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ON THE SUSPENSION ISOMORPHISM FOR INDEX BRAIDS IN A SINGULAR PERTURBATION PROBLEM

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Abstract. We consider the singularly perturbed system of ordinary differential equations

$$(E_\varepsilon) \qquad \begin{aligned} \varepsilon\dot y &= f(y,x,\varepsilon),\\ \dot x &= h(y,x,\varepsilon) \end{aligned}$$

on $Y\times\mathcal{M}$, where Y is a finite dimensional normed space and $\mathcal{M}$ is a smooth manifold. We assume that there is a reduced manifold of (E_ε) given by the graph of a function $\phi\colon\mathcal{M}\to Y$ and satisfying an appropriate hyperbolicity assumption with unstable dimension $k\in\mathbb{N}_0$. We prove that every Morse decomposition $(M_p)_{p\in P}$ of a compact isolated invariant set S_0 of the reduced equation

$$\dot x = h(\phi(x),x,0)$$

gives rises, for $\varepsilon>0$ small, to a Morse decomposition $(M_{p,\varepsilon})_{p\in P}$ of an isolated invariant set S_ε of (E_ε) such that $(S_\varepsilon,(M_{p,\varepsilon})_{p\in P})$ is close to $(\{0\}\times S_0,(\{0\}\times M_p)_{p\in P})$ and the (co)homology index braid of $(S_\varepsilon,(M_{p,\varepsilon})_{p\in P})$ is isomorphic to the (co)homology index braid of $(S_0,(M_p)_{p\in P})$ shifted by k to the left.

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1. Introduction

Consider the following singularly perturbed system of ordinary differential equations

$$\begin{aligned} \varepsilon\dot y_1 &= f_1((y_1,y_2,x),\varepsilon),\\ \varepsilon\dot y_2 &= f_2((y_1,y_2,x),\varepsilon),\\ \dot x &= h((y_1,y_2,x),\varepsilon) \end{aligned} \tag{1.1}$$

and assume the following

Hypothesis 1.1.

(a) *Y_1, Y_2 and X are finite dimensional normed linear spaces with $k := \dim Y_2$, U is open in X, $\overline{\varepsilon} \in]0,\infty[$ is arbitrary, Z_0 is open in $Y_1\times Y_2\times U$ and $W_0 := Z_0 \times [0,\bar\varepsilon]$.*

(b) *$f_1: W_0 \to Y_1$, $f_2: W_0 \to Y_2$ and $h: W_0 \to X$ are maps such that, for each $\varepsilon \in]0,\bar\varepsilon]$, $f_1(\,\cdot\,,\varepsilon)$, $f_2(\,\cdot\,,\varepsilon)$ and $h(\,\cdot\,,\varepsilon)$ are locally Lipschitzian.*

(c) *$\phi_1: U \to Y_1$ and $\phi_2: U \to Y_2$ are C^2-maps such that for all $x \in U$, $(\phi_1(x),\phi_2(x),x) \in Z_0$ and*

$$f_1((\phi_1(x),\phi_2(x),x),0) = 0, \quad f_2((\phi_1(x),\phi_2(x),x),0) = 0.$$

(d) *The maps $f_1(\,\cdot\,,0)$, $f_2(\,\cdot\,,0)$ are of class C^2 and the map $h(\,\cdot\,,0)$ is locally Lipschitzian.*

(e) *For every $(y_1,y_2,x) \in Z_0$, the maps f_1, f_2 are continuous at the point $((y_1,y_2,x),0)$ and for every $x \in U$ the map h is continuous at the point $((\phi_1(x),\phi_2(x),x),0)$.*

(f) *For all $x \in U$, $\operatorname{re}\sigma(B_{11}(x)) < 0$, $\operatorname{re}\sigma(B_{22}(x)) > 0$, $B_{12}(x) \equiv 0$ and $B_{21}(x) \equiv 0$, where*

$$B_{jl}(x) = D_j f_l((\phi_1(x),\phi_2(x),x),0), \quad j,l \in \{1,2\},\ x \in U.$$

In singular perturbation theory the set

$$\{(\phi_1(x),\phi_2(x),x) \mid x \in U\}$$

is called the *reduced manifold* of (1.1). The corresponding *reduced equation* is given by

$$\dot x = h((\phi_1(x),\phi_2(x),x),0). \tag{1.2}$$

Part (f) of Hypothesis 1.1 is a *hyperbolicity* assumption on the reduced manifold with respect to equation (1.1).

A natural question is whether the dynamics of the reduced equation (1.2) 'survives' in the dynamics of (1.1) for $\varepsilon > 0$ small.

In this paper, this question is considered in the context of Conley index theory. In particular, we prove that every isolated invariant set S_0 of the reduced equation (1.2) gives rise to a family of isolated invariant sets S_ε, $\varepsilon > 0$ small, of (1.1) whose Conley index $h(S_\varepsilon)$ is equal to the wedge product of the pointed k-sphere with the Conley index $h(S_0)$ of S_0. Moreover, every (partially) ordered Morse decomposition $(M_p)_{p\in P}$ of S_0 gives rise to a family $(M_{p,\varepsilon})_{p\in P}$, $\varepsilon > 0$ small, such that, for all such ε, $(M_{p,\varepsilon})_{p\in P}$ is a Morse decomposition of S_ε and the (co)homology index braid of $(S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ is isomorphic to the (co)homology index braid of $(S_0, (M_p)_{p\in P})$ shifted by k to the left.

Let us now describe our results more precisely. By π_ε denote the local (semi)flow on Z_0 generated by the solutions of the differential equation (1.1) and by π_0 denote the local (semi)flow generated on U by the ordinary differential equation (1.2).

Our first result reads as follows.

Theorem 1.2. *Assume Hypothesis* 1.1. *Let $S_0 \subset U$ be a compact isolated invariant set relative to π_0 and $N \subset U$ be a compact isolating neigbourhood of S_0. Then there is an $\eta_0 \in]0,\infty[$ such that for every $\eta \in]0,\eta_0]$, there exists an $\varepsilon_0 = \varepsilon_0(\eta) \in]0,\overline{\varepsilon}]$ such that for every $\varepsilon \in]0,\varepsilon_0]$, the set*

$$N_\eta := \{(y_1,y_2,x) \in Z_0 \mid x \in N,\ |y_1 - \phi_1(x)|_{Y_1} \le \eta \text{ and } |y_2 - \phi_2(x)|_{Y_2} \le \eta\},$$

is an isolating neigbourhood relative to π_ε and

$$h(\pi_\varepsilon, S_\varepsilon) = \Sigma^k \wedge h(\pi_0, S_0),$$

where $S_\varepsilon = S_{\varepsilon,N,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(N_\eta)$ and k is the dimension of Y_2.

Now, for the rest of this paper, let P be a finite set and $\prec$ be a strict partial order on P.

Using the notation of the papers [5], [4], [17] we can state our second result as follows.

Theorem 1.3. *Assume Hypothesis* 1.1. *Let $S_0 \subset U$ be a compact isolated invariant set relative to π_0 and $N \subset U$ be a compact isolating neigbourhood of S_0. Moreover, let $(M_p)_{p\in P}$ be a $\prec$-ordered Morse decomposition of S_0 relative to π_0. For each $p \in P$, let $V_p \subset N$ be an isolating neigbourhood of M_p relative to π_0. For every $\eta \in]0,\infty[$, every $\varepsilon \in]0,\overline{\varepsilon}]$ and every $p \in P$, define*

$$S_\varepsilon = S_{\varepsilon,N,\eta} := \mathrm{Inv}_{\pi_\varepsilon}(N_\eta) \quad \text{and} \quad M_{p,\varepsilon} = M_{p,\varepsilon,V_p,\eta} := \mathrm{Inv}_{\pi_\varepsilon}((V_p)_\eta),$$

where,

$$(V_p)_\eta := \{(y_1,y_2,x) \in Z_0 \mid x \in V_p,\ |y_1 - \phi_1(x)|_{Y_1} \le \eta \text{ and } |y_2 - \phi_2(x)|_{Y_2} \le \eta\}.$$

Then there exists an $\eta_0 \in \,]0,\infty[$ *such that for every* $\eta \in \,]0,\eta_0]$ *there is an* $\varepsilon_0 = \varepsilon_0(\eta) \in \,]0,\overline{\varepsilon}]$ *such that for every* $\varepsilon \in \,]0,\varepsilon_0]$*, the family* $(M_{p,\varepsilon})_{p\in P}$ *is a* $\prec$*-ordered Morse decomposition for* S_ε *relative to* π_ε*. Moreover, for every* $\varepsilon \in \,]0,\varepsilon_0]$*, for every* $K \in \mathcal{I}(\prec)$ *and for every* $q \in \mathbb{Z}$*, there exist isomorphisms*

$$\Theta^\varepsilon_q(K): H_q(\pi_\varepsilon, M_\varepsilon(K)) \to H_{q-k}(\pi_0, M(K))$$

and

$$\Theta_\varepsilon^q(K): H^{q-k}(\pi_0, M(K)) \to H^q(\pi_\varepsilon, M_\varepsilon(K))$$

such that given $(I,J) \in \mathcal{I}_2(\prec)$ *the diagrams*

$$\begin{array}{ccccccccc}
\longrightarrow & H_q(M_\varepsilon(I)) & \longrightarrow & H_q(M_\varepsilon(IJ)) & \longrightarrow & H_q(M_\varepsilon(J)) & \longrightarrow & H_{q-1}(M_\varepsilon(I)) & \longrightarrow \\
& \downarrow{\scriptstyle \Theta^\varepsilon_q(I)} & & \downarrow{\scriptstyle \Theta^\varepsilon_q(IJ)} & & \downarrow{\scriptstyle \Theta^\varepsilon_q(J)} & & \downarrow{\scriptstyle \Theta^\varepsilon_{q-1}(I)} & \\
\longrightarrow & H_{q-k}(M(I)) & \longrightarrow & H_{q-k}(M(IJ)) & \longrightarrow & H_{q-k}(M(J)) & \longrightarrow & H_{q-k-1}(M(I)) & \longrightarrow
\end{array}$$

$$\begin{array}{ccccccccc}
\longleftarrow & H^q(M_\varepsilon(I)) & \longleftarrow & H^q(M_\varepsilon(IJ)) & \longleftarrow & H^q(M_\varepsilon(J)) & \longleftarrow & H^{q-1}(M_\varepsilon(I)) & \longleftarrow \\
& {\scriptstyle \Theta_\varepsilon^q(I)}\uparrow & & {\scriptstyle \Theta_\varepsilon^q(IJ)}\uparrow & & {\scriptstyle \Theta_\varepsilon^q(J)}\uparrow & & {\scriptstyle \Theta_\varepsilon^{q-1}(I)}\uparrow & \\
\longleftarrow & H^{q-k}(M(I)) & \longleftarrow & H^{q-k}(M(IJ)) & \longleftarrow & H^{q-k}(M(J)) & \longleftarrow & H^{q-k-1}(M(I)) & \longleftarrow
\end{array}$$

commute, where for every $K \in \mathcal{I}(\prec)$*, for every* $\varepsilon \in \,]0,\varepsilon_0]$ *and for every* $q \in \mathbb{Z}$*,* $H_q(M(K)) := H_q(\pi_0, M(K))$*,* $H^q(M(K)) := H^q(\pi_0, M(K))$*,* $H_q(M_\varepsilon(K)) := H_q(\pi_\varepsilon, M_\varepsilon(K))$ *and* $H^q(M_\varepsilon(K)) := H^q(\pi_\varepsilon, M_\varepsilon(K))$*. Thus, the* (*co*)*homology index braid of* $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ *is isomorphic to the graded module braid obtained by shifting the* (*co*)*homology index braid of* $(\pi_0, S_0, (M_p)_{p\in P})$ *to the left by* k*.*

In addition, we show that the sets $S_\varepsilon = S_{\varepsilon,N,\eta}$ are asymptotically independent of N and η and the family $\widetilde{S}_\varepsilon$, $\varepsilon \in [0,\varepsilon_0]$, where $\widetilde{S}_0 = \{0_{Y_1}\} \times \{0_{Y_2}\} \times S_0$ and $\widetilde{S}_\varepsilon = S_\varepsilon$, $\varepsilon > 0$, is upper-semicontinuous at $\varepsilon = 0$ in the topology of $Y_1 \times Y_2 \times X$. In this sense, the sets S_ε are close to $\{0_{Y_1}\} \times \{0_{Y_2}\} \times S_0$ for $\varepsilon > 0$ small. Analogously, the sets $M_{p,\varepsilon} = M_{p,\varepsilon,N,\eta}$ are asymptotically independent of N and η and close to $\{0_{Y_1}\} \times \{0_{Y_2}\} \times M_p$ for $\varepsilon > 0$ small.

In particular, the above results show that the Conley index of S_0 completely determines the Conley index of S_ε and the (co)homology index braid of $(\pi_0, S_0, (M_p)_{p\in P})$ completely determines the (co)homology index braid of $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$. This answers the question posed above from the point of view of Conley index theory.

Theorems 1.2 and 1.3 are special cases of the main result of this paper, Theorem 4.3. A crucial step in the proof of that theorem is an application of the suspension isomorphism results for (co)homology index braids established in [4] and [17].

This paper is organized as follows. In Section 2 we establish an isomorphism result for (co)homology index braids in the case of the product of an arbitrary local semiflow with an asymptotically stable linear flow. This result is required in the proof of Theorem 4.3. In Section 3 we recall some useful facts about ordinary differential equations on manifolds. In Section 4 we introduce a generalization of problems (1.1) and (1.2) (see (4.1) and (4.2)) in which the open set $U \subset X$ is replaced by a finite dimensional differentiable manifold $\mathcal{M}$ and Hypothesis 1.1 is replaced by more general assumptions (see Hypotheses 4.1 and 4.2). We also state our main result, Theorem 4.3. We then prove this theorem and the upper-semicontinuity results alluded to before. We then discuss two cases in which Hypothesis 4.2 is satisfied. We end the paper with an example showing that Hypothesis 4.1 alone is not sufficient for the validity of Theorem 4.3.

We refer the reader to the papers [1], [2], [4], [5], [17] for various notations and results used implicitly throughout this paper. The interested reader is also referred to the recent paper [6] for a continuation result of (co)homology index braids in singularly perturbed hyperbolic equations.

2. A special product case

In this section let E be a Banach space and Π be the global semiflow generated by a C_0-semigroup $(T(t))_{t\in[0,\infty[}$ satisfying, for some constants M, $\beta \in\,]0,\infty[$, the estimate

$$|T(t)u|_E \leq Me^{-\beta t}|u|_E, \quad t \in [0,\infty[\,,\ u \in E. \tag{2.1}$$

Moreover, let X be a metric space and π be a local semiflow on X. Let $\pi' = \pi\times\Pi$ be the product of π with Π. Unless specified otherwise, whenever M is a subset of X, we write $M' = M \times \{0_E\} \subset X \times E$. We will prove in this section that, under the usual admissibility assumptions, whenever, relative to π, S is an isolated invariant set and $(M_p)_{p\in P}$ is a (partially) ordered Morse decomposition of S, then, relative to π', S' is an isolated invariant set, $(M'_p)_{p\in P}$ is a Morse decomposition of S' and the (co)homology index braid of $(\pi, S, (M_p)_{p\in P})$ is isomorphic to the (co)homology index braid of $(\pi', S', (M'_p)_{p\in P})$. Together with the suspension isomorphism results established in [4], [17] this will be a crucial step in the proof of Theorem 4.3.

We will first prove the following result.

Theorem 2.1. *Let B be a closed ball in E centered at $0 = 0_E$.*

(a) *Let S be an isolated π-invariant set and (Y, Z) be an FM-index pair for (π, S) such that $\mathrm{Cl}_X(Y \setminus Z)$ is strongly π-admissible. Then S' is an isolated π'-invariant set and $(Y \times B, Z \times B)$ is an FM-index pair for (π', S') such that $\mathrm{Cl}_{X\times E}((Y \times B) \setminus (Z \times B))$ is strongly π'-admissible. Let $f_{Y,Z}\colon Y/Z \to (Y \times B)/(Z \times B)$ be the (base point preserving) map* *induced by* $f\colon Y \to Y \times B$, $x \mapsto (x,0)$, *and, for* $q \in \mathbb{Z}$, *let*

$$F_q := H_q(f_{Y,Z})\colon H_q(Y/Z,\{[Z]\}) \to H_q((Y\times B)/(Z\times B),\{[Z\times B]\}),$$

resp.

$$F^q := H^q(f_{Y,Z})\colon H^q((Y\times B)/(Z\times B),\{[Z\times B]\}) \to H^q(Y/Z,\{[Z]\})$$

be the induced homology, resp. cohomology, map. The map $f_{Y,Z}$ *is a homotopy equivalence of pointed spaces so* F_q, *resp.* F^q, *is an* Γ*-module isomorphism for all* $q \in \mathbb{Z}$.

(b) *For all* $q \in \mathbb{Z}$, *the map*

$$\langle F_q\rangle = \langle F_q\rangle_{\mathcal{C},\Phi,\mathcal{C}',\widehat{\Phi'}}\colon \widehat{\Phi}(\mathcal{C}) \to \widehat{\Phi'}(\mathcal{C}'),$$

resp. the map

$$\langle F^q\rangle = \langle F^q\rangle_{\mathcal{C},\Phi,\mathcal{C}',\widehat{\Phi'}}\colon \widehat{\Phi'}(\mathcal{C}') \to \widehat{\Phi}(\mathcal{C}),$$

is independent of the choice of (Y,Z). *Here,* $\mathcal{C}$ *(resp.* $\mathcal{C}'$*) is the categorial Conley–Morse index of* (π,S) *(resp.* (π',S')*) as defined in* [5] *and* Φ *(resp.* Φ'*) is the restriction of* H_q, *resp.* H^q, *to* $\mathcal{C}$ *(resp.* $\mathcal{C}'$*). For all* $q \in \mathbb{Z}$, *define the morphism* $\kappa_q(\pi,S)\colon H_q(\pi,S) \to H_q(\pi',S')$ *by*

$$\kappa_q(\pi,S) = \langle F_q\rangle$$

and the morphism $\kappa^q(\pi,S)\colon H^q(\pi',S') \to H^q(\pi,S)$ *by*

$$\kappa^q(\pi,S) = \langle F^q\rangle.$$

$\kappa_q(\pi,S)$ *and* $\kappa^q(\pi,S)$, $q \in \mathbb{Z}$, *are* Γ*-module isomorphisms.*

(c) *Given an isolated* π*-invariant set* S *having a strongly* π*-admissible isolating neigbourhood and an attractor-repeller pair* (A,A^*) *of* S *relative to* π, *then* S' *is an isolated* π'*-invariant set having a strongly* π'*-admissible isolating neigbourhood,* $(A',(A^*)')$ *is an attractor-repeller pair of* S' *relative to* π' *and the diagrams*

$$\begin{array}{ccccccccc} \longrightarrow & H_q(\pi,A) & \longrightarrow & H_q(\pi,S) & \longrightarrow & H_q(\pi,A^*) & \longrightarrow & H_{q-1}(\pi,A) & \longrightarrow \\ & \downarrow{\scriptstyle \kappa_q(\pi,A)} & & \downarrow{\scriptstyle \kappa_q(\pi,S)} & & \downarrow{\scriptstyle \kappa_q(\pi,A^*)} & & \downarrow{\scriptstyle \kappa_{q-1}(\pi,A)} & \\ \longrightarrow & H_q(\pi',A') & \longrightarrow & H_q(\pi',S') & \longrightarrow & H_q(\pi',(A^*)') & \longrightarrow & H_{q-1}(\pi',A') & \longrightarrow \end{array} \tag{2.2}$$

$$\begin{array}{ccccccccc} \longleftarrow & H^q(\pi,A) & \longleftarrow & H^q(\pi,S) & \longleftarrow & H^q(\pi,A^*) & \longleftarrow & H^{q-1}(\pi,A) & \longleftarrow \\ & \uparrow{\scriptstyle \kappa^q(\pi,A)} & & \uparrow{\scriptstyle \kappa^q(\pi,S)} & & \uparrow{\scriptstyle \kappa^q(\pi,A^*)} & & \uparrow{\scriptstyle \kappa^{q-1}(\pi,A)} & \\ \longleftarrow & H^q(\pi',A') & \longleftarrow & H^q(\pi',S') & \longleftarrow & H^q(\pi',(A^*)') & \longleftarrow & H^{q-1}(\pi',A') & \longleftarrow \end{array} \tag{2.3}$$

commute.

Proof. Let S be an isolated π-invariant set and (Y, Z) be an FM-index pair for (π, S) such that $\mathrm{Cl}_X(Y \setminus Z)$ is strongly π-admissible. Since, by (2.1), $(B, \emptyset)$ is an FM-index pair for $(\Pi, \{0\})$ with B strongly Π-admissible, an application of [4, Proposition 2.2] shows that $(Y \times B, Z \times B)$ is an FM-index pair for (π', S') such that $\mathrm{Cl}_{X\times E}((Y \times B) \setminus (Z \times B))$ is strongly π'-admissible. Now, working with the homotopy $((x, b), \theta) \mapsto (x, \theta b)$ from $(Y \times B) \times [0, 1]$ to $Y \times B$ we easily show that $f_{Y,Z}$ is a homotopy equivalence of pointed spaces. This proves part (a).

To prove the independence of $\langle F_q \rangle$ of the choice of (Y, Z), let $(\widehat{Y}, \widehat{Z})$ be another FM-index pair for (π, S) with $\mathrm{Cl}_X(\widehat{Y} \setminus \widehat{Z})$ strongly π-admissible. By [5, Proposition 4.6, Lemma 4.8 and Proposition 2.5] we obtain sets L_1, L_2, W and $\widehat{W}$ such that $(L_1, L_2) \subset (Y \cap \widehat{Y}, W \cap \widehat{W})$, $Z \subset W$, $\widehat{Z} \subset \widehat{W}$ and (L_1, L_2), (Y, W) and $(\widehat{Y}, \widehat{W})$ are FM-index pairs for (π, S) such that $\mathrm{Cl}_X(L_1 \setminus L_2)$, $\mathrm{Cl}_X(Y \setminus Z)$ and $\mathrm{Cl}_X(\widehat{Y} \setminus \widehat{W})$ are strongly π-admissible. We thus obtain the commutative diagram

$$\begin{array}{ccc}
H_q(Y/Z, \{[Z]\}) & \xrightarrow{H_q(f_{Y,Z})} & H_q((Y \times B)/(Z \times B), \{[Z \times B]\}) \\
\downarrow & & \downarrow \\
H_q(Y/W, \{[W]\}) & \xrightarrow{H_q(f_{Y,W})} & H_q((Y \times B)/(W \times B), \{[W \times B]\}) \\
\uparrow & & \uparrow \\
H_q(L_1/L_2, \{[L_2]\}) & \xrightarrow{H_q(f_{L_1,L_2})} & H_q((L_1 \times B)/(L_2 \times B), \{[L_2 \times B]\}) \\
\downarrow & & \downarrow \\
H_q(\widehat{Y}/\widehat{W}, \{[\widehat{W}]\}) & \xrightarrow{H_q(f_{\widehat{Y},\widehat{W}})} & H_q((\widehat{Y} \times B)/(\widehat{W} \times B), \{[\widehat{W} \times B]\}) \\
\uparrow & & \uparrow \\
H_q(\widehat{Y}/\widehat{Z}, \{[\widehat{Z}]\}) & \xrightarrow{H_q(f_{\widehat{Y},\widehat{Z}})} & H_q((\widehat{Y} \times B)/(\widehat{Z} \times B), \{[\widehat{Z} \times B]\})
\end{array}$$

whose vertical maps are inclusion induced. Hence, by [5, Proposition 4.5], these maps are induced by the unique morphisms in $\mathcal{C}$ (resp. in $\mathcal{C}'$) between the corresponding objects of these connected simple systems. In particular, the vertical maps are all bijective, and so we may invert the upward pointing arrows and then compose the columns to obtain the commutative diagram

$$\text{(2.4)} \qquad \begin{array}{ccc}
H_q(Y/Z, \{[Z]\}) & \xrightarrow{H_q(f_{Y,Z})} & H_q((Y \times B)/(Z \times B), \{[Z \times B]\}) \\
\downarrow & & \downarrow \\
H_q(\widehat{Y}/\widehat{Z}, \{[\widehat{Z}]\}) & \xrightarrow[H_q(f_{\widehat{Y},\widehat{Z}})]{} & H_q((\widehat{Y} \times B)/(\widehat{Z} \times B), \{[\widehat{Z} \times B]\})
\end{array}$$

where the vertical maps are induced by the corresponding morphism in $\mathcal{C}$ (resp. in $\mathcal{C}'$). Now an application of [4, Proposition 2.4] to diagram (2.4) completes the proof of part (b) of the theorem in the homology case. The proof of the cohomology case is analogous (reversing the arrows). To prove part (c) let (N_1, N_2, N_3) be an FM-index triple for (π, S, A, A^*) with $\mathrm{Cl}_X(N_1 \setminus N_3)$ strongly π-admissible. It follows that $(N_1', N_2', N_3') := (N_1 \times B, N_2 \times B, N_3 \times B)$ is an FM-index triple for $(\pi', S', A', (A^*)')$ such that $\mathrm{Cl}_{X\times E}((N_1 \times B) \setminus (N_3 \times B))$ is strongly π'-admissible. In the notation of [4] we thus have the following commutative diagram

$$\begin{array}{ccccc}
\Delta(N_2/N_3)/\Delta(\{[N_3]\}) & \to & \Delta(N_1/N_3)/\Delta(\{[N_3]\}) & \to & \Delta(N_1/N_2)/\Delta(\{[N_2]\}) \\
\downarrow{\scriptstyle \Delta(f_{N_2,N_3})} & & \downarrow{\scriptstyle \Delta(f_{N_1,N_3})} & & \downarrow{\scriptstyle \Delta(f_{N_1,N_2})} \\
\Delta(N_2'/N_3')/\Delta(\{[N_3']\}) & \to & \Delta(N_1'/N_3')/\Delta(\{[N_3']\}) & \to & \Delta(N_1'/N_3')/\Delta(\{[N_3']\})
\end{array} \tag{2.5}$$

with inclusion induced weakly exact rows (in view of [4, Proposition 2.8]). Applying [4, Proposition 2.7] to diagram (2.5) we obtain the induced long commutative ladder with exact rows. An application of the $\langle\,\cdot\,,\,\cdot\,\rangle$-operation to that ladder and using part (b) we obtain diagram (2.2). This proves part (c) in the homology case.

Now, in the notation of [17] and using [17, Proposition 3.4] we obtain the following commutative diagram of cochain maps with weakly coexact rows

$$\begin{array}{ccccc}
\overline{C}^*(N_1/N_2, \{[N_2]\}) & \to & \overline{C}^*(N_1/N_3, \{[N_3]\}) & \to & \overline{C}^*(N_2/N_3, \{[N_3]\}) \\
{\scriptstyle f^\sharp_{N_1,N_2}}\uparrow & & {\scriptstyle f^\sharp_{N_1,N_3}}\uparrow & & {\scriptstyle f^\sharp_{N_2,N_3}}\uparrow \\
\overline{C}^*(N_1'/N_2', \{[N_2']\}) & \to & \overline{C}^*(N_1'/N_3', \{[N_3']\}) & \to & \overline{C}^*(N_2'/N_3', \{[N_3']\})
\end{array} \tag{2.6}$$

Applying [17, Proposition 2.2] to diagram (2.6) we obtain the induced long commutative ladder with exact rows. An application of the $\langle\,\cdot\,,\,\cdot\,\rangle$-operation to that ladder and using part (b) we obtain diagram (2.3). This proves part (c) in the cohomology case. □

Let $(M_p)_{p\in P}$ be a $\prec$-ordered Morse decomposition of S relative to π. It follows that $(M_p')_{p\in P}$ is a $\prec$-ordered Morse decomposition of S' relative to π'. Given $(I,J) \in \mathcal{I}_2(\prec)$, $(M(I), M(J))$ is an attractor-repeller pair in $M(IJ)$ (where $IJ = I \cup J$) relative to π, so $(M'(I), M'(J))$ is an attractor-repeller pair in $M'(IJ)$ relative to π'.

Setting, for each $K \in \mathcal{I}(\prec)$ and for each $q \in \mathbb{Z}$, $H_q(M(K)) := H_q(\pi, M(K))$, $H^q(M(K)) := H^q(\pi, M(K))$, $H_q(M'(K)) := H_q(\pi', M'(K))$, $H^q(M'(K)) := H^q(\pi', M'(K))$, $\kappa_q(K) := \kappa_q(\pi, M(K))$ and $\kappa^q(K) := \kappa^q(\pi, M(K))$ and using

Theorem 2.1 we thus arrive at the commutative diagrams

$$\begin{array}{ccccccccc}
\longrightarrow H_q(M(I)) & \longrightarrow & H_q(M(IJ)) & \longrightarrow & H_q(M(J)) & \longrightarrow & H_{q-1}(M(I)) & \longrightarrow \\
\downarrow{\scriptstyle \kappa_q(I)} & & \downarrow{\scriptstyle \kappa_q(IJ)} & & \downarrow{\scriptstyle \kappa_q(J)} & & \downarrow{\scriptstyle \kappa_{q-1}(I)} & \\
\longrightarrow H_q(M'(I)) & \longrightarrow & H_q(M'(IJ)) & \longrightarrow & H_q(M'(J)) & \longrightarrow & H_{q-1}(M'(I)) & \longrightarrow
\end{array}$$

and

$$\begin{array}{ccccccccc}
\longleftarrow H^q(M(I)) & \longleftarrow & H^q(M(IJ)) & \longleftarrow & H^q(M(J)) & \longleftarrow & H^{q-1}(M(I)) & \longleftarrow \\
\uparrow{\scriptstyle \kappa^q(I)} & & \uparrow{\scriptstyle \kappa^q(IJ)} & & \uparrow{\scriptstyle \kappa^q(J)} & & \uparrow{\scriptstyle \kappa^{q-1}(I)} & \\
\longleftarrow H^q(M'(I)) & \longleftarrow & H^q(M'(IJ)) & \longleftarrow & H^q(M'(J)) & \longleftarrow & H^{q-1}(M'(I)) & \longleftarrow
\end{array}$$

Here, the lower horizontal sequence of the first (resp. second) diagram is the homology (resp. cohomology) index sequence of $(\pi', M'(IJ), M'(I), M'(J))$ and the upper horizontal sequence of the first (resp. second) diagram is the homology (resp. cohomology) index sequence of $(\pi, M(IJ), M(I), M(J))$. We thus obtain the following result.

Theorem 2.2. *The family $(\kappa_q(J))_{q\in\mathbb{Z}}$, $J \in \mathcal{I}(\prec)$, is an isomorphism from the homology index braid of $(\pi, S, (M_p)_{p\in P})$ to the homology index braid of $(\pi', S', (M'_p)_{p\in P})$. The family $(\kappa^q(J))_{q\in\mathbb{Z}}$, $J \in \mathcal{I}(\prec)$, is an isomorphism from the cohomology index braid of $(\pi', S', (M'_p)_{p\in P})$ to the cohomology index braid of $(\pi, S, (M_p)_{p\in P})$.*

3. Ordinary differential equations on manifolds

In this section we will recall a few facts about ordinary differential equations on manifolds.

3.1. Let $\mathcal{M}$ be a differentiable manifold of class C^p ($p \geq 1$) modeled on some Banach space E. The set of all charts of $\mathcal{M}$ is denoted by $\mathrm{Chart}(\mathcal{M})$. Let $x \in \mathcal{M}$ be arbitrary. A chart $\alpha: U \to E$ of $\mathcal{M}$ is called *a chart at x* if $x \in U$. The set of all charts at x is denoted by $\mathrm{Chart}_x(\mathcal{M})$. A *tangent vector at x* is a map $\underline{u}: \mathrm{Chart}_x(\mathcal{M}) \to E$ such that for every α, $\widetilde{\alpha} \in \mathrm{Chart}_x(\mathcal{M})$

$$\underline{u}(\widetilde{\alpha}) = D(\widetilde{\alpha} \circ \alpha^{-1})(\alpha(x)).\underline{u}(\alpha).$$

The set of all tangent vectors at x is called the *tangent space to $\mathcal{M}$ at x* and is denoted by $T_x(\mathcal{M})$.

Let I be an arbitrary subset of $\mathbb{R}$ and $t_0 \in I$ be such that $t_0 \in \mathrm{Cl}_{\mathbb{R}}(I \setminus \{t_0\})$. A map $\gamma: I \to \mathcal{M}$ is called *differentiable at t_0* if γ is continuous at t_0 and for some, hence (by the chain rule) every, chart α of $\mathcal{M}$ at $x = \gamma(t_0)$ the map $\alpha \circ \gamma$ is differentiable at t_0 into E. In this case the chain rule implies that the map

$\underline{u}\colon \mathrm{Chart}_x(\mathcal{M}) \to E$, $\alpha \mapsto (\alpha \circ \gamma)'(t_0)$, is a tangent vector to $\mathcal{M}$ at x. We denote $\underline{u}$ by $\dot{\gamma}^{\mathcal{M}}(t_0)$ or simply by $\dot{\gamma}(t_0)$.

Let Y be a Banach space. A map $f\colon V \to Y$, where V is a neigbourhood of x in $\mathcal{M}$ (resp. V is open in $\mathcal{M}$), is called *differentiable at* x (resp. of class C^p) if for some, and hence every, chart α of $\mathcal{M}$ at x, the map $f \circ \alpha^{-1}$ is differentiable at $\alpha(x)$ (resp. of class C^p), as a map from the Banach space E to the Banach space Y. We then define the map $Df(x) = D^{\mathcal{M}} f(x)\colon T_x(\mathcal{M}) \to Y$ by

$$Df(x).\underline{u} = D(f \circ \alpha^{-1})(\alpha(x)).\underline{u}(\alpha), \quad \underline{u} \in T_x(\mathcal{M}).$$

It follows from the chain rule and the definition of a tangent vector that this definition is independent of the choice of $\alpha \in \mathrm{Chart}_x(\mathcal{M})$.

Let $\widetilde{\mathcal{M}}$ be differentiable manifold of class C^p modeled on a Banach space $\widetilde{E}$. A map $f\colon \mathcal{M} \to \widetilde{\mathcal{M}}$ is called *differentiable at* x (resp. *of class* C^p) if f is continuous at x (resp. f is continuous) and for every $\beta \in \mathrm{Chart}_{f(x)}(\widetilde{\mathcal{M}})$ the map $\beta \circ f$ (defined, by continuity of f at x resp. by continuity of f, on a neigbourhood of x, resp. on an open subset of $\mathcal{M}$) is differentiable at x, resp. is of class C^p. We define the map $T_x f\colon T_x(\mathcal{M}) \to T_{f(x)}(\widetilde{\mathcal{M}})$ by

$$T_x f(\underline{u}) = \underline{v}$$

where $\underline{v}(\beta) = D^{\mathcal{M}}(\beta \circ f)(x)(\underline{u})$ for $\beta \in \mathrm{Chart}_{f(x)}(\widetilde{\mathcal{M}})$.

3.2. If I, t_0 and Y are as in Subsection 3.1, $\gamma\colon I \to \mathcal{M}$ is differentiable at t_0, $\gamma(I) \subset V$, V is a neigbourhood of $x = \gamma(t_0)$ in $\mathcal{M}$ and $f\colon V \to Y$ is differentiable at x, then an application of the chain-rule shows that $\widetilde{\gamma} := f \circ \gamma$ is differentiable at t_0 as a map from $\mathbb{R}$ to Y and

$$\widetilde{\gamma}'(t_0) = D^{\mathcal{M}} f(x).\dot{\gamma}(t_0).$$

3.3. The set

$$T(\mathcal{M}) := \bigcup_{x \in \mathcal{M}} (\{x\} \times T_x(\mathcal{M}))$$

is called the *tangent bundle of* $\mathcal{M}$. If $\alpha\colon U \to E$ is a chart of $\mathcal{M}$ at x, then define the map

$$\chi_\alpha\colon \bigcup_{x \in U} (\{x\} \times T_x(\mathcal{M})) \to \alpha(U) \times E, \quad (x, \underline{u}) \mapsto (\alpha(x), \underline{u}(\alpha)).$$

The set of all the maps χ_α, $\alpha \in \mathrm{Chart}(\mathcal{M})$, is a C^{p-1}-atlas of $T(\mathcal{M})$, making $T(\mathcal{M})$ into a differentiable manifold of class C^{p-1} if $p \geq 2$ and a topological manifold if $p = 1$, modeled on the Banach space $E \times E$.

If $f\colon \mathcal{M} \to \widetilde{\mathcal{M}}$ is of class C^p then we define the map $Tf\colon T(\mathcal{M}) \to T(\widetilde{\mathcal{M}})$ by

$$Tf(x, \underline{u}) = (f(x), T_x f(\underline{u})), \quad (x, \underline{u}) \in T(\mathcal{M}).$$

It follows that Tf is of class C^{p-1}.

3.4. A map $f\colon \mathcal{M} \to \widetilde{\mathcal{M}}$ is called *locally Lipschitzian* if f is continuous and for some and hence every choice of charts $\alpha \in \mathrm{Chart}(\mathcal{M})$ and $\beta \in \mathrm{Chart}(\widetilde{\mathcal{M}})$ the map $\beta \circ f \circ \alpha^{-1}$ is locally Lipschitzian, as a map from the Banach space E to the Banach space $\widetilde{E}$.

3.5. Suppose $\mathcal{M}$ is Hausdorff, let $p \geq 2$ and F be a locally Lipschitzian vector field on $\mathcal{M}$, i.e. a locally Lipschitzian map $F\colon \mathcal{M} \to T(\mathcal{M})$ such that for every $x \in \mathcal{M}$, $F(x) = (x, F_1(x))$ where $F_1(x) \in T_x(\mathcal{M})$. $F_1(x)$ is called *the principal part of* F. By slightly modifying the proofs of [12, IV. 2, Theorems 2, 3 and 5] we can show that under these assumptions the initial value problem for the *ordinary differential equation*

$$\dot{x} = F_1(x) \tag{3.1}$$

generated by F on $\mathcal{M}$ is well-posed. This means that for every $x_0 \in \mathcal{M}$ there are uniquely determined numbers $\alpha_{x_0} \in [-\infty, 0[$ and $\omega_{x_0} \in]0, \infty]$ and a unique, maximally defined differentiable function $x_{x_0}(\,\cdot\,)\colon]\alpha_{x_0}, \omega_{x_0}[\ \to \mathcal{M}$, $t \mapsto x_{x_0}(t)$ such that

$$\dot{x}_{x_0}(t) = F_1(x_{x_0}(t)), \quad t \in]\alpha_{x_0}, \omega_{x_0}[$$

with $x_{x_0}(0) = x_0$. Moreover, the set

$$\Omega_\Pi = \bigcup_{x_0 \in \mathcal{M}}]\alpha_{x_0}, \omega_{x_0}[\times \{x_0\}$$

is open in $\mathbb{R} \times \mathcal{M}$ and the map $\Pi\colon \Omega_\Pi \to \mathcal{M}$, $(t, x_0) \mapsto x_{x_0}(t)$ is continuous. Π is a local flow on $\mathcal{M}$ and $\pi := \Pi_{|(\Omega_\Pi \cap ([0,\infty[\times \mathcal{M}))}$ is a local semiflow on $\mathcal{M}$. Π, resp. π is called *the local flow*, resp. *the local semiflow, generated by* (3.1). We write $x_0\Pi t$ (resp. $x_0\pi t$) instead of $\Pi(t, x_0)$ (resp. $(\pi(t, x_0))$).

3.6. Let Y be a Banach space and Z_0 be an open set in the product manifold $Y \times \mathcal{M}$. Then Z_0 has a canonical structure of differentiable manifold of class C^p. Suppose $f\colon Z_0 \to Y$ and $h\colon Z_0 \to T(\mathcal{M})$ are locally Lipschitzian maps such that for all $(y, x) \in Z_0$, $h(y, x) = (y, h_1(y, x))$, where $h_1(y, x) \in T_x(\mathcal{M})$. Then there is a unique locally Lipschitzian vector field F on the manifold Z_0 such that for every $(y, x) \in Z_0$ and every chart β of Z_0 at (y, x) of the form $\beta = \mathrm{id}_U \times \alpha$, with U open in Y, $y \in U$ and $\alpha \in \mathrm{Chart}_x(\mathcal{M})$, the principle part $F_1(y, x)$ of $F(y, x)$ has the form

$$F_1(y, x)(\beta) = (f(y, x), h_1(y, x)(\alpha)).$$

Thus Subsection 3.5 implies that the ordinary differential equation

$$\begin{aligned} \dot{y} &= f(y, x) \\ \dot{x} &= h_1(y, x) \end{aligned}$$

regarded, by definition, as the ordinary differential equation generated by F on Z_0, generates a local (semi)flow on Z_0.

4. A singular perturbation result

Consider the following hypotheses:

Hypothesis 4.1.

(a) *Y is a finite dimensional normed linear space, $\mathcal{M}$ is a finite dimensional (boundaryless) second countable paracompact differentiable manifold of class C^2, $\overline{\varepsilon} \in]0,\infty[$ is arbitrary, Z_0 is open in $Y \times \mathcal{M}$ and $W_0 := Z_0 \times [0,\overline{\varepsilon}]$.*

(b) *$f\colon W_0 \to Y$ and $h\colon W_0 \to T(\mathcal{M})$ are maps such that, for each $\varepsilon \in]0,\overline{\varepsilon}]$, $f(\,\cdot\,,\varepsilon)$ and $h(\,\cdot\,,\varepsilon)$ are locally Lipschitzian.*

(c) *For $((y,x),\varepsilon) \in W_0$, $h((y,x),\varepsilon) = (x, h_1((y,x),\varepsilon))$ with $h_1((y,x),\varepsilon) \in T_x(\mathcal{M})$.*

(d) *$\phi\colon \mathcal{M} \to Y$ is a C^2-map such that for all $x \in \mathcal{M}$, $(\phi(x),x) \in Z_0$ and $f((\phi(x),x),0) = 0$.*

(e) *The map $f(\,\cdot\,,0)$ is of class C^2 and the map $h(\,\cdot\,,0)$ is locally Lipschitzian.*

(f) *For every $(y,x) \in Z_0$ the map f is continuous at $((y,x),0)$ and for every $x \in \mathcal{M}$, the map h is continuous at $((\phi(x),x),0)$.*

Hypothesis 4.2. *a_0, $b_0 \in \mathbb{R}$ are such that $a_0 < 0 < 1 < b_0$ and $B\colon \mathcal{M} \times]a_0,b_0[\to \mathcal{L}(Y,Y)$ is a locally Lipschitzian map such that $B(x,\lambda)$ is hyperbolic for every $(x,\lambda) \in \mathcal{M} \times [0,1]$, $B(x,0) = Df((\phi(x),x),0)$ and $B(x,1) = \overline{B}$ for every $x \in \mathcal{M}$, where $\overline{B} \in \mathcal{L}(Y,Y)$ has Morse-index $k \in \mathbb{N}_0$.*

Here, for normed spaces Z_1 and Z_2, $\mathcal{L}(Z_1,Z_2)$ is the normed space of all bounded linear maps from Z_1 to Z_2.

By Subsection 3.6, for every $\varepsilon \in]0,\overline{\varepsilon}]$, the ordinary differential equation

$$\begin{aligned} \varepsilon\dot{y} &= f((y,x),\varepsilon), \\ \dot{x} &= h_1((y,x),\varepsilon). \end{aligned} \tag{4.1}$$

generates a local (semi)flow π_ε on Z_0.

In the same way the ordinary differential equation

$$\dot{x} = h_1((\phi(x),x),0). \tag{4.2}$$

generates a local (semi)flow π_0 on $\mathcal{M}$.

Given $M \subset \mathcal{M}$ and $\eta \in]0,\infty[$ define

$$[M]^\phi_\eta := \{(y,x) \in Z_0 \mid\ x \in M \text{ and } |y - \phi(x)|_Y \le \eta\}.$$

We can now state the main result of this paper.

Theorem 4.3. *Assume Hypotheses* 4.1 *and* 4.2. *Let $S_0 \subset \mathcal{M}$ be a compact isolated invariant set relative to π_0 and $N \subset \mathcal{M}$ be a compact isolating neigbourhood of S_0. Then there is an $\eta_0 \in]0,\infty[$ such that for every $\eta \in]0,\eta_0]$, there exists an $\varepsilon_0 = \varepsilon_0(\eta) \in]0,\overline{\varepsilon}]$ such that for every $\varepsilon \in]0,\varepsilon_0]$, the set $[N]^\phi_\eta$ is an isolating neigbourhood relative to π_ε and*

$$h(\pi_\varepsilon, S_\varepsilon) = \Sigma^k \wedge h(\pi_0, S_0),$$

where $S_\varepsilon = S_{\varepsilon,N,\eta} := \mathrm{Inv}_{\pi_\varepsilon}([N]^\phi_\eta)$. In addition, let $(M_p)_{p\in P}$ be a $\prec$-ordered Morse decomposition for S_0 relative to π_0. For each $p \in P$, let $V_p \subset N$ be an isolating neigbourhood of M_p relative to π_0. For every $\eta \in]0,\infty[$, every $\varepsilon \in]0,\overline{\varepsilon}]$ and every $p \in P$, define

$$M_{p,\varepsilon} = M_{p,\varepsilon,V_p,\eta} := \mathrm{Inv}_{\pi_\varepsilon}([V_p]^\phi_\eta).$$

Then, for every $\eta \in]0,\eta_0]$, there is an $\overline{\varepsilon}_0 = \overline{\varepsilon}_0(\eta) \in]0,\overline{\varepsilon}]$ such that for every $\varepsilon \in]0,\overline{\varepsilon}_0]$, the family $(M_{p,\varepsilon})_{p\in P}$ is a $\prec$-ordered Morse decomposition for S_ε relative to π_ε. For every $\varepsilon \in]0,\overline{\varepsilon}_0]$ and for every $K \in \mathcal{I}(\prec)$, set

$$M_\varepsilon(K) := \bigcup_{(p,q)\in K\times K} \mathrm{CS}_{\pi_\varepsilon}(M_{p,\varepsilon}, M_{q,\varepsilon}).$$

Then for every $\varepsilon \in]0,\overline{\varepsilon}_0]$, for every $K \in \mathcal{I}(\prec)$ and for every $q \in \mathbb{Z}$, there exist isomorphisms

$$\Theta^\varepsilon_q(K)\colon H_q(\pi_\varepsilon, M_\varepsilon(K)) \to H_{q-k}(\pi_0, M(K))$$

and

$$\Theta_\varepsilon^q(K)\colon H^{q-k}(\pi_0, M(K)) \to H^q(\pi_\varepsilon, M_\varepsilon(K))$$

such that given $(I,J) \in \mathcal{I}_2(\prec)$ the following diagrams

$$\begin{array}{ccccccccc}
\longrightarrow H_q(M_\varepsilon(I)) & \longrightarrow & H_q(M_\varepsilon(IJ)) & \longrightarrow & H_q(M_\varepsilon(J)) & \longrightarrow & H_{q-1}(M_\varepsilon(I)) \longrightarrow \\
\downarrow \Theta^\varepsilon_q(I) & & \downarrow \Theta^\varepsilon_q(IJ) & & \downarrow \Theta^\varepsilon_q(J) & & \downarrow \Theta^\varepsilon_{q-1}(I) \\
\to H_{q-k}(M(I)) & \to & H_{q-k}(M(IJ)) & \to & H_{q-k}(M(J)) & \to & H_{q-k-1}(M(I)) \to
\end{array}$$

$$\begin{array}{ccccccccc}
\longleftarrow H^q(M_\varepsilon(I)) & \longleftarrow & H^q(M_\varepsilon(IJ)) & \longleftarrow & H^q(M_\varepsilon(J)) & \longleftarrow & H^{q-1}(M_\varepsilon(I)) \longleftarrow \\
\Theta_\varepsilon^q(I) \uparrow & & \Theta_\varepsilon^q(IJ) \uparrow & & \Theta_\varepsilon^q(J) \uparrow & & \Theta_\varepsilon^{q-1}(I) \uparrow \\
\leftarrow H^{q-k}(M(I)) & \leftarrow & H^{q-k}(M(IJ)) & \leftarrow & H^{q-k}(M(J)) & \leftarrow & H^{q-k-1}(M(I)) \leftarrow
\end{array}$$

commute, where for every $K \in \mathcal{I}(\prec)$, for every $\varepsilon \in]0,\overline{\varepsilon}_0]$ and for every $q \in \mathbb{Z}$, $H_q(M(K)) := H_q(\pi_0, M(K))$, $H^q(M(K)) := H^q(\pi_0, M(K))$, $H_q(M_\varepsilon(K)) :=$

$H_q(\pi_\varepsilon, M_\varepsilon(K))$ and $H^q(M_\varepsilon(K)) := H^q(\pi_\varepsilon, M_\varepsilon(K))$. Thus, the (co)homology index braid of $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P})$ is isomorphic to the graded module braid obtained by shifting the (co)homology index braid of $(\pi_0, S_0, (M_p)_{p\in P})$ to the left by k.

The proof of Theorem 4.3 requires various auxiliary results.

Proposition 4.4. *Let $\widetilde{Z}_0$ be the set of all $(u,x) \in Y \times \mathcal{M}$ such that $(u+\phi(x), x) \in Z_0$. Then $\widetilde{Z}_0$ is open in $Y \times \mathcal{M}$. The map $\Phi\colon Z_0 \to \widetilde{Z}_0$ defined by $\Phi(y,x) = (u,x) := (y - \phi(x), x)$ is a C^2-diffeomorphism with inverse $\Phi^{-1}\colon \widetilde{Z}_0 \to Z_0$ given by $\Phi^{-1}(u,x) = (y,x) := (u + \phi(x), x)$. For $\varepsilon \in \,]0, \overline{\varepsilon}]$, let $\widetilde{\pi}_\varepsilon$ be the conjugate of π_ε via Φ i.e.*

$$(u,x)\widetilde{\pi}_\varepsilon t := \Phi((\Phi^{-1}(u,x))\pi_\varepsilon t),$$

where $(u,x) \in \widetilde{Z}_0$ and $t \in [0,\infty[$ is such that $(\Phi^{-1}(u,x))\pi_\varepsilon t$ is defined. Then $\widetilde{\pi}_\varepsilon$ is the local (semi)flow generated on $\widetilde{Z}_0$ by the equation

$$\begin{aligned} \varepsilon\dot{u} &= \widetilde{f}((u,x),\varepsilon), \\ \dot{x} &= \widetilde{h}_1((u,x),\varepsilon), \end{aligned} \tag{4.3}$$

where, for $((u,x),\varepsilon) \in \widetilde{W_0} := \widetilde{Z}_0 \times [0,\overline{\varepsilon}]$,

$$\begin{aligned} \widetilde{f}((u,x),\varepsilon) &= f((u+\phi(x),x),\varepsilon) - \varepsilon D^{\mathcal{M}}\phi(x)h_1((u+\phi(x),x),\varepsilon), \\ \widetilde{h}_1((u,x),\varepsilon) &= h_1((u+\phi(x),x),\varepsilon). \end{aligned}$$

Proof. This is a simple calculation using Subsections 3.2 and 3.6. □

Remark 4.5. Since semiflow conjugation leads to the same Conley-index and isomorphic (co)homology index braids (cf. [15, Proposition II.3.2], [4, Theorem 3.2] and [17, Theorem 4.2]), it follows from Proposition 4.4 that we may and will assume without loss of generality that $\phi = 0$ in Hypothesis 4.1. We will also write $[M]_\eta$ for $[M]_\eta^\phi$, i.e.

$$[M]_\eta := \{(y,x) \in Z_0 \mid x \in M \text{ and } |y|_Y \le \eta\}.$$

Our hypotheses on $\mathcal{M}$ and Whitney Imbedding Theorem imply that there is a finite dimensional normed space $\mathbf{E}$ and an imbedding $\mathbf{e}\colon \mathcal{M} \to \mathbf{E}$ of class C^2. We define the metric $d_{\mathcal{M}}$ on $\mathcal{M}$ such that $\mathbf{e}$ is an isometry.

Let $\beta = \mathrm{id}_{\mathbf{E}}$ and $\chi_\beta\colon T(\mathbf{E}) \to \mathbf{E}\times\mathbf{E}$ be as in Subsection 3.3. It follows that χ_β is of class C^∞ and so $\chi_\beta \circ T\mathbf{e}\colon T(\mathcal{M}) \to \mathbf{E}\times\mathbf{E}$ is of class C^1. In particular, $\chi_\beta \circ T\mathbf{e}$ is continuous. Moreover, Subsections 3.1 and 3.3 imply that, for $(x,\underline{u}) \in T(\mathcal{M})$

$$\begin{aligned} \chi_\beta T\mathbf{e}(x,\underline{u}) &= \chi_\beta(\mathbf{e}(x), T_x\mathbf{e}(\underline{u})) \\ &= (\beta\mathbf{e}(x), D^{\mathcal{M}}(\beta\circ\mathbf{e})(x)(\underline{u})) = (\mathbf{e}(x), D^{\mathcal{M}}\mathbf{e}(x)(\underline{u})). \end{aligned}$$

It follows that the map $\Gamma\colon T(\mathcal{M}) \to \mathbf{E}$, $(x,\underline{u}) \mapsto D^{\mathcal{M}}\mathbf{e}(x)(\underline{u})$, is continuous.

Proposition 4.6. *Let* $g\colon W_0 \to T(\mathcal{M})$ *be a map such that*

(a) *for each* $\varepsilon \in]0,\overline{\varepsilon}]$, $g(\,\cdot\,,\varepsilon)$ *is continuous,*
(b) g *is continuous at* $((0,x),0)$ *for every* $x \in \mathcal{M}$,
(c) *for each* $((u,x),\varepsilon) \in W_0$,

$$g((u,x),\varepsilon) = (x, g_1((u,x),\varepsilon)) \text{ with } g_1((u,x),\varepsilon) \in T_x(\mathcal{M}).$$

Let M *be compact in* $\mathcal{M}$. *Then there is an* $\eta_1' \in]0,\infty[$ *and an* $\varepsilon' \in]0,\overline{\varepsilon}]$ *such that* $[M]_{\eta_1'} \subset Z_0$ *and*

$$\sup\{|\Gamma(g((u,x),\varepsilon))|_{\mathbf{E}} \mid |u|_Y \le \eta_1',\ x \in M,\ \varepsilon \in]0,\varepsilon']\} < \infty.$$

For each $n \in \mathbb{N}$, *let* $\varepsilon_n \in]0,\varepsilon']$, a_n, $b_n \in [0,1]$, $u_n\colon \mathbb{R} \to Y$ *and* $x_n\colon \mathbb{R} \to M$ *be such that* $\varepsilon_n \to 0$, $\sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} |u_n(t)|_Y \le \eta_1'$ *and for every* $n \in \mathbb{N}$, x_n *is differentiable into* $\mathcal{M}$ *and* $((u_n(t), x_n(t)), \varepsilon_n) \in W_0$. *Moreover, assume that one of the following alternatives holds:*

(i) $\lim_{n\to\infty} u_n(t) = 0$ *for all* $t \in \mathbb{R}$ *and* $\dot{x}_n(t) = g_1((a_n u_n(t), x_n(t)), b_n\varepsilon_n)$ *for all* $n \in \mathbb{N}$ *and* $t \in \mathbb{R}$;
(ii) $\dot{x}_n(t) = \varepsilon_n g_1((a_n u_n(t), x_n(t)), b_n\varepsilon_n)$ *for all* $n \in \mathbb{N}$ *and* $t \in \mathbb{R}$.

Under these assumptions there is a subsequence of $(x_n)_n$ *which converges in* $(\mathcal{M}, d_{\mathcal{M}})$, *uniformly on compact subsets of* $\mathbb{R}$, *to a function* $x\colon \mathbb{R} \to M$ *which is differentiable into* $\mathcal{M}$ *and such that, in case* (i),

$$\dot{x}(t) = g_1((0,x(t)),0), \quad t \in \mathbb{R}$$

and, in case (ii),

$$\dot{x}(t) = 0, \quad t \in \mathbb{R}.$$

Proof. Assumption (b) and compactness of M imply the existence of η_1' and ε' with the desired properties. Set $y_n = \mathbf{e} \circ x_n$ for $n \in \mathbb{N}$. By Subsection 3.2 we have that, for each $n \in \mathbb{N}$, y_n is differentiable into $\mathbf{E}$ and, in case (i),

$$\begin{aligned} y_n'(t) &= D^{\mathcal{M}}\mathbf{e}(x_n(t)).g_1((a_n u_n(t), x_n(t)), b_n\varepsilon_n) \\ &= \Gamma(g((a_n u_n(t), x_n(t)), b_n\varepsilon_n)), \quad t \in \mathbb{R}, \end{aligned}$$

while in case (ii)

$$\begin{aligned} y_n'(t) &= D^{\mathcal{M}}\mathbf{e}(x_n(t)).\varepsilon_n g_1((a_n u_n(t), x_n(t)), b_n\varepsilon_n) \\ &= \varepsilon_n \Gamma(g((a_n u_n(t), x_n(t)), b_n\varepsilon_n)), \quad t \in \mathbb{R}. \end{aligned}$$

By our assumptions,

$$\sup_{n\in\mathbb{N}} \sup_{t\in\mathbb{R}} |\Gamma(g((a_n u_n(t), x_n(t)), b_n\varepsilon_n))|_{\mathbf{E}} < \infty. \tag{4.4}$$

This together with the fact that all functions y_n lie in the compact set $\mathbf{e}(M)$ implies, by Arzelà–Ascoli Theorem, that there is a subsequence of $(y_n)_n$ again denoted by $(y_n)_n$ and a continuous function $y:\mathbb{R}\to\mathbf{E}$ such that $(y_n)_n$ converges to y in $\mathbf{E}$, uniformly on compact subsets of $\mathbb{R}$. For every $t\in\mathbb{R}$ a subsequence of $(x_n(t))_n$ (depending on t) converges to some point $x(t)\in\mathcal{M}$ (as M is compact in the metric space $(\mathcal{M},d_{\mathcal{M}})$). Continuity of $\mathbf{e}$ implies that $\mathbf{e}(x(t))=y(t)$. Since $\mathbf{e}$ is a homeomorphism of $\mathcal{M}$ onto the topological subspace $\mathbf{e}(\mathcal{M})$ of $\mathbf{E}$, it follows that $x:\mathbb{R}\to\mathcal{M}$ is defined and continuous and $\mathbf{e}\circ x=y$. Moreover, $x_n(t)\to x(t)$ in $(\mathcal{M},d_{\mathcal{M}})$, uniformly for t lying in compact subsets of $\mathbb{R}$. Thus, for each $t\in\mathbb{R}$, $\Gamma(g((a_nu_n(t),x_n(t)),b_n\varepsilon_n))\to\Gamma(g((0,x(t)),0))$. Together with (4.4) this implies that for all t, $t_0\in\mathbb{R}$

$$y_n(t)-y_n(t_0)=\int_{t_0}^t\Gamma(g((a_nu_n(s),x_n(s)),b_n\varepsilon_n))\,ds\to\int_{t_0}^t\Gamma(g((0,x(s)),0))\,ds$$

in case (i) and

$$y_n(t)-y_n(t_0)=\int_{t_0}^t\varepsilon_n\Gamma(g((a_nu_n(s),x_n(s)),b_n\varepsilon_n))\,ds\to 0$$

in case (ii). Thus

$$y(t)-y(t_0)=\int_{t_0}^t\Gamma(g((0,x(s)),0))\,ds,\quad t,t_0\in\mathbb{R}$$

in case (i) and

$$y(t)-y(t_0)=0,\quad t,t_0\in\mathbb{R}$$

in case (ii). It follows that y is differentiable into $\mathbf{E}$ and

$$y'(t)=\Gamma(g((0,x(t)),0))=D^{\mathcal{M}}\mathbf{e}(x(t))(g_1((0,x(t)),0)),\quad t\in\mathbb{R}\tag{4.5}$$

in case (i) and

$$y'(t)=0,\quad t\in\mathbb{R}\tag{4.6}$$

in case (ii).

Since $\mathbf{e}(\mathcal{M})$ is a C^2-submanifold of $\mathbf{E}$, it follows that y is differentiable into $\mathbf{e}(\mathcal{M})$ and since $\mathbf{e}$ is a C^2-diffeomorphism from $\mathcal{M}$ to $\mathbf{e}(\mathcal{M})$ it follows that x is differentiable into $\mathcal{M}$. By Subsection 3.2

$$y'(t)=D^{\mathcal{M}}\mathbf{e}(x(t))(\dot{x}(t)),\quad t\in\mathcal{M}.\tag{4.7}$$

Since $\mathbf{e}$ is an imbedding, it follows that, for every $x\in\mathcal{M}$, the map $T_x\mathbf{e}:T_x(\mathcal{M})\to T_{\mathbf{e}(x)}(\mathbf{E})$ is injective. Since $D^{\mathcal{M}}\mathbf{e}(x)(\underline{u})=(T_x\mathbf{e}(\underline{u}))(\beta)$ for all $\underline{u}\in T_x(\mathcal{M})$ (where, as before, $\beta=\mathrm{id}_{\mathbf{E}}$) it follows that the map $D^{\mathcal{M}}\mathbf{e}(x):T_x(\mathcal{M})\to\mathbf{E}$ is injective. Thus (4.5), (4.7) and (4.6) imply that

$$\dot{x}(t)=g_1((0,x(t)),0),\quad t\in\mathbb{R}$$

in case (i) and, in case (ii),

$$\dot{x}(t) = 0, \quad t \in \mathbb{R}. \qquad \square$$

For each $\varepsilon \in \,]0, \overline{\varepsilon}]$ and $\lambda \in [0,1]$, by Subsection 3.6, the solutions of the differential equation

$$\begin{aligned} \varepsilon\dot{u} &= (1-\lambda)(f((u,x),\varepsilon) - Df((0,x),0)u) + B(x,\lambda)u, \\ \dot{x} &= h_1(((1-\lambda)u, x), (1-\lambda)\varepsilon) \end{aligned} \tag{4.8}$$

generate a local (semi)flow $\pi'_{\varepsilon,\lambda}$ on Z_0.

Proposition 4.7. *Let $\varepsilon \in \,]0, \overline{\varepsilon}]$ be arbitrary and $(\lambda_n)_n$ be an arbitrary sequence in $[0,1]$ converging to some $\lambda \in [0,1]$. Then $\pi'_{\varepsilon,\lambda_n} \to \pi'_{\varepsilon,\lambda}$ as $n \to \infty$.*

Proof. Consider the differential equation

$$\begin{aligned} \varepsilon\dot{u} &= (1-\lambda)(f((u,x),\varepsilon) - Df((0,x),0)u) + B(x,\lambda)u, \\ \dot{x} &= h_1(((1-\lambda)u, x), (1-\lambda)\varepsilon), \\ \dot{\lambda} &= 0. \end{aligned} \tag{4.9}$$

Since the right hand side of (4.9) defines a locally Lipschitzian vector field on $Z_0 \times \,]a_0, b_0[$, it follows that the solutions of (4.9) generate a local (semi)flow $\Pi = \Pi_\varepsilon$ on $Z_0 \times \,]a_0, b_0[$. The definition of Π_ε shows that, for all $((u,x),\lambda) \in Z_0 \times [0,1]$ and all $t \in [0,\infty[$, $((u,x),\lambda)\Pi t$ is defined if and only if $(u,x)\pi'_{\varepsilon,\lambda}t$ is defined and then $((u,x),\lambda)\Pi t = ((u,x)\pi'_{\varepsilon,\lambda}t, \lambda)$. Now continuity of Π and openness of the domain of definition of Π in $Z \times \mathbb{R}$ imply the assertion of the proposition. $\square$

Define the maps $T_1 \colon W_0 \to Y$ and $T_2 \colon Z_0 \to Y$ by

$$T_1((u,x),\varepsilon) = f((u,x),\varepsilon) - f((u,x),0), \quad ((u,x),\varepsilon) \in W_0$$

and

$$T_2(u,x) = f((u,x),0) - f((0,x),0) - Df((0,x),0)(u), \quad (u,x) \in Z_0.$$

Since $f((0,x),0) = 0$ for all $x \in \mathcal{M}$ it follows that

$$f((u,x),\varepsilon) = T_1((u,x),\varepsilon) + T_2(u,x) + Df((0,x),0)(u), \quad ((u,x),\varepsilon) \in W_0.$$

Lemma 4.8. *Let M be compact in $\mathcal{M}$. Then there is an $\eta'_2 \in \,]0,\infty[$ such that $[M]_{\eta'_2} \subset Z_0$ and whenever $x \in M$, $\lambda \in [0,1]$ and $u \colon \mathbb{R} \to Y$ is a solution of the equation*

$$\dot{u} = (1-\lambda)T_2(u,x) + B(x,\lambda)u$$

lying in $[M]_{\eta_2'}$*, then* $u \equiv 0$.

Proof. Since M is compact in $\mathcal{M}$ and Z_0 is open in $Y \times \mathcal{M}$, there exists an $\overline{\eta} = \overline{\eta}(M) \in \,]0, \infty[$ such that $[M]_{\overline{\eta}} \subset Z_0$. For each $\eta \in \,]0, \overline{\eta}]$ define

$$C(\eta) = \sup_{(u,x)\in[M]_\eta} |T_2(u,x)|_Y.$$

The mean-value theorem implies that

$$\lim_{\eta\to 0^+} C(\eta)/\eta = 0. \tag{4.10}$$

If the lemma does not hold, then there are sequences $(\eta_n)_n$, $(x_n)_n$ and $(\lambda_n)_n$ in $]0, \overline{\eta}]$, M and $[0,1]$, respectively, with $\eta_n \to 0$, and there is a sequence $(u_n)_n$ such that for each $n \in \mathbb{N}$, $u_n\colon \mathbb{R} \to Y$ is differentiable,

$$\dot{u}_n(t) = (1-\lambda_n)T_2(u_n(t), x_n) + B(x_n, \lambda_n)u_n(t), \quad t \in \mathbb{R}$$

and $|u_n(0)|_Y = \eta_n$. We may assume that $x_n \to x \in M$ and $\lambda_n \to \lambda \in [0,1]$ as $n \to \infty$. Set $v_n := u_n/\eta_n$, $n \in \mathbb{N}$. Then

$$\sup_{t\in\mathbb{R}} |\dot{v}_n(t)|_Y \le C(\eta_n)/\eta_n + \sup_{n\in\mathbb{N}} |B(x_n,\lambda_n)|_{\mathcal{L}(Y,Y)}. \tag{4.11}$$

Since, for each $t \in \mathbb{R}$, $\{v_n(t) \mid n \in \mathbb{N}\}$ lies in a compact subset of Y, using (4.10), (4.11) and Arzelà–Ascoli theorem we see that a subsequence of $(v_n)_n$, again denoted by $(v_n)_n$, converges, uniformly on compact subsets of $\mathbb{R}$, to a bounded function $v\colon \mathbb{R} \to Y$ which is differentiable and such that

$$\dot{v}(t) = B(x,\lambda)v(t), \quad t \in \mathbb{R}.$$

Since $B(x,\lambda)$ is hyperbolic, it follows that $v \equiv 0$. However, $|v(0)| = 1$, a contradiction which proves the lemma. □

Let $M \subset \mathcal{M}$ be compact and $\overline{\eta} = \overline{\eta}(M) \in \,]0,\infty[$ be such that $[M]_{\overline{\eta}} \subset Z_0$. Let $\mathcal{T}_0(M)$ be the set of functions $\sigma\colon \mathbb{R} \to Y \times \mathcal{M}$ such that $\sigma(t) = (0, x(t))$, $t \in \mathbb{R}$ where x is a full solution of π_0 lying in $\mathrm{Inv}_{\pi_0}(M)$. Moreover, for $\eta \in \,]0, \overline{\eta}]$, $\varepsilon \in \,]0, \overline{\varepsilon}]$ and $\lambda \in [0,1]$, let $\mathcal{T}'(M, \eta, \varepsilon, \lambda)$ be the set of all full solutions of $\pi'_{\varepsilon,\lambda}$ lying in $\mathrm{Inv}_{\pi'_{\varepsilon,\lambda}}([M]_\eta)$. Since $\mathrm{Inv}_{\pi_0}(M)$ and $\mathrm{Inv}_{\pi'_{\varepsilon,\lambda}}([M]_\eta)$ are compact in $\mathcal{M}$ and $Y \times \mathcal{M}$ respectively, it follows from [2, Proposition 2.7] that

Lemma 4.9. *The set* $\mathcal{T}_0(M)$ *is compact in* $C(\mathbb{R} \to Y \times \mathcal{M})$ *and translation and cut-and-glue invariant. Moreover, for* $\eta \in \,]0, \overline{\eta}]$*,* $\varepsilon \in \,]0, \overline{\varepsilon}]$ *and* $\lambda \in [0,1]$*, the set* $\mathcal{T}'(M, \eta, \varepsilon, \lambda)$ *is compact in* $C(\mathbb{R} \to Y \times \mathcal{M})$ *and translation and cut-and-glue invariant.*

Proposition 4.10. *Let M be compact in $\mathcal{M}$. Then there is an $\eta' = \eta'(M) \in]0, \overline{\eta}(M)]$ such that whenever $\eta \in]0, \eta']$, $(\varepsilon_\kappa)_\kappa$ is a sequence in $]0, \overline{\varepsilon}]$ converging to* 0 *and $(\lambda_\kappa)_\kappa$ is an arbitrary sequence in* $[0, 1]$ *then $\mathcal{T}_\kappa \to \mathcal{T}_0 = \mathcal{T}_0(M)$, where*

$$\mathcal{T}_\kappa = \mathcal{T}'(M, \eta, \varepsilon_\kappa, \lambda_\kappa), \quad \kappa \in \mathbb{N}.$$

Proof. Let $\eta' = \max(\eta'_1, \eta'_2)$, where η'_1 and ε' are as in Proposition 4.6 with $g = h$ and η'_2 is as in Lemma 4.8. Let $\eta \in]0, \eta']$ be arbitrary. It is enough to prove that whenever $(\varepsilon_n)_n$ is a sequence in $]0, \varepsilon']$ converging to 0, $(\lambda_n)_n$ is a sequence in $[0, 1]$ converging to $\lambda \in [0, 1]$ and $(\sigma_n)_n$ is a sequence such that, for each $n \in \mathbb{N}$, σ_n is a full solution of $\pi'_{\varepsilon_n, \lambda_n}$ lying in $[M]_\eta$ and $\sigma_n(t) =: (u_n(t), x_n(t))$, $t \in \mathbb{R}$ then (i) $(u_n)_n$ converges to $u \equiv 0$ in Y, uniformly on $\mathbb{R}$ and (ii) $(x_n)_n$ has a subsequence converging in $(\mathcal{M}, d_{\mathcal{M}})$, uniformly on compact subsets of $\mathbb{R}$, to a full solution of π_0 lying in M.

Suppose (i) is not true. Then by translation invariance and passing to a subsequence if necessary, we may assume that there is a $\delta \in]0, \infty[$ such that $|u_n(0)|_Y \geq \delta$ for all $n \in \mathbb{N}$. Define functions $v_n: \mathbb{R} \to Y$ and $\xi_n: \mathbb{R} \to \mathcal{M}$, $n \in \mathbb{N}$, by

$$v_n(t) = u_n(\varepsilon_n t), \quad \xi_n(t) = x_n(\varepsilon_n t), \quad t \in \mathbb{R}.$$

It follows that

$$\dot{\xi}_n(t) = \varepsilon_n h_1(((1 - \lambda_n) v_n(t), \xi_n(t)), (1 - \lambda_n)\varepsilon_n), \quad n \in \mathbb{N},\ t \in \mathbb{R}.$$

An application of Proposition 4.6 (with $g = h$) shows that, by passing to subsequences if necessary, we may assume that $(\xi_n)_n$ converges $(\mathcal{M}, d_{\mathcal{M}})$, uniformly on compact subsets of $\mathbb{R}$, to a constant $\overline{\xi} \in M$. We also have that

$$(4.12) \quad \dot{v}_n(t) = (1 - \lambda_n) T_1((v_n(t), \xi_n(t)), \varepsilon_n) + (1 - \lambda_n) T_2(v_n(t), \xi_n(t)) + B(\xi_n(t), \lambda_n) v_n(t), \quad t \in \mathbb{R}.$$

By our assumptions

$$(4.13) \quad \lim_{\varepsilon \to 0^+} \sup_{(u,x) \in [M]_\eta} |T_1((u, x), \varepsilon)|_Y = 0.$$

Since, for each $t \in \mathbb{R}$, $\{v_n(t) \mid n \in \mathbb{N}\}$ lies in a compact subset of Y, it follows from (4.13), (4.12) and Arzelà–Ascoli Theorem, passing to subsequences if necessary, that $(v_n)_n$ converges in Y, uniformly on compact subsets of $\mathbb{R}$ to a function $v: \mathbb{R} \to Y$ which is is differentiable into Y and

$$\dot{v}(t) = (1 - \lambda) T_2(v(t), \overline{\xi}) + B(\overline{\xi}, \lambda) v(t), \quad t \in \mathbb{R}.$$

It follows from Lemma 4.8 that $v = 0$, a contradiction as $|v(0)|_Y \geq \delta$. This shows that (i) is satisfied.

Now (i) and an application of Proposition 4.6 with $g = h$ shows that there is a subsequence of $(x_n)_n$ which converges in $(\mathcal{M}, d_{\mathcal{M}})$, uniformly on compact subsets of $\mathbb{R}$, to a function $x:\mathbb{R} \to M$ which is differentiable into $\mathcal{M}$ and such that

$$\dot{x}(t) = h_1((0, x(t)), 0), \quad t \in \mathbb{R}.$$

Thus x is a full solution of π_0 lying in M. This proves (ii). □

Proposition 4.10 has several important corollaries.

Corollary 4.11. *Let M be compact in $\mathcal{M}$, $\eta'(M)$ be as in Proposition 4.10 and η_1, $\eta_2 \in]0, \eta'(M)]$ be arbitrary. Then there is an $\widehat{\varepsilon} \in]0, \overline{\varepsilon}]$ such that for all $\varepsilon \in]0, \widehat{\varepsilon}]$ and all $\lambda \in [0, 1]$*

$$\mathcal{T}'(M, \eta_1, \varepsilon, \lambda) = \mathcal{T}'(M, \eta_2, \varepsilon, \lambda).$$

Proof. Suppose e.g. that $\eta_1 \leq \eta_2$. If the corollary is not true then we may assume that there is a sequence $(\varepsilon_\kappa)_\kappa$ in $]0, \overline{\varepsilon}]$ converging to zero, there is a sequence $(\lambda_\kappa)_\kappa$ in $[0, 1]$ and there is a sequence $(\sigma_\kappa)_\kappa$ with $\sigma_\kappa \in \mathcal{T}'(M, \eta_2, \varepsilon_\kappa, \lambda_\kappa)$ and $\sigma_\kappa(0) \notin [M]_{\eta_1}$ for all $\kappa \in \mathbb{N}$. If $(u_\kappa(t), x_\kappa(t)) := \sigma_\kappa(t)$ for $\kappa \in \mathbb{N}$ and $t \in \mathbb{R}$ then it follows that $|u_\kappa(0)| > \eta_1$ for all $\kappa \in \mathbb{N}$. However, Proposition 4.10 implies that a subsequence of $(u_\kappa)_\kappa$ converges to zero in Y. This contradiction proves the corollary. □

Corollary 4.12. *Let M_1, M_2 be compact in $\mathcal{M}$, $\eta'(M_1)$, $\eta'(M_2)$ be as in Proposition 4.10 and η_1, $\eta_2 \in]0, \min(\eta'(M_1), \eta'(M_2))]$ be arbitrary. Suppose that both M_1 and M_2 are isolating neigbourhoods of the same isolated invariant set S relative to π_0. Then there is an $\widehat{\varepsilon} \in]0, \overline{\varepsilon}]$ such that for all $\varepsilon \in]0, \widehat{\varepsilon}]$ and all $\lambda \in [0, 1]$*

$$\mathcal{T}'(M_1, \eta_1, \varepsilon, \lambda) = \mathcal{T}'(M_2, \eta_2, \varepsilon, \lambda).$$

Proof. If the corollary is not true then there is a sequence $(\varepsilon_\kappa)_\kappa$ in $]0, \overline{\varepsilon}]$ converging to zero, there is a sequence $(\lambda_\kappa)_\kappa$ in $[0, 1]$ and there is a sequence $(\sigma_\kappa)_\kappa$ with $\sigma_\kappa \in \mathcal{T}'(M_2, \eta_2, \varepsilon_\kappa, \lambda_\kappa)$ and $\sigma_\kappa(0) \notin [M_1]_{\eta_1}$ for all $\kappa \in \mathbb{N}$. Set $(u_\kappa(t), x_\kappa(t)) := \sigma_\kappa(t)$ for $\kappa \in \mathbb{N}$ and $t \in \mathbb{R}$. Using Proposition 4.10 and taking subsequences if necessary, we may assume that $(u_\kappa)_\kappa$ converges to zero in Y and $(x_\kappa)_\kappa$ converges in $\mathcal{M}$, uniformly on compact subsets of $\mathbb{R}$, to a full solution x of π_0 lying in S. In particular, $x(0) \in \mathrm{Int}_{\mathcal{M}}(M_1)$ so $x_\kappa(0) \in \mathrm{Int}_{\mathcal{M}}(M_1)$ and $|u_\kappa(0)|_Y \leq \eta_1$ for $\kappa \in \mathbb{N}$ large enough. It follows that $\sigma_\kappa(0) \in [M_1]_{\eta_1}$ for all such κ, a contradiction which proves the corollary. □

Corollary 4.13. *Let S_0 and N be as in Theorem 4.3. Let $\eta' = \eta'(N)$ be as in Proposition 4.10 with $M = N$. Then for every $\eta \in]0, \eta']$ there is an $\varepsilon_1(\eta) \in]0, \overline{\varepsilon}]$ such that for every $\varepsilon \in]0, \varepsilon_1(\eta)]$ and for every $\lambda \in [0, 1]$ the set $[N]_\eta$ is a $\pi'_{\varepsilon,\lambda}$-isolating neigbourhood of $S_{\varepsilon,\lambda} = S_{\varepsilon,\lambda,N,\eta} := \mathrm{Inv}_{\pi'_{\varepsilon,\lambda}}([N]_\eta)$.*

Proof. If the corollary is not true, then there is an $\eta \in]0, \eta']$ and sequences $(\varepsilon_\kappa)_\kappa$ and $(\lambda_\kappa)_\kappa$ in $]0, \overline{\varepsilon}]$ and $[0, 1]$ respectively such that $(\varepsilon_\kappa)_\kappa$ converges to zero and $[N]_\eta$ is not a $\pi'_{\varepsilon_\kappa,\lambda_\kappa}$-isolating neigbourhood for all $\kappa \in \mathbb{N}$. For $\kappa \in \mathbb{N}$ set $\pi_\kappa = \pi'_{\varepsilon_\kappa,\lambda_\kappa}$ and $\mathcal{T}_\kappa = \mathcal{T}'(N,\eta,\varepsilon_\kappa,\lambda_\kappa)$. Moreover, set $\mathcal{T}_0 = \mathcal{T}_0(N)$. Then

$$\mathrm{Inv}_{\mathcal{T}_\kappa}([N]_\eta) = \mathrm{Inv}_{\pi_\kappa}([N]_\eta) \not\subset \mathrm{Int}_{Y\times\mathcal{M}}([N]_\eta)$$

for all $\kappa \in \mathbb{N}$. Now, by Proposition 4.10, $\mathcal{T}_\kappa \to \mathcal{T}_0$. Since $\mathrm{Inv}_{\mathcal{T}_0}([N]_\eta) = \{0\} \times \mathrm{Inv}_{\pi_0}(N) \subset \mathrm{Int}_{Y\times\mathcal{M}}([N]_\eta)$, it follows from [1, Proposition 2.4] that, for all $\kappa \in \mathbb{N}$ large enough, $\mathrm{Inv}_{\mathcal{T}_\kappa}([N]_\eta) \subset \mathrm{Int}_{Y\times\mathcal{M}}([N]_\eta)$, a contradiction which proves the corollary. □

Corollary 4.14. *Let S_0, N, $(M_p)_{p\in P}$ and $(V_p)_{p\in P}$ be as in Theorem* 4.3. *Let $\eta' = \eta'(N)$ be as in Proposition* 4.10 *with $M = N$. For all $\eta \in \,]0,\infty[$, $\varepsilon \in \,]0,\overline{\varepsilon}]$, $\lambda \in [0,1]$ and every $p \in P$, define $S_{\varepsilon,\lambda} = S_{\varepsilon,\lambda,N,\eta} := \mathrm{Inv}_{\pi'_{\varepsilon,\lambda}}([N]_\eta)$ and*

$$M_{p,\varepsilon,\lambda} = M_{p,\varepsilon,\lambda,V_p,\eta} := \mathrm{Inv}_{\pi'_{\varepsilon,\lambda}}([V_p]_\eta).$$

Then for every $\eta \in \,]0,\eta']$ there is an $\varepsilon_2(\eta) \in \,]0,\overline{\varepsilon}]$ such that for all $\varepsilon \in \,]0,\varepsilon_2(\eta)]$ and $\lambda \in [0,1]$ the family $(M_{p,\varepsilon,\lambda})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $S_{\varepsilon,\lambda}$ relative to $\pi'_{\varepsilon,\lambda}$ and for every $p \in P$ the set $[V_p]_\eta$ is a $\pi'_{\varepsilon,\lambda}$-isolating neigbourhood of $M_{p,\varepsilon,\lambda}$.

Proof. If the corollary is not true, then there is an $\eta \in \,]0,\eta']$ and sequences $(\varepsilon_\kappa)_\kappa$ and $(\lambda_\kappa)_\kappa$ in $]0,\overline{\varepsilon}]$ and $[0,1]$ respectively such that $(\varepsilon_\kappa)_\kappa$ converges to zero and, for every $\kappa \in \mathbb{N}$, either the family $(M_{p,\varepsilon_\kappa,\lambda_\kappa})_{p\in P}$ is not a $\prec$-ordered Morse decomposition of $S_{\varepsilon_\kappa,\lambda_\kappa}$ relative to $\pi'_{\varepsilon_\kappa,\lambda_\kappa}$ or else, for some $p \in P$, the set $[V_p]_\eta$ is not a $\pi'_{\varepsilon_\kappa,\lambda_\kappa}$-isolating neigbourhood of $M_{p,\varepsilon_\kappa,\lambda_\kappa}$.

For $\kappa \in \mathbb{N}$ set $\pi_\kappa = \pi'_{\varepsilon_\kappa,\lambda_\kappa}$ and $\mathcal{T}_\kappa = \mathcal{T}'(N,\eta,\varepsilon_\kappa,\lambda_\kappa)$. Moreover, set $\mathcal{T}_0 = \mathcal{T}_0(N)$.

Our hypotheses imply that $(\{0\} \times M_p)_{p\in P}$ is a $\prec$-ordered $\mathcal{T}_0$-Morse decomposition. Moreover, for every $p \in P$,

$$\mathrm{Inv}_{\mathcal{T}_0}([V_p]_\eta) = \{0\} \times M_p \subset \mathrm{Int}_{Y\times\mathcal{M}}([V_p]_\eta)$$

and

$$\mathrm{Inv}_{\mathcal{T}_\kappa}([V_p]_\eta) = \mathrm{Inv}_{\pi_\kappa}([V_p]_\eta) = M_{p,\varepsilon_\kappa,\lambda_\kappa}, \quad \kappa \in \mathbb{N}.$$

Now, by Proposition 4.10, $\mathcal{T}_\kappa \to \mathcal{T}_0$. Therefore, it follows from [2, Theorem 3.3] that, for all $\kappa \in \mathbb{N}$ large enough, the family $(M_{p,\varepsilon_\kappa,\lambda_\kappa})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $S_{\varepsilon_\kappa,\lambda_\kappa}$ relative to $\pi'_{\varepsilon_\kappa,\lambda_\kappa}$ and, for all $p \in P$, the set $[V_p]_\eta$ is a $\pi'_{\varepsilon_\kappa,\lambda_\kappa}$-isolating neigbourhood of $M_{p,\varepsilon_\kappa,\lambda_\kappa}$, a contradiction which proves the corollary. □

We can now give a

Proof of Theorem 4.3. Let N be as in Theorem 4.3. Let $\eta'(N)$ and for every $\eta \in \,]0,\eta'(N)]$ let $\eta_1(\eta)$ be as in Corollary 4.13. Set $\eta_0 = \eta'(N)$ and $\varepsilon_0(\eta) = \varepsilon_1(\eta)$, $\eta \in \,]0,\eta_0]$. Let $\eta \in \,]0,\eta_0]$ and $\varepsilon \in \,]0,\varepsilon_0(\eta)]$ be arbitrary.

By Corollary 4.13 for every $\lambda \in [0,1]$ the set $[N]_\eta$ is a $\pi'_{\varepsilon,\lambda}$-isolating neigbourhood of $S_{\varepsilon,\lambda}$. Being compact, the set $[N]_\eta$ is strongly $\pi'_{\varepsilon,\lambda}$-admissible for all $\lambda \in [0,1]$ and $(\pi'_{\varepsilon,\lambda_\kappa})_\kappa$-admissible for every sequence $(\lambda_\kappa)_\kappa$ in $[0,1]$. Thus Proposition 4.7 and the Conley-index continuation principle, see e.g. [15, Theorem I.12.2], imply that

$$h(\pi_\varepsilon, S_\varepsilon) = h(\pi'_{\varepsilon,0}, S_{\varepsilon,0}) h(\pi'_{\varepsilon,1}, S_{\varepsilon,1}).$$

Now $\pi'_{\varepsilon,1} = \widetilde{\pi}_\varepsilon \times \pi_0$, where $\widetilde{\pi}_\varepsilon$ is the (semi)flow generated by the linear differential equation

$$\varepsilon\dot{y} = \overline{B}y.$$

Since $\overline{B}$ is hyperbolic with Morse-index k, it follows that $S_{\varepsilon,1} = \{0\} \times S_0$ and $h(\widetilde{\pi}_\varepsilon, \{0\}) = \Sigma^k$. Thus $h(\pi'_{\varepsilon,1}, S_{\varepsilon,1}) = \Sigma^k \wedge h(\pi_0, S_0)$ so

$$h(\pi_\varepsilon, S_\varepsilon) = \Sigma^k \wedge h(\pi_0, S_0).$$

This proves the first part of Theorem 4.3. Now let M_p and V_p, $p \in P$ be as in Theorem 4.3. For $\eta \in]0,\eta_0]$ let $\varepsilon_2(\eta)$ be as in Corollary 4.14. Set $\overline{\varepsilon}_0(\eta) = \min(\varepsilon_0(\eta), \varepsilon_2(\eta))$, $\eta \in]0,\eta_0]$.

Let $\eta \in]0,\eta_0]$ and $\varepsilon \in]0,\overline{\varepsilon}_0(\eta)]$ be arbitrary. By Corollary 4.14 for every $\lambda \in [0,1]$ the family $(M_{p,\varepsilon,\lambda})_{p\in P}$ is a $\prec$-ordered Morse decomposition of $S_{\varepsilon,\lambda}$ relative to $\pi'_{\varepsilon,\lambda}$ and, for every $p \in P$, $[V_p]_\eta$ is a $\pi'_{\varepsilon,\lambda}$-isolating neigbourhood of $M_{p,\varepsilon,\lambda}$. Together with what we have established so far it follows that all assumptions of the continuation principle for (co)homology index braids ([3, Theorem 3.7] with $\Lambda = [0,1]$) are satisfied. Now that continuation principle implies that the (co)homology index braid of $(\pi_\varepsilon, S_\varepsilon, (M_{p,\varepsilon})_{p\in P}) = (\pi'_{\varepsilon,0}, S_{\varepsilon,0}, (M_{p,\varepsilon,0})_{p\in P})$ is isomorphic to the (co)homology index braid of $(\pi'_{\varepsilon,1}, S_{\varepsilon,1}, (M_{p,\varepsilon,1})_{p\in P})$. The (semi)flow $\widetilde{\pi}_\varepsilon$ is clearly conjugate to the product semiflow $\widetilde{\pi}_\varepsilon^- \times \widetilde{\pi}_\varepsilon^+$ where $\widetilde{\pi}_\varepsilon^-$ resp. $\widetilde{\pi}_\varepsilon^+$ is the (semi)flow on a finite-dimensional Y^- resp. Y^+ generated by the linear differential equation

$$\varepsilon\dot{y} = B^- y \quad \text{resp. } \varepsilon\dot{y} = B^+ y$$

where $B^- \in \mathcal{L}(Y^-, Y^-)$ resp. $B^+ \in \mathcal{L}(Y^+, Y^+)$ is a linear operator with all eigenvalues having negative resp. positive real parts. Thus $\pi'_{\varepsilon,1}$ is conjugate to the (semi)flow $(\pi_0 \times \widetilde{\pi}_\varepsilon^+) \times \widetilde{\pi}_\varepsilon^-$. Now Theorem 2.2 implies that the (co)homology index braid of $(\pi'_{\varepsilon,1}, S_{\varepsilon,1}, (M_{p,\varepsilon,1})_{p\in P})$ is isomorphic to the (co)homology index braid of $(\pi_0 \times \widetilde{\pi}_\varepsilon^+, S_0 \times \{0_{Y^+}\}, (M_p \times \{0_{Y^+}\})_{p\in P})$.

Since $k = \dim Y^+$, an application of [4, Theorem 3.1] and [17, Theorem 4.1] now completes the proof of Theorem 4.3. □

The sets $S_{\varepsilon,N,\eta}$ and $M_{p,\varepsilon,N,\eta}$ in Theorem 4.3 are asymptotically independent of N and η. More precisely, the following result holds.

Proposition 4.15. *Let S_0 and $(M_p)_{p\in P}$ be as in Theorem 4.3. Let $N_1 \subset \mathcal{M}$ and $N_2 \subset \mathcal{M}$ be two compact isolating neigbourhoods of S_0, $\eta'(N_1)$, $\eta'(N_2)$ be as in Proposition 4.10 and η_1, $\eta_2 \in]0, \min(\eta'(N_1), \eta'(N_2))]$ be arbitrary. Moreover, for $p \in P$ let $V_{1,p} \subset N_1$ and $V_{2,p} \subset N_2$ be two compact isolating neigbourhoods of M_p, relative to π_0. Then there is an $\widehat{\varepsilon} \in]0, \overline{\varepsilon}]$ such that for all $\varepsilon \in]0, \widehat{\varepsilon}]$*

$$S_{\varepsilon,N_1,\eta_1} = S_{\varepsilon,N_2,\eta_2} \quad \text{and} \quad M_{p,\varepsilon,V_{1,p},\eta_1} = M_{p,\varepsilon,V_{2,p},\eta_2}, \quad p \in P.$$

Proof. This is an immediate consequence of Corollary 4.12. □

Moreover, the following upper-semicontinuity result obtains.

Proposition 4.16. *In the notation of Theorem 4.3*

$$\lim_{\varepsilon\to 0^+} \sup_{(y,x)\in S_\varepsilon} \inf_{z\in S_0} (|y|_Y + d_{\mathcal{M}}(x,z)) = 0$$

and for every $p \in P$,

$$\lim_{\varepsilon\to 0^+} \sup_{(y,x)\in M_{p,\varepsilon}} \inf_{z\in M_p} (|y|_Y + d_{\mathcal{M}}(x,z)) = 0.$$

Proof. This follows easily from Proposition 4.10. □

Let us discuss two special cases of Theorem 4.3. In the first result we use the following common notation: if, for $i \in \{1, 2\}$, C_i and E_i are sets and $\alpha_i : C_i \to E_i$ is a map, then $\alpha_1 \times \alpha_2 : C_1 \times C_2 \to E_1 \times E_2$ is the product map defined by

$$(\alpha_1 \times \alpha_2)(c_1, c_2) = (\alpha_1(c_1), \alpha_2(c_2)), \quad (c_1, c_2) \in C_1 \times C_2.$$

Corollary 4.17. *Assume Hypothesis 4.1. In addition, assume that $Y = Y_1 \times Y_2$ where Y_1 and Y_2 are finite-dimensional normed linear spaces. Suppose that, for all $x \in \mathcal{M}$,*

$$Df((\phi(x), x), 0) = B_1(x) \times B_2(x)$$

where $B_i(x) \in \mathcal{L}(Y_i, Y_i)$, $i \in \{1, 2\}$, $\operatorname{re}\sigma(B_1(x)) < 0$ and $\operatorname{re}\sigma(B_2(x)) > 0$. Let k be the dimension of Y_2. Then Hypothesis 4.2 is satisfied and so, in particular, the assertions of Theorem 4.3 hold.

Proof. Let $a_0 = -1$ and $b_0 = 2$. Define $B : \mathcal{M} \times]a_0, b_0[\to \mathcal{L}(Y, Y)$ by

$$B(x, \lambda) = (1 - \lambda)Df((\phi(x), x), 0) + \lambda\overline{B}, \quad (x, \lambda) \in \mathcal{M} \times]a_0, b_0[$$

where $\overline{B} = (-\operatorname{id}_{Y_1}) \times \operatorname{id}_{Y_2}$. Then, clearly, $B : \mathcal{M} \times]a_0, b_0[\to \mathcal{L}(Y, Y)$ is a locally Lipschitzian map. An easy calculation shows that $B(x, \lambda)$ is hyperbolic for every $(x, \lambda) \in \mathcal{M} \times [0, 1]$, $B(x, 0) = Df((\phi(x), x), 0)$ and $B(x, 1) = \overline{B}$ for every $x \in \mathcal{M}$, where $\overline{B} \in \mathcal{L}(Y, Y)$ has Morse-index $k \in \mathbb{N}_0$. Thus, indeed, Hypothesis 4.2 is satisfied. □

In particular, Corollary 4.17 implies Theorems 1.2 and 1.3 stated in the Introduction.

Corollary 4.18. *Assume Hypothesis* 4.1. *In addition, assume $\mathcal{M}$ is contractible to a point $x_0 \in \mathcal{M}$. Moreover, let $A(x) := Df((\phi(x), x), 0)$ be hyperbolic for every $x \in \mathcal{M}$. Then Hypothesis* 4.2 *is satisfied with k being the Morse index of $\overline{B} := Df((\phi(x_0), x_0), 0)$. In particular, the assertions of Theorem* 4.3 *hold.*

Proof. Our hypothesis implies that there is a continuous map $G: \mathcal{M} \times [0, 1] \to \mathcal{M}$ such that $G(x, 0) = x$ and $G(x, 1) = x_0$ for all $x \in \mathcal{M}$. By well-known results from the theory of differentiable manifolds we may assume that G is of class C^2 and that, for some a_0, $b_0 \in \mathbb{R}$ with $a_0 < 0 < 1 < b_0$, G has an extension to a C^2-map from $\mathcal{M} \times]a_0, b_0[$ to $\mathcal{M}$ again denoted by G. Define $B: \mathcal{M} \times]a_0, b_0[\to \mathcal{L}(Y, Y)$ by

$$B(x, \lambda) = A(G(x, \lambda)), \quad (x, \lambda) \in \mathcal{M} \times]a_0, b_0[\,.$$

It is clear that $B: \mathcal{M} \times]a_0, b_0[\to \mathcal{L}(Y, Y)$ is a locally Lipschitzian map such that $B(x, \lambda)$ is hyperbolic for every $(x, \lambda) \in \mathcal{M} \times [0, 1]$, $B(x, 0) = Df((\phi(x), x), 0)$ and $B(x, 1) = \overline{B}$ for every $x \in \mathcal{M}$, where $\overline{B} \in \mathcal{L}(Y, Y)$ has Morse-index $k \in \mathbb{N}_0$. Thus, indeed, Hypothesis 4.2 is satisfied. □

We will now show that, in general, Hypothesis 4.1 alone does not suffice for the assertions of Theorem 4.3 to hold.

Let $Y = \mathbb{R}^2$ and $\langle \cdot, \cdot \rangle$ be the canonical scalar product on Y. Let $\mathcal{M} = S^1 \subset \mathbb{R}^2$ be the one-dimensional sphere endowed with the canonical differentiable structure of a submanifold of $\mathbb{R}^2$. For $\theta \in \mathbb{R}$ and $i \in \{1, 2\}$ let $e_i(\theta) \in Y$ be defined by

$$e_1(\theta) = (\cos(\theta/2), \sin(\theta/2)), \quad e_2(\theta) = (-\sin(\theta/2), \cos(\theta/2)).$$

Notice that, for $\theta \in \mathbb{R}$, $(e_1(\theta), e_2(\theta))$ is an orthonormal basis of Y and $e_i(\theta + 2\pi) = -e_i(\theta)$, $i \in \{1, 2\}$.

Let $\overline{\varepsilon} \in]0, \infty[$ be arbitrary and define the map $g: (Y \times \mathbb{R}) \times [0, \overline{\varepsilon}] \to Y$ by

$$g((y, \theta), \varepsilon) = \langle y, e_1(\theta) \rangle e_1(\theta) - \langle y, e_2(\theta) \rangle e_2(\theta), \quad ((y, \theta), \varepsilon) \in (Y \times \mathbb{R}) \times [0, \overline{\varepsilon}]\,.$$

Then g is of class C^∞ and 2π-periodic in θ. Thus there is a unique map $f: (Y \times \mathcal{M}) \times [0, \overline{\varepsilon}] \to Y$ such that

$$f((y, (\cos\theta, \sin\theta)), \varepsilon) = g((y, \theta), \varepsilon), \quad ((y, \theta), \varepsilon) \in (Y \times \mathcal{M}) \times [0, \varepsilon]\,.$$

The map f is of class C^∞. Define the map $h: (Y \times \mathcal{M}) \times [0, \overline{\varepsilon}] \to T\mathcal{M}$ by

$$h((y, x), \varepsilon) = (x, 0_x), \quad ((y, x), \varepsilon) \in (Y \times \mathcal{M}) \times [0, \overline{\varepsilon}]\,,$$

where, for $x \in \mathcal{M}$, 0_x is the zero tangent vector to $\mathcal{M}$ at x. Finally, let $\phi: \mathcal{M} \to Y$ be defined by $\phi(x) = 0$ for all $x \in \mathcal{M}$.

With these definitions we see that Hypothesis 4.1 is satisfied with $Z_0 = Y \times \mathcal{M}$. Let π_0 and π_ε, $\varepsilon \in \,]0,\overline{\varepsilon}]$, be the local semiflows defined by the differential equations (4.2) and (4.1), respectively.

The $N = \mathcal{M}$ is a compact π_0-isolating neigbourhood of the isolated π_0-invariant set $S_0 = N$. Since $(N,\emptyset)$ is an index pair in N, it follows that for $q \in \mathbb{Z}$ the homology Conley index $H_q(\pi_0, S_0)$ is represented by $H_q(N \cup \{p\}, \{p\})$, where $p \notin N$, $\{p\}$ is endowed the discrete topology $N \cup \{p\}$ is endowed with the sum topology. By the excision property it follows that $H_q(\pi_0, S_0)$ is represented by $H_q(N)$.

Now suppose that the assertions of Theorem 4.3 hold for some $k \in \mathbb{N}_0$ and let $\eta_0 = \eta_0(N)$ be as in that theorem. Choose $\eta \in \,]0, \eta_0/\sqrt{2}]$ arbitrarily. Then by Theorem 4.3 and Corollary 4.11 there is an $\widehat{\varepsilon} \in \,]0,\overline{\varepsilon}]$ such that for every $\varepsilon \in \,]0,\widehat{\varepsilon}]$, the sets $[N]_\eta$ and $[N]_{\sqrt{2}\cdot\eta}$ are isolating neigbourhoods of the same isolated invariant set S_ε relative to π_ε and for each $q \in \mathbb{Z}$ there is an isomorphism from $H_q(\pi_\varepsilon, S_\varepsilon)$ to $H_{q-k}(\pi_0, S_0)$.

Let $\varepsilon \in \,]0,\widehat{\varepsilon}]$ be arbitrary. Define the following sets:

$$\begin{aligned}
L_1 = \{(y,x) \in Y \times \mathcal{M} \mid & \text{ there exists a } \theta \in \mathbb{R} \\
& \text{with } x = (\cos\theta, \sin\theta),\ |\langle y, e_1(\theta)\rangle| \le \eta \text{ and } |\langle y, e_2(\theta)\rangle| \le \eta\},\\
L_2 = \{(y,x) \in Y \times \mathcal{M} \mid & \text{ there exists a } \theta \in \mathbb{R} \\
& \text{with } x = (\cos\theta, \sin\theta),\ |\langle y, e_1(\theta)\rangle| = \eta \text{ and } |\langle y, e_2(\theta)\rangle| \le \eta\},\\
\widehat{L}_1 = \{(y,x) \in Y \times \mathcal{M} \mid & \text{ there exists a } \theta \in \mathbb{R} \\
& \text{with } x = (\cos\theta, \sin\theta),\ |\langle y, e_1(\theta)\rangle| \le \eta \text{ and } \langle y, e_2(\theta)\rangle = 0\},\\
\widehat{L}_2 = \{(y,x) \in Y \times \mathcal{M} \mid & \text{ there exists a } \theta \in \mathbb{R} \\
& \text{with } x = (\cos\theta, \sin\theta),\ |\langle y, e_1(\theta)\rangle| = \eta \text{ and } \langle y, e_2(\theta)\rangle = 0\}.
\end{aligned}$$

Any solution $t \mapsto (y(t), x(t))$ of the (semi)flow π_ε satisfies $x(t) \equiv$ constant. Therefore we easily see that L_1 is an isolating block relative to the (semi)flow π_ε with exit set L_2. Since $[N]_\eta \subset L_1 \subset [N]_{\sqrt{2}\cdot\eta}$, the set L_1 is an isolating neigbourhood of S_ε relative to π_ε. It follows that $H_q(\pi_\varepsilon, S_\varepsilon)$ is represented by $H_q(L_1/L_2, \{[L_2]\})$ so

$$H_q(L_1/L_2, \{[L_2]\}) \cong H_{q-k}(N) = H_{q-k}(S^1), \quad q \in \mathbb{Z}.$$

The map $G\colon (Y \times \mathbb{R}) \times [0,1] \to (Y \times \mathcal{M})$ given by

$$G((y,\theta),t) = (y - t\langle y, e_2(\theta)\rangle e_2(\theta), (\cos\theta, \sin\theta)) \quad ((y,\theta),t) \in (Y \times \mathbb{R}) \times [0,1]$$

is continuous and 2π-periodic in θ. Therefore there is a unique map $F\colon (Y \times \mathcal{M}) \times [0,1] \to Y \times \mathcal{M}$ such that

$$F((y,(\cos\theta,\sin\theta)),t) = G((y,\theta),t), \quad ((y,\theta),t) \in (Y \times \mathbb{R}) \times [0,1]\,.$$

The map F is continuous and $F(L_i \times [0,1]) \subset L_i$ and $F(\widehat{L}_i \times [0,1]) \subset \widehat{L}_i$ for $i \in \{1,2\}$.

Using the map F we easily see that $(L_1/L_2, [L_2])$ is homotopy equivalent to $(\widehat{L}_1/\widehat{L}_2, [\widehat{L}_2])$. Hence, using integer coefficients, we obtain, for all $q \in \mathbb{Z}$,

$$H_q(\widehat{L}_1/\widehat{L}_2, \{[\widehat{L}_2]\}) \cong H_{q-k}(S^1) \cong \begin{cases} \mathbb{Z} & \text{if } q-k \in \{0,1\}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.14}$$

Since $\widehat{L}_1$ is a Möbius strip and $\widehat{L}_2$ is its geometric boundary, it follows that $\widehat{L}_1/\widehat{L}_2$ is homeomorphic to the projective plane. In particular,

$$H_1(\widehat{L}_1/\widehat{L}_2, \{[\widehat{L}_2]\}) \cong H_1(\widehat{L}_1/\widehat{L}_2) \cong \mathbb{Z}/2\mathbb{Z},$$

a contradiction to (4.14). Thus, indeed, the assertions of Theorem 4.3 do not hold in this case.

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