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TOPOLOGICAL CHARACTERISTIC OF FULLY NONLINEAR PARABOLIC BOUNDARY VALUE PROBLEMS

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Abstract. A general nonlinear initial boundary value problem

$$\frac{\partial u}{\partial t} - F(x,t,u,D^1u,\dots,D^{2m}u) = f(x,t), \quad (x,t)\in Q_T \equiv \Omega\times(0,T), \tag{1}$$

$$G_j(x,t,u,\dots,D^{m_j}u) = g_j(x,t), \quad (x,t)\in S_T \equiv \partial\Omega\times(0,T),\ j=\overline{1,m}, \tag{2}$$

$$u(x,0) = h(x), \quad x\in\Omega \tag{3}$$

is being considered, where Ω is a bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary. The problem (1)–(3) is then reduced to an operator equation $Au=0$, where the operator A satisfies $(\mathrm{S})_+$ condition. The local and global solvability of the problem (1)–(3) are achieved via topological methods developed by the first author. Further applications involving the convergence of Galerkin approximations are also given.

1. Introduction

This paper is devoted to the study of fully nonlinear parabolic problems via topological methods based on the degree theory for operators satisfying condition $(\mathrm{S})_+$.

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Let Ω denote a bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. We consider an initial-boundary value problem of the form

$$\Phi[u] \equiv \frac{\partial u}{\partial t} - F(x,t,u,D^1u,\dots,D^{2m}u) = f(x,t), \quad (x,t)\in Q_T \equiv \Omega\times(0,T), \tag{1.1}$$

$$\Psi_j[u] \equiv G_j(x,t,u,\dots,D^{m_j}u) = g_j(x,t), \quad (x,t)\in S_T = \partial\Omega\times(0,T),\ j=1,\dots,m, \tag{1.2}$$

$$u(x,0) = h(x), \quad x\in\Omega \tag{1.3}$$

in the Sobolev space $W_p^{(4m,2)}(Q_T)$ for $p\ge 2$, $p>(2m+n)/2m$, $p\ne(2m+1)/(2m-m_j)$, $m_j\le 2m-1$, $j=1,\dots,m$.

We assume that $F(x,t,\xi)$, $G_j(x,\tau,\zeta_j)$ are sufficiently smooth functions satisfying the parabolicity and the Lopatinskiĭ conditions, functions on the right-hand sides of (1.1)–(1.3) satisfy the inclusions

$$\begin{aligned} f &\in W_p^{(2m,1)}(Q_T),\\ g_j &\in W_p^{(4m-m_j-1/p,2-m_j/2m-1/2mp)}(S_T),\\ h &\in W_p^{(4m-2m/p)}(\Omega), \end{aligned}$$

as well as some compatibility conditions for $x\in\partial\Omega$, $t=0$.

In a standard way it is possible to reduce the problem (1.1)–(1.3) to the analogous problem in the space $W_p^{(4m,2),0}(Q_T)$ with zero initial condition. The main result of this paper is the reduction of the problem (1.1)–(1.3) with zero initial condition to the operator equation

$$Au = 0 \tag{1.4}$$

in the space $W_p^{(4m,2),0}(Q_T)$ with an operator A acting from $W_p^{(4m,2),0}(Q_T)$ into the dual space $[W_p^{(4m,2),0}(Q_T)]^*$. This operator for the problem (1.1)–(1.3) with $h(x)\equiv 0$ is defined by the following equality

$$\langle Au,\varphi\rangle = \frac{1}{p}\frac{d}{ds}\bigg\{(\|\Phi[u+s\varphi]-f\|_{p,Q_T}^{(2m,1)})^p + \sum_{j=1}^m(\|\Psi_j[u+s\varphi]-g_j\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/2m-1/(2mp))})^p\bigg\}\bigg|_{s=0} \tag{1.5}$$

and norms in the right-hand side of (1.5) are norms in spaces $W_p^{(2m,1)}(Q_T)$, $W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}(S_T)$, respectively.

We prove that this operator is bounded, continuous and it satisfies following condition:

(S$_+$) for every sequence of functions $u_j \in W_p^{(4m,2),0}(Q_T)$ which converges weakly to some function u_0 and is such that

$$\limsup_{j\to\infty}\langle Au_j, u_j - u_0\rangle \le 0$$

we have that u_j converges strongly to u_0.

Here $\langle Au_j, u_j - u_0\rangle$ is the value of the functional $Au_j \in [W_p^{(4m,2),0}(Q_T)]^*$ on the function $u_j(x) - u_0(x) \in W_p^{(4m,2),0}(Q_T)$.

These properties of the operator A allow as to study the solvability of the equation (1.4) by the use of the degree theory for such operators, that was developed in [6], [7].

This paper is organized in the following way. In Section 2 we introduce the considered function spaces. We also formulate assumptions for data functions in the problem (1.1)–(1.3) and formulate a priori estimates for linear parabolic problems with coefficients from Sobolev spaces that are principal for the study of properties of the operator A. In Section 3, we reduce the initial-boundary value problem (1.1)–(1.3) to the operator equation (1.4) and we establish the properties of the operator A. Some applications of the topological approach to the proof of local and global solvability of nonlinear parabolic problems and to the study of strong convergence of the Galerkin approximants are contained in Section 4.

For the case of Dirichlet boundary condition the solvability of initial-boundary value problems for fully nonlinear parabolic equations was established by other methods in the papers of Hudjaev, Kruzhkov, Castro and Lopes, Sopolov, Krylov, Lunardi, Wang. The local approach for problems with nonlinear boundary condition was developed by Amann and Acquistapace and Terreni. The list of corresponding papers there is given the bibliography of [2].

2. Problem formulation

2.1. Functional spaces. Let n, m, $\{m_j\}_{j=1}^m$ be positive integers such that $0 \le m_j \le 2m-1$ and p, T be positive real numbers. In what follows Ω denotes a bounded open set in $\mathbb{R}^n$ with sufficiently smooth boundary $\partial\Omega$. We'll use notations $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be multi-index with non-negative integer components. For $x \in \mathbb{R}^n$ we denote $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

For $u\colon Q_T \to \mathbb{R}$ and positive integer $k \ge 0$ we'll use notations

$$D^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \ldots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u, \quad D^k u = \{D^\alpha u : |\alpha| = k\}.$$

By $M(k)$ we denote the collection of all different multiindices of order that is less or equal k.

We fix further some notations and definitions of norms in anisotropic Hölder and Sobolev spaces that are analogous to corresponding spaces in the monograph [1].

For a positive integer b and a positive non-integer k the space $C^{(bk,k)}(\overline{Q_T})$ is defined to be the Banach space of all functions u, that have continuous derivatives $(\partial/\partial t)^s D^\alpha u(x,t)$, $|\alpha|+bs\le bk$ $(x,t)\in\overline{Q_T}$ and the finite norm

$$|u|^{(bk,k)}_{Q_T}=\sum_{|\alpha|+bs\le[bk]}\left|\left(\frac{\partial}{\partial t}\right)^s D^\alpha u(x,t)\right|^{(0)}_{Q_T}+|u|^{(bk)}_{b,Q_T},$$

where $|u|^{(0)}_{Q_T}=\max\{|u(x,t)|:(x,t)\in\overline{Q_T}\}$, $[k]$ is greatest integer function of k and

$$\begin{aligned}|u|^{(bk)}_{b,Q_T}&=\sum_{|\alpha|+bs=[bk]}\left|\left(\frac{\partial}{\partial t}\right)^s D^\alpha u(x,t)\right|^{(bk-[bk])}_{x,Q_T}\\&\quad+\sum_{0<bk-|\alpha|-bs<b}\left|\left(\frac{\partial}{\partial t}\right)^s D^\alpha u(x,t)\right|^{((bk-|\alpha|-bs)/b)}_{t,Q_T},\\|u|^{(l)}_{x,Q_T}&=\sup\left\{\frac{|u(x,t)-u(y,t)|}{|x-y|^l}:x,y\in\Omega,\ x\ne y,\ t\in(0,T)\right\},\\|u|^{(l)}_{t,Q_T}&=\sup\left\{\frac{|u(x,t)-u(x,\tau)|}{|t-\tau|^l}:x\in\Omega,\ t,\tau\in(0,T),\ t\ne\tau\right\},\end{aligned}$$

where $l\in(0,1)$.

For $p>1$ and positive integers b, k, by $W^{(bk,k)}_p(Q_T)$ we denote the Banach space of all functions u that have generalized derivatives $(\partial/\partial t)^s D^\alpha u\in L_p(Q_T)$, $|\alpha|+bs\le bk$. The norm of the space $W^{(bk,k)}_p(Q_T)$ will be defined in the following way:

$$\|u\|^{(bk,k)}_{p,Q_T}=\left\{\sum_{|\alpha|+bs\le bk}\left\|\left(\frac{\partial}{\partial t}\right)^s D^\alpha u\right\|^p_{p,Q_T}\right\}^{1/p},$$

$$\text{where } \|u\|_{p,Q_T}=\left\{\int_{Q_T}|u|^p\,dx\,dt\right\}^{1/p}.$$

In the case of a positive non-integer k such that bk is not integer, we denote by $W^{(bk,k)}_p(Q_T)$ the Banach space of all functions u that have generalized derivatives $(\partial/\partial t)^s D^\alpha u\in L_p(Q_T)$, $|\alpha|+bs\le bk$ and the finite norm

$$\|u\|^{(bk,k)}_{p,Q_T}=\left\{\sum_{|\alpha|+bs\le[bk]}\left\|\left(\frac{\partial}{\partial t}\right)^s D^\alpha u\right\|^p_{p,Q_T}+(\|u\|^{(bk)}_{b,p,Q_T})^p\right\}^{1/p},$$

$$\|u\|^{(bk)}_{b,p,Q_T} = \Big\{ \sum_{|\alpha|+bs=[bk]} \Big(\Big\| \Big(\frac{\partial}{\partial t}\Big)^s D^\alpha u \Big\|^{(bk-[bk])}_{x,p,Q_T} \Big)^p + \sum_{0<bk-|\alpha|-bs<b} \Big(\Big\| \Big(\frac{\partial}{\partial t}\Big)^s D^\alpha u \Big\|^{((bk-|\alpha|-bs)/b)}_{t,p,Q_T} \Big)^p \Big\}^{1/p},$$

$$\|u\|^{(l)}_{x,p,Q_T} = \Big\{ \int_0^T dt \iint_{\Omega^2} \frac{|u(x,t)-u(y,t)|^p}{|x-y|^{n+pl}}\, dx\, dy \Big\}^{1/p}, \qquad 0<l<1,$$

$$\|u\|^{(l)}_{t,p,Q_T} = \Big\{ \int_\Omega dx \iint_{[0,T]^2} \frac{|u(x,t)-u(x,\tau)|^p}{|t-\tau|^{1+pl}}\, dt\, d\tau \Big\}^{1/p}, \qquad 0<l<1.$$

Let S be the $n-1$-dimentional surface in $\mathbb{R}^n$ and $l_0 \ge 0$. We will say that S belongs to class C^{l_0} if there exists a finite collection of open sets $\{U_i\}_{i=1}^I$ and $d>0$ such that

- (S$_1$) $S \subset \bigcup_{i=1}^I U_i$;
- (S$_2$) for each i there exists $\xi^{(i)} \in S \cap U_i$ such that the set $S \cap U_i$ in local Cartesian system $\{y\}$ with origin at $\xi^{(i)}$ is given by the equation $y_n = h_i(y')$, $y' \in D(d)$, where $y' = (y_1,\dots,y_{n-1})$, $D(d) = (-d,d)^{n-1}$;
- (S$_3$) $h_i \in C^{l_0}(D(d))$ for each i.

Let b be a positive integer, $k>0$, $p>1$, and $\partial\Omega \in C^{l_0}$, where $l_0 \ge \max\{bk,1\}$. Let $D_T(d) = D(d)\times(0,T)$. For $u\colon S_T \to \mathbb{R}$ we will use the notation $u^{(i)}(y',t) = u(\phi_i(y',h_i(y')),t)$, where $(y',t)\in D_T(d)$, $i=\overline{1,I}$ and $\phi_i(y)$ is the transformation from the local coordinate system $\{y\}$ to the system $\{x\}$.

We define the space $C^{(bk,k)}(S_T)$ as the set of all functions $u\colon S_T\to\mathbb{R}$ such that $u^{(i)} \in C^{(bk,k)}(D_T(d))$, $i=\overline{1,I}$ with the norm

$$|u|^{(bk,k)}_{S_T} = \max\{|u^{(i)}|^{(bk,k)}_{D_T(d)},\ i=\overline{1,I}\}.$$

We define the space $W_p^{(bk,k)}(S_T)$ as the set of all functions $u\colon S_T\to\mathbb{R}$ such that $u^{(i)} \in W_p^{(bk,k)}(D_T(d))$, $i=\overline{1,I}$ with the norm

$$\|u\|^{(bk,k)}_{p,S_T} = \Big\{ \sum_{i=1}^I (\|u^{(i)}\|^{(bk,k)}_{p,D_T(d)})^p \Big\}^{1/p}.$$

It is a simple task to check that norms corresponding to different covers of $\partial\Omega$ by sets U_i are equivalent.

For positive integer k we denote

$$W_p^{(bk,k),0}(Q_T) := \Big\{ u \in W_p^{(bk,k)}(Q_T) : \frac{\partial^s u}{\partial t^s} = 0,\ 0 \le s \le k-1 \Big\}.$$

2.2. Some results from degree theory. We recall some results from the operator degree theory developed, for example, in [6], [7]. These results are used in paper to obtain the theorems of uniqueness and solvability for problem (1.1)–(1.3).

Definition (Degree for continuous mapping in finite-dimentional space). Let Ω be finite domain from $\mathbb{R}^n$. Suppose that f is continuous mapping from Ω to $\mathbb{R}^n$ such that $f(u) \neq 0$, $u \in \partial\Omega$. We call the degree of f the *integer-value function* $\deg(f, \overline{\Omega}, 0)$ such that satisfies following conditions:

(a) If $f(x) = x - x_0$ where $x_0 \in \Omega$ then $\deg(f, \overline{\Omega}, 0) = 1$.
(b) Let Ω_1, Ω_2 be subsets of Ω such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $f(x) \neq 0$, $x \in \overline{\Omega} \setminus (\Omega_1 \cup \Omega_2)$. Then $\deg(f, \overline{\Omega}, 0) = \deg(f, \overline{\Omega_1}, 0) + \deg(f, \overline{\Omega_2}, 0)$.
(a) Let $h\colon [0,1] \times \Omega \to \mathbb{R}^n$ be continuous mapping such that $h(t,x) \neq 0$ when $t \in [0,1]$, $x \in \partial\Omega$. We denote $f_0(x) = h(0,x)$, $f_1(x) = h(1,x)$. Then $\deg(f_0, \overline{\Omega}, 0) = \deg(f_1, \overline{\Omega}, 0)$.

The definition of the degree could be extended to some classes of contiguous mappings A acting from reflexive separable Banach space X to X^*.

Definition ($(\mathrm{S})_+$ condition). We say that an operator $A\colon X \to X^*$ satisfies the condition $(\mathrm{S})_+$ if, for arbitrary sequence $\{u_j\} \subset X$, which converges weakly to some $u_0 \in X$ and such that

$$\limsup_{j\to\infty} \langle Au_j, u_j - u_0\rangle \leq 0$$

we have that u_j converges strongly to u_0.

Let $\Omega \in X$ be bounded domain. The mapping $A\colon \overline{\Omega} \to X$, $Au \neq 0$, $u \in \Omega$ satisfying the $(\mathrm{S})_+$ on condition on $\overline{\Omega}$ could be approximated by the finite-dimensional mappings

$$A_n u = \sum_{i=1}^{n} \langle Au, v_i\rangle v_i, \quad u \in \Omega_n = \left\{ u \in \Omega : u = \sum_{i=1}^{n} \alpha_i v_i,\ \alpha_i \in \mathbb{R} \right\},$$

where $\{v_i\}_{i\geq 1}$ is a basis in X.

It is proved that the limit $\lim_{n\to\infty} \deg(A_n, \bar{(}\Omega_n), 0)$ exists and does not depend on the choice of the basis $\{v_i\}_{i\geq 1}$. We call the degree of the operator A in the domain Ω the limit $\mathrm{Deg}(A, \overline{\Omega}, 0) = \lim_{n\to\infty} \deg(A_n, \bar{(}\Omega_n), 0)$.

The definition of the degree for mapping in unbounded space preserves all properties of the degree in the case of finite-dimensional space.

Definiton (Homotopy). Let mappings A_0, A_1 satisfy $(\mathrm{S})_+$ condition on $\overline{\Omega}$. We say that A_0, A_1 are homotopial if there exists one-parameter family of mappings $A_t\colon \Omega \to X^*$, $t \in [0,1]$ such that

(a) for every sequence $u_n \in \partial\Omega$ and $t_n \in [0,1]$ such that u_n converges weakly to u_0, $A_{t_n}(u_n)$ converges weakly to 0 and $\lim_{n\to\infty} < A_{t_n}(u_n), u_n - u_0 >= 0$ it follows that $u_n \to u_0$ in X;
(b) $A_t(u) \neq 0$ when $u \in \partial\Omega$, $t \in [0,1]$;

(c) for every sequence $t_n \in [0,1]$, $u_n \in \overline{\Omega}$ such that $t_n \to t_0$, $u_n \to u_0$ it follows that the sequence $A_{t_n}(u_n)$ converges weakly to $A_{t_0}(u_0)$.

Theorem 2.1 ([10, Theorem 4.1]). *Let A_0, A_1 be homotopical mappings on Ω. Then* $\operatorname{Deg}(A_0,\overline{\Omega},0) = \operatorname{Deg}(A_1,\overline{\Omega},0)$.

2.3. Main problem and assumptions. We consider a boundary value problem

$$\Phi[u] \equiv \frac{\partial u}{\partial t} - F(x,t,u,D^1u,\dots,D^{2m}u) = f(x,t), \quad (x,t)\in Q_T, \tag{2.1}$$

$$\Psi_j[u] \equiv G_j(x,t,u,\dots,D^{m_j}u) = g_j(x,t), \quad (x,t)\in S_T,\ j=\overline{1,m}, \tag{2.2}$$

$$u(x,0) = h(x), \quad x\in\Omega. \tag{2.3}$$

Solvability of the problem (2.1)–(2.3) will be considered in space $W_p^{(4m,2)}(Q_T)$. We assume that numbers p, n, m, m_j satisfy inequalities

$$p\ge 2, \quad p>\frac{2m+n}{2m}, \quad p\ne\frac{2m+1}{2m-m_j}, \quad m_j\le 2m-1, \quad j=\overline{1,m} \tag{2.4}$$

and boundary $\partial\Omega$ of domain Ω satisfies the condition

$$\partial\Omega\in C^{4m}. \tag{2.5}$$

We define

$$F_\alpha(x,t,\xi) := \frac{\partial}{\partial\xi_\alpha}F(x,t,\xi), \qquad |\alpha|\le 2m,\ \xi=\{\xi_\alpha\in\mathbb{R} : |\alpha|\le 2m\},$$
$$G_{j\beta}(x,t,\zeta_j) := \frac{\partial}{\partial\zeta_\beta}G_j(x,t,\zeta_j), \quad |\beta|\le m_j,\ \zeta_j=\{\zeta_\beta\in\mathbb{R} : |\beta|\le m_j\},\ j=\overline{1,m}.$$

and suppose that the following conditions for the functions F, G_j are fulfilled:

(F_1) the function $F(x,t,\xi)$ have all mixed continuous derivatives by ξ up to the order 2m+1, $F(x,t,0)\equiv 0$.

(F_2) there exists a continuous function $\nu:\mathbb{R}^+\to\mathbb{R}^+$ such that for each $\xi\in\mathbb{R}^{M(2m)}$, $\eta\in\mathbb{R}^n$ the inequality

$$(-1)^{m+1}\sum_{|\alpha|=2m}F_\alpha(x,t,\xi)\eta^\alpha \ge \nu(|\xi|)|\eta|^{2m}$$

holds.

(G_1) for each $j=1,\dots,m$ the function $G_j(x,t,\zeta_j)$ has all mixed continuous derivatives with respect to variables ζ up to the order $4m-m_j+1$, $G_j(x,t,0)\equiv 0$.

For $(x,t)\in S_T$, $\xi\in\mathbb{R}^{M(2m)}$, $\zeta_j=\{\xi_\beta:|\beta|\le m_j\}$ (here $j=1,\dots,m$), we define

$$L(x,t,\xi,\delta+\tau\eta,q):=q-(-1)^m\sum_{|\alpha|=2m}F_\alpha(x,t,\xi)(\delta+\tau\eta)^\alpha,$$
$$B_j(x,t,\zeta_j,\delta+\tau\eta):=\sum_{|\beta|=m_j}G_{j\beta}(x,t,\zeta_j)(\delta+\tau\eta)^\beta,\quad j=\overline{1,m},$$

where η is the unit vector in the direction of the outward normal to $\partial\Omega$ at the point x; δ means an arbitrary vector from the plane tangent to $\partial\Omega$ at the point x; τ is complex variable and q is real number.

If $q\ge-\widetilde{\nu}|\delta|^{2m}$, $0<\widetilde{\nu}<\nu(|\xi|)$ and $|q|+|\delta|>0$, then $L(x,t,\xi,\delta+\tau\eta,q)$ as a polynomial of τ has m roots τ_s^+ with positive real part, other roots are with negative real part (see [1]). We denote

$$L^+(x,t,\xi,\delta,\tau,q):=\prod_{s=1}^m(\tau-\tau_s^+)$$

and assume that the following condition (the Lopatynsky condition) is fulfilled:

(G_2) for each $(x,t)\in S_T$, $\xi\in\mathbb{R}^{M(2m)}$ and δ, belonging to the tangent plane to $\partial\Omega$ at the point x, inequalities $q\ge-\widetilde{\nu}|\delta|^{2m}$, $0<\widetilde{\nu}<\nu(|\xi|)$ and $|q|+|\delta|>0$ imply the linear independence of B_j by modulus of L^+.

We assume that the following inclusions for the functions from the right side of (2.1)–(2.3) are fullfilled

$$\begin{aligned}&f\in W_p^{(2m,1)}(Q_T),\\ &g_j\in W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}(S_T),\quad j=\overline{1,m},\\ &h\in W_p^{(4m-2m/p)}(\Omega).\end{aligned}\tag{2.6}$$

We also assume that the compatibility condititons for the problem (2.1)–(2.3) are fullfilled.

To this end we will use notations

$$u^{(0)}(x):=u(x,0),\quad u^{(1)}(x):=\frac{\partial u}{\partial t}(x,0).$$

From (2.1), (2.3) we determine that

$$\begin{aligned}u^{(0)}(x)&:=h(x),\\ u^{(1)}(x)&:=\frac{\partial u}{\partial t}(x,0)=f(x,0)+F(x,0,h,D^1h,\dots,D^{2m}h).\end{aligned}$$

Formulating the compatibility conditions for (2.1)–(2.3) we shall use the following equalities (here $j=1,\dots,m$)

$$G_j(x,0,u^{(0)},\dots,D^{m_j}u^{(0)})=g_j(x,0),\tag{2.7}$$

$$\text{(2.8)}\quad \frac{\partial}{\partial t}G_j(x,0,u^{(0)},\dots,D^{m_j}u^{(0)}) + \sum_{|\beta|\le m_j} G_{j\beta}(x,0,u^{(0)},\dots D^{m_j}u^{(0)})D^{\beta}u^{(1)}(x,0) = \frac{\partial}{\partial t}g_j(x,0).$$

We say that the compatibility conditions for (2.1)–(2.2) are fullfilled if

(C) for each $j = 1,\dots,m$ the condition (2.7) is satisfied and the condition (2.8) is fulfilled for such j that $p > (2m+1)/(2m-m_j)$.

2.4. Linear initial boundary value problem. In a further investigation we will need some facts concerning solvability and a priori estimates for linear initial boundary value problems of the type

$$\text{(2.9)}\qquad Lu \equiv \frac{\partial u}{\partial t} - \sum_{|\alpha|\le 2m} a_{\alpha}(x,t)D^{\alpha}u = f(x,t), \quad (x,t)\in Q_T,$$

$$\text{(2.10)}\qquad B_j u \equiv \sum_{|\beta|\le m_j} b_{j\beta}(x,t)D^{\beta}u = g_j(x,t), \quad (x,t)\in S_T,\ j=\overline{1,m},$$

$$\text{(2.11)}\qquad u(x,0) = h(x), \quad x\in\Omega.$$

Assume that the following conditions for the functions a_{α}, $b_{j\beta}$ are fulfilled:

(a_1) for $|\alpha|\le 2m$ coefficients $a_{\alpha}\in W_p^{(2m,1)}(Q_T)$;

(a_2) there exists $\nu_0 > 0$ such that, for arbitrary $(x,t)\in Q_T$, $\eta\in\mathbb{R}^n$, an inequality

$$(-1)^{m+1}\sum_{|\alpha|=2m} a_{\alpha}(x,t)\eta^{a} \ge \nu_0|\eta|^{2m}$$

holds;

(b_1) for each $j=1,\dots,m$ and $|\beta|\le m_j$, the inclusions

$$b_{j\beta}\in W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}(S_T)$$

hold;

(b_2) the left-side functions in the equation and boundary conditions of (2.9)–(2.11) satisfy the Lopatinsky condition, that is formulated analogously to condition (G_2).

Assume that the compatibility conditions analogous to condition (C) for (2.9)–(2.11) are fullfilled.

The facts we need are included in the following theorem:

Theorem 2.2. *Assume that condition* (2.5) *for the boundary* $\partial\Omega$ *of a bounded domain* Ω *is satisfied and conditions* (2.4), (2.6), (a_1), (a_2), (b_1), (b_2) *and the compatibility conditions for the problem* (2.9)–(2.11) *hold. Suppose that*

$T \in (0,\overline{T}]$. Then (2.9)–(2.11) *has a unique solution $u \in W_p^{(4m,2)}(Q_T)$ which satisfies an apriori estimate*

$$
\begin{aligned}
(2.12)\quad K^{(1)}\bigg\{\|f\|_{p,Q_T}^{(2m,1)} &+ \sum_{j=1}^{m}\|g_j\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))} \\
&+ \|h\|_{p,\Omega}^{(4m-2m/p)}\bigg\} \le \|u\|_{p,Q_T}^{(4m,2)} \\
\le K^{(2)}\bigg\{\|f\|_{p,Q_T}^{(2m,1)} &+ \sum_{j=1}^{m}\|g_j\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))} \\
&+ \|h\|_{p,\Omega}^{(4m-2m/p)}\bigg\}.
\end{aligned}
$$

Numbers $K^{(1)}$, $K^{(2)}$ in (2.12) *depend only on Ω, $\overline{T}$, n, p, ν_0 and the norms of functions $a_\alpha(x,t)$ in spaces $C(Q_T)$, $W_p^{(2m,1)}(Q_T)$, and norms of functions $b_{j\beta}(x,t)$ in spaces $C(S_T)$, $W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}(S_T)$.*

The theorem is proved in [8].

3. Reduction to operator equation

3.1. Reduction to the problem with zero initial conditions. We start with a reduction of problem (2.1)–(2.3) to an equivalent one with zero initial conditions. For that we need next two auxiliary lemmas.

Lemma 3.1. *Assume that Ω is a bounded domain in $\mathbb{R}^n$ with the boundary that satisfies condition* (2.5) *and inequalities* (2.4)*, for m, p, n, m_j, hold. Let a function $P(x,t,v)$ be defined for $(x,t) \in Q_T$, $v \in \mathbb{R}^M$ for some $M \in N$ and have all continuous derivatives $(\partial/\partial t)^s D_x^\alpha D_v^\beta F(x,t,v)$ of the order $bs+|\alpha|+|\beta|$ up to bk, when bk is an integer and up to $[bk]+1$, when bk is not integer. Here α, β mean multiindices and $[\,\cdot\,]$ is greatest integer function. Then the operator*

$$
(3.1)\qquad [\widetilde{P}(v)](x,t) = P(x,t,v(x,t))
$$

acts from space $[W_p^{(bk,k),0}(Q_T)]^M$ to space $W_p^{(bk,k)}(Q_T)$ and appears to be bounded and continuous.

Proof. The proof is based on the direct calculation of the derivatives

$$
\left(\frac{\partial}{\partial t}\right)^s D^\alpha P(x,t,v(x,t))
$$

and use of the embedding results for the considered functitonal spaces. □

Lemma 3.2. *Assume that conditions of Lemma* 3.1 *are fullfilled. Then an operator*

$$
(3.2)\qquad [\widetilde{P}(v)](x,t) = P(x,t,v(x,t))
$$

acts from the space $[W_p^{(bk,k),0}(S_T)]^M$ *to* $W_p^{(bk,k)}(S_T)$ *and appears to be bounded and continuous.*

Proof. Lemma is proved analogously to Lemma 3.1 by passing to covering $\{U_i\}_{i=1}^I$ of $\partial\Omega$ and analising of operator in the local coordinate system of ξ_i. $\square$

In a standard way (see [2]) we can construct a function $v \in W_p^{(4m,2)}(Q_T)$ that satisfies conditions

$$v(x,0) = u^{(0)}(x), \quad \frac{\partial v}{\partial t}(x,0) = u^{(1)}(x) \quad \text{for } x \in \Omega.$$

Now we introduce a new function $u_1 = u - v$. If u is the solution of problem (2.1)–(2.3) then u_1 is the solution of boundary value problem

$$\frac{\partial u_1}{\partial t} - F^{(1)}(x,t,u_1,D^1u_1,\dots,D^{2m}u_1) = f^{(1)}(x,t), \quad (x,t) \in Q_T, \tag{3.3}$$

$$G_j^{(1)}(x,t,u_1,\dots,D^{m_j}u_1) = g_j^{(1)}(x,t), \quad (x,t) \in S_T, \ j = \overline{1,m}, \tag{2.4}$$

$$u_1(x,0) = 0, \quad x \in \Omega, \tag{2.5}$$

where

$$\begin{aligned} F^{(1)}(x,t,u_1,D^1u_1,\dots,D^{2m}u_1) &= F(x,t,u_1+v,D^1(u_1+v),\dots,D^{2m}(u_1+v)) \\ &\quad - F(x,t,v,D^1v,\dots,D^{2m}v), \\ f^{(1)}(x,t) &= f(x,t) - \widetilde{\Phi}[v], \\ G_j^{(1)}(x,t,u_1,\dots,D^{m_j}u_1) &= G_j(x,t,u_1+v,\dots,D^{m_j}u_1+v) - \widetilde{\Psi_j}[v], \\ g_j^{(1)}(x,t) &= g_j(x,t) - \widetilde{B_j}[v], \quad j = \overline{1,m}. \end{aligned}$$

Additionaly to (F_1), (F_2), (G_1), (G_2), we introduce conditions:

(F_3) the function $F(x,t,\xi)$ has all continuous derivatives by variables ξ_β up to the order $2m+1$, $F(x,t,0) \equiv 0$;

(F_4) the operators

$$F_\alpha(\,\cdot\,,\,\cdot\,,u,D^1u,\dots,D^{2m}u)\colon W_p^{(4m,2)}(Q_T) \to W_p^{(2m,1)}(Q_T)$$

are bounded and continuous;

(G_3) for each j, the function $G_j(x,t,\zeta_j)$ has all mixed continuous derivatives by η_β up to the order $4m - m_j + 1$, $G_j(x,t,0) \equiv 0$;

(G_4) the operators

$$G_{j\beta}(\,\cdot\,,\,\cdot\,,u,D^1u,\dots,D^{m_j}u)\colon\\ W_p^{(4m-1/p,2-1/(2mp)),0}(S_T) \to W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}(S_T)$$

are bounded and continuous.

Lemma 3.3. *Assume that conditions* (2.4)–(2.7), (F$_1$), (F$_2$), (G$_1$), (G$_2$) *for the problem* (2.1)–(2.3) *are satisfied and* $u \in W_p^{(4m,2)}$ *is the soluton of* (2.1)–(2.3). *Then*

(a) *the function* $F^{(1)}(x,t,\xi)$ *satisfies conditions* (F$_2$) (*with some function* $\nu^{(1)}$ *that, possibly, differs from* ν), (F$_3$), (F$_4$), *and* $G_j^{(1)}(x,t,\zeta_j)$ *satisfy conditions* (G$_2$), (G$_3$), (G$_4$);

(b) *the following inclusions for* $u_1(x,t)$, $f^{(1)}(x,t)$, $g_j^{(1)}(x,t)$ *are fulfilled:*

$$u_1 \in W_p^{(4m,2),0}(Q_T), \quad f^{(1)} \in W_p^{(2m,1),0}(Q_T),$$
$$g_j^{(1)} \in W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp)),0}(S_T), \quad j=\overline{1,m}.$$

Proof. Part (a) follows from (F$_1$), (F$_2$), (G$_1$), (G$_2$), the definition of $F^{(1)}$, $G_j^{(1)}$ and Lemmas 3.1, 3.2.

Inclusion for u_1 in formula (b) follows from the definition of v and Lemma 1 from [8].

Definition of $f^{(1)}$, embedding theorems and Lemma 3.1 imply inclusion $f^{(1)} \to W_p^{(2m,1),0}(Q_T)$.

Definition of $g_j^{(1)}$ together with conditions (2.7) and Lemma 3.2 imply the inclusions for $g_j^{(1)}$ in (b). □

Lemma 3.4. *Assume that conditions of Lemma* 3.3 *are satisfied, and* $u_1 \in W_p^{(4m,2),0}(Q_T)$ *is the solution of* (3.3)–(3.5). *Then function* $u(x,t) = u_1(x,t) + v(x,t)$ *is the solution of* (2.1)–(2.3).

Proof. Follows immediately from definition of $F^{(1)}$, $G_j^{(1)}$, $f^{(1)}$, $g_j^{(1)}$ and v.□

3.2. Definition of operator. Thus, instead of the problem (2.1)–(2.3), we can analyze the equivalent problem

$$\widetilde{\Phi}[u] \equiv \frac{\partial u}{\partial t} - F(x,t,u,D^1u,\dots,D^{2m}u) = f(x,t), \quad (x,t) \in Q_T, \tag{3.6}$$

$$\widetilde{\Psi_j}[u] \equiv G_j(x,t,u,\dots,D^{m_j}u) = g_j(x,t), \quad (x,t) \in S_T,\ j=\overline{1,m}, \tag{3.7}$$

$$u \in W_p^{(4m,2),0}(Q_T), \tag{3.8}$$

where F satisfies conditions (F$_2$) functions G_j satisfy conditons (G$_2$)–(G$_4$), and for the functions f, g_j the following inclusions are fullfilled

$$\begin{aligned} &f \in W_p^{(2m,1),0}(Q_T), \\ &g_j \in W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp)),0}(S_T), \quad j=\overline{1,m}. \end{aligned} \tag{3.9}$$

A nonlinear operator corresponding to (3.6)–(3.8) will be defined by the following equality

$$(3.10)\quad \langle Au,\phi\rangle := \frac{1}{p}\frac{d}{ds}\Big[(\|\widetilde{\Phi}[u+s\phi]-f\|_{p,Q_T}^{(2m,1)})^p + \sum_{j=1}^{m}(\|\widetilde{\Psi_j}[u+s\phi]-g_j\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))})^p\Big]\Big|_{s=0}.$$

In (3.10) $\{u,\phi\}\in W_p^{(4m,2),0}(Q_T)$ and the symbol $\langle Au,\phi\rangle$ means the value of functional Au on function ϕ.

It follows from (3.10) that

$$(3.11)\qquad Au := A_E(u,\widetilde{\Phi}[u]-f(x,t)) + \sum_{j=1}^{m} A_{B_j}(u,\widetilde{\Psi_j}[u]-g_j(x,t)),$$

where

$$(3.12)\quad \langle A_E(u,v),\phi\rangle := \sum_{|\alpha|+2ms\le 2m}\int_{Q_T}\psi_p\Big[\Big(\frac{\partial}{\partial t}\Big)^s D^\alpha v(x,t)\Big] \cdot\Big(\frac{\partial}{\partial t}\Big)^s D^\alpha\Big[\frac{\partial\phi}{\partial t}-L(u)\phi(x,t)\Big]\,dx\,dt,$$

$\psi_p(s)=s|s|^{p-2}$ for $s\in\mathbb{R}$ and for $j=1,\dots,m$, $\mu(j)=4m-m_j-1$

$$(3.13)\qquad \langle A_{B_j}(u,w),\phi\rangle := \sum_{k=1}^{3}\langle A_{B_j}^{(k)}(u,w),\phi\rangle,$$

$$\langle A_{B_j}^{(1)}(u,w),\phi\rangle := \sum_{i=1}^{I}\ \sideset{}{'}\sum_{|\beta|+2ms\le\mu(j)}\int_{D_T(d)}\psi_p\Big[\Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta w^{(i)}(y',t)\Big] \cdot\Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta[B_j(u)\phi]^{(i)}(y',t)\,dy'\,dt,$$

$$(3.14)\quad \langle A_{B_j}^{(2)}(u,w),\phi\rangle := \sum_{i=1}^{I}\ \sideset{}{'}\sum_{|\beta|+2ms=\mu(j)}\int_0^T dt\iint_{[D(d)]^2}\psi_p \cdot\Big[\Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta w^{(i)}(y',t)-\Big(\frac{\partial}{\partial t}\Big)^s D_z^\beta w^{(i)}(z',t)\Big] \cdot\Big\{\Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta[B_j(u)\phi]^{(i)}(y',t)-\Big(\frac{\partial}{\partial t}\Big)^s D_z^\beta[B_j(u)\phi]^{(i)}(z',t)\Big\} \cdot\frac{dy'dz'}{|y'-z'|^{n+p-2}},$$

$$(3.15)\qquad \langle A^{(3)}_{B_j}(u,w),\phi\rangle := \sum_{i=1}^{I}\sum_{l=2m-m_j}^{\mu(j)}\sum_{|\beta|+2ms=l}{}' \int_{D(d)} dy' \cdot \int_0^T\int_0^T \psi_p\bigg[\Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta w^{(i)}(y',t) - \Big(\frac{\partial}{\partial \tau}\Big)^s D_y^\beta w^{(i)}(y',\tau)\bigg] \cdot\bigg\{\Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta [B_j(u)\phi]^{(i)}(y',t) - \Big(\frac{\partial}{\partial \tau}\Big)^s D_y^\beta [B_j(u)\phi]^{(i)}(y',\tau)\bigg\}\frac{dt\,d\tau}{|t-\tau|^{p(l)}},$$

where

$$(3.16)\qquad L(u)\phi := \sum_{|\alpha|\le 2m} F_\alpha(x,t,u,D^1u,\dots,D^{2m}u)D^\alpha\phi,\quad (x,t)\in Q_T,$$

$$(2.17)\qquad B_j(u)\phi = \sum_{|\beta|\le m_j} G_{j\beta}(x,t,u,\dots,D^{m_j}u)D^\beta\phi,$$

$$p(l) = 1 + \bigg(2 - \frac{m_j+l}{2m} - \frac{1}{2mp}\bigg)p,\quad l = \overline{2m-m_j,\mu(j)},$$

and noting $\sum'$ we mean summation over multiindices β such that $\beta_n = 0$.

The following theorem formulates the main properties the operator A, defined by (3.10).

Theorem 3.5. *Assume that conditions* (2.4), (2.5), (3.9), (F_2) *for the problem* (3.6)–(3.8) *are fulfilled. Then*

(a) *for each $u \in W_p^{(4m,2),0}(Q_T)$, Au appears to be linear and continuous functional on $W_p^{(4m,2),0}(Q_T)$.*
(b) *the operator A is bounded, continuous and satisfies the* $(S)_+$ *condition on $W_p^{(4m,2),0}(Q_T)$.*

Remark 3.6. We recall the definition of the condition $(S)_+$ for the operator A acting from the Banach space X into the dual space X^* (see, for example, [7]). Analogous condition in [6] was called the condition α).

We say that an operator $A\colon X \to X^*$ satisfies the condition $(S)_+$ if, for arbitrary sequence $\{u_j\}\subset X$, which converges weakly to some $u_0 \in X$ and such that

$$\limsup_{j\to\infty}\langle Au_j, u_j - u_0\rangle \le 0$$

we have that u_j converges strongly to u_0.

To prove theorem we need following auxiliary lemmas.

Lemma 3.7. *Assume that conditions* (2.4)*,* (2.5)*,* (F_2)–(F_4)*,* (G_2)–(G_4) *are satisfied. Then there exists a continuous nondecreasing function* $\omega_1 \colon \overline{\mathbb{R}_+} \to \mathbb{R}_+$ *such that, for each* $u, \phi \in W_p^{(4m,2),0}(Q_T)$ *and* $j = 1, \dots, m$*, the following inequalities hold:*

$$\begin{aligned} \|L(u)\phi\Big\|^{(2m,1)}_{p,Q_T} &\le \omega_1(\|u\|^{(4m,2)}_{p,Q_T})\|\phi\|^{(4m,2)}_{p,Q_T}, \\ \|B_j(u)\phi\|^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}_{p,S_T} &\le \omega_1(\|u\|^{(4m,2)}_{p,Q_T})\|\phi\|^{(4m,2)}_{p,Q_T}, \\ \|F(\,\cdot\,,\,\cdot\,,u,D^1u,\dots,D^{2m}u)\|^{(2m,1)}_{p,Q_T} &\le \omega_1(\|u\|^{(4m,2)}_{p,Q_T})\|\phi\|^{(4m,2)}_{p,Q_T}, \\ \|G_j(\,\cdot\,,\,\cdot\,,u,\dots,D^{m_j}u)\|^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}_{p,S_T} & \\ &\le \omega_1(\|u\|^{(4m,2)}_{p,Q_T})\|\phi\|^{(4m,2)}_{p,Q_T}, \end{aligned}$$

where the operators $L(u)$*,* $B_j(u)$ *are defined by equalities* (3.16)*,* (3.17)*.*

Proof. We denote $\|u\|^{(4m,2)}_{p,Q_T} R$. Conditions (F_4), (G_4) imply that the norms of operators $L(u)$, $B_j(u)$ in the corresponding spaces

$$W_p^{(2m,1),0}(Q_T), \quad W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp)),0}(S_T)$$

are bounded by some nondecreasing function $C'(R)$. From the definition of $L(u)$, $B_j(u)$ and the norms in spaces $W_p^{(bk,k)}(Q_T)$, using Lemmas 2–6 from [7] we get

$$\|L(u)\phi\|^{(2m,1)}_{p,Q_T} \le C''(R), \quad \|B_j(u)\phi\|^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}_{p,S_T} \le C''(R),$$

where $\phi \in W_p^{(4m,2),0}(Q_T)$ and $\|\phi\|^{(4m,2)}_{p,Q_T} = 1$. That, in fact, proves first two inequalities.

Another two inequalities could be obtained from the first two ones and the identities

$$\begin{aligned} F(\,\cdot\,,\,\cdot\,,u,D^1u,\dots,D^{2m}u) &= \int_0^1 L(su)u\,ds, \\ G_j(x,t,u,\dots,D^{m_j}u) &= \int_0^1 B_j(su)u\,ds, \quad j = \overline{1,m}. \end{aligned}$$ □

Lemma 3.8. *Assume that conditions of Lemma* 3.7 *are satisfied. Then there exists a continuous nondecreasing function* $\omega_2 \colon \overline{\mathbb{R}_+} \to \mathbb{R}_+$ *such that for each*

$$\begin{gathered} u, \phi \in W_p^{(4m,2),0}(Q_T), \quad v \in W_p^{(2m,1)}(Q_T), \\ w \in W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp)),0}(S_T) \end{gathered}$$

and $j \in 1, \dots, m$*, the following inequalities hold:*

$$\begin{aligned} &|\langle A_E(u,v),\phi\rangle| \le \omega_2(\|u\|^{(4m,2)}_{p,Q_T})[\|v\|^{(2m,1)}_{p,Q_T}]^{p-1}\|\phi\|^{(4m,2)}_{p,Q_T}, \\ &|\langle A_{B_j}(u,w),\phi\rangle| \\ &\qquad \le \omega_2(\|u\|^{(4m,2)}_{p,Q_T})[\|w\|^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}_{p,S_T}]^{p-1}\|\phi\|^{(4m,2)}_{p,Q_T}, \end{aligned}$$

Proof. The proof follows from Lemma 3.7 and the Hölder inequality. □

Lemma 3.9. *Assume that conditions* (2.4) *and* (2.5) *are satisfied and* $P \in C(\overline{Q_T} \times \mathbb{R}^M, \mathbb{R})$ *for some positive integer* M*. Let operator*

$$[\widetilde{P}(v)](x,t) := P(x,t,v(x,t)) \colon [W_p^{(2m,1),0}(Q_T)]^M \to W_p^{(2m,1)}(Q_T)$$

for $v \in [W_p^{(2m,1),0}(Q_T)]^M$ *be bounded and continuous. Then, for a sequence* $\{v^{(k)}\} \subset [W_p^{(2m,1),0}(Q_T)]^M$ *weakly convergent in* $[W_p^{(2m,1),0}(Q_T)]^M$ *to* $v^{(0)}$ *and for* $\{w_k\} \subset W_p^{(2m,1),0}(Q_T)$ *that weakly converges to* w_0 *in* $W_p^{(2m,1),0}(Q_T)$ *we obtain that sequence*

$$\rho_k(x,t) := \{[\widetilde{P}(v^{(k)})](x,t) - [\widetilde{P}(v^{(0)})](x,t)\}(w_k(x,t) - w_0(x,t))$$

converges strongly to 0 *in* $W_p^{(2m,1)}(Q_T)$*.*

Proof. Using Lemma 3 from [8] and compactness of the embedding of the space $W_p^{(2m,1),0}(Q_T)$ into $C^{(\delta,\delta/(2m))}(Q_T)$ for some $\delta > 0$, we obtain $\rho'_k := \widetilde{P}(v^{(k)}) - \widetilde{P}(v^{(0)}) \to 0$ in $C(Q_T)$. By the definition, we have

$$(\|\rho_k\|_{p,Q_T}^{2m,1})^p = \left\|\frac{\partial \rho_k}{\partial t}\right\|_{p,Q_T}^p + \sum_{|\alpha| \le 2m} \|D^\alpha \rho_k\|_{p,Q_T}^p,$$

and we need to establish the convergence of the right hand side to zero.

Both summands from the right side of the equality

$$\frac{\partial \rho_k}{\partial t} = \frac{\partial \rho'_k}{\partial t}(w_k - w_0) + \frac{\partial}{\partial t}(w_k - w_0)\rho'_k$$

tend to 0 in $L_p(Q_T)$ as a result of boundedness of first multipliers and convergence to 0 of second factors.

For $|\alpha| \le 2m$, we can estimate

$$|D^\alpha \rho_k| \le C_1 \bigg\{ |D^\alpha \rho'_k||w_k - w_0| + \sum_{\substack{\alpha' + \alpha'' = \alpha \\ |\alpha'||\alpha''| > 0}} |D^{\alpha'} \rho'_k||D^{\alpha''}(w_k - w_0)| + |\rho'_k||D^\alpha (w_k - w_0)| \bigg\}.$$

We can obtain convergence to 0 in $L_p(Q_T)$ for the first and the last summand from the right side of equality in the way analogous to investigating derivatives by t. The convergence to 0 in $L_p(Q_T)$ of other summands follows in view of the Hölder inequality and Lemma 4 from [8]. □

Lemma 3.10. *Assume that conditions of the Theorem 3.5 are satisfied and the sequence* $\{u_k\} \subset W_p^{(4m,2),0}(Q_T)$ *converges weakly to* $u_0 \in W_p^{(4m,2),0}(Q_T)$. *Then the sequence*

$$\begin{aligned} r_k(x,t) := F(x,t,u_k,D^1u_k,\dots,D^{2m}u_k) \\ - F(x,t,u_k,D^1u_k,\dots,D^{2m-1}u_k) - L(u_0)(u_k-u_0) \end{aligned}$$

converges strongly to 0 *in* $W_p^{(2m,1)}(Q_T)$.

Proof. We can rewrite our sequence

$$r_k(x,t) = \sum_{|\alpha|\le 2m} \int_0^1 \{[\widetilde{F}_\alpha(u_k^{(s)})](x,t) - [\widetilde{F}_\alpha(u_0)](x,t)\}\, ds\, D^\alpha(u_k-u_0),$$

where

$$\begin{aligned} u_k^{(s)}(x,t) &:= s\,u_k(x,t) + (1-s)u_0(x,t), \\ [\widetilde{F}_\alpha(u)](x,t) &:= F_\alpha(x,t,u(x,t),D^1u(x,t),\dots,D^{2m}u(x,t)). \end{aligned}$$

Then the assertion of the lemma is a consequence of conditions (F_1), (F_3) and Lemma 3.9. □

Lemma 3.11. *Assume that conditions* (2.4), (2.5) *are satisfied and* $Q \in C(\overline{S_T}\times\mathbb{R}^M,\mathbb{R})$ *for some positive integer* M. *Let there exist a positive integer* $\widetilde{m}$ *such that* $0 \le \widetilde{m} \le 2m-1$, *and for* $v \in [W^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp)),0}(S_T)]^M$ *the operator*

$$\begin{aligned} [\widetilde{Q}(v)](x,t) := Q(x,t,v(x,t))\colon [W_p^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp)),0}(S_T)]^M \\ \to W^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp))}(S_T) \end{aligned}$$

is bounded and continuous and the function $\Psi(x,t,\zeta)$ *has all mixed continuous derivatives up to the order* $4m-\widetilde{m}$ *with respect to* ζ_l. *Then, for each sequence*

$$\{v^{(k)}\} \subset [W_p^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp)),0}(S_T)]^M,$$

weakly convergent in $[W_p^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp)),0}(S_T)]^M$ *to* $v^{(0)}$, *and for every sequence*

$$\{w_k\} \subset W_p^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp)),0}(S_T),$$

that converges weakly to w_0 *in* $W_p^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp)),0}(S_T)$, *the sequence*

$$\sigma_k(x,t) := \{[\widetilde{Q}(v^{(k)})](x,t) - [\widetilde{Q}(v^{(0)})](x,t)\}(w_k(x,t)-w_0(x,t))$$

converges strongly to 0 *in* $W_p^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp))}(S_T)$.

Proof. From the definition of the norm in the space

$$W_p^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp))}(S_T)$$

it follows that

$$\|\sigma_k\|_{p,S_T}^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp))} = \sum_{i=1}^{I} \|\sigma_k^{(i)}\|_{D_T(d)}^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp))},$$

where

$$(\|\sigma_k^{(i)}\|_{D_T(d)}^{(4m-\widetilde{m}-1/p,2-\widetilde{m}/(2m)-1/(2mp))})^p = \sideset{}{'}\sum_{2ms+|\alpha|\le 4m-\widetilde{m}-1} \bigg\| \Big(\frac{\partial}{\partial t}\Big)^s D^\alpha \sigma_k^{(i)} \bigg\|_{p,D_T(d)}^p + (\|\sigma_k^{(i)}\|_{2m,p,D_T(d)}^{(4m-\widetilde{m}-1/p)})^p,$$

for $i=\overline{1,I}$, and notation $\sum'$ means summation over multiindices α such that $\alpha_n = 0$.

Convergence to 0 for summands from sum $\sum'$ can be proved analogously as in Lemma 3.9. We consider one summand from norm $(\|\sigma_k^{(i)}\|_{2m,p,D_T(d)}^{(4m-\widetilde{m}-1/p)})^p$

$$I_k^{(i)} := \int_0^T dt \iint_{D(d)^2} \frac{|\frac{\partial}{\partial t} D_y^\alpha \sigma_k^{(i)}(y',t) - \frac{\partial}{\partial t} D_z^\alpha \sigma_k^{(i)}(z',t)|^p}{|y'-z'|^{n+p-2}}\, dy'\, dz',$$

where $|\alpha| \le 2m-\widetilde{m}-1$, $\alpha_n = 0$.

We note $\sigma_k'(x,t) := [\widetilde{Q}(v^{(k)})](x,t) - [\widetilde{Q}(v^{(0)})](x,t)$, $w_k'(x,t) := w_k(x,t) - w_0(x,t)$. Using Lemmas 2, 3 from [8] we can obtain the convergence for sequences

$${\sigma_k'}^{(i)} \to 0, \quad {w_k'}^{(i)} \to 0 \quad \text{in } C^{(2m-\widetilde{m}+\delta,1-(\widetilde{m}-\delta)/(2m))}(D_T(d))$$

for some $\delta > 0$. Then, for such multiindices α', α'' such that $\alpha'+\alpha''=\alpha$, obviously we obtain

$$D^{\alpha'} {\sigma_k'}^{(i)} \to 0, \quad D^{\alpha''} {w_k'}^{(i)} \to 0 \quad \text{in } C^{(1+\delta,(1+\delta)/(2m))}(D_T(d)).$$

Using the proved convergence together with an inequality

$$\begin{aligned}
&\bigg|\frac{\partial}{\partial t} D_y^\alpha \sigma_k'(y',t) - \frac{\partial}{\partial t} D_z^\alpha \sigma_k'(z',t)\bigg| \\
&\quad \le K \sum_{\alpha'+\alpha''=\alpha} \bigg\{ \bigg|\frac{\partial}{\partial t} D_y^{\alpha'} \sigma_k'(y',t) - \frac{\partial}{\partial t} D_z^{\alpha'} \sigma_k'(z',t)\bigg| |D_y^{\alpha''} w_k'(y',t)| \\
&\qquad + |D_z^{\alpha'} \sigma_k'(z',t)| \bigg|\frac{\partial}{\partial t} D_y^{\alpha''} w_k'(y',t) - \frac{\partial}{\partial t} D_z^{\alpha''} w_k'(z',t)\bigg| \\
&\qquad + \bigg[|D^{\alpha'} \sigma_k'|_{D_T(d)}^{(1+\delta,(1+\delta)/(2m))} \bigg|\frac{\partial}{\partial t} D_y^{\alpha''} w_k'(y',t)\bigg| \\
&\qquad + \bigg|\frac{\partial}{\partial t} D^{\alpha'} \sigma_k'(z',t)\bigg| |D^{\alpha''} w_k'|_{D_T(d)}^{(1+\delta,(1+\delta)/(2m))} \bigg] |y'-z'|^{1+\delta} \bigg\}
\end{aligned}$$

it is easy to check that $I_k^{(i)} \to 0$, $k \to \infty$. The rest of summands from norm $(\|\sigma_k^{(i)}\|_{2m,p,D_T(d)}^{(4m-\widetilde{m}-1/p)})^p$ could be estimated in the same way. □

Lemma 3.12. *Assume that conditions of Theorem* 3.5 *are satisfied and the sequence* $\{u_k\} \subset W_p^{(4m,2),0}(Q_T)$ *converges weakly to* $u_0 \in W_p^{(4m,2),0}(Q_T)$. *Then, for fixed* $j = 1, \dots, m$, *the sequence*

$$r_k^{(j)}(x,t) := G_j(x,t,u_k,\dots,D^{m_j}u_k) - G_j(x,t,u_0,\dots,D^{m_j}u_0) - B_j(u_0)(u_k - u_0)$$

converges strongly to 0 *in* $W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp)),0}(S_T)$.

Proof. We can rewrite $r_k^{(j)}$ in the form

$$\begin{aligned} r_k^{(j)}(x,t) = \sum_{|\beta|\le m_j} \int_0^1 \{ &G_{j\beta}(x,t,u_k^{(s)}(x,t), D^1 u_k^{(s)}(x,t),\dots, D^{m_j}u_k^{(s)}(x,t)) \\ &- G_{j\beta}(x,t,u_0(x,t), D^1u_0(x,t),\dots,D^{m_j}u_0(x,t))\}\, ds\, D^\beta(u_k-u_0), \end{aligned}$$

where $u_k^{(s)}(x,t) := su_k(x,t) + (1-s)u_0(x,t)$. Then the assertion of lemma follows from (G_1), (G_3) and Lemma 3.11. □

Proof of Theorem 3.5. It is obvious that for each fixed $u \in W_p^{(4m,2),0}(Q_T)$ functional $\langle Au, \phi\rangle$, defined by (3.11)–(3.15), is linear with respect to ϕ. Boundedness for functional Au follows from Lemma 3.8.

Lemma 3.8 also gives us the boundedness for the operator A, defined by (3.10). The continuity for A could be proved in a standard way.

Now we prove that A satisfies $(S)_+$ condition on $W_p^{(4m,2),0}(Q_T)$. We assume that the sequence $\{u_k\} \subset W_p^{(4m,2),0}(Q_T)$ converges weakly to a function $u_0 \in W_p^{(4m,2),0}(Q_T)$ and $\limsup_{k\to\infty}\langle Au_k, u_k - u_0\rangle \le 0$.

We consider the numerical sequence

$$\begin{aligned} E_k &:= \left\langle A_E\left(u_k, \frac{\partial u_k}{\partial t} - F(x,t,u_k,D^1u_k,\dots,D^{2m}u_k) - f(x,t)\right), u_k - u_0\right\rangle \\ &= \sum_{|\alpha|+2ms\le 2m} \int_{Q_T} \psi_p\left[\left(\frac{\partial}{\partial t}\right)^s D^\alpha\left\{\frac{\partial u_k}{\partial t} - F_k(x,t)\right\}\right] \\ &\qquad \cdot \left(\frac{\partial}{\partial t}\right)^s D^\alpha\left[\frac{\partial}{\partial t}(u_k-u_0) - L(u_k)(u_k-u_0)\right] dx\, dt, \end{aligned}$$

where

$$F_k(x,t) := F(x,t,u_k,D^1u_k,\dots,D^{2m}u_k) + f(x,t), \quad k \ge 0,$$

and A_E is defined by (3.12). Using Lemmas 3.9, 3.10 we obtain that the sequences

$$F_k(x,t) - F_0(x,t) - L(u_0)(u_k-u_0), \quad L(u_k)(u_k-u_0) - L(u_0)(u_k-u_0)$$

converge strongly to 0 in $W_p^{(2m,1)}(Q_T)$. That implies

$$E_k - \sum_{|\alpha|+2ms\le 2m} \int_{Q_T} \psi_p \cdot \bigg\{ \bigg(\frac{\partial}{\partial t}\bigg)^s D^\alpha \bigg[\frac{\partial u_k}{\partial t} - F_0(x,t) - L(u_0)(u_k-u_0)\bigg]\bigg\} \cdot \bigg(\frac{\partial}{\partial t}\bigg)^s D^\alpha \bigg[\frac{\partial}{\partial t}(u_k-u_0) - L(u_0)(u_k-u_0)\bigg]\,dx\,dt \to 0, \tag{3.18}$$

as $k\to\infty$. From the weak convergence u_k to u_0 in $W_p^{(4m,2),0}(Q_T)$ we obtain

$$\sum_{|\alpha|+2ms\le 2m} \int_{Q_T} \psi_p \bigg\{ \bigg(\frac{\partial}{\partial t}\bigg)^s D^\alpha \bigg[\frac{\partial}{\partial t}u_0 - F_0(x,t)\bigg]\bigg\} \cdot \bigg(\frac{\partial}{\partial t}\bigg)^s D^\alpha \bigg[\frac{\partial}{\partial t}(u_k-u_0) - L(u_0)(u_k-u_0)\bigg] dxdt \to 0, \tag{3.19}$$

as $k\to\infty$. Using (3.18), (3.19) and the elementary inequality

$$[\psi_p(a)-\psi_p(b)](a-b) \ge C_p|a-b|^p, \quad a,b\in\mathbb{R}, \tag{3.20}$$

we can get

$$E_k \ge C_p\bigg(\bigg\|\frac{\partial}{\partial t}(u_k-u_0) - L(u_0)(u_k-u_0)\bigg\|_{p,Q_T}^{(2m,1)}\bigg)^p + \varepsilon_k^{(E)}, \tag{3.21}$$

where $\varepsilon_k^{(E)}\to 0$, $k\to\infty$. For fixed $j\in 1,\dots,m$, we consider the sequences

$$B_{jk}^{(l)} := \langle A_{B_j}^{(l)}(u_k, G_j(x,t,u_k,\dots,D^{m_j}u_k) - g_j(x,t)), u_k-u_0\rangle,$$

for $l=1,2,3$, where $A_{B_j}^{(l)}$ are defined by (3.13)–(3.15). We will prove that

$$\sum_{l=1}^{3} B_{jk}^{(l)} \ge C_p(\|B_j(u_0)(u_k-u_0)\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp)),0})^p + \varepsilon_k^{(j)}, \tag{3.22}$$

where $\varepsilon_k^{(j)}\to 0$, $k\to\infty$. Let us, for example, consider

$$B_{jk}^{(2)} = \sum_{i=1}^{I} {\sum_{|\beta|+2ms=4m-m_j-1}}' \int_0^T dt \iint_{D(d)^2} \psi_p \cdot \bigg[\bigg(\frac{\partial}{\partial t}\bigg)^s D_y^\beta G_{jk}^{(i)}(y',t) - \bigg(\frac{\partial}{\partial t}\bigg)^s D_z^\beta G_{jk}^{(i)}(z',t)\bigg] \cdot \bigg\{\bigg(\frac{\partial}{\partial t}\bigg)^s D_y^\beta [B_j(u_k)(u_k-u_0)]^{(i)}(y',t) - \bigg(\frac{\partial}{\partial t}\bigg)^s D_z^\beta [B_j(u_k)(u_k-u_0)]^{(i)}(z',t)\bigg\} \frac{dy'\,dz'}{|y'-z'|^{n+p-2}},$$

where $G_{jk}(x,t) := G_j(x,t,u_k,\dots,D^{m_j}u_k) - g_j(x,t)$.

From Lemmas 3.11, 3.12 we get the convergence

$$\begin{aligned} B_j(u_k)(u_k-u_0) - B_j(u_0)(u_k-u_0) &\to 0, \quad k\to\infty,\\ G_{jk}(x,t) - G_{j0}(x,t) - B_j(u_0)(u_k-u_0) &\to 0, \quad k\to\infty \end{aligned}$$

in $W_p^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp)),0}(S_T)$. Then for $k\to\infty$

$$\begin{aligned} B_{jk}^{(2)} - \sum_{i=1}^{I} \sideset{}{'}\sum_{|\beta|+2ms=4m-m_j-1} \int_0^T dt \iint_{D(d)^2} \cdot\psi_p \Big\{ \Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta (G_{j0}^{(i)}(y',t) \\ + [B_j(u_0)(u_k-u_0)]^{(i)}(y',t)) \\ - \Big(\frac{\partial}{\partial t}\Big)^s D_z^\beta (G_{j0}^{(i)}(z',t) + [B_j(u_0)(u_k-u_0)]^{(i)}(z',t)) \Big\} \\ \cdot \Big\{ \Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta [B_j(u_0)(u_k-u_0)]^{(i)}(y',t) \\ - \Big(\frac{\partial}{\partial t}\Big)^s D_z^\beta [B_j(u_0)(u_k-u_0)]^{(i)}(z',t) \Big\} \frac{dy'\,dz'}{|y'-z'|^{n+p-2}} \to 0. \end{aligned} \tag{3.23}$$

If follows from the weak convergence u_k to u_0 in $W_p^{(4m,2),0}(Q_T)$ that

$$\begin{aligned} \sum_{i=1}^{I} \sideset{}{'}\sum_{|\beta|+2ms=4m-m_j-1} \int_0^T dt \iint_{D(d)^2} \psi_p \\ \cdot \Big[\Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta G_{j0}^{(i)}(y',t) - \Big(\frac{\partial}{\partial t}\Big)^s D_z^\beta G_{j0}^{(i)}(z',t) \Big] \\ \cdot \Big\{ \Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta [B_j(u_0)(u_k-u_0)]^{(i)}(y',t) \\ - \Big(\frac{\partial}{\partial t}\Big)^s D_z^\beta [B_j(u_0)(u_k-u_0)]^{(i)}(z',t) \Big\} \frac{dy'\,dz'}{|y'-z'|^{n+p-2}} \to 0, \end{aligned} \tag{3.24}$$

when $k\to\infty$. Using (3.23), (3.24) we obtain that

$$\begin{aligned} B_{jk}^{(2)} \ge C_p \sum_{i=1}^{I} \sideset{}{'}\sum_{|\beta|+2ms=4m-m_j-1} \int_0^T dt \\ \cdot \iint_{D(d)^2} \Big| \Big(\frac{\partial}{\partial t}\Big)^s D_y^\beta [B_j(u_0)(u_k-u_0)]^{(i)}(y',t) \\ - \Big(\frac{\partial}{\partial t}\Big)^s D_z^\beta [B_j(u_0)(u_k-u_0)]^{(i)}(z',t) \Big|^p \frac{dy'\,dz'}{|y'-z'|^{n+p-2}} + \varepsilon_k^{(j,2)}, \end{aligned} \tag{3.25}$$

where $\varepsilon_k^{(j,2)} \to 0$, $k\to\infty$. By the same reasons we can obtain estimates similar to (3.23) for $B_{jk}^{(1)}$, $B_{jk}^{(3)}$ and, as a result, the inequality (3.22).

It follows from the definitions of operator A and sequences E_k, $B_{jk}^{(l)}$ that

$$\langle Au_k, u_k - u_0\rangle = E_k + \sum_{j=1}^{m}\sum_{l=1}^{3} B_{jk}^{(l)}.$$

Applying estimates (3.21), (3.22) to last equality we obtain

$$(3.26)\quad \limsup_{k\to\infty}\bigg[\bigg(\bigg\|\frac{\partial}{\partial t}(u_k-u_0) - L(u_0)(u_k-u_0)\bigg\|_{p,Q_T}^{(2m,1)}\bigg)^p + \sum_{j=1}^{m}(\|B_j(u_0)(u_k-u_0)\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))})^p\bigg] = 0.$$

From a priori estimate (2.12) for linear parabolic operators we get

$$(3.27)\quad \|u_k-u_0\|_{p,Q_T}^{(4m,2)} \le K\bigg(\bigg\|\frac{\partial}{\partial t}(u_k-u_0) - L(u_0)(u_k-u_0)\bigg\|_{p,Q_T}^{(2m,1)} + \sum_{j=1}^{m}\|B_j(u_0)(u_k-u_0)\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}\bigg),$$

where K is independent of k. Inequality (3.27) together with (3.26) proves that $u_k \to u_0$ in $W_p^{(4m,2),0}(Q_T)$. □

3.3. Reduction to operator equation. Now we can introduce an operator equation

$$(3.28)\qquad Au = 0, \quad u \in W_p^{(4m,2),0}(Q_T),$$

where the operator A is defined by (3.10). The following theorem shows connection between the equation (3.28) and the boundary value problem (3.6)–(3.8).

Theorem 3.13. *Assume that problem* (3.6)–(3.8) *satisfies conditions* (2.4), (2.5), (3.9), (F_2)–(F_4), (G_2)–(G_4). *A function* $u \in W_p^{(4m,2),0}(Q_T)$ *is a solution for the problem* (3.6)–(3.8) *if and only if it is a solution for the equation* (3.28).

Proof. It follows from the definition of A_E, A_{B_j} that

$$A_E(u,0) = 0, \quad A_{B_j}(u,0) = 0, \quad j = \overline{1,m}$$

for $u \in W_p^{(4m,2),0}(Q_T)$. Then the solution u for the problem (3.6)–(3.8) will be a solution of the equation (3.28).

Let $u \in W_p^{(4m,2),0}(Q_T)$ be the solution for equation (3.28). From Lemma 3.3 we obtain that

$$f_0(x,t) := \frac{\partial u}{\partial t} - \sum_{|\alpha|=2m} a_\alpha(x,t,u,D^1u,\dots D^{2m-1}u)D^\alpha u - F(x,t,u,D^1u,\dots,D^{2m}u) \in W_p^{(2m,1),0}(Q_T),$$

$$g_{j0}(x,t) := G_j(x,t,u,\dots,D^{m_j}u) - g_j(x,t) \\ \in W^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp)),0}(S_T), \quad j = 1,m.$$

Lemma 3.1 guarantees the existence of a solution $v \in W_p^{(4m,2),0}(Q_T)$ for linear problem

$$\begin{aligned} \frac{\partial v}{\partial t} - L(u)v(x,t) &= f_0(x,t), \quad (x,t) \in Q_T, \\ B_j(u)v(x,t) &= g_{j0}(x,t), \quad (x,t) \in S_T,\ j = \overline{1,m}. \end{aligned}$$

Functions u, v satisfy equality $\langle Au, v\rangle = 0$ which, together with equations (3.10)–(3.17) implies

$$\begin{aligned} 0 &= \langle A_E(u,f_0), v\rangle + \sum_{j=1}^{m} \langle A_{B_j}(u,g_{j0}), v\rangle \\ &= (\|f_0\|_{p,Q_T}^{(2m,1)})^p + \sum_{j=1}^{m} (\|g_{j0}\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))})^p. \end{aligned}$$

Then $f_0 \equiv 0$, $g_{j0} \equiv 0$, $j = \overline{1,m}$, and the function u will be the solution of the problem (3.6)–(3.8). □

3.4. Topological characteristic of parabolic problem. Using the notion of the degree for $(\mathrm{S})_+$ operators (see [6], [7]) we can introduce a topological characteristic for the problem (3.6)–(3.7). Namely, for an arbitrary bounded domain D in $W_p^{(4m,2),0}(Q_T)$, we define an integer number $\operatorname{Deg}(A,\overline{D},0)$ (see Section 2, Chapter 2 from [7]) provided the following condition is satisfied

$$Au \neq 0, \quad u \in \partial D. \tag{3.29}$$

Some applications of this characteristic to the study of solvability of initial boundary value problem (3.6)–(3.8) will be given in Section 4.

4. Some applications

Having reduced problem (3.6)–(3.8) to the operator equation with the operator satisfying $(\mathrm{S})_+$ condition, we can investigate solvability of the operator equation (3.28) instead of studying solvablitity of problem (3.6)–(3.8). Then we can apply topological methods developed in [6], [7].

4.1. Uniqueness of solution.

Theorem 4.1. *Let conditons* (2.4), (2.5), (3.9), (F_2)–(F_4), (G_2)–(G_4) *for the problem* (3.6)–(3.8) *be fulfilled. Then the problem* (3.6)–(3.8) *can have at most one solution.*

Proof. Let $\{u_0, u_1\} \in W_p^{(4m,2),0}(Q_T)$ be two solutions of the problem (3.6)–(3.8). Then

$$(4.1)\quad \frac{\partial}{\partial t}(u_1 - u_0) - F(x,t,u_1,D^1u_1,\dots,D^{2m-1}u_1) + F(x,t,u_0,D^1u_0,\dots,D^{2m-1}u_0) = 0, \quad (x,t) \in Q_T,$$

$$G_j(x,t,u_1,\dots,D^{m_j}u_1) - G_j(x,t,u_0,\dots,D^{m_j}u_0) = 0, \quad (x,t) \in S_T,\ j = \overline{1,m},$$
$$u_1 - u_0 \in W_p^{(4m,2),0}(Q_T).$$

From (4.1) we can discover that $u_1 - u_0$ is the solution for the problem

$$(4.2)\quad \frac{\partial}{\partial t}(u_1 - u_0) + \sum_{|\alpha|\le 2m} \widetilde{a}_\alpha(x,t)D^\alpha(u_1 - u_0) = 0, \quad (x,t) \in Q_T,$$

$$(4.3)\quad \sum_{|\beta|\le m_j} \widetilde{b}_{j\beta}(x,t)D^\beta(u_1 - u_0) = 0, \quad (x,t) \in S_T,\ j = \overline{1,m},$$

$$(4.4)\quad u_1 - u_0 \in W_p^{(4m,2),0}(Q_T),$$

where, for $|\alpha| \le 2m$, $|\beta| \le m_j$,

$$\widetilde{a}_\alpha(x,t) = \int_0^1 F_\alpha(x,t,u_s(x,t),D^1u_s(x,t),\dots,D^{2m}u_s(x,t))\,ds,$$
$$\widetilde{b}_{j\beta}(x,t) := \int_0^1 G_{j\beta}(x,t,u_s(x,t),\dots,D^{m_j}u_s(x,t))\,ds, \quad j = \overline{1,m},$$
$$u_s(x,t) = su_0(x,t) + (1-s)u_1(x,t).$$

The problem (4.2)–(4.4) satisfies conditions of Theorem 2.2. Applying a priori estimate (2.12) to solution of (4.2)–(4.4) we get that $u_1 - u_0 \equiv 0$. □

Corollary 4.2. *Assume that conditions* (2.4)–(2.7), (F_1), (F_2), (G_1), (G_2) *for the problem* (2.1)–(2.3) *are fulfilled. Then the problem* (2.1)–(2.3) *can have at most one solution.*

4.2. Local existence of solution.

Theorem 4.3. *Assume that conditions* (2.4), (2.5), (3.9), (F_2)–(F_4), (G_2)–(G_4) *for the problem* (3.6)–(3.8) *are satisfied and K is some positive number. Then, there exists a positive number T_0, depending on K, but independent on functions from the right side of the problem* (3.6)–(3.8), *such that the problem* (3.6)–(3.8) *has a solution $u \in W_p^{(4m,2),0}(Q_T)$ for $0 < T < T_0$ provided the following inequalities hold:*

$$(4.5)\quad \|f\|_{p,Q_T}^{(2m,1)} \le K, \quad \|g_j\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))} \le K, \quad j = \overline{1,m}.$$

Proof. Let $\mathbb{R}$ be some positive number. We can choose a number T_1, dependent on $\mathbb{R}$, such that, for $u \in W_p^{(4m,2),0}(Q_T)$ with $\|u\|_{p,Q_T}^{(4m,2)} \le \mathbb{R}$ for $T \in (0, T_1]$, the following inequalities hold

$$\|L(u)u\|_{p,Q_T}^{(2m,1)} \le C^{(1)} \|u\|_{p,Q_T}^{(4m,2)}, \tag{4.6}$$

$$\|F(\,\cdot\,,\,\cdot\,,u,D^1u,\dots,D^{2m}u)\|_{p,Q_T}^{(2m,1)} \le C^{(1)} \|u\|_{p,Q_T}^{(4m,2)}, \tag{4.7}$$

$$\|L(u)u - L(0)u\|_{p,Q_T}^{(2m,1)} \le C^{(2)}(T) \|u\|_{p,Q_T}^{(4m,2)}, \tag{4.8}$$

$$\|F(\,\cdot\,,\,\cdot\,,u,D^1u,\dots,D^{2m}u) - L(0)u\|_{p,Q_T}^{(2m,1)} \le C^{(2)}(T) \|u\|_{p,Q_T}^{(4m,2)}, \tag{4.9}$$

where $L(u)\phi$ is defined by (3.16), numbers $C^{(1)}$, $C^{(2)}$ are independent of u, $\mathbb{R}$, and $C^{(2)}(T)$ tends to 0 as T tends to 0.

Using estimates (4.6)–(4.9), we can estimate the difference

$$d_1 := \Big\langle A_E\Big(u, \frac{\partial u}{\partial t} - F(x,t,u,D^1u,\dots,D^{2m}u) - f(x,t)\Big), u\Big\rangle - \Big\langle A_E\Big(0, \frac{\partial u}{\partial t} - L(0)u\Big), u\Big\rangle,$$

where $T \in (0, T_1]$, $u \in W_p^{(4m,2),0}(Q_T)$, the A_E is defined by (3.12). Using the last inequality we can prove the following estimates for $u \in W_p^{(4m,2),0}(Q_T)$ such that $\|u\|_{p,Q_T}^{(4m,2)} = R$ and $T \in (0, T_1]$:

Case 1. Non-integer p

$$|d_1| \le C_1(T)\mathbb{R}^p + +C_2\Big(\sum_{k=1}^{[p]-1} K(R+K)^{p-k-1}\mathbb{R}^k + K^{p-[p]}\mathbb{R}^{[p]}\Big), \tag{4.10}$$

Case 2. Integer p

$$|d_1| \le C_3(T)\mathbb{R}^p + C_4 \sum_{k=1}^{p-1} K\,(R+K)^{p-k-1}\,\mathbb{R}^k. \tag{4.11}$$

Here $C_1,\dots,C_4$ are independent of u, $\mathbb{R}$ and $C_1(T)$, $C_3(T)$ tend to 0 as $T \to 0$.

Analogously to (4.6)–(4.9), we obtain that it is possible to choose a positive number T_2, depending on R, such that for $u \in W_p^{(4m,2),0}(Q_T)$, $\|u\|_{p,Q_T}^{(4m,2)} \le R$, $T \in (0, T_2]$ and $j = 1,\dots,m$, the following inequalities hold

$$\|B_j(u)u\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))} \le C^{(3)} \|u\|_{p,Q_T}^{(4m,2)}, \tag{4.12}$$

$$\|G_j(\,\cdot\,,\,\cdot\,,u,\dots,D^{m_j}u)\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))} \le C^{(2)} \|u\|_{p,Q_T}^{(4m,2)}, \tag{4.13}$$

$$\|B_j(u)u - B_j(0)u\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))} \le C^{(4)}(T) \|u\|_{p,Q_T}^{(4m,2)}, \tag{4.14}$$

$$\|G_j(\,\cdot\,,\,\cdot\,,u,\dots,D^{m_j}u)-B_j(0)u\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))}\le C^{(4)}(T)\|u\|_{p,Q_T}^{(4m,2)}, \tag{4.15}$$

where $B_j(u)\phi$ is defined by (3.17), $C^{(3)}$, $C^{(4)}$ does not depend on u, R, and $C^{(4)}(T)$ tends to 0 as T tends to 0.

Using (4.12)–(4.15) we can evaluate the difference

$$d_{2j}:=\langle A_{B_j}(u,G_j(x,t,u,\dots,D^{m_j}u)-g_j(x,t)),u\rangle-\langle A_{B_j}(0,B_j(0)u),u\rangle,$$

where A_{B_j} is defined by (3.13) and prove the following inequalities for $T\in(0,T_2]$, $u\in W_p^{(4m,2),0}(Q_T)$, $\|u\|_{p,Q_T}^{(4m,2)}=R$, $j\in 1,\dots,m$:

Case 1. Non-integer p

$$|d_{2j}|\le C_5(T)\mathbb{R}^p+C_6\bigg(\sum_{k=1}^{[p]-1}K(R+K)^{p-k-1}\mathbb{R}^k+K^{p-[p]}\mathbb{R}^{[p]}\bigg), \tag{4.16}$$

Case 2. Integer p

$$|d_{2j}|\le C_7(T)\mathbb{R}^p+C_8\sum_{k=1}^{p-1}K\,(R+K)^{p-k-1}\,\mathbb{R}^k. \tag{4.17}$$

Here C_5–C_8 are independent of u, R, and $C_5(T)$, $C_7(T)$ tend to 0 as T tends to 0.

Choosing $T_3:=\min\{T_1,T_2\}$ we get that inequalities (4.10), (4.11), (4.16), (4.17) are satisfied for $T\in(0,T_3]$ $u\in W_p^{(4m,2),0}(Q_T)$, $\|u\|_{p,Q_T}^{(4m,2)}\le R$. Thus, using definitions of d_1, d_{2j}, we have

$$\begin{aligned}\langle Au,u\rangle\ge{}&\Big\langle A_E\Big(0,\frac{\partial u}{\partial t}-L(0)u\Big),u\Big\rangle\\&+\sum_{j=1}^m\langle A_{B_j}(0,B_j(0)u),u\rangle-C_9(T)\mathbb{R}^p-C_{10}\;\varepsilon(K,R),\end{aligned} \tag{4.18}$$

where constants are independent of u, R, $C_9(T)\to0$, $C_{10}(T)\to0$ as $T\to0$ and $\varepsilon(K,R)/\mathbb{R}^p\to0$ when $R\to\infty$.

A priori estimate (2.12) gives us the inequality

$$\begin{aligned}&\Big\langle A_E\Big(0,\frac{\partial u}{\partial t}-L(0)u\Big),u\Big\rangle+\sum_{j=1}^m\langle A_{B_j}(0,B_j(0)u),u\rangle\\&=\bigg(\Big\|\frac{\partial u}{\partial t}-L(0)u\Big\|_{p,Q_T}^{(2m,1)}\bigg)^p+\sum_{j=1}^m(\|B_j(0)u\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))})^p\\&\ge C_{11}(\|u\|_{p,Q_T}^{(4m,2)})^p,\end{aligned} \tag{4.19}$$

where C_{11} is independent of u.

Inequalities (4.18), (4.19) allow to get estimate

$$\langle Au, u\rangle \geq C_{11}\mathbb{R}^p - C_9(T)\mathbb{R}^p - C_{10}\varepsilon(K,R)$$

that implies inequality

$$\langle Au, u\rangle \geq 0$$

for $u \in W_p^{(4m,2),0}(Q_T)$, $T \in (0,T_0]$, $\|u\|_{p,Q_T}^{(4m,2)} = R_0$ for sufficiently small $T_0 \leq T_3$ and sufficiently large R_0.

Thus we have conditions of Theorem 4.4 from Chapter 2 of [7] satisfied. Then

$$\operatorname{Deg}(A, \overline{B_{R_0}(0)}, 0) = 1,$$

where $B_{R_0}(0) := \{\|u \in W_p^{(4m,2),0}(Q_T)\| : \|u\|_{p,Q_T}^{(4m,2)} = R_0\}$.

Using Corollary 4.1 from Chapter 2 of [7] we assert that the operator equation (3.28) has a solution in $B_{R_0}(0)$. The solvability of problem (3.6)–(3.8) follows then from Theorem 3.13. □

Corollary 4.4. *Assume that conditions* (1.4)–(1.7), (F$_1$), (F$_2$), (G$_1$), (G$_2$) *for the problem* (2.1)–(2.3) *are fulfilled and K is some positive number. Then there exists a positive T_0 that depends on K, such that the problem* (2.1)–(2.3) *has a solution $u \in W_p^{(4m,2)}(Q_T)$ for $0 < T < T_0$ if the following inequalities hold:*

$$\|f\|_{p,Q_T}^{(2m,1)} \leq K, \quad \|g_j\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))} \leq K, \quad j = \overline{1,m},$$
$$\|h\|_{p,\Omega}^{4m-2m/p} \leq K.$$

4.3. Conditional solvability of initial boundary value problems. We include initial boundary value problem (3.6)–(3.8) into a one-parameter family of problems

$$\text{(4.20)}\quad \widetilde{\Phi_\tau}[u] = \frac{\partial u}{\partial t} - F_\tau(x,t,u,D^1u,\dots,D^{2m}u) = \tau f(x,t), \quad (x,t) \in Q_T,$$
$$\text{(4.21)}\quad \widetilde{\Psi_{j,\tau}}[u] = G_{j,\tau}(x,t,u,\dots,D^{m_j}u) = \tau g_j(x,t), \quad (x,t) \in S_T,\ j = \overline{1,m},$$
$$\text{(4.22)}\quad u \in W_p^{(4m,2),0}(Q_T),$$

where

$$F_\tau(x,t,\xi) := F(\tau,x,t,\xi), \qquad \tau \in [0,1],\ (x,t) \in Q_T,\ \xi \in \mathbb{R}^{M(2m)},$$
$$G_{j,\tau}(x,t,\zeta_j) := G_j(\tau,x,t,\zeta_j), \quad \tau \in [0,1],\ (x,t) \in S_T,\ \zeta_j \in \mathbb{R}^{M(m_j)},\ j = \overline{1,m}.$$

We assume that $F(x,t,\xi) = F_1(x,t,\xi)$, $G_j(x,t,\zeta_j) = G_{j,1}(x,t,\zeta_j)$, where the functions $F(x,t,\xi)$, $G_j(x,t,\zeta_j)$ appear in the left side of equations in problem (3.6)–(3.8).

Theorem 4.5. *Let functions $F_\tau(x,t,\xi)$ together with all their derivatives w.r.t. ξ_β up to the order $2m+1$, be continuous for $\tau \in [0,1]$, $(x,t) \in Q_T$, $\xi \in \mathbb{R}^{M(2m)}$, $F_\tau(x,t,0) \equiv 0$ and we assume that, for every $\tau \in [0,1]$, function $F_\tau(x,t,\xi)$ satisfies conditions* (F$_2$), (F$_4$). *Let function $G_{j,\tau}(x,t,\zeta_j)$, $j \in 1,\dots,m$ and all their derivatives up to the order $4m-m_j+1$ w.r.t. ζ_β, be continuous for $\tau \in [0,1]$, $(x,t) \in S_T$, $\zeta_j \in \mathbb{R}^{M(m_j)}$, $G_{j,\tau}(x,t,0) \equiv 0$ and functions $G_{j,\tau}$ satisfy conditions* (G$_2$), (G$_4$), *for every $\tau \in [0,1]$. Assume that conditions* (2.4), (2.5) *are fullfilled and, for each $\tau \in [0,1]$, inclusions* (3.9) *are valid. We suppose that there exists a number $R = R(f, g_1, \dots, g_m)$ independent of τ and such that the problem* (3.20)–(3.22), *for each $\tau \in [0,1]$, has no solutions outside the ball*

$$\{u \in W_p^{(4m,2),0}(Q_T) : \|u_\tau\|_{p,Q_T}^{(4m,2)} \le R\}. \tag{4.23}$$

Then the problem (3.6)–(3.8) *has the unique solution $u \in W_p^{(4m,2),0}(Q_T)$.*

Proof. We can reduce parametrical family of problems (4.20)–(4.22) to one-parametrical family of operator equations

$$A_\tau u = 0, \quad u \in W_p^{(4m,2),0}(Q_T), \tag{4.24}$$

where A_τ is defined by equality

$$\langle A_\tau u, \phi\rangle := \frac{1}{p}\frac{d}{ds}\bigg[(\|\widetilde{\Phi_\tau}[u+s\phi] - \tau f\|_{p,Q_T}^{(2m,1)})^p + \sum_{j=1}^m (\|\widetilde{\Psi_{j,\tau}}[u+s\phi] - \tau g_j\|_{p,S_T}^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp))})^p\bigg]\bigg|_{s=0}.$$

Using results of Theorem 3.5, we can assert that for each $\tau \in [0,1]$, operator A_τ is bounded, continuous and satisfies $(\mathrm{S})_+$ condition on $W_p^{(4m,2),0}(Q_T)$. It follows from Theorem 3.13 that solvability of the problem (3.6)–(3.8) is equivalent to solvability of operator equation $A_1 u = 0$, $u \in W_p^{(4m,2),0}(Q_T)$.

We define

$$D_R := \{u \in W_p^{(4m,2),0}(Q_T) : \|u\|_{p,Q_T}^{(4m,2)} < R+1\},$$

where R is the number from (4.23). Then, for every $\tau \in [0,1]$, the inequality

$$A_\tau u \ne 0, \quad u \in \partial D_R, \tag{4.25}$$

holds and the operator degree $\mathrm{Deg}(A_\tau, \overline{D_R}, 0)$ is well defined.

Similarly to the proof of Theorem 3.5, we can prove that the operator family A_τ has the following properties:

(1) for each sequence $\{u_k\}_{k=1}^\infty \subset W_p^{(4m,2),0}(Q_T)$, that converges to $u_0 \in W_p^{(4m,2),0}(Q_T)$, and for every sequence $\{\tau_k\}_{k=1}^\infty \subset [0,1]$ such that $\tau_k \to \tau_0$, it follows that $A_{\tau_k} u_k \to A_{\tau_0} u_0$;

(2) for each sequence $\{u_k\}_{k=1}^{\infty} \subset W_p^{(4m,2),0}(Q_T)$, weakly convergent to $u_0 \in W_p^{(4m,2),0}(Q_T)$, and and for every sequence $\{\tau_k\}_{k=1}^{\infty} \subset [0,1]$ $\{\tau_k\}_{k=1}^{\infty} \subset [0,1]$, such that $\tau_k \to \tau_0$, condition

$$\lim_{k\to\infty} \langle A_{\tau_k} u_k, u_k - u_0 \rangle = 0$$

implies the strong convergence u_k to u_0.

Using properties (1), (2) and inequality (4.25) we prove homotopy of mappings A_0 and A_1 on D_R. From the Theorem 4.1 from [7] it follows that

$$\operatorname{Deg}(A_0, \overline{D_R}, 0) = \operatorname{Deg}(A_1, \overline{D_R}, 0).$$

In full analogy to the proof theorem 4.3 we obtain that inequality

$$\langle A_0 u, u \rangle > 0,$$

is valid for $u \in W_p^{(4m,2),0}(Q_T)$, $\|u\|_{p,Q_T}^{(4m,2)} = r_0$ when $r_0 > 0$ is sufficiently small. In fact, it implies that

$$\operatorname{Deg}(A_0, \overline{B_{r_0}(0)}, 0) = 1, \tag{4.26}$$

where $B_{r_0}(0) := \{u \in W_p^{(4m,2),0}(Q_T) : \|u\|_{p,Q_T}^{(4m,2)} \le r_0\}$.

It is easy to check that the function $u \equiv 0$ will be solution of operator equation $A_0 u = 0$. From the Theorem 4.1, (4.26) and Theorem 5.1 (Chapter 2 of [7]) we have

$$\operatorname{Deg}(A_0, \overline{D_R}, 0) = 1.$$

Using Corollary 4.1 ([7, Chapter 2]) we obtain solvability for (3.6)–(3.8). □

4.4. Theorem of domain preservation.

Theorem 4.6. *Assume that initial boundary value problem* (3.6)–(3.8) *satisfies conditions of Theorem* 3.5 *and D is the open set in $W_p^{(4m,2),0}(Q_T)$. Then the set*

$$R(D) := \left\{ \left(\frac{\partial u}{\partial t} - F(\,\cdot\,,\,\cdot\,, u, D^1 u, \dots, D^{2m} u), G_1(\,\cdot\,,\,\cdot\,, u, \dots, D^{m_1} u), \dots, G_m(\,\cdot\,,\,\cdot\,, u, \dots, D^{m_m} u) \right) : u \in D \right\}$$

will be open in space

$$W_p^{(2m,2\{m_j\}),0}(Q_T, S_T) := W_p^{(2m,1),0}(Q_T) \times \left(\prod_{j=1}^{m} W^{(4m-m_j-1/p,2-m_j/(2m)-1/(2mp)),0}(S_T) \right).$$

Proof. The proof of theorem is fully analogous to the proof of Theorem 6.1 from [2]. □

Corollary 4.7. *Assume that initial boundary value problem* (3.6)–(3.8) *satisfies conditions of Theorem* 3.5. *Then the set*

$$\mathbb{R}^{(0)} := \{(f, g_1, \dots, g_m) \in W_p^{(2m,\{m_j\}),0}(Q_T, S_T) : \text{problem (3.6)–(3.8) has a solution } u \in W_p^{(4m,2),0}(Q_T)\}$$

is open in $W_p^{(2m,\{m_j\}),0}(Q_T, S_T)$.

Corollary 4.8. *Assume that initial boundary value problem* (3.6)–(3.8) *satisfies conditions of Theorem* 2.5 *and* $U: \mathbb{R}^{(0)} \to W_p^{(4m,2),0}(Q_T)$ *is operator which maps a vector of functions* $(f, g_1, \dots, g_m) \in \mathbb{R}^{(0)}$ (*where* $\mathbb{R}^{(0)}$ *is defined in Corollary* 4.7) *onto the solution of problem* (2.6)–(2.8). *Then, the operator* U *is continuous on* $\mathbb{R}^{(0)}$.

4.5. Convergence of the Galerkin approximants. Let $\{v_k\}_{k=1}^{\infty}$ be the complete system of functions in $W_p^{(4m,2),0}(Q_T)$. Assume that initial boundary value problem (3.6)–(3.8) satisfies conditions of Theorem 3.5. By a $\mathfrak{K}$-*approximate solution* of boundary value problem (3.6)–(3.8) we mean the function $u_{\mathfrak{K}}$ such that

$$u_{\mathfrak{K}} = \sum_{k=1}^{\mathfrak{K}} c_k^{(\mathfrak{K})} v_k(x,t)$$

and

$$\langle Au_{\mathfrak{K}}, v_k \rangle = 0, \quad k = \overline{1, \mathfrak{K}},$$

where $c_k^{(\mathfrak{K})}$ are real numbers and the operator A is defined by (3.10).

We say that the problem (3.6)–(3.8) has a *bounded sequence of* $\mathfrak{K}$-*approximate solutions* if there exists a number $\mathfrak{K}_0$ such that for $\mathfrak{K} \geq \mathfrak{K}_0$ the problem (3.6)–(3.8) has an $\mathfrak{K}$ - approximate solution and the sequence $\{u\}_{\mathfrak{K}=\mathfrak{K}_0}^{\infty}$ is bounded.

Theorem 4.9. *Assume that conditions* (2.4), (2.5), (3.9), (F_2)–(F_4), (G_2)–(G_4) *for the problem* (3.6)–(3.8) *are fulfilled. The problem* (3.6)–(3.8) *has a solution* $u_0 \in W_p^{(4m,2),0}(Q_T)$ *if and only if it has bounded sequence of* $\mathfrak{K}$-*approximate solutions* $\{u_{\mathfrak{K}}\}_{\mathfrak{K}=\mathfrak{K}_0}^{\infty}$. *The sequence* $u_{\mathfrak{K}}$ *strongly converges to* u_0 *in* $W_p^{(4m,2),0}(Q_T)$.

Proof. The proof of the theorem is identical to the proof of the Theorem 7.1 from [2]. □

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