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NEW APPLICATION OF HOMOTOPY PERTURBATION METHOD TO ZK-MEW EQUATION

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Abstract. The work presents a derivation of solitary solutions of the two-dimensional Zakharov–Kuznetsov Modified Equal Width (ZK-MEW) equation using the homotopy perturbation method.

1. Introduction

The discussed Zakharov–Kuznetsov (ZK) equation has been studied by many authors via different approaches. The ZK equation governs the behaviour of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [18]. In [15], the ZK equation is solved by the sine-cosine and the tanh-function methods. The numbers of solitary waves, periodic waves and kink waves of the modified Zakharov–Kuznetsov equation are obtained by A. M. Wazwaz [16]. Recently, Mustafa Inc [10] obtained some exact solutions for the ZK-MEW equation by using extended tanh-method. The modified equal width (MEW) equation is given by

$$u_t + 3u^2u_x - \alpha u_{xxt} = 0$$

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has been discussed in [10], [15], [16], [18]. The MEW equation is related to the RLW equation. This equation has solitary waves with both positive and negative amplitudes. We will consider two-dimensional ZK-MEW equation in the following form:

$$u_t + \alpha(u^3)_x + (bu_{xt} + ru_{yy})_x = 0, \tag{1.1}$$

where a, b and r are constants.

The homotopy perturbation method (HPM) was first proposed by He [5]–[9]. The HPM does not depend on a small parameter in the equation. Using homotopy technique in topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$ which is considered as a "small parameter". Recently, many researcher do a lot of significant work about the homotopy perturbation method [11], [14].

In this paper we further extend the method to solve the nonlinear ZK-MEW equation. By using the HPM, we get the explicit solutions of the nonlinear ZK-MEW equation without using any extended-tanh method. The method presented here is also simple to use for obtaining numerical solution of the equations without using any discrete techniques. Furthermore, we will show that considerably better approximations related to the accuracy level are obtained.

2. Analysis of He's homotopy perturbation method

The principles of the HPM and its applicability for various kinds of differential equations are given in [4]–[8], [11], [14]. To illustrate the basic ideas of this method, we consider the following nonlinear differential equation [5]:

$$A(u) - f(r) = 0, \quad r \in \Omega,$$

with the boundary conditions of

$$B(u, \partial u/\partial n) = 0, \quad r \in \Gamma,$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω.

Generally speaking, the operator A can be decomposed into two operators, L and N, where L is linear, and N is a nonlinear operator. Equation (1.1) can therefore be rewritten as follows:

$$L(u) + N(u) - f(r) = 0.$$

By the homotopy technique, we construct a homotopy $V(r, p)\colon \Omega \times [0, 1] \to R$ which satisfies:

$$H(V, p) = (1 - p)[L(V) - L(u_0)] + p[A(V) - f(r)] = 0, \quad p \in [0, 1],\ r \in \Omega, \tag{2.1}$$

or

$$H(V,p) = L(u) - L(u_0) + pL(u_0) + p[N(V) - f(r)] = 0, \tag{2.2}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of equation (1.1), which satisfies the boundary conditions. Obviously, from equations (2.1) and (2.2), we will have:

$$H(V,0) = L(V) - L(u_0) = 0, \quad H(V,1) = A(V) - f(r) = 0.$$

The changing process of p from zero to unity is just that of $V(r,p)$ from $u_0(r)$ to $u(r)$. According to the HPM, we can first use the embedding parameter p as a "small parameter", and assume that the solution of equations (2.1) and (2.2) can be written as a power series in p:

$$V = V_0 + pV_1 + p^2V_2 + \dots$$

Setting $p = 1$ results in the approximate solution of equation (1.1):

$$u = \lim_{p=1} V = V_0 + V_1 + V_2 + \dots$$

3. Soliton solutions for the two-dimensional ZK-MEW equation

To investigate the traveling wave solution of equation (1.1), we first construct a homotopy as follows:

$$(1-p)\left(\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial u}{\partial t} + 3\alpha u^2\frac{\partial u}{\partial x} + b\frac{\partial^3 u}{\partial x^2 \partial t} + r\frac{\partial^3 u}{\partial y^2 \partial x}\right) = 0. \tag{3.1}$$

Suppose the solution of equation (3.1) and the initial approximations are as follows:

$$u_0(x,y,t) = u(x,y,0), \tag{3.2}$$

$$u(x,y,t) = U(x,y,t) = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots \tag{3.3}$$

where $u_i (i = 1, 2, \dots)$ are functions of (x,y,t) yet to be determined. Substituting equation (3.3) into equation (3.1), and equating the coefficients of the terms with the identical powers of p, we have

$$\begin{aligned}
&\left(\left(\frac{\partial}{\partial t}u_1\right) + \left(\frac{\partial}{\partial t}u_0\right) + r\left(\frac{\partial^3}{\partial y^2 \partial x}u_0\right) + 3au_0^2\left(\frac{\partial}{\partial x}u_0\right) + b\left(\frac{\partial^3}{\partial x^2 \partial t}u_0\right)\right)p \\
&\quad + \left(b\left(\frac{\partial^3}{\partial x^2 \partial t}u_1\right) + 6au_0u_1\left(\frac{\partial}{\partial x}u_0\right) + \left(\frac{\partial}{\partial t}u_2\right)\right. \\
&\quad \left. + 3au_0^2\left(\frac{\partial}{\partial x}u_1\right) + r\left(\frac{\partial^3}{\partial y^2 \partial x}u_1\right)\right)p^2 \\
&\quad + \left(r\left(\frac{\partial^3}{\partial y^2 \partial x}u_2\right) + 3au_1^2\left(\frac{\partial}{\partial x}u_0\right) + 6au_0u_1\left(\frac{\partial}{\partial x}u_1\right)\right.
\end{aligned}$$

$$+\left(\frac{\partial}{\partial t}u_3\right)+b\left(\frac{\partial}{\partial t}u_3\right)+b\left(\frac{\partial}{\partial x^2\partial t}u_2\right)$$
$$+6au_0u_2\left(\frac{\partial}{\partial x}u_0\right)+3au_0^2\left(\frac{\partial}{\partial x}u_2\right)\Big)p^3+\dots=0.$$

In order to obtain the unknowns of $u_i (i = 1, 2, \dots)$, we must construct and solve the following system which includes three equations with three unknowns, considering the initial approximations of equation (3.1)

$$\left(\frac{\partial}{\partial t}u_1\right)+\left(\frac{\partial}{\partial t}u_0\right)+r\left(\frac{\partial^3}{\partial y^2\partial x}u_0\right)+3au_0^2\left(\frac{\partial}{\partial t}u_0\right)+b\left(\frac{\partial^3}{\partial x^2\partial t}u_0\right)=0,$$
$$b\left(\frac{\partial^3}{\partial x^2\partial t}u_1\right)+6au_0u_1\left(\frac{\partial}{\partial x}u_0\right)+\left(\frac{\partial}{\partial t}u_2\right)$$
$$+3au_0^2\left(\frac{\partial}{\partial x}u_1\right)+r\left(\frac{\partial^3}{\partial y^2\partial x}u_1\right)=0,$$
$$r\left(\frac{\partial^3}{\partial y^2\partial x}u_2\right)+3au_1^2\left(\frac{\partial}{\partial x}u_0\right)+6au_0u_1\left(\frac{\partial}{\partial x}u_1\right)+\left(\frac{\partial}{\partial t}u_3\right)$$
$$+b\left(\frac{\partial}{\partial x^2\partial t}u_2\right)+6au_0u_2\left(\frac{\partial}{\partial t}u_0\right)+3au_0^2\left(\frac{\partial}{\partial x}u_2\right)=0.$$

If the first three approximations are sufficient, we will obtain:

$$u(x,y,t)=\lim_{p\to 1}U(x,y,t)=\sum_{k=0}^{3}u_k(x,y,t). \tag{3.4}$$

4. Application

Firstly, we consider the solutions of equation (1.1) with the initial condition [10]:

$$u(x,y,0)=-\sqrt{-\frac{2c}{a}}\tanh\left[\sqrt{-\frac{c}{bc-r}}(x+y)\right],\quad \text{where } \frac{c}{bc-r}<0. \tag{4.1}$$

To calculate the terms of the homotopy series (3.4) for $u(x, y, t)$, we substitute the initial conditions (4.1) into the system (4.2), and finally using Maple, the solutions of the equation can be obtained as follows:

$$u_0=-\sqrt{-\frac{2c}{a}}\tanh\left[\sqrt{-\frac{c}{bc-r}}(x+y)\right], \tag{4.2}$$

$$\begin{aligned}u_1=&(32\sqrt{2}r\tanh(\sqrt{2}x+\sqrt{2}y)^2+24\sqrt{2}a\tanh(\sqrt{2}x+\sqrt{2}y)^2\\&-24\sqrt{2}a\tanh(\sqrt{2}x+\sqrt{2}y)^4-8r\sqrt{2}-24\sqrt{2}r\tanh(\sqrt{2}x+\sqrt{2}y)^4)t\end{aligned} \tag{4.3}$$

$$\begin{aligned}u_2=&-32(192a\sinh(\sqrt{2}(x+y))\cosh(\sqrt{2}(x+y))^2r\\&-54a^2\sinh(\sqrt{2}(x+y))+120r^2\sinh(\sqrt{2}(x+y))\cosh(\sqrt{2}(x+y))^2)\\&-8r^2\sinh(\sqrt{2}(x+y))\cosh(\sqrt{2}(x+y))^4\end{aligned} \tag{4.4}$$

$$+ 72a^2 \sinh(\sqrt{2}(x+y)) \cosh(\sqrt{2}(x+y))^2 - 180r^2 \sinh(\sqrt{2}(x+y))$$
$$- 18a^2 \sinh(\sqrt{2}(x+y)) \cosh(\sqrt{2}(x+y))^4$$
$$- 24r \sinh(\sqrt{2}(x+y)) \cosh(\sqrt{2}(x+y))^4 a$$
$$- 234a \sinh(\sqrt{2}(x+y))r)t^2/$$
$$\cosh(\sqrt{2}(x+y))^7 - 32(-30b\sqrt{2}\cosh(\sqrt{2}(x+y))^3)r$$
$$- 33b\sqrt{2}\cosh(\sqrt{2}(x+y))^3 a + 6b\sqrt{2}\cosh(\sqrt{2}(x+y))^5 a$$
$$+ 4b\sqrt{2}\cosh(\sqrt{2}(x+y))^5 r + 30b\sqrt{2}\cosh(\sqrt{2}(x+y))r$$
$$+ 30b\sqrt{2}\cosh(\sqrt{2}(x+y))a)t/\cosh(\sqrt{2}(x+y))^7,$$

$$\begin{aligned}
(4.5)\quad u_3 = {} & \frac{1}{3}((110592ra^2\sqrt{2} + 73728r^2a\sqrt{2} + 55296a^3\sqrt{2} + 16384r^3\sqrt{2}) \\
& \cdot \cosh(8\sqrt{2}x + 8\sqrt{2}y) \\
& + (-11280384ra^2\sqrt{2} - 8224768r^3\sqrt{2} - 16957440r^2a\sqrt{2} \\
& - 2543616a^3\sqrt{2})\cosh(6\sqrt{2}x + 6\sqrt{2}y) \\
& + (-669966336ra^2\sqrt{2} - 61157376a^3\sqrt{2} - 1445625856r^3\sqrt{2} \\
& - 2054504448r^2a\sqrt{2})\cosh(2\sqrt{2}x + 2\sqrt{2}y) \\
& + (239337472r^3\sqrt{2} + 378667008r^2a\sqrt{2} + 20791296a^3\sqrt{2} \\
& + 160137216ra^2\sqrt{2})\cosh(4\sqrt{2}x + 4\sqrt{2}y) + 42854400a^3\sqrt{2} \\
& + 1764679680r^2a\sqrt{2} + 528076800ra^2\sqrt{2} \\
& + 12795088480r^3\sqrt{2}t^3/(\cosh(10\sqrt{2}x + 10\sqrt{2}y) \\
& + 10\cosh(8\sqrt{2}x + 8\sqrt{2}y) + 45\cosh(6\sqrt{2}x + 6\sqrt{2}y) \\
& + 120\cosh(4\sqrt{2}x + 4\sqrt{2}y) + 210\cosh(2\sqrt{2}x + 2\sqrt{2}y) + 126) \\
& + \frac{1}{3}(556695552r^2b + 648806400rba + 96657408ba^2) \\
& \cdot \sinh(2\sqrt{2}x + 2\sqrt{2}y)/(\cosh(10\sqrt{2}x + 10\sqrt{2}y) \\
& + 10\cosh(8\sqrt{2}x + 8\sqrt{2}y) + 45\cosh(6\sqrt{2}x + 6\sqrt{2}y) \\
& + 120\cosh(4\sqrt{2}x + 4\sqrt{2}y) + 210\cosh(2\sqrt{2}x + 2\sqrt{2}y) + 126) \\
& + \frac{1}{3}(18874368rba + 12091392r^2b + 6414336ba^2) \\
& \cdot \cosh(8\sqrt{2}x + 8\sqrt{2}y)/(\cosh(10\sqrt{2}x + 10\sqrt{2}y) \\
& + 10\cosh(8\sqrt{2}x + 8\sqrt{2}y) + 45\cosh(6\sqrt{2}x + 6\sqrt{2}y) \\
& + 120\cosh(4\sqrt{2}x + 4\sqrt{2}y) + 210\cosh(2\sqrt{2}x + 2\sqrt{2}y) + 126) \\
& + \frac{1}{3}(-110592ba^2 - 147456rba - 49152r^2b)\sinh(8\sqrt{2}x + 8\sqrt{2}y)/ \\
& (\cosh(10\sqrt{2}x + 10\sqrt{2}y) + 10\cosh(8\sqrt{2}x + 8\sqrt{2}y) \\
& + 45\cosh(6\sqrt{2} + 6\sqrt{2} + 120\cosh(4\sqrt{2}x + 4\sqrt{2}y) \\
& + 210\cosh(2\sqrt{2}x + 2\sqrt{2}) + 126)t^2
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{3}(12288b^2\sqrt{2}r + 18432b^2\sqrt{2}a)\cosh(8\sqrt{2}x + 8\sqrt{2}y)\\
&+(-1585152b^2\sqrt{2}a - 1449984b^2\sqrt{2}r)\cosh(6\sqrt{2}x + 6\sqrt{2}y)\\
&+(-1892352b^2\sqrt{2}r - 1363968b^2\sqrt{2}a)\cosh(2\sqrt{2}x + 2\sqrt{2}y)\\
&+(11501568b^2\sqrt{2}a + 11698176b^2\sqrt{2}r)\cosh(4\sqrt{2}x + 4\sqrt{2}y)\\
&-14469120b^2\sqrt{2}a - 15052800b^2\sqrt{2}r)t/(\cosh(10\sqrt{2}x + 10\sqrt{2}y)\\
&+10\cosh(8\sqrt{2}x + 8\sqrt{2}y) + 45\cosh(6\sqrt{2}x + 6\sqrt{2}y)\\
&+120\cosh(4\sqrt{2}x + 4\sqrt{2}y) + 210\cosh(2\sqrt{2}x + 2\sqrt{2}y) + 126).
\end{aligned}
$$

In this manner the other components can be easily obtained. Substituting equations (4.2)–(4.5) into (3.4):

$$
u(x,y,t) = u_0(x,y,t) + u_1(x,y,t) + u_2(x,y,t) + u_3(x,y,t) + \ldots
$$

Using Taylor series, we obtain the closed form solutions as follows:

$$
u(x,y,t) = -\sqrt{-\frac{2c}{a}}\tanh\left[\sqrt{-\frac{c}{bc-r}}(x+y-ct)\right], \quad \frac{c}{bc-r} < 0.
$$

With initial conditions (4.1), the solitary wave solutions of equation (1.1) are in full agreement with the ones constructed by Mustafa Inc [10].

(x,y,t)	$\lvert u_{\mathrm{exact}} - u_{\mathrm{homotopy}}\rvert$	$\lvert u_{\mathrm{exact}} - u_{\mathrm{homotopy}}\rvert/u_{\mathrm{homotopy}}$
(8,8,0.1)	1E–09	5E–10
(8,8,0.2)	9E–09	4.5E–09
(8,8,0.3)	1E–09	5E–10
(8,8,0.4)	2E–09	1E–09
(8,8,0.5)	2E–09	1E–10
(13,13,0.1)	1E–09	5E–10
(13,13,0.2)	9E–09	4.5E–09
(13,13,0.3)	1E–09	5E–10
(13,13,0.4)	2E–09	1E–09
(13,13,0.5)	2E–09	1E–10
(10,10,0.1)	1E–09	5E–10
(10,10,0.2)	9E–09	4.5E–09
(10,10,0.3)	1E–09	5E–10
(10,10,0.4)	2E–09	1E–09
(10,10,0.5)	2E–09	1E–09

Table 1. The HPM results for $u(x,y,t)$ for the first three approximations in comparison with the analytical solutions when $a = 1$, $b = -1$, $c = -2$, $r = 1$, for the solitary wave solutions with the initial conditions (3.2) of equation (1.1).

5. Discussion and conclusion

In this paper, the homotopy perturbation method (HPM) was used for finding soliton solutions of a Zakharov–Kuznetsov Modified Equal Width (ZK-MEW) equation with initial conditions. The obtained solutions are compared with the extended tanh-method [10]. To demonstrate the convergence of the HPM, the results of the numerical example are presented and only few terms are required to obtain accurate solutions. The accuracy of the HPM for the Zakharov–Kuznetsov Modified Equal Width (ZK-MEW) equation is controllable, and absolute errors are very small with the present choice of t and x, y. These results are listed in Tables 1, it is seen that the implemented method achieves a minimum accuracy for the first three approximations for the initial conditions (4.1). It is also evident that when more terms for the HPM are computed the numerical results get much more closer to the corresponding exact solutions with the initial conditions (4.1) of equation (1.1).

REFERENCES

[1] J. BIAZAR, M. ESLAMI AND H. GHAZVINI, *Homotopy perturbation method for systems of partial differential equations*, Internat. J. Nonlinear Sci. **8** (2007), 413–418.

[2] D. D. GANJI AND A. SADIGHI, *Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations*, Internat. J. Nonlinear Sci. **7** (2006), 411–418.

[3] M. GORJI, D. D. GANJI AND S. SOLEIMANI, *New Application of He's Homotopy Perturbation Method*, Int. J. Nonlinear Sci. **8(3)** (2007), 319–328.

[4] J. H. HE, *Homotopy perturbation technique J. Comput*, Math. Appl. Mech. Eng (1999), 257–262.

[5] ———, *The homotopy perturbation method for nonlinear oscillators with discontinuities*, Appl. Math. Comput. **151** (2004), 287–292.

[6] ———, *Comparison of homotopy perturbation method and homotopy analysis method*, Appl. Math. Comput. **156** (2004), 527–539.

[7] ———, *Homotopy perturbation method for bifurcation of nonlinear problems*, Internat. J. Nonlinear Sci. Numer. Simul. **6** (2005), 207–208.

[8] ———, *Homotopy perturbation method for bifurcation of nonlinear problems*, Internat. J. Nonlinear Sci. Numer. Simul. (2006), 87–88.

[9] ———, *Some asymptotic methods for strongly nonlinear equations*, Internat. J. Modern Phys. **20** (2006), 2561–2568.

[10] M. INC, *New exact solutions for the ZK-MEW equations by using symbolic computation*, Appl. Math. Comput. (2007), in press.

[11] T. OZIS AND A. YILDIRIM, *Traveling wave solution of Korteweg-de Vries equation using He's homotopy perturbation method*, Int. J. Nonlinear Sci. **8(2)** (2007), 239–242.

[12] M. RAFEI AND D. D. GANJI, *Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method*, Int. J. Nonlinear Sci. **7** (2006), 321–328.

[13] A. SADIGHI AND D. D. GANJI, *Solution of the generalized nonlinear Boussinesq equation using homotopy perturbation and variational iteration methods*, Internat. J. Nonlinear Sci. **8** (2007), 435–450.

[14] H. Tari, D. D. Ganji and M. Rostamian, *Approximate solutions of $K(2,2)$, KdV and modified KdV equations by variational iteration method, homotopy perturbation method and homotopy analysis method*, Internat. J. Nonlinear Sci. **8** (2007), 203–210.

[15] A. M. Wazwaz, *Compact and noncompact physical structures for ZK-BBM equation*, Appl. Math.Comput **169** (2005), 713–725.

[16] ______, *The tanh and the sine-cosine methods for a reliable of the modified equal width equation and itsvariants*, Commun Nonlinear Sci. and Numer. Simu **1** (2006), 148–160.

[17] E. Yusufoglu, *Homotopy Perturbation Method for Solving a Nonlinear System of Second Order Boundary Value Problems*, Int. J. Nonlinear Sci. **8(3)** (2007), 353–358.

[18] V. E. Zakharov and E. A. Kuznetsov, *On three-dimensional solitons*, Soviet Phys. **39** (1974), 285–288.

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