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THE IMPLICIT FUNCTION THEOREM FOR CONTINUOUS FUNCTIONS

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Abstract. In the present paper we obtain a new homological version of the implicit function theorem and some versions of the Darboux theorem. Such results are proved for continuous maps on topological manifolds. As a consequence, some versions of these classic theorems are proved when we consider differenciable (not necessarily C^1) maps.

1. Introduction

This paper deals with a generalization of the classical Implicit Function Theorem. In this respect, C. Biasi and E. L. dos Santos proved *a homological version of the implicit function theorem* for continuous functions on general topological spaces which has interesting applications in the theory of topological groups. More specifically, Theorem 2.1 of [1] states that: "If X, Y, Z are Hausdorff spaces, with X locally connected, Y locally compact and $f\colon X \times Y \to Z$ such that $(f_{x_0})_* = (f|_{(x_0 \times Y)})_*\colon H_n(Y, Y - y_0) \to H_n(Z, Z - z_0)$ is a nontrivial homomorphism, for some natural $n > 0$, where $\{y_0\} = (f_{x_0})^{-1}(\{z_0\})$, then there exists an open neighbourhood V of x_0 and a function $g\colon V \subset X \to Y$ satisfying the equation $f(x, g(x)) = z_0$, for each $x \in V$. Moreover, g is continuous at x_0". This result establishes the existence of an implicit function g, however, the continuity of such function is guaranteed only at the point x_0. In this paper, under

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little stronger assumptions, we can ensure the continuity of g at a neighbourhood of x_0. More precisely,

Theorem 4.1. *Let X be a locally pathwise connected Hausdorff space and let Y, Z be oriented connected topological manifolds of dimension n. Let $f\colon X \times Y \to Z$ be a continuous map such that, for all $x \in X$, the map $f_x\colon Y \to Z$ given by $f_x(y) = f(x,y)$ is open and discrete. Suppose that for some $(x_0,y_0) \in X \times Y$, $|\deg(f_{x_0},y_0)| = 1$. Then there exists a neighbourhood V of x_0 in X and a continuous function $g\colon V \to Y$ such that $f(x,g(x)) = f(x_0,y_0)$, for all $x \in V$.*

As a consequence we obtain

Corollary 4.2. *Let X be a locally pathwise connected Hausdorff space. Let $U \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and let $f\colon X \times U \to \mathbb{R}^n$ be a continuous map. Suppose that $f_x\colon U \to \mathbb{R}^n$ is a differentiable (not necessarily C^1) map without critical points, for each $x \in X$. Given $(x_0,y_0) \in X \times U$ and $z_0 = f(x_0,y_0)$, there exist a neighbourhood V of x_0 and a continuous function $g\colon V \to U$ such that $g(x_0) = y_0$ and, for every $x \in V$, $f(x,g(x)) = z_0$.*

Corollary 4.3. *Let $I \times V \times W$ be an open neighbourhood of (t_0,x_0,y_0) in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ and let $F\colon (I \times V \times W, (t_0,x_0,y_0)) \mapsto (\mathbb{R}^m, 0)$ be a continuous map. Suppose that, for all $(t,x) \in I \times V$, the map $y \in W \to F(t,x,y)$ is differentiable (but not necessarily C^1) and without critical points. Then the differential equation*

$$F(t,x,x') = 0, \quad x(t_0) = x_0, \quad x'(t_0) = y_0$$

has a solution in some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.

We also prove the following versions of Darboux Theorem.

Theorem 5.1. *Let M and N be oriented connected topological manifolds of dimension n and let $f\colon M \to N$ be a continuous map. Suppose that there exist x_0 and x_1 in M such that $\deg(f,x_0) < 0$ and $\deg(f,x_1) > 0$. Then there exists x_2 in M such that $\deg(f,x_2) = 0$*

Corollary 5.2. *Let M and N be oriented connected topological manifolds of dimension n and let $f\colon M \to N$ be a differentiable map. Suppose that there exist x_0 and x_1 in M such that $\det[f'(x_0)] < 0$ and $\det[f'(x_1)] > 0$. Then f has a critical point.*

A fundamental step to establish the versions of the Implicit Function Theorem and Darboux Theorem is Lemma 3.1 (Key Lemma), which is connected with the Yu Yu Trohimčuk conjecture (see [3], [7]), known as the problem of sign constancy of the local degree.

2. Preliminares

In this section we introduce some basic notions, notations and results that will be used throughout this paper. All considered singular homology groups will always have coefficients in $\mathbb{Z}$. By *dimension* we understand the usual topological dimension in the sense of [5].

In the following definitions, X and Y will be oriented connected topological manifolds of dimension $n \geq 1$ and $f\colon X \to Y$ will be a continuous map. The definition of orientation for topological manifolds can be found, for instance, in [4].

Definition 2.1. A map $f\colon X \to Y$ is said to be *discrete* at a point x_0, if there exists a neighbourhood V of x_0 such that $f(x) \neq f(x_0)$, for any $x \in V - x_0$, that is, $f^{-1}(f(x_0)) \cap (V - x_0) = \emptyset$.

An example: the mapping $f(z) = z^2$ is open and discrete at every point of the complex plane.

Definition 2.2. Let $y \in Y$ such that $L_y = f^{-1}(y)$ is a compact subset of X. Let $\alpha_{L_y} \in H_n(X, X - L_y)$ and $\beta_y \in H_n(Y, Y - y)$ be the orientation classes along L_y and y, respectively. There exists an integer number $\deg(f, y)$ such that $f_*(\alpha_{L_y}) = \deg(f, y) \cdot \beta_y$, where $f_*\colon H_n(X, X - L_y) \to H_n(Y, Y - \{y\})$ is the homomorphism induced by f. The number $\deg(f, y)$ is called *degree of f at y.*

Definition 2.3. Let $f\colon X \to Y$ be a *discrete map* at a point x_0 and let us denote by $y_0 = f(x_0)$. Consider $\alpha_{x_0} \in H_n(V, V - x_0)$ and $\beta_{y_0} \in H_n(Y, Y - y_0)$ the orientation classes at x_0 and y_0, respectively. There exists an integer number $\deg(f, x_0)$ such that $f_*(\alpha_{x_0}) = \deg(f, x_0)\cdot\beta_{y_0}$, where the homomorphism $f_*\colon H_n(V, V - x_0) \to H_n(Y, Y - y_0)$ is induced by f. The number $\deg(f, x_0)$ is called *local degree of f at x_0.*

Definition 2.4. Suppose that $f\colon X \to Y$ is not necessarily a discrete map. Define

$$\deg(f, x) = \begin{cases} 0 & \text{if } f \text{ is not discrete at } x, \\ \deg(f, x) & \text{if } f \text{ is discrete at } x \text{ (as in Definition 2.3).} \end{cases}$$

The proof of the following two propositions can be found in [4].

Proposition 2.5. *Let X and Y be oriented connected topological manifolds of dimension $n \geq 1$ and let $f\colon X \to Y$ be a continuous map. Consider $y \in Y$ such that $f^{-1}(y)$ is finite. Then,*

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} \deg(f, x).$$

Proposition 2.6. *Let X and Y be oriented connected topological manifolds of dimension $n \geq 1$ and $f\colon X \to Y$ be a continuous map. Let us consider a compact connected subset K of Y such that $L_K = f^{-1}(K)$ is compact. Then,* $\deg(f, y)$ *does not depend of $y \in K$.*

As an immediate consequence of Proposition 2.6 we have the following:

Corollary 2.7. *Let X and Y be oriented connected topological manifolds of dimension $n \geq 1$ and $f\colon X \to Y$ be a proper continuous map. Then, for all $y \in Y$,* $\deg(f, y) = \deg(f)$.

In [8], Väisälä proved the following version of the Černavskiĭ's theorem (see [2] and [3]):

Theorem 2.8. *Let X and Y be topological manifold of dimension n. Suppose that $f\colon X \to Y$ is an open and discrete map. Then* $\dim(f(B_f)) \leq n - 2$ *and* $\dim(B_f) \leq n - 2$, *where B_f denotes the set of points x of X such that f is not a local homeomorphism at x.*

3. Persistence of the sign of $\deg(f, x)$

The following lemma will be fundamental in the proof of the versions of the Implicit Function Theorem and Darboux Theorem.

Lemma 3.1 (Key Lemma). *Let X and Y be oriented connected topological manifolds of dimension n. Suppose that $f\colon X \to Y$ is an open and discrete map. Then, for any $x \in X$, one has that* $\deg(f, x) \neq 0$*; moreover,* $\deg(f, x)$ *has always the same sign.*

Proof. It follows from Theorem 2.8 that $\dim B_f \leq n - 2$, then $X - B_f$ is connected. Since f is a local homeomorphism on $X - B_f$, one has that

$$\deg(f, x) = c, \quad \text{for all } x \in X - B_f, \text{ where either } c = 1 \text{ or } c = -1. \tag{3.1}$$

Let $x_0 \in B_f$ and denote by $y_0 = f(x_0)$. Since f is a discrete map and X is locally compact, we can choose an open connected neighbourhood V of x_0 such that $\overline{V}$ is compact and $f(x) \neq f(x_0)$ for all $x \in \overline{V} - x_0$. Then, there exists an open neighbourhood W of $y_0 = f(x_0)$ such that $\overline{W}$ is a compact and connected subset of Y and $(f|_{\overline{V}})^{-1}(\overline{W}) = (f|_V)^{-1}(\overline{W}) \subset V$. Therefore, $(f|_V)^{-1}(\overline{W})$ is compact and it follows from Proposition 2.6 that

$$\deg(f|_V, y) = \deg(f|_V, y_0), \quad \text{for all } y \in \overline{W} \tag{3.2}$$

On the other hand, since $U = (f|_V)^{-1}(W)$ is an open set in X and f is a open map, we have that $f|_V(U) \subset \overline{W}$ is an open set in Y and since $Y - f(B_f)$ is dense in Y (recall that $\dim(f(B_f)) \leq n - 2$ by Theorem 2.8), there exists $y_1 \in f|_V(U) - f(B_f)$. Therefore, it follows from (3.2) that $\deg(f|_V, y_1) = \deg(f|_V, y_0)$.

Let $(f|_V)^{-1}(y_1) = \{x_1, \dots , x_k\}$ in $X - B_f$. Then, by Proposition 2.5 we have that

$$\deg(f, x_0) = \deg(f|_V, y_0) = \deg(f|_V, y_1) = \sum_{x_i \in (f|_V)^{-1}(y_1)} \deg(f|_V, x_i). \tag{3.3}$$

On the other hand, since $x_i \in X - B_f$ for $i = 1, \dots , k$, by formula (3.1), $\deg(f, x_i) = c$ where either $c = 1$ or $c = -1$. Then, it follows from (3.3) that $\deg(f, x_0) = kc$ and therefore $\deg(f, x)$ has always the same sign, for any $x \in X$. ☐

4. The Implicit Function Theorem

In this section we shall show a new homological version of the Implicit Function Theorem.

Theorem 4.1 (Implicit Function Theorem). *Let X be a locally pathwise connected Hausdorff space and let Y, Z be oriented connected topological manifolds of dimension n. Let $f: X \times Y \to Z$ be a continuous map such that, for all $x \in X$, the map $f_x: Y \to Z$ given by $f_x(y) = f(x, y)$ is open and discrete. Suppose that for some $(x_0, y_0) \in X \times Y$, $|\deg(f_{x_0}, y_0)| = 1$. Then there exists a neighbourhood V of x_0 in X and a continuous function $g: V \to Y$ such that $f(x, g(x)) = f(x_0, y_0)$, for all $x \in V$.*

Proof. Let $z_0 = f(x_0, y_0)$. Since Y is locally compact and f_{x_0} is a discrete map, we can choose a compact neighbourhood $W \subset Y$ of $y_0 \in Y$ satisfying

$$(f_{x_0})^{-1}(z_0) \cap W = y_0.$$

We will first show that *for any compact neighbourhood $K \subset \mathrm{Int}(W)$ containing y_0, there exists a neighbourhood V of x_0 such that*

$$(f_{x|_W})^{-1}(\{z_0\}) \subseteq K, \quad \text{for all } x \in V. \tag{4.1}$$

In fact, suppose that for each neighbourhood V of x_0 there exists (x_V, y_V) in $V \times (W - K)$ such that $f(x_V, y_V) = z_0$. Let us consider a generalized sequence, called a net, $((x_V, y_V))_{V\in\mathcal{V}}$, where $\mathcal{V}$ is the collection of all the neighbourhoods of x_0, partially ordered by reverse inclusion (for details see [6, p. 187–188]). Therefore, $\lim x_V = x_0$ and since (y_V) is a net contained in the compact subset $W - \mathrm{Int}(K)$, there exists $y_1 \in W - \mathrm{Int}(K)$ which is a limit point of some convergent subnet of (y_V). Hence, (x_0, y_1) is a limit point of some subnet of (x_V, y_V). Since f is a continuous map, one has $f(x_0, y_1) = z_0$ which implies that $y_1 = y_0$, contradicting the fact that $y_1 \in W - \mathrm{Int}(K)$. Therefore, there exists a neighbourhood V of x_0 satisfying (4.1).

Choose a compact neighbourhood $K \subset \mathrm{Int}(W)$ of y_0. It follows from (4.1) that for each $x \in V$, the map of pairs $f_x: (W, W-K) \to (Z, Z-z_0)$ is well defined.

Since X is locally pathwise connected, we can assume that V is a pathwise connected neighbourhood of x_0 and therefore, for each x in V, there exists a path α in V joining x_0 to x. We define the homotopy of pairs $H\colon (I\times W, I\times(W-K)) \to (Z, Z-z_0)$ given by

$$H(t,y) = f(\alpha(t), y) = f_{\alpha(t)}(y), \quad \text{for all } (t,y) \in I \times W.$$

Since W is compact and H is an appropriate homotopy between $f_{x_0}|_W$ and $f_x|_W$, we obtain $\deg(f_x|_W, z_0) = \deg(f_{x_0}|_W, z_0)$. Now, as $\deg(f_{x_0}|_W, z_0) = \deg(f_{x_0}, y_0)$ and as, by hypotheses, $|\deg(f_{x_0}, y_0)| = 1$ we obtain

$$|\deg(f_x|_W, z_0)| = |\deg(f_{x_0}|_W, z_0)| = |\deg(f_{x_0}, y_0)| = 1. \tag{4.2}$$

Since f_x is open and discrete, it follows from Lemma 3.1 that $\deg(f_x, y) \neq 0$, for any $y \in Y$, and $\deg(f_x, y)$ has always the same sign. Thus, if $(f_x|_W)^{-1}(z_0) = \{y_1, \cdots, y_k\} \subset W$ with $k \geq 2$, we have that

$$|\deg(f_x|_W, z_0)| = \left|\sum_{i=1}^{k} \deg(f_x, y_i)\right| = \sum_{i=1}^{k} |\deg(f_x, y_i)| > 1,$$

which contradicts (4.2). Therefore, for each $x \in V$ there exists a unique $y \in K \subset W$ such that $(f_x|_W)^{-1}(z_0) = y$; in other words, for each $x \in V$ there exists a unique $y = g(x) \in K$ such that $f_x(g(x)) = f(x, g(x)) = z_0$ for each $x \in V$.

We will show that the map $g\colon V \to K \subset Y$ is continuous. Let A be a neighbourhood of $y = g(x)$ such that $A \subset K$. Let us assume that for any neighbourhood U of x, there exists x_U in U such that $g(x_U)$ belongs to the compact subset $K - A$. Let $\overline{y}$ in $K - A$ be the limit point of the some convergent subnet of $(g(x_U))$. Thus, $(x, \overline{y})$ is the limit point of the some convergent subnet of $(x_U, g(x_U))$. Since f is continuous and g is given implicitly by the equation $f(x_U, g(x_U)) = z_0$ we have that $f(x, \overline{y}) = z_0$, which implies that $y = \overline{y}$, contradicting the fact that $\overline{y} \in K - A$. $\square$

Corollary 4.2. *Let X be a locally pathwise connected Hausdorff space. Let $U \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and let $f\colon X \times U \to \mathbb{R}^n$ be a continuous map. Suppose that $f_x\colon U \to \mathbb{R}^n$ is a differentiable (not necessarily C^1) map without critical points, for each $x \in X$. Given $(x_0, y_0) \in X \times U$ and $z_0 = f(x_0, y_0)$, there exist a neighbourhood V of x_0 and a continuous function $g\colon V \to U$ such that $g(x_0) = y_0$ and, for every $x \in V$, $f(x, g(x)) = z_0$.*

Proof. Since $f_x\colon U \to \mathbb{R}^n$ is a differentiable (not necessarily C^1) map without critical points, we have that f_x is an open and *discrete* map, for each x in X. Therefore, the result follows from Theorem 4.1. $\square$

Corollary 4.3. *Let* $I \times V \times W$ *be an open neighbourhood of* (t_0, x_0, y_0) *in* $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$ *and let* $F\colon (I \times V \times W, (t_0, x_0, y_0)) \mapsto (\mathbb{R}^m, 0)$ *be a continuous map. Suppose that, for all* $(t, x) \in I \times V$, *the map* $y \in W \to F(t, x, y)$ *is differentiable* (*but not necessarily* C^1) *and without critical points. Then the differential equation*

$$F(t, x, x') = 0, \quad x(t_0) = x_0, \quad x'(t_0) = y_0$$

has a solution in some interval $(t_0 - \varepsilon, t_0 + \varepsilon)$.

Proof. By Corollary 4.2, we have that there exists a neighbourhood $(t_0 - \varepsilon_1, t_0 + \varepsilon_1) \times V_1$ of (t_0, x_0) and a function $g\colon (t_0 - \varepsilon_1, t_0 + \varepsilon_1) \times V_1 \to W$ such that

$$F(t, x, g(t, x)) = 0$$

By Peano Theorem, there exists a solution $\varphi\colon (t_0 - \varepsilon, t_0 + \varepsilon) \to V_1$ of the differential equation

$$x' = g(t, x), \quad x(t_0) = x_0, \quad x'(t_0) = y_0.$$

This implies the corollary. □

5. Generalizations of the Darboux theorem

The classical Darboux theorem states that if $f\colon [a, b] \to \mathbb{R}$ is a differentiable map which has the property that $f'(a) < 0$ and $f'(b) > 0$, then there exists a point $c \in (a, b)$ such that $f'(c) = 0$.

In order to obtain some versions of the Darboux theorem we apply the results previously obtained.

Theorem 5.1 (A homological version of the Darboux theorem). *Let* M, N *be oriented connected topological manifolds of dimension* n *and let* $f\colon M \to N$ *be a continuous map. Suppose that there exist* x_0, x_1 *in* M *such that* $\deg(f, x_0) < 0$ *and* $\deg(f, x_1) > 0$, *then there exists* x_2 *in* M *such that* $\deg(f, x_2) = 0$.

Proof. If f is not a *discrete map* at some $x \in M$, it follows from Definition 2.4 that $\deg(f, x) = 0$. In this way, suppose that f is a *discrete map* at x, for every $x \in M$. Therefore, if $\deg(f, x) \neq 0$ for any $x \in M$, then f is also an open map and from Lemma 3.1 we conclude that $\deg(f, x)$ has always the same sign, for any $x \in M$, which is a contradiction. □

The following theorems are differentiable versions of the Darboux theorem. Let us observe that when f is of class C^1, these results are trivial.

Corollary 5.2. *Let* M *and* N *be oriented connected topological manifolds of dimension* n *and let* $f\colon M \to N$ *be a differentiable map. Suppose that there exist* x_0 *and* x_1 *in* M *such that* $\det[f'(x_0)] < 0$ *and* $\det[f'(x_1)] > 0$. *Then* f *has a critical point.*

Proof. Suppose that f does not have critical points. In this case, one has that for any $x \in M$, $|\deg(f,x)| = 1 \neq 0$. Thus, f is an open and *discrete* map and it follows from Lemma 3.1 that $\deg(f,x)$ has always the same sign, which is a contradiction. $\square$

Corollary 5.3. *Consider $f, g\colon M \to \mathbb{R}^n$ differentiable maps, where M is an oriented connected topological manifold of dimension n. Suppose that there exist $\alpha \in \mathbb{R}$ and $x_0, x_1 \in M^n$ such that*

$$\det[f'(x_0) - \alpha g'(x_0)] < 0 \quad \textit{and} \quad \det[f'(x_1) - \alpha g'(x_1)] > 0.$$

Then, there exists $x_2 \in M$ such that $\det[f'(x_2) - \alpha g'(x_2)]$ is equal to zero.

Proof. It suffices to apply Theorem 5.2 for the map $h = f - \alpha g$. $\square$

As a direct consequence of Theorem 5.3 one has the following version of the classical Darboux theorem for differentiable maps from $\mathbb{R}^n$ into $\mathbb{R}^n$.

Corollary 5.4 (Differentiable Darboux theorem). *Let $f\colon U \to \mathbb{R}^n$ be a differentiable map, where U is an open connected subset of $\mathbb{R}^n$. Suppose that there exist $\alpha \in \mathbb{R}$ and $x_0, x_1 \in U$ such that $\det[f'(x_0) - \alpha I] < 0$ and $\det[f'(x_1) - \alpha I] > 0$. Then, there exists $x_2 \in U$ such that $\det[f'(x_2) - \alpha I]$ is equal to zero (i.e. α is an eigenvalue of $f'(x_2)$).*

Now consider f and U under the same assumptions of Corollary 5.4 and let us denote by $p_0(\lambda) = \det[f'(x_0)) - \lambda I]$ and by $p_1(\lambda) = \det[f'(x_1)) - \lambda I]$. Let n_0 and n_1 natural numbers. In these conditions, we prove the following

Corollary 5.5. *Let $x_0, x_1 \in U$ and $\alpha \in \mathbb{R}$ such that $p_0(\lambda) = q_0(\lambda)\cdot(\alpha-\lambda)^{n_0}$ and $p_1(\lambda) = q_1(\lambda)(\alpha - \lambda)^{n_1}$, where $q_0(\lambda)$ and $q_1(\lambda)$ are not null polynomials. Suppose that n_0 is odd and n_1 is even. If $q_0(\alpha)q_1(\alpha) > 0$ then there exists $\delta > 0$ satisfying the following condition: for each $\lambda \in (\alpha, \alpha+\delta)$ there exists $x_2 = x_2(\lambda)$ such that $\det[f'(x_2) - \lambda I]$ is equal to zero.*

Proof. Since n_0 is odd and n_1 is even, one has that, for $\lambda > \alpha$ close to α, the polynomials $p_0(\lambda)$ and $p_1(\lambda)$ have different signs. It follows from Corollary 5.4 that if $\delta > 0$ is small enough and $\lambda \in (\alpha, \alpha + \delta)$, there exists $x_2 = x_2(\lambda)$ such that $p_2(\lambda) = \det[f'(x_2) - \lambda I]$ is equal to zero. $\square$

Remark 5.6. Theorem 5.5 remains the same in the case that $q_0(\alpha)q_1(\alpha) < 0$ and $\lambda \in (\alpha - \delta, \alpha)$.

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