hence, if φ_0 denotes the principal eigenfunction of $\mathcal{L}(\widetilde{\lambda}_0)$ associated with $\Sigma(\widetilde{\lambda}_0) = 0$, then

$$N[\mathcal{D}_u\mathcal{H}(\widetilde{\lambda}_0, 0)] = N[\mathcal{L}(\widetilde{\lambda}_0)] = \operatorname{span}[\varphi_0], \tag{4.8}$$

$$W\varphi_0 \notin R[\mathcal{L}(\widetilde{\lambda}_0)]. \tag{4.9}$$

Moreover,

$$\mathcal{D}_{u\lambda}\mathcal{H}(\widetilde{\lambda}_0,0)\varphi_0 \notin R[D_u\mathcal{H}(\widetilde{\lambda}_0,0)], \tag{4.10}$$

because if (4.10) is not satisfied, then $W\varphi_0 \in R[\mathcal{L}(\widetilde{\lambda}_0)]$ and this contradicts (4.9). Thus, due to the fact that $\mathcal{D}_u\mathcal{H}(\widetilde{\lambda}_0,0)$ is a Fredholm operator of index 0 and owing to (4.8) and (4.10), we find that

$$\begin{gathered}\dim N[\mathcal{D}_u\mathcal{H}(\widetilde{\lambda}_0,0)] = \operatorname{codim} R[\mathcal{D}_u\mathcal{H}(\widetilde{\lambda}_0,0)] = 1,\\ \mathcal{D}_{u\lambda}\mathcal{H}(\widetilde{\lambda}_0,0)\varphi_0 \notin R[\mathcal{D}_u\mathcal{H}(\widetilde{\lambda}_0,0)].\end{gathered}$$

Therefore, 0 is a $\mathcal{D}_{u\lambda}\mathcal{H}(\widetilde{\lambda}_0,0)$-simple eigenvalue of $\mathcal{D}_u\mathcal{H}(\widetilde{\lambda}_0,0)$ and [8, Lemma 1.1, Definition 1.2] implies that $\widetilde{\lambda}_0$ is a bifurcation value to positive solutions of (1.1) from the trivial branch $(\lambda,u)=(\lambda,0)$. This completes the proof. □

5. A priori bounds for the positive solutions of (1.1)

In this section we adapt to our general setting some of the re-scaling arguments showed in the previous work of H. Amann and J. López-Gómez [2, Section 4], to get a priori bounds for the positive solutions of (1.1). To deal with this problem, firstly we show that if the positive and negative part of $a \in L_\infty(\Omega)$ satisfy certain structural conditions, then the existence of a priori bounds for the positive solutions of (1.1) in $\overline{\Omega}_{a^-}$ implies the existence of a priori bounds for them in Ω. Throughout this section we will suppose that the positive and negative part of $a \in L_\infty(\Omega)$, a^+ and a^- respectively, satisfy the assumptions of Proposition 4.4.

THEOREM 5.1. *Let $a = a^+ - a^- \in L_\infty(\Omega)$ be with a^+ and a^- satisfying the assumptions of Proposition* 4.4. *Assume in addition that there exists a constant $C > 0$ such that*

$$\sup_{\Omega_{a^-}} u \le C \tag{5.1}$$

for any positive solution u of (1.1). *Then, there exists a constant $C_1 > 0$ such that*

$$\sup_{\Omega} u \le C_1 \tag{5.2}$$

for any positive solution of (1.1). *Moreover, if* (5.1) *is satisfied for all λ in a compact subinterval $[\alpha,\beta] \subset \Lambda_0^+$, then* (5.2) *also holds uniformly in $[\alpha,\beta]$.*

PROOF. Let (λ, u_λ) be a positive solution of (1.1). Then, Proposition 4.3, implies that

$$\Sigma_0(\lambda) > 0, \tag{5.3}$$

and Lemma 4.2 gives that u_λ is strongly positive in Ω. Then, it follows from (5.1) and the above arguments that u_λ satisfies the following sublineal elliptic boundary value problem in $\Omega^0_{a^-}$

$$\begin{cases} \mathcal{L}u = \lambda W u - a^+(x)F(x,u)u & \text{in } \Omega^0_{a^-}, \\ \partial u + bu = 0 & \text{on } \partial\Omega^0_{a^-} \cap \Gamma_1, \\ u = 0 & \text{on } \partial\Omega^0_{a^-} \cap \Gamma_0, \\ 0 < u \le C & \text{on } \partial\Omega^0_{a^-} \cap \Omega. \end{cases} \tag{5.4}$$

Thus, u_λ is a positive subsolution of the following sublineal elliptic mixed boundary value problem

$$\begin{cases} \mathcal{L}u = \lambda W u - a^+(x)F(x,u)u & \text{in } \Omega^0_{a^-}, \\ \partial u + bu = 0 & \text{on } \partial\Omega^0_{a^-} \cap \Gamma_1, \\ u = 0 & \text{on } \partial\Omega^0_{a^-} \cap \Gamma_0, \\ u = C & \text{on } \partial\Omega^0_{a^-} \cap \Omega. \end{cases} \tag{5.5}$$

On the other hand, since $a^+ \in \mathcal{A}(\Omega^0_{a^-})$, due to the fact that $F(x,u)$ satisfies $(\mathcal{F}1)$ and $(\mathcal{F}2)$ and since (5.3) is satisfied, it follows from Theorem 3.6 that the sublineal elliptic problem

$$\begin{cases} \mathcal{L}u = \lambda W u - a^+(x)F(x,u)u & \text{in } \Omega^0_{a^-}, \\ \mathcal{B}(b, \Omega^0_{a^-})u = 0 & \text{on } \partial\Omega^0_{a^-}, \end{cases} \tag{5.6}$$

possesses a positive strict supersolution arbitrarily large and bounded away from zero in $\overline{\Omega}^0_{a^-}$, since

$$\Omega_0 = [\Omega^0_{a^-}]^0_{a^+} \subset \Omega^0_{a^+},$$

and by Proposition 3.2 we have that

$$\sigma_1^{\Omega_0}[\mathcal{L}(\lambda), \mathcal{B}(b,\Omega_0)] > \Sigma_0(\lambda) > 0.$$

Let $\theta_C \in W^2_p, p > N$, a positive strict supersolution of (5.6) satisfying

$$\theta_C > C \quad \text{in } \overline{\Omega}^0_{a^-}. \tag{5.7}$$

Then θ_C is a positive strict supersolution of (5.5) and since u_λ is a positive subsolution of (5.5), it follows from $(\mathcal{F}3)$ and Theorem 3.1 that

$$u_\lambda \le \theta_C \quad \text{in } \overline{\Omega}^0_{a^-}. \tag{5.8}$$

Therefore, (5.1) and (5.8) imply that $\sup_\Omega u_\lambda \le C_1$, by taking

$$C_1 := \max\{\|\theta_C\|_{L^\infty(\Omega^0_{a^-})}, C\}.$$

This completes the proof. □

Corollary 5.2. *Let $I := [\alpha, \beta] \subset \Lambda_0^+$, with $\alpha < \beta$, such that the positive solutions of* (1.1) *do not have uniform a priori bounds in I. Then, there exists a sequence (λ_k, u_k) of solutions of* (1.1) *with $\lambda_k \in I$, $k \geq 1$ and a sequence of points $x_k \in \Omega_{a^-}$, $k \geq 1$, such that*

$$\lim_{k\to\infty} u_k(x_k) = \infty. \tag{5.9}$$

Proof. It follows from Theorem 5.1 taking into account that if u_k grows to infinity on $\partial\Omega_{a^-}$, then it grows to infinity close to the boundary and therefore the $x_k's$ can be taken in Ω_{a^-}. This completes the proof. □

Proposition 5.3. *Suppose that either*

$$r > 1 \quad \text{and} \quad N = 1, 2, \tag{5.10}$$

or

$$r < \frac{N+2}{N-2} \quad \text{and} \quad N \geq 3. \tag{5.11}$$

Let $I := [\alpha, \beta] \subset \Lambda_0^+$ be such that the positive solutions of (1.1) *do not have uniform a priori bounds for $\lambda \in I$ and let (λ_k, u_k), $k \geq 1$, be a sequence of positive solutions of* (1.1) *with*

$$\lim_{k\to\infty} \|u_k\|_{L^\infty(\Omega)} = \infty. \tag{5.12}$$

For each $k \geq 1$, let $x_k \in \overline{\Omega}_{a^-}$ be such that

$$u_k(x_k) = \sup_{\Omega_{a^-}} u_k. \tag{5.13}$$

Then,

$$\lim_{k\to\infty} u_k(x_k) = \infty, \tag{5.14}$$

and for any compact subset $K \subset \Omega_{a^-}$, K contains at most a finite number of x_k. In particular, by choosing a subsequence, if necessary, we can assume that

$$\lim_{k\to\infty} (\lambda_k, x_k) = (\lambda_\infty, x_\infty) \in I \times \partial\Omega_{a^-}. \tag{5.15}$$

Proof. Thanks to Theorem 5.1, (5.12) and (5.13) imply (5.14). Let K be any compact subset of Ω_{a^-}. To show that K contains at most a finite number of points x_k, $k \geq 1$ we argue by contradiction. Then, by taking a subsequence, if necessary, we can assume that

$$\lim_{k\to\infty} x_k = x_\infty \in K \subset \Omega_{a^-},$$

and hence,

$$d := \frac{\text{dist }(x_\infty, \partial\Omega_{a^-})}{2} > 0.$$

Now, for each $k \geq 1$, we set

$$M_k := u_k(x_k), \quad \rho_k := M_k^{(1-r)/2}. \tag{5.16}$$

Then, thanks to (5.14), $\lim_{k\to\infty} M_k = \infty$ and since $r > 1$,

$$\lim_{k\to\infty} \rho_k = 0. \tag{5.17}$$

It can be easily seen that the change of variables

$$y := \frac{x - x_k}{\rho_k}, \quad v_k(y) := \rho_k^{2/(r-1)} u_k(x),$$

transforms the differential equation of (1.1) into

$$\mathcal{L}_k v_k = \rho_k^2 \lambda_k W v_k + a^-(x_k + \rho_k y) F(x_k + \rho_k y, \rho_k^{-2/(r-1)} v_k(y)) \rho_k^2 v_k(y), \tag{5.18}$$

provided $x_k + \rho_k y \in \Omega_{a^-}$, where the operator $\mathcal{L}_k$ is defined by

$$\mathcal{L}_k := -\sum_{i,j=1}^N \alpha_{ij}(x_k + \rho_k y) \frac{\partial^2}{\partial y_i \partial y_j} + \sum_{j=1}^N \rho_k \alpha_j(x_k + \rho_k y) \frac{\partial}{\partial y_j} + \rho_k^2 \alpha_0(x_k + \rho_k y).$$

Moreover, by definition of d, for k sufficiently large we have that $|x - x_k| \leq d$ implies $x \in \Omega_{a^-}$. Thus, $|y| \leq d/\rho_k$ implies $x_k + \rho_k y \in \Omega_{a^-}$ and therefore, (5.18) holds. Hereafter, $\mathcal{Q}_\delta$ will stand for the ball of radius $\delta > 0$ centered at the origin. Note that since $\lim_{k\to\infty} d/\rho_k = \infty$, given any radius $R > 0$, $\mathcal{Q}_R \subset \mathcal{Q}_{d/\rho_k}$ for k sufficiently large. Moreover, it follows from (5.16) that

$$0 < v_k \leq \rho_k^{2/(r-1)} M_k = 1 \quad \text{in } \mathcal{Q}_{d/\rho_k}. \tag{5.19}$$

Thus, by the growth condition (1.7),

$$\lim_{k\to\infty} |\rho_k^2 F(x_k + \rho_k y, \rho_k^{-2/(r-1)} v_k(y)) v_k(y) - l(x_k + \rho_k y) v_k^r(y)| = 0.$$

Fix $R > 0$. Thanks to (5.19), by the elliptic L^p estimates we have uniform bounds for v_k in $W_p^2(\mathcal{Q}_R)$ for each $p \geq 2$. Thus, by Morrey's theorem (see [16]), we also uniform bounds for v_k in $\mathcal{C}^{1,\nu}(\mathcal{Q}_R)$ for each $\nu \in (0,1)$. Therefore, by taking a subsequence, if necessary, we can assume that $v_k \to v$ in $W^{p,2}(\mathcal{Q}_R) \cap \mathcal{C}^{1,\nu}(\mathcal{Q}_R), p > N$. By Holder continuity $v(0) = 1$. Moreover, since $x_k \to x_\infty$ and $\rho_k \to 0$ as $k \to \infty$, the following relations are satisfied

$$\begin{aligned}
\lim_{k\to\infty} \alpha_{ij}(x_k + \rho_k y) &= \alpha_{ij}(x_\infty), \\
\lim_{k\to\infty} \rho_k \alpha_j(x_k + \rho_k y) &= \lim_{k\to\infty} \rho_k^2 \alpha_0(x_k + \rho_k y) = 0, \\
\lim_{k\to\infty} l(x_k + \rho_k y) &= l(x_\infty) > 0, \\
\lim_{k\to\infty} a^-(x_k + \rho_k y) &= a^-(x_\infty) > 0,
\end{aligned}$$

uniformly in $y \in \mathcal{Q}_R$. Therefore, passing to the limit as $k \to \infty$ in (5.18) we find that $v(y)$ is a non-negative solution of

$$-\sum_{i,j=1}^{N} \alpha_{ij}(x_\infty) \frac{\partial^2 v}{\partial y_i \partial y_j} = a^-(x_\infty) l(x_\infty) v^r, \quad v(0) = 1, \tag{5.20}$$

in $\mathcal{Q}_R$. The same argument of the last paragraph of [11, p. 889] shows that in fact v is well defined in all of $\mathbb{R}^N$. By [11, Theorem 1.2] necessarily $v = 0$, a contradiction, since $v(0) = 1$. This completes the proof. □

Theorem 5.4. *Let $a \in L_\infty(\Omega)$ with*

$$a^+, a^- \in \mathcal{A}(\Omega), \quad \Omega^0_{a^-} \text{ connected}, \quad a^+ \in \mathcal{A}(\Omega^0_{a^-}), \quad \partial\Omega_{a^-} \in \mathcal{C}^1,$$

where

$$a^-(x) = C(x) \operatorname{dist}(x, \partial\Omega_{a^-})^\gamma, \quad x \in \Omega_{a^-}, \tag{5.21}$$

with $\gamma > 0$ and $C\colon \overline{\Omega}_{a^-} \to [0,\infty)$ is a continuous function bounded away from zero near $\partial\Omega_{a^-}$. Assume in addition that either

$$N \le 2 \quad \text{and} \quad r > 1 \tag{5.22}$$

or

$$N \ge 3 \quad \text{and} \quad r < \min\left\{\frac{N+2}{N-2}, \frac{N+1+\gamma}{N-1}\right\}. \tag{5.23}$$

Let $I \subset \Lambda_0^+$ be a compact interval. Then, the positive solutions of (1.1) *have uniform a priori bounds for $\lambda \in I$.*

Proof. We argue by contradiction. Assume that (1.1) does not admit uniform a priori bounds in I. Then, due to Corollary 5.2 and Proposition 5.3, there exists a sequence (λ_k, u_k) of positive solutions of (1.1) with $\lambda_k \in I$ and a sequence of points $x_k \in \overline{\Omega}_{a^-}$, $k \ge 1$ such that

$$\lim_{k\to\infty} x_k = x_\infty \in \partial\Omega_{a^-},$$
$$u_k(x_k) = \sup_{\Omega_{a^-}} u_k \to \infty \quad \text{as } k \to \infty.$$

Now, for each $k \ge 1$ we consider M_k and ρ_k defined by

$$M_k = u_k(x_k), \quad \rho_k^{(2+\gamma)/(r-1)} M_k = 1. \tag{5.24}$$

Then, the change of variables

$$y = \frac{x - x_k}{\rho_k}, \quad v_k(y) = \rho_k^{(2+\gamma)/(r-1)} u_k(x), \tag{5.25}$$

transforms the differential equation of (1.1) into

$$\mathcal{L}_k v_k = \rho_k^2 \lambda_k W v_k - \rho_k^2 a(x_k + \rho_k y) F(x_k + \rho_k y, \rho_k^{-(2+\gamma)/(r-1)} v_k(y)) v_k(y). \tag{5.26}$$

Since $\partial\Omega_{a^-}$ is of class $\mathcal{C}^1$, by a change of variable we can assume that near $x_\infty \in \partial\Omega_{a^-}$, $\partial\Omega_{a^-}$ is contained in the hyperplane $x^N = 0$ and that $x^N > 0$ if, and only if $x \in \Omega_{a^-}$. This can be achieved by straightening $\partial\Omega_{a^-}$ in a neighbourhood of x_∞ by a non-singular change of coordinates. Hereafter, we shall denote by x^j the j-th coordinate of x. Given $\varepsilon > 0$ small enough, assume that $x \in \mathcal{Q}_\varepsilon(x_\infty) \cap \Omega_{a^-}$. Then, for k large enough, $|x - x_k| < 2\varepsilon$ and hence

$$y \in H_k := \mathcal{Q}_{2\varepsilon/\rho_k} \cap \left\{ y^N \geq -\frac{x_k^N}{\rho_k} \right\}. \tag{5.27}$$

Thus, for large k, v_k is well defined in H_k defined by (5.27). Moreover, if $y \in H_k$, $x_k + \rho_k y \in \Omega_{a^-}$ and so, $a^+(x_k + \rho_k y) = 0$. Therefore, in H_k (5.26) reduces to

$$\mathcal{L}_k v_k = \rho_k^2 \lambda_k W v_k + \rho_k^2 a^-(x_k + \rho_k y) F(x_k + \rho_k y, \rho_k^{-(2+\gamma)/(r-1)} v_k(y)) v_k(y). \tag{5.28}$$

In H_k, we have $0 < v_k \leq v_k(0) = u_k(x_k) = 1$, and by the assumption (5.21),

$$a^-(x_k + \rho_k y) = C(x_k + \rho_k y)(x_k^N + \rho_k y^N)^\gamma,$$

where we have used that $x^N = \operatorname{dist}(x, \partial\Omega_{a^-})$. Setting

$$d_k := x_k^N = \operatorname{dist}(x_k, \partial\Omega_{a^-}^+) \tag{5.29}$$

yields

$$a^-(x_k + \rho_k y) = C(x_k + \rho_k y)\rho_k^\gamma \left(y^N + \frac{d_k}{\rho_k} \right)^\gamma,$$

and therefore, (5.28) can be written as

$$\begin{aligned} \mathcal{L}_k v_k = \rho_k^2 \lambda_k W v_k &+ \rho_k^{2+\gamma} C(x_k + \rho_k y) \left(y^N + \frac{d_k}{\rho_k} \right)^\gamma \\ &\cdot F(x_k + \rho_k y, \rho_k^{-(2+\gamma)/(r-1)} v_k(y)) v_k(y). \end{aligned} \tag{5.30}$$

In the sequel we shall distinguish three different situations according to the behaviour of d_k/ρ_k as $k \to \infty$.

Case 1. The sequence d_k/ρ_k, $k \geq 1$ is not bounded away from zero. Then, by choosing a subsequence, if necessary, we can assume that

$$\lim_{k\to\infty} \frac{d_k}{\rho_k} = 0. \tag{5.31}$$

Then, using (5.29) and taking into account that $\lim_{k\to\infty} \rho_k = 0$, it is clear that the H_k defined by (5.27) approaches the half-space $y^N > 0$, and the same compactness argument of the proof of Proposition 5.3 shows that along some subsequence $v_k \to v$, where v is a non-negative regular solution of

$$-\sum_{i,j=1}^N \alpha_{ij}(x_\infty) \frac{\partial^2 v_k}{\partial y_i \partial y_j} = C(x_\infty) l(x_\infty) (y^N)^\gamma v^r \quad \text{in } y^N > 0, \tag{5.32}$$

such that $v(0) = 1$. By assumption either (5.22) or (5.23) this is impossible, since due to [3, Corollary 2.1], $v = 0$ is the unique non-negative solution of (5.32).

Case 2. The sequence d_k/ρ_k, $k \geq 1$ is not bounded above. Then, by choosing a subsequence, if necessary, we can assume that

$$\lim_{k\to\infty} \frac{d_k}{\rho_k} = \infty. \tag{5.33}$$

Now, the H_k defined by (5.27) converges to $\mathbb{R}^N$ and so, v_k is defined on arbitrarily large balls for k large enough. Moreover, the change of variables

$$\beta_k := \left(\frac{\rho_k}{d_k}\right)^{\gamma/2}, \quad z := \frac{y}{\beta_k}, \quad w_k(z) := v_k(y),$$

transforms (5.30) into

$$\begin{aligned} \mathcal{A}_k w_k = {}& \rho_k^2 \beta_k^2 \lambda_k w_k + \rho_k^{2+\gamma} C(x_k + \rho_k \beta_k z)(\beta_k^{2/\gamma+1} z^N + 1)^\gamma \\ & \cdot F(x_k + \rho_k \beta_k z, \rho_k^{-(2+\gamma)/(r-1)} w_k(z)) w_k(z) \end{aligned} \tag{5.34}$$

where $\mathcal{A}_k$ is the differential operator defined by

$$\begin{aligned} \mathcal{A}_k := {}& -\sum_{i,j=1}^N \alpha_{ij}(x_k + \rho_k \beta_k z) \frac{\partial^2}{\partial z_i \partial z_j} \\ & + \sum_{j=1}^N \rho_k \beta_k \alpha_j (x_k + \rho_k \beta_k z) \frac{\partial}{\partial z_j} + \rho_k^2 \beta_k^2 \alpha_0 (x_k + \rho_k \beta_k z). \end{aligned}$$

By (5.33), $\lim_{k\to\infty} \beta_k = 0$. Thus, the same compactness argument of the proof of Proposition 5.3 shows that along some subsequence $w_k \to w$, where w is a non-negative regular solution of

$$-\sum_{i,j=1}^N \alpha_{ij}(x_\infty) \frac{\partial^2 w_k}{\partial z_i \partial z_j} = C(x_\infty) l(x_\infty) w^r \quad \text{in } \mathbb{R}^N, \tag{5.35}$$

such that $w(0) = 1$. Since $r < (N+2)/(N-2)$, it follows from [11, Theorem 1.2] that $w = 0$, which is impossible.

Case 3. The sequence d_k/ρ_k, $k \geq 1$ is bounded above and bounded away from zero. Then, by choosing a subsequence, if necessary, we can assume that

$$\lim_{k\to\infty} \frac{d_k}{\rho_k} = s > 0. \tag{5.36}$$

Then, passing to the limit as $k \to \infty$ in (5.30), we find that there exists a positive solution v of

$$-\sum_{i,j=1}^N \alpha_{ij}(x_\infty) \frac{\partial^2 v}{\partial y_i \partial y_j} = C(x_\infty) l(x_\infty) (y^N + s)^\gamma v^r \quad \text{in } y^N > -s, \tag{5.37}$$

such that $v(0) = 1$. The change of variable $z^j = y^j$, $1 \le j \le N-1$, $z^N = y^N + s$, $w(z) = v(y)$ transforms (5.37) into

$$-\sum_{i,j=1}^{N} \alpha_{ij}(x_\infty)\frac{\partial^2 w}{\partial z_i \partial z_j} = C(x_\infty)l(x_\infty)(z^N)^\gamma w^r, \quad \text{in } z^N > 0, \tag{5.38}$$

with $w(0,\dots,0,s) = 1$. Since $r < (N+\gamma+1)/(N-1)$, of [3, Corollary 2.1] implies $w = 0$, which again is a contradiction. This completes the proof of the theorem. $\square$

Theorem 5.4 shows that if $N \le 2$, then we always have uniform a priori bounds on compact subintervals of λ. When $N \ge 3$, it provides us with the folllowing result

Corollary 5.5. *Let $a \in L_\infty(\Omega)$ with*

$$a^+, a^- \in \mathcal{A}(\Omega), \quad \Omega^0_{a^-} \text{ connected}, \quad a^+ \in \mathcal{A}(\Omega^0_{a^-}), \quad \partial\Omega_{a^-} \in \mathcal{C}^1.$$

Assume in addition that $N \ge 3$,

$$r < \frac{N+2}{N-2}, \tag{5.39}$$

and

$$a^-(x) = C(x)\operatorname{dist}(x, \partial\Omega_{a^-})^\gamma, \quad x \in \Omega_{a^-},$$

where

$$\gamma \ge \frac{2N}{N-2} \tag{5.40}$$

and $C: \Omega_{a^-} \to (0,\infty)$ is a continuous function bounded away from zero near $\partial\Omega_{a^-}$. Then, the positive solutions of (1.1) *have uniform a priori bounds for λ varying on any compact subinterval of Λ^+_0.*

Proof. Under condition (5.40), (5.23) becomes into (5.39). This completes the proof. $\square$

6. Compact components of positive solutions of (1.1)

Throughout this section, we will assume that we are working under the assumptions of either Theorem 5.4, or Corollary 5.5. In any of these cases, we have a priori bounds for the positive solutions of (1.1) in compact subset of Λ^+_0.

Theorem 6.1. *Let $a = a^+ - a^- \in L_\infty(\Omega)$ be with a^+ and a^- satisfying*

$$a^+, a^- \in \mathcal{A}(\Omega), \quad \Omega^0_{a^-} \text{ is connected}, \quad a^+ \in \mathcal{A}(\Omega^0_{a^-}), \quad \partial\Omega_{a^-} \in \mathcal{C}^1.$$

Assume in addition that the potential W changes of sign in $\Omega^0_{a^+}$ in the sense of Definition 3.4 *with D_+ satisfying $D_+ \cap \Omega_{a^-} \ne \emptyset$ and*

$$\sup_{\lambda\in\mathbb{R}} \Sigma(\lambda) > 0. \tag{6.1}$$

Set $\sigma_1^1 < \sigma_1^2$ the unique roots of the map $\Sigma(\lambda)$ and $\lambda_1^0 < \lambda_2^0$ the unique roots of the map $\Sigma_0(\lambda)$, whose existence will be guaranteed. Then:

(a) $\Lambda(a,F) \subset (\lambda_1^0, \lambda_2^0)$ *and therefore,* $\overline{\Lambda}(a,F)$ *is bounded.*

(b) *The roots σ_1^i, $i = 1,2$ of the map $\Sigma(\lambda)$ are simple eigenvalues of $(\mathcal{L}, W, \mathcal{B}(b), \Omega)$ in the sense of Definition* 1.3 *and they are bifurcation values to positive solutions of* (1.1) *from the trivial branch* $(\lambda, u) = (\lambda, 0)$.

(c) *The global continuum $\mathcal{C}^+(\sigma_1^1)$ of positive solutions emanating from the trivial branch at $(\lambda,0) = (\sigma_1^1, 0)$, it is bounded in $\mathbb{R} \times L_\infty(\Omega)$ and comes back again to the trivial branch at the bifurcation point $(\lambda, 0) = (\sigma_1^2, 0)$. Therefore,*

$$\mathcal{C}^+(\sigma_1^1) = \mathcal{C}^+(\sigma_1^2). \tag{6.2}$$

(d) $(\sigma_1^1, \sigma_1^2) \subset \Lambda(a,F) \subset (\lambda_1^0, \lambda_2^0)$.

PROOF. (a) Indeed, thanks to (6.1), it follows from (3.3) that

$$\sup_{\lambda\in\mathbb{R}} \Sigma_0(\lambda) \geq \sup_{\lambda\in\mathbb{R}} \Sigma(\lambda) > 0. \tag{6.3}$$

Moreover, since W changes of sign in $\Omega_{a^+}^0$, W changes of sign in Ω and hence, thanks to Theorem 3.5, it follows that

$$\lim_{\lambda\to\infty} \Sigma(\lambda) = \lim_{\lambda\to-\infty} \Sigma(\lambda) = -\infty$$

and

$$\lim_{\lambda\to\infty} \Sigma_0(\lambda) = \lim_{\lambda\to-\infty} \Sigma_0(\lambda) = -\infty.$$

Thus, thanks to (6.3) and the concavity of $\Sigma(\lambda)$ and $\Sigma_0(\lambda)$ guaranteed by Theorem 3.5, it follows that eachone of the maps $\Sigma(\lambda)$ and $\Sigma_0(\lambda)$ have two real roots denoted by σ_1^1, σ_1^2 and λ_1^0, λ_2^0, respectively. Then,

$$\Lambda_0^+ = (\lambda_1^0, \lambda_2^0), \tag{6.4}$$

and owing to Proposition 4.3, $\Lambda(a,F) \subset (\lambda_1^0, \lambda_2^0)$. In particular $\Lambda(a,F)$ is bounded. Moreover, thanks to the fact that $D_+ \cap \Omega_{a^-} \neq \emptyset$, arguing as in Lemma 3.4 and Theorem 3.5 of [2], we have that $\Lambda(a,f)$ is strongly contained in $(\lambda_1^0, \Lambda_2^0)$, that is to say, $\overline{\Lambda}(a,F) \subset (\lambda_1^0, \lambda_2^0)$.

(b) Due to the fact that W changes of sign in Ω and thanks to (6.1), the result it is straighforward from Theorems 3.5 and 4.6.

(c) Set $\mathcal{C}^+(\sigma_1^i)$, $i = 1,2$ the global continuum of positive solutions of (1.1) emanating from the trivial branch at the bifurcation point $(\lambda, 0) = (\sigma_1^i, 0)$, $i = 1,2$, whose existence is guaranteed by (b). Since we are assuming the assumptions of either Theorem 5.4 or Corollary 5.5, the existence of a priori bounds in compact subset of Λ_0^+ for the positive solutions of (1.1) is guaranteed. Hence, thanks to thanks to the fact that (a) is satisfied, the continuum

$$\mathcal{C}^+(\sigma_1^i) \subset (\lambda_1^0, \lambda_2^0) \times [0, M], \quad i = 1, 2$$

for $M > 0$ sufficiently large. Therefore, owing to Rabinowitz global bifurcation theorem (see [18], [14]), the continuum $\mathcal{C}^+(\sigma_1^1)$ must necessarily come back again to the trivial branch at the bifurcation point $(\lambda, u) = (\sigma_1^2, 0)$, since σ_1^1 and σ_1^2 are the unique bifurcation values to positive solutions of (1.1) from the trivial branch and they are simple eigenvalues for the problem $(\mathcal{L}, W, \mathcal{B}(b), \Omega)$ in the sense of Definition 1.3. The same occur with the continuum $\mathcal{C}^+(\sigma_1^2)$, which must necessarily come back to the trivial branch at the bifurcation point $(\sigma_1^1, 0)$. Therefore, (6.2) is followed.

(d) Since (6.2) is satisfied,

$$\mathcal{C}^+(\sigma_1^1) \cap \{(\lambda, 0)\} = \{(\sigma_1^1, 0)\} \cup \{(\sigma_1^2, 0)\},$$

and the λ-projection $\mathcal{P}_\lambda(\mathcal{C}^+(\sigma_1^1))$ of the continuum $\mathcal{C}^+(\sigma_1^1)$ on the λ-axes is connected, we have that $(\sigma_1^1, \sigma_1^2) \subset \mathcal{P}_\lambda(\mathcal{C}^+(\sigma_1^1))$, and hence, $(\sigma_1^1, \sigma_1^2) \subset \Lambda(a, F)$. Now, the result it follows from (a). This completes the proof. □

Theorem 6.2. *Under the assumptions of Theorem* 6.1*, set* $\mathcal{C}^+$ *the global bounded continuum of positive solutions of* (1.1) *emanating from the trivial branch at the bifurcation points* $(\lambda, u) = (\sigma_1^1, 0)$ *and* $(\lambda, u) = (\sigma_1^2, 0)$*, whose existence is guaranteed by Theorem* 6.1*. Then, the following is satisfied:*

(a) *If* $\mathcal{C}^+$ *emanates subcritically from the bifurcation point* $(\lambda, u) = (\sigma_1^1, 0)$*, then there exists* λ_1^* *satisfying* $\lambda_1^0 < \lambda_1^* < \sigma_1^1$ *such that*

$$[\lambda_1^*, \sigma_1^2) \subset \Lambda(a, F) \subset (\lambda_1^0, \lambda_2^0). \tag{6.5}$$

In particular, (1.1) *possesses a positive solution for* $\lambda = \sigma_1^1$.

(b) *If* $\mathcal{C}^+$ *emanates supercritically from the bifurcation point* $(\lambda, u) = (\sigma_1^2, 0)$*, then there exists* λ_2^* *satisfying* $\sigma_1^2 < \lambda_2^* < \lambda_2^0$ *such that*

$$(\sigma_1^1, \lambda_2^*] \subset \Lambda(a, F) \subset (\lambda_1^0, \lambda_2^0). \tag{6.6}$$

In particular, (1.1) *possesses a positive solution for* $\lambda = \sigma_1^2$.

Proof. Set $\mathcal{P}_\lambda(\mathcal{C}^+)$ the λ-projection of $\mathcal{C}^+$.

(a) Indeed, by taking $\lambda_1^* := \min \mathcal{P}_\lambda(\mathcal{C}^+)$, and due to the fact that $\mathcal{C}^+$ emanates subcritically from the bifurcation point $(\sigma_1^1, 0)$, we have that $\lambda_1^* < \sigma_1^1$. Moreover, thanks to the existence of a priori bounds for the positive solutions of (1.1) it follows by an standart compactness arguments the existence of a non-negative solution u_1^* of (1.1) for $\lambda = \lambda_1^*$. Now taking into account that σ_1^1 and σ_1^2 are the unique bifurcation values to positive solutions of (1.1) from the trivial branch and thanks to Proposition 4.3 and (a) of Theorem 6.1, it follows that u_1^* is a positive solution of (1.1) for $\lambda = \lambda_1^* \in (\lambda_1^0, \sigma_1^1)$. Then, since $\mathcal{C}^+$ is connected, it follows (6.5) and thanks to the simplicity of the eigenvalue σ_1^1, (1.1) possesses a positive solution for $\lambda = \sigma_1^1$.

(b) It follows arguing as in (a) by taking $\lambda_2^* := \max \mathcal{P}_\lambda(\mathcal{C}^+)$. This completes the proof. □

Remark 6.3. It should be noted that it might exist either λ^* or μ^* satisfying

$$\lambda_1^0 < \lambda^* < \min \mathcal{P}_\lambda(\mathcal{C}^+) \quad \text{or} \quad \sup \mathcal{P}_\lambda(\mathcal{C}^+) < \mu^* < \lambda_2^0,$$

for which (1.1) possesses a positive solution u_{λ^*} or u_{μ^*}, since the existence of other continuum of positive solutions of (1.1) bounded away from the trivial branch are not discarded. Nevertheless, under the assumptions either Theorem 5.4 or Corollary 5.5, and thanks to Proposition 4.3, any global continuum $\widetilde{\mathcal{C}}^*$ of positive solutions of (1.1) bounded away from the trivial branch $(\lambda, u) = (\lambda, 0)$ is bounded in $\mathbb{R} \times L_\infty(\Omega)$ and exactly

$$\widetilde{\mathcal{C}}^* \subset (\lambda_1^0, \lambda_2^0) \times (0, M)$$

for $M > 0$ sufficiently large.

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