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ERGODIC COCYCLES FOR GAUSSIAN ACTIONS

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Abstract. Ergodic Gaussian cocycles for rigid Gaussian actions are constructed. It is also shown when any isomorphism between Gaussian actions is Gaussian.

1. Introduction

Throughout $\mathbb{G}$ denotes a countable Abelian group with identity element e and discrete topology. Assume that $(X,\mathcal{B},\mu)$ is a standard probability space and $\mathcal{T}\colon\mathbb{G}\times X\to X$ ($T^g(\,\cdot\,)=\mathcal{T}(g,\,\cdot\,)$) is a free $\mathbb{G}$-action on $(X,\mathcal{B},\mu)$. Let $\mathbb{A}$ be a locally compact metric Abelian group with Borel σ-algebra $\mathcal{B}_{\mathbb{A}}$ and Haar measure λ. Recall that a Borel map $F\colon\mathbb{G}\times X\to\mathbb{A}$ is called a cocycle for $\mathcal{T}$ if $F(g_1+g_2,x)=F(g_1,x)+F(g_2,T^{g_1}x)$ for all $g_1,g_2\in\mathbb{G}$ and for a.e. $x\in X$. A cocycle F is said to be a coboundary if there exists a Borel map $\xi\colon X\to\mathbb{A}$ such that $F(g,x)=\xi(x)-\xi(T^gx)$. To a cocycle F we associate the corresponding skew product $\mathcal{T}_F\colon\mathbb{G}\times(X\times\mathbb{A},\mathcal{B}\otimes\mathcal{B}_{\mathbb{A}},\mu\times\lambda)\to(X\times\mathbb{A},\mathcal{B}\otimes\mathcal{B}_{\mathbb{A}},\mu\times\lambda)$ given by $T_F^g(x,a)=(T^gx,F(g,x)+a)$. We say that F is ergodic if $\mathcal{T}_F$ is ergodic.

An action $\mathcal{T}$ is called Gaussian if there exists $\mathcal{H}\subset L^2(X,\mathcal{B},\mu)$ a $\mathcal{T}$-invariant closed subspace of the zero mean real functions such that each nonzero $h\in\mathcal{H}$ is a Gaussian variable and the smallest σ-algebra $\mathcal{B}(\mathcal{H})$ which makes all variables of $\mathcal{H}$ measurable equals $\mathcal{B}$. We call $\mathcal{H}$ a Gaussian space of $\mathcal{T}$. The maximal spectral type of $\mathcal{T}$ on $\mathcal{H}$ is called the spectral measure of $\mathcal{T}$. We will consider $\mathcal{T}$ with

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continuous spectral measure. It implies that $\mathcal{T}$ will be ergodic and even weakly mixing. We call a cocycle F Gaussian if $F(g,\,\cdot\,) \in \mathcal{H}$ for all $g \in \mathbb{G}$. A Gaussian cocycle is a Gaussian coboundary if it is a coboundary with transfer function ξ in $\mathcal{H}$.

Motivated by some strong dichotomies in the theory of Gaussian dynamical systems (see [1], [2], [4], [5], [12]) in 1999 Lemańczyk proposed the following conjecture: every Gaussian cocycle either is ergodic or is a Gaussian coboundary. There are Gaussian actions with trivial solutions of this, that is, every Gaussian cocycle is a coboundary (see Section 2). So the first step to verify the conjecture is a construction of ergodic Gaussian cocycles. Such a construction is done in [3] for Gaussian $\mathbb{Z}$-actions, where cocycles are identified with single measurable functions. But if we replace $\mathbb{Z}$-actions by $\mathbb{G}$-actions then we have an additional difficulty, namely, we do not know whether there exist nontrivial cocycles, because of the more complicated structure of them. There are results for different types of $\mathbb{Z}^d$-actions ($d > 1$) showing, in contrast to $\mathbb{Z}$-actions, that if we impose some restriction on cocycles (for instance continuity) then the only cocycles are trivial (see [9] and the references given there). We show that, as a rule, this is not true for Gaussian $\mathbb{G}$-actions and Gaussian cocycles. Section 3 is devoted to constructing ergodic Gaussian cocycles for some rigid Gaussian $\mathbb{G}$-actions. The validity of the conjecture was not decided in [3] but authors proved it in its multiplicative version considering cocycles of the form $\mathrm{e}^{2\pi\mathrm{i}h}$, where h is a Gaussian cocycle. It is easy to check that the analogous result holds for Gaussian $\mathbb{G}$-actions.

Section 4 can be treated as an appendix. We give some condition, under which, any isomorphism between Gaussian actions is Gaussian, i.e. it sends the Gaussian structure of one action to the other. We generalize Theorem 5 from [14] (given without proof). Although our result has not a direct connection with Gaussian cocycles, we think, it is sufficiently interesting to placing in the paper about Gaussian actions.

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2. Preliminaries

Let $\mathcal{T}$ be a free ergodic $\mathbb{G}$-action on a nonatomic standard probability space $(X, \mathcal{B}, \mu)$, and let $F\colon \mathbb{G} \times X \to \mathbb{A}$ be a cocycle for $\mathcal{T}$. Since $\mathbb{G}$ is Abelian, it follows that

$$F(g_1, x) - F(g_1, T^{g_2}x) = F(g_2, x) - F(g_2, T^{g_1}x) \tag{2.1}$$

for all $g_1, g_2 \in \mathbb{G}$ and $x \in X$.

Conversely, suppose that (2.1) holds and $F(g,\,\cdot\,)$ are real functions with zero mean. Consider $N_{g_1,g_2}(x) = F(g_1+g_2,x) - F(g_1,x) - F(g_2,T^{g_1}x)$. We have

$$\begin{aligned}
N_{g_1,g_2}(x) - N_{g_1,g_2}(T^g x) &= (F(g_1+g_2,x) - F(g_1+g_2,T^g x)) \\
&\quad - (F(g_1,x) - F(g_1,T^g x)) \\
&\quad - (F(g_2,T^{g_1}x) - F(g_2,T^g T^{g_1}x)) \\
&= (F(g,x) - F(g,T^{g_1+g_2}x)) - (F(g,x) - F(g,T^{g_1}x)) \\
&\quad - (F(g,T^{g_1}x) - F(g,T^{g_2}T^{g_1}x)) = 0
\end{aligned}$$

for all $g \in \mathbb{G}$. Hence $N_{g_1,g_2} = \text{const} = 0$ for all $g_1,g_2 \in \mathbb{G}$, and consequently F is a cocycle.

Let $\overline{\mathbb{A}} = \mathbb{A} \cup \{\infty\}$ be the one-point Alexandroff compactification of $\mathbb{A}$ (for compact $\mathbb{A}$, $\overline{\mathbb{A}} = \mathbb{A}$). Recall from [13] that $a \in \overline{\mathbb{A}}$ is called an essential value of F if for every open neighbourhood U of a, and for every $B \in \mathcal{B}$ with $\mu(B) > 0$, there exists $g \in \mathbb{G}$ such that $\mu(B \cap T^g B \cap \{x \in X : F(g,x) \in U\}) > 0$. The set of essential values of F will be denoted by $\overline{E}(F)$. The set $E(F) = \overline{E}(F) \cap \mathbb{A}$ is a closed subgroup of $\mathbb{A}$. It is shown in [13] that F is ergodic if and only if $E(F) = \mathbb{A}$. A sequence $(g_t)_{t=1}^{\infty} \subset \mathbb{G}$ is said to be a rigidity time for $\mathcal{T}$ if for each measurable function f on X, $f \circ T^{g_t} \to f$ in measure. We need an easy generalization of Proposition 12 from [6] and we briefly prove it for reader's convenience.

Proposition 2.1. *Let $(g_t)_{t=1}^{\infty}$ be a rigidity time for $\mathcal{T}$. If $F\colon \mathbb{G} \times X \to \mathbb{A}$ is a cocycle satisfying*

(a) *for all $\varepsilon > 0$ there exists a compact set $K \subset \mathbb{A}$ such that, for all $t \in \mathbb{N}$, $\mu(\{x \in X : F(g_t,x) \in K\}) > 1 - \varepsilon$,*

(b) *for all $\chi \in \widehat{\mathbb{A}}$, $\chi \not\equiv 1$ there exists $C > 0$ such that $|\int_X \chi(F(g_t,x))\,d\mu(x)| \le C < 1$ for almost all $t \in \mathbb{N}$,*

then F is ergodic.

Proof. We can assume that the sequence of measures $\mu \circ F(g_t,\,\cdot\,)^{-1}$ converges weakly to some probability measure ν on $\overline{\mathbb{A}}$. We first claim that for each continuous function φ on $\overline{\mathbb{A}}$ and each $h \in L^2(X,\mathcal{B},\mu)$,

$$\int_X \varphi(F(g_t,x)) \cdot h(x)\,d\mu(x) \to \int_{\overline{\mathbb{A}}} \varphi(a)\,d\nu(a) \int_X h(x)\,d\mu(x). \tag{2.2}$$

Indeed, (2.2) holds for constant functions h and we can restrict to the case $h = \xi - \xi \circ T^g$ for $\xi \in L^2(X,\mathcal{B},\mu)$ with zero mean. Then

$$\int_X \varphi(F(g_t,x)) \cdot h(x)\,d\mu(x) = \int_X (\varphi(F(g_t,T^g x)) - \varphi(F(g_t,x))) \cdot \xi(T^g x)\,d\mu(x).$$

Since $F(g_t,x) - F(g_t,T^g x) = F(g,x) - F(g,T^{g_t}x)$ and $(g_t)_{t=1}^{\infty}$ is a rigidity time for $\mathcal{T}$, the integral goes to 0, as φ is uniformly continuous.

Now let us take $a_0 \in \operatorname{supp}\nu$, an open neighbourhood U of a_0 in $\overline{\mathbb{A}}$ and $B \in \mathcal{B}$ with $\mu(B) > 0$. Choose a continuous function φ on $\overline{\mathbb{A}}$ with $0 \leq \varphi \leq 1_U$ and $\int_{\overline{\mathbb{A}}} \varphi(a)\, d\nu(a) > 0$. Since $(g_t)_{t=1}^{\infty}$ is a rigidity time for $\mathcal{T}$,

$$\begin{aligned}\liminf & \mu(B \cap T^{g_t} B \cap \{x \in X : F(g_t, x) \in U\}) \\ &= \liminf \mu(B \cap \{x \in X : F(g_t, x) \in U\}) = \liminf \int_B 1_U(F(g_t, x))\, d\mu(x) \\ &\geq \liminf \int_B \varphi(F(g_t, x))\, d\mu(x) = \mu(B) \int_{\overline{\mathbb{A}}} \varphi(a)\, d\nu(a) > 0.\end{aligned}$$

Hence $a_0 \in \overline{E}(F)$. Consequently $\operatorname{supp}\nu \subset \overline{E}(F)$. From (a) it follows that $\nu(\{\infty\}) = 0$. Thus $\operatorname{supp}\nu \subset E(F)$. If F is not ergodic then there exists $\chi_0 \in \widehat{\mathbb{A}}$ such that $\chi_0 \not\equiv 1$ and $\chi_0(a) = 1$ for all $a \in E(F)$. Hence

$$\int_X \chi_0(F(g_t, x))\, d\mu(x) \to \int_{\mathbb{A}} \chi_0(a)\, d\nu(a) = 1,$$

contrary to (b). □

Remark 2.2. In the case of $\mathbb{A} = \mathbb{R}$, the condition (a) is satisfied if, for example, the sequence $(F(g_t, \cdot))_{t=1}^{\infty}$ is bounded in $L^1(X, \mathcal{B}, \mu)$.

Now we recall basic definitions from the spectral theory of unitary operators. Let U be a unitary representation of $\mathbb{G}$ on a real Hilbert space H. Given $h \in H$, we denote by $\mathbb{G}(h)$ the cyclic space generated by h, that is, the smallest closed subspace containing $U^g h$, $g \in \mathbb{G}$. By the spectral measure of h we mean the measure σ_h on $\widehat{\mathbb{G}}$, the dual group of $\mathbb{G}$, determined by $\int_{\widehat{\mathbb{G}}} \chi(g) d\sigma_h(\chi) = \langle U^g h, h\rangle$. There exists $h_0 \in H$ such that $\sigma_h \ll \sigma_{h_0}$ for every $h \in H$. The type of σ_{h_0} is called the maximal spectral type of U. A measure σ, absolutely continuous with respect to σ_{h_0}, has the multiplicity n if there exists a maximal sequence $\mathbb{G}(h_1) \oplus \ldots \oplus \mathbb{G}(h_n)$ such that $\sigma_{h_i} \equiv \sigma$, i.e. there is no element of type σ orthogonal to the sum. A number $n \in \{1, 2, \ldots\} \cup \{\infty\}$ is an essential value of the multiplicity function of U if there exists $\sigma \ll \sigma_{h_0}$ with multiplicity n. We say that U has simple spectrum if 1 is the only essential value (further details can be easily adapted from the case of $\mathbb{Z}$-representation, see [10], [11]).

We will consider a standard Gaussian action. If σ is a finite symmetric Borel measure on $\widehat{\mathbb{G}}$, then on the space $X_\sigma = \mathbb{R}^{\mathbb{G}}$ endowed with the natural Borel structure $\mathcal{B}_\sigma$ there exists μ_σ, a measure such that projections $\{Y_g\}_{g \in \mathbb{G}}$ ($Y_g(x) = x(g)$, $x \in X_\sigma$) form a real stationary centered Gaussian process whose spectral measure is σ, i.e. $\int_{\widehat{\mathbb{G}}} \chi(g)\, d\sigma(\chi) = \int_{X_\sigma} Y_g Y_e\, d\mu_\sigma$ for all $g \in \mathbb{G}$. If we denote by $\mathcal{T}_\sigma$ the $\mathbb{G}$-action on $(X_\sigma, \mathcal{B}_\sigma, \mu_\sigma)$ given by $(T^g x)(s) = x(s + g)$ then $\mathcal{T}_\sigma$ is a Gaussian action with Gaussian space $\mathcal{H}_\sigma = \mathbb{G}(Y_e)$. In the case of $\mathbb{Z}$-action we speak about a Gaussian automorphism T_σ. Write $L^2_{\mathrm{her}}(\widehat{\mathbb{G}}, \sigma) = \{f \in L^2(\widehat{\mathbb{G}}, \sigma) : f(\overline{\chi}) = \overline{f(\chi)}\}$. The corresponding Koopman representation $U_{\mathcal{T}_\sigma}$ ($U^g_{\mathcal{T}_\sigma} f = f \circ T^g$)

on $\mathcal{H}_\sigma$ is unitarilly equivalent to the representation V on $L^2_{\mathrm{her}}(\widehat{\mathbb{G}},\sigma)$ given by $V^g f(\chi) = \chi(g) f(\chi)$.

LEMMA 2.3. *Assume that*

$$\frac{1-\chi(g)}{1-\chi(g_0)} \in L^2_{\mathrm{her}}(\widehat{\mathbb{G}},\sigma)$$

for all $g \in \mathbb{G}$ *and some* $g_0 \in \mathbb{G}$. *There exists a Gaussian cocycle* F *such that* $F(g,\,\cdot\,)$ *corresponds to the function* $(1-\chi(g))/(1-\chi(g_0))$ *by the unitary equivalence between* $U_{\mathcal{T}_\sigma}$ *and* V.

PROOF. Let $f_g(\chi) = (1-\chi(g))/(1-\chi(g_0))$. We have

$$f_{g_1}(\chi) - \chi(g_2) f_{g_1}(\chi) = f_{g_2}(\chi) - \chi(g_1) f_{g_2}(\chi).$$

Therefore functions $F(g,\,\cdot\,) \in \mathcal{H}_\sigma$ corresponding to f_g satisfy (2.1). Hence F is a cocycle. □

LEMMA 2.4. *Assume that*

$$\frac{1}{1-\chi(g_0)} \in L^\infty(\widehat{\mathbb{G}},\sigma)$$

for some $g_0 \in \mathbb{G}$. *Then each Gaussian cocycle for* $\mathcal{T}_\sigma$ *is a coboundary.*

PROOF. Let F be a Gaussian cocycle. By (2.1), the corresponding functions $f_g \in L^2_{\mathrm{her}}(\widehat{\mathbb{G}},\sigma)$ satisfy $f_g(\chi)(1-\chi(g_0)) = f_{g_0}(\chi)(1-\chi(g))$. Thus

$$f_g(\chi) = \frac{f_{g_0}(\chi)}{1-\chi(g_0)} - \chi(g)\frac{f_{g_0}(\chi)}{1-\chi(g_0)}.$$

Since $(f_{g_0}(\chi))/(1-\chi(g_0)) \in L^2_{\mathrm{her}}(\widehat{\mathbb{G}},\sigma)$, F is a coboundary. □

As an example, we take a generalization of well known Gaussian–Kronecker $\mathbb{Z}$-action (see [4]). A subset E of $\widehat{\mathbb{G}}$ is called a Kronecker set if for every continuous function f on E, $|f| = 1$, and for every $\varepsilon > 0$, there exist $g \in \mathbb{G}$ such that $\sup_{\chi\in E} |f(\chi) - \chi(g)| < \varepsilon$. Assume that σ is concentrated on $E \cup \overline{E}$, where $E \subset \widehat{\mathbb{G}}$ is a Kronecker set. Let f be the constant function equals -1. Then there exists $g_0 \in \mathbb{G}$ such that, for all $\chi \in E$, $|1+\chi(g_0)| = |f(\chi) - \chi(g_0)| < 1$. Since $|2 - |1-\chi(g_0)|| \le |1+\chi(g_0)|$, we have $|1-\chi(g_0)| > 1$. It follows that $1/(1-\chi(g_0)) \in L^\infty(\widehat{\mathbb{G}},\sigma)$, and each Gaussian cocycle for $\mathcal{T}_\sigma$ is a coboundary.

We will need an auxiliary result on L^2 spaces generated by processes (see [3, Corollary 1]).

LEMMA 2.5. *Let* $H \subset L^2(X,\mathcal{B},\mu)$ *be a real subspace such that* $\mathcal{B}(H) = \mathcal{B}$. *Then*

$$\operatorname{span}(\{e^{2\pi i h} : h \in H\}) = L^2(X,\mathcal{B},\mu).$$

3. Construction of ergodic cocycles

Assume that $\mathbb{G}$ is not a torsion group and fix an element $g_0 \in \mathbb{G}$ of infinite order.

Proposition 3.1. *Let σ be a finite continuous symmetric Borel measure on $\widehat{\mathbb{G}}$. If there exists an increasing sequence $(n_t)_{t=1}^{\infty} \subset \mathbb{N}$ such that*

(a) *for all $g \in \mathbb{G}$ there exists $M_g > 0$ such that*

$$\frac{|\chi(g)-1|}{|\chi(g_0)-1|} \le M_g \quad \sigma\text{-a.e.}$$

(b) $\int_{\widehat{\mathbb{G}}} |\chi^{n_t}(g_0)-1|^2 \, d\sigma(\chi) \to 0,$

(c) *there exist $C, D > 0$ such that, for all $t \in \mathbb{N}$,*

$$D \le \int_{\widehat{\mathbb{G}}} \left(\frac{|\chi^{n_t}(g_0)-1|}{|\chi(g_0)-1|} \right)^2 d\sigma(\chi) \le C,$$

then there exists an ergodic Gaussian cocycle for the Gaussian $\mathbb{G}$-action $\mathcal{T}_\sigma$.

Proof. According to (a), we obtain

$$f_g(\chi) = \frac{1-\chi(g)}{1-\chi(g_0)} \in L^2_{\mathrm{her}}(\widehat{\mathbb{G}}, \sigma)$$

for all $g \in \mathbb{G}$, and from Lemma 2.3 it follows that corresponding functions $F(g, \cdot\,) \in \mathcal{H}_\sigma$ form a cocycle. From (b), we have $V^{n_t g_0} f \to f$ in $L^2_{\mathrm{her}}(\widehat{\mathbb{G}}, \sigma)$. Hence $U^{n_t g_0}_{\mathcal{T}_\sigma} h \to h$ for each $h \in \mathcal{H}_\sigma$, and we conclude from Lemma 2.5 that $(n_t g_0)$ is a rigidity time for $\mathcal{T}_\sigma$. The condition (c) means that, for all $t \in \mathbb{N}$,

$$D \le \|F(n_t g_0, \cdot\,)\|^2_{\mathcal{H}_\sigma} \le C.$$

Since $F(n_t g_0, \cdot\,)$ is a Gaussian variable,

$$\left| \int_X \mathrm{e}^{2\pi \mathrm{i} r F(n_t g_0, x)} \, d\mu(x) \right| = \mathrm{e}^{-2(\pi r)^2 \|F(n_t g_0, \cdot\,)\|^2_{\mathcal{H}_\sigma}} \le \mathrm{e}^{-2(\pi r)^2 D} < 1$$

for all $r \neq 0$. It follows from Proposition 2.1 that F is an ergodic cocycle. □

Now we construct a class of measures satisfying the assumptions of Proposition 3.1. We will often replace the Euclidean distance of two points from the circle $\mathbb{T}$ by the equivalent distance ϱ given by the length of the shorter arc joining them. Let $(g_t)_{t=1}^{\infty}$ be a sequence of all elements of $\mathbb{G} \setminus \{g_0\}$. Assume that $(a_t)_{t=1}^{\infty} \subset \mathbb{R}$ is a sequence such that $a_t \to \infty$. For $t \in \mathbb{N}$ and $a, b \in \mathbb{R}$ satisfying $0 < (b/2) \le a < b < \pi$, write

$$B^{(a,b)} = \{\chi \in \widehat{\mathbb{G}} : a < \varrho(\chi(g_0), 1) < b\}, \quad A_t^{(a)} = \{\chi \in \widehat{\mathbb{G}} : \varrho(\chi(g_t), 1) < a a_t\}.$$

The set

$$B^{(a,b)} \cap \bigcap_{t \in \mathbb{N}} A_t^{(a)} = B^{(a,b)} \cap \bigcap_{\{t \in \mathbb{N} : a_t \le \frac{\pi}{a}\}} A_t^{(a)} \tag{3.1}$$

is open as finite intersection of open sets. We will find a sequence $(a_t)_{t=1}^{\infty}$ such that the set (3.1) is not empty for all $b \in (0,\pi)$ and $a \in [(b/2), b)$. Fix $t \in \mathbb{N}$. Let g_{i_1} be an element of infinite order of $\{g_1, \dots, g_{t-1}\}$ with the smallest index such that $\{g_0, g_{i_1}\}$ is independent, g_{i_2} be an element of infinite order of $\{g_{i_1+1}, \dots, g_{t-1}\}$ with the smallest index such that $\{g_0, g_{i_1}, g_{i_2}\}$ is independent, and we continue this procedure maximal times getting the independent set $\{g_0, \dots, g_{i_{m_t}}\}$, $m_t \le t-1$. If $\{g_0, \dots, g_{i_{m_t}}, g_t\}$ is independent then $a_t = t$, otherwise there exist $k_0^{(t)}, \dots, k_{i_{m_t}}^{(t)}, k_t^{(t)} \in \mathbb{Z}$ such that

$$k_0^{(t)} g_0 + k_{i_1}^{(t)} g_{i_1} + \ldots + k_{i_{m_t}}^{(t)} g_{i_{m_t}} + k_t^{(t)} g_t = e, \quad k_t^{(t)} \neq 0, \tag{3.2}$$

and we put $a_t = \max\{t, 2((|k_0^{(t)}|)/(|k_t^{(t)}|))\}$. In general, if $\{g_1, \dots, g_t\}$ is a dependent set of elements of infinite order such that $\{g_1, \dots, g_{t-1}\}$ is independent and we assume that

$$k_1 g_1 + \ldots + k_{t-1} g_{t-1} + k_t g_t = e = l_1 g_1 + \ldots + l_{t-1} g_{t-1} + l_t g_t,$$

then

$$l_t(k_1 g_1 + \ldots + k_{t-1} g_{t-1}) = k_t(l_1 g_1 + \ldots + l_{t-1} g_{t-1}).$$

Hence $(k_i/k_t) = (l_i/l_t)$ for each $i = 1, \dots, t-1$. It follows that $(a_t)_{t=1}^{\infty}$ is well defined.

Next we define some character χ_0 of the set (3.1) describing principal arguments α_t of $\chi_0(g_t)$, $t \ge 0$. As α_0 we take some element of (a, b). Fix $t \in \mathbb{N}$. If $\{g_0, g_{i_1}, \dots, g_{i_{m_t}}, g_t\}$ is independent then $\alpha_t = 0$, otherwise $\alpha_t = -(k_0^{(t)}/k_t^{(t)})\alpha_0 \pmod{2\pi}$. Let us check that $\chi_0 \in \widehat{\mathbb{G}}$. Let

$$k_1 g_{j_1} + \ldots + k_s g_{j_s} = e, \quad (k_1, \dots, k_s \in \mathbb{Z} \setminus \{0\},\ j_1, \dots, j_s \in \mathbb{N}; s \ge 1). \tag{3.3}$$

It suffices to show that $k_1 \alpha_{j_1} + \ldots + k_s \alpha_{j_s} = 0 \pmod{2\pi}$. We can assume that $\alpha_{j_1}, \dots, \alpha_{j_s}$ are different from zero. Then the last equality may be written as

$$k_1\Bigg(-\frac{k_0^{(j_1)}}{k_{j_1}^{(j_1)}}\alpha_0\Bigg) + \ldots + k_s\Bigg(-\frac{k_0^{(j_s)}}{k_{j_s}^{(j_s)}}\alpha_0\Bigg) = 0 \pmod{2\pi}.$$

Hence it is enough to show that

$$-k_1\frac{k_0^{(j_1)}}{k_{j_1}^{(j_1)}} - \ldots - k_s\frac{k_0^{(j_s)}}{k_{j_s}^{(j_s)}} = 0. \tag{3.4}$$

From (3.3), we obtain $k_{j_1}^{(j_1)} \ldots k_{j_s}^{(j_s)}(k_1 g_{j_1} + \ldots + k_s g_{j_s}) = e$. Then from (3.2). we have

$$\begin{aligned} &k_{j_2}^{(j_2)} \ldots k_{j_s}^{(j_s)} k_1(-k_0^{(j_1)} g_0 - k_{i_1}^{(j_1)} g_{i_1} - \ldots - k_{i_{m_{j_1}}}^{(j_1)} g_{i_{m_{j_1}}}) + \ldots \\ &\quad + k_{j_1}^{(j_1)} \ldots k_{j_{s-1}}^{(j_{s-1})} k_s(-k_0^{(j_s)} g_0 - k_{i_1}^{(j_s)} g_{i_1} - \ldots - k_{i_{m_{j_s}}}^{(j_s)} g_{i_{m_{j_s}}}) = e. \end{aligned}$$

There is a combination of elements of the independent set on the left-hand side. Therefore taking the sum of coefficients of g_0 we get

$$-k_{j_2}^{(j_2)}\dots k_{j_s}^{(j_s)}k_1k_0^{(j_1)}-\dots-k_{j_1}^{(j_1)}\dots k_{j_{s-1}}^{(j_{s-1})}k_sk_0^{(j_s)}=0,$$

and dividing the last equality by $k_{j_1}^{(j_1)}\dots k_{j_s}^{(j_s)}$ we obtain (3.4).

Let us notice that

$$\varrho(\chi_0(g_t),1)\le\left|\frac{k_0^{(t)}}{k_t^{(t)}}\alpha_0\right|<\left|\frac{k_0^{(t)}}{k_t^{(t)}}\right|b\le aa_t$$

for suitable t, and then $\varrho(\chi_0(g_t),1)\le aa_t$ for all $t\in\mathbb{N}$. Hence $\chi_0\in B^{(a,b)}\cap\bigcap_{t\in\mathbb{N}}A_t^{(a)}$. Thus, if we put $M_{g_0}=1$, $M_{g_t}=a_t$ and write

$$R=\left\{\chi\in\widehat{\mathbb{G}}:\frac{\varrho(\chi(g),1)}{\varrho(\chi(g_0),1)}\le M_g\text{ for all }g\in\mathbb{G}\right\},$$

then we conclude that

($\star$) $B^{(a,b)}\cap R$ is the set of positive Haar measure for all $b\in(0,\pi)$ and $a\in[(b/2),b)$.

Our next goal is a construction of a sequence $(\sigma_t)_{t=1}^{\infty}$ of absolutely continuous measures with respect to Haar measure on $\widehat{\mathbb{G}}$. Let $\widehat{\mathbb{G}}^{(z)}=\{\chi\in\widehat{\mathbb{G}}:\chi(g_0)=z\}$ for $z\in\mathbb{T}$, and let $\overline{G}=\{\chi\in\widehat{\mathbb{G}}:\chi=\chi_1^{-1},\ \chi_1\in G\}$ for $G\subset\widehat{\mathbb{G}}$. We decompose $\widehat{\mathbb{G}}^{(1)}$ into pairwise disjoint subsets $G_0^{(1)}$, $G_1^{(1)}$, $G_2^{(1)}$, where $G_0^{(1)}=\{\chi\in\widehat{\mathbb{G}}^{(1)}:\chi=\chi^{-1}\}$, $G_2^{(1)}=\overline{G_1^{(1)}}$. Similarly, we decompose $\widehat{\mathbb{G}}^{(-1)}$ into $G_0^{(-1)}$, $G_1^{(-1)}$, $G_2^{(-1)}$. Finally, we decompose $\widehat{\mathbb{G}}$ into two disjoint subsets $\widehat{\mathbb{G}}^{+}$, $\widehat{\mathbb{G}}^{-}$, where $\widehat{\mathbb{G}}^{+}=G_0^{(1)}\cup G_1^{(1)}\cup G_0^{(-1)}\cup G_1^{(-1)}\cup\bigcup_{z\in\mathbb{T}^+}\widehat{\mathbb{G}}^{(z)}$, $\widehat{\mathbb{G}}^{-}=G_2^{(1)}\cup G_2^{(-1)}\cup\bigcup_{z\in\mathbb{T}^-}\widehat{\mathbb{G}}^{(z)}$, and $\mathbb{T}^+$, $\mathbb{T}^-$ denote the upper and lower half of the circle $\mathbb{T}$ $(1,-1\notin\mathbb{T}^+,\mathbb{T}^-)$. Fix $n_1\in\mathbb{N}$ and two constants $0<D<E<\pi n_1$. Choose k_1 different n_1-roots of 1

$$\varepsilon_1^{(1)},\dots,\varepsilon_{k_1}^{(1)}\in\mathbb{T}^+\cup\{-1,1\},\quad \varepsilon_1^{(1)}=1,\quad 1\le k_1<\frac{n_1+3}{2}.$$

Let $I_1^{(1)},\dots,I_{k_1}^{(1)}\subset\mathbb{T}^+\cup\{-1,1\}$ be pairwise disjoint closed intervals such that $\varepsilon_l^{(1)}$ is the centre (1 and if need be -1 is the suitable end) of $I_l^{(1)}$. We denote by $d_l^{(1)}$ the length of $I_l^{(1)}$. We require that

$$n_1^2\max_{1\le l\le k_1}\{d_l^{(1)}\}\le E.$$

Let $J_l^{(1)}=\{\chi\in\widehat{\mathbb{G}}^{+}:\chi(g_0)\in I_l^{(1)}\},l=1,2,\dots,k_1$. These sets are pairwise disjoint of positive Haar measure. The measure σ_1 on $\widehat{\mathbb{G}}^{+}$ will be concentrated on $\bigcup_{l=1}^{k_1}J_l^{(1)}$. On each $J_l^{(1)}$, σ_1 is equal its Haar measure multiplied by some positive constant and we require that

$$D\le n_1^2\sigma_1(J_1^{(1)})\le E.$$

Moreover, we put $\sigma_1(G) = \sigma_1(\overline{G})$ for each Borel subset $G \subset \widehat{\mathbb{G}}^-$. Let us formulate the induction hypothesis: we have positive integers n_t, k_t; we have a set $\{\varepsilon_1^{(t)}, \dots, \varepsilon_{k_t}^{(t)}\}$ of n_t-roots of 1; for $1 \le l \le k_t$ we have a family $I_l^{(t)}$ of pairwise disjoint closed intervals (called t-intervals) of the length $d_l^{(t)}$ and a family $J_l^{(t)}$ of pairwise disjoint sets (called t-sets) with positive Haar measure; we have a finite absolutely continuous symmetric measure σ_t on $\widehat{\mathbb{G}}$. We choose now $n_{t+1} > n_t$ such that

$$n_{t+1} > \frac{4\pi}{\min_{1 \le l \le k_t}\{d_l^{(t)}\}}.$$

Therefore there are at least two n_{t+1}-roots of 1 in each t-interval. We choose k_{t+1} different n_{t+1}-roots of 1

$$\varepsilon_1^{(t+1)}, \dots, \varepsilon_{k_{t+1}}^{(t+1)} \in \mathbb{T}^+ \cup \{-1, 1\}, \quad \varepsilon_1^{(t+1)} = 1, \quad 1 \le k_{t+1} < \frac{n_{t+1} + 3}{2}$$

and pairwise disjoint closed intervals $I_1^{(t+1)}, \dots, I_{k_{t+1}}^{(t+1)} \subset \mathbb{T}^+ \cup \{-1, 1\}$ with centres $\varepsilon_1^{(t+1)}, \dots, \varepsilon_l^{(t+1)}$ respectively (1 and if need be -1 be suitable ends). We choose them so that each $(t+1)$-interval is contained in a t-interval and there are at least two $(t+1)$-intervals in each t-interval. Moreover, if we denote by $d_l^{(t+1)}$ the length of $I_l^{(t+1)}$ and write $d_{t+1} = \max_{1 \le l \le k_{t+1}}\{d_l^{(t+1)}\}$ then we require that

$$n_{t+1}^2 d_{t+1} \le E.$$

Let $J_l^{(t+1)} = \{\chi \in \widehat{\mathbb{G}}^+ : \chi(g_0) \in I_l^{(t+1)}\}$, $l = 1, 2, \dots, k_{t+1}$. These sets are pairwise disjoint of positive Haar measure. The measure σ_{t+1} on $\widehat{\mathbb{G}}^+$ will be concentrated on $\bigcup_{l=1}^{k_{t+1}} J_l^{(t+1)}$. It equals a multiple of Haar measure on $J_1^{(t+1)}$ so that

$$\sigma_{t+1}(J_1^{(t+1)}) < \sigma_t(J_1^{(t)}), \quad D \le n_{t+1}^2 \sigma_{t+1}(J_1^{(t+1)}) \le E$$

and on the remaining $(t+1)$-sets contained in $J_1^{(t)}$ we distribute the mass $\sigma_t(J_1^{(t)}) - \sigma_{t+1}(J_1^{(t+1)})$ in equal parts. Moreover, for $2 \le l \le k_t$, in all $(t+1)$-sets contained in $J_l^{(t)}$ we distribute the mass $\sigma_t(J_l^{(t)})$ in equal parts. Finally we complete the definition of σ_{t+1} on $\widehat{\mathbb{G}}^-$ by symmetrization.

Choosing a subsequence, if necessary, we can assume that $(\sigma_t)_{t=1}^\infty$ converges weakly to a symmetric measure $\widetilde{\sigma}$ on $\widehat{\mathbb{G}}$. By the construction, it follows that the support of $\widetilde{\sigma}$ is contained in a disjoint sum of sets of the form $\bigcap_{t=1}^\infty J_{l_t}^{(t)}$ with $J_{l_1}^{(1)} \supset J_{l_2}^{(2)} \supset \dots$. The measure $\widetilde{\sigma}$ is finite continuous, because $\sigma_t(J_{l_t}^{(t)}) \to 0$, and $\sigma_s(J_l^{(t)}) = \sigma_t(J_l^{(t)})$ for each $s \ge t$, and $\sigma_s(\widehat{\mathbb{G}}) = \sigma_t(\widehat{\mathbb{G}}) > 0$ for all $s, t \in \mathbb{N}$. Notice that $\widetilde{\sigma}$-a.e.

$$\varrho(\chi^{n_t}(g_0), 1) \le n_t d_t \to 0.$$

Hence (b) of Proposition 3.1 holds for the measure $\widetilde{\sigma}$. If $\varrho(\chi(g_0), 1) > (\pi/n_t)$ then $\widetilde{\sigma}$-a.e.

$$\frac{\varrho(\chi^{n_t}(g_0), 1)}{\varrho(\chi(g_0), 1)} < \frac{n_t d_t}{\pi/n_t} = \frac{1}{\pi} n_t^2 d_t \le \frac{1}{\pi} E$$

If $\varrho(\chi(g_0),1) \le (\pi/n_t)$ then $\varrho(\chi^{n_t}(g_0),1)/\varrho(\chi(g_0),1) = n_t$. Moreover, since $d_t \le (E/n_t^2) \le ((\pi n_1)/n_t^2) \le ((\pi n_t)/n_t^2) = (\pi/n_t)$, the sets $\{\chi \in \widehat{\mathbb{G}}^+ : \varrho(\chi(g_0),1) \le (\pi/n_t)\}$, $J_1^{(t)}$ are equal σ_t-a.e., and consequently σ_s-a.e. for all $s \ge t$. This implies that

$$\int_{\{\chi\in\widehat{\mathbb{G}}^+:\varrho(\chi(g_0),1)\le(\pi/n_t)\}} \left(\frac{\varrho(\chi^{n_t}(g_0),1)}{\varrho(\chi(g_0),1)}\right)^2 d\sigma_s(\chi) = n_t^2\sigma_s(J_1^{(t)}) = n_t^2\sigma_t(J_1^{(t)}) \in [D,E].$$

Therefore

$$\int_{\widehat{\mathbb{G}}} \left(\frac{\varrho(\chi^{n_t}(g_0),1)}{\varrho(\chi(g_0),1)}\right)^2 d\widetilde{\sigma}(\chi) = \lim_{s\to\infty} \int_{\widehat{\mathbb{G}}} \left(\frac{\varrho(\chi^{n_t}(g_0),1)}{\varrho(\chi(g_0),1)}\right)^2 d\sigma_s(\chi) \in [D, 2E + (E/\pi)^2\widetilde{\sigma}(\widehat{\mathbb{G}})],$$

and (c) of Proposition 3.1 holds for the measure $\widetilde{\sigma}$. Finally, let $\sigma(G) = \widetilde{\sigma}(G\cap R)$ for every Borel set $G \subset \widehat{\mathbb{G}}$. From $(\star)$ it follows that $J_1^{(t)} \cap R$ is the set of positive Haar measure for all $t \in \mathbb{N}$. Hence the measure σ satisfies the assumptions of Proposition 3.1 if we replace $D \le n_t^2\sigma_t(J_1^{(t)})$ by $D \le n_t^2\sigma_t(J_1^{(t)} \cap R)$ in the definition of σ_t.

4. Gaussian isomorphisms of Gaussian actions

We consider a Gaussian $\mathbb{G}$-action $\mathcal{T}_\sigma$ with Gaussian space $\mathcal{H}_\sigma$. There exists the decomposition of $L^2_{0\mathbb{R}}(X_\sigma, \mathcal{B}_\sigma, \mu_\sigma)$, the space of real square-integrable functions with zero mean, into Wiener chaos:

$$L^2_{0\mathbb{R}}(X_\sigma, \mathcal{B}_\sigma, \mu_\sigma) = \bigoplus_{m=1}^{\infty} \mathcal{H}^{(m)}$$

where $\mathcal{H}^{(1)} = \mathcal{H}_\sigma$, $\mathcal{H}^{(m)}$ is real closed $\mathcal{T}_\sigma$-invariant subspace. The maximal spectral type on $\mathcal{H}^{(m)}$ is $\sigma^{(m)}$, the mth convolution power of σ (see [1]). Moreover, if $f \in L^2_{0\mathbb{R}}(X_\sigma, \mathcal{B}_\sigma, \mu_\sigma)$ is a Gaussian variable then either $f \in \mathcal{H}^{(1)}$ or $f = \sum_{m=1}^{\infty} f_m$, $f_m \in \mathcal{H}^{(m)}$ with infinitely many f_m different from zero (see e.g. [7]).

The centralizer of $\mathcal{T}_\sigma$, denoted by $C(\mathcal{T}_\sigma)$, is defined to be the set of all automorphisms of $(X_\sigma, \mathcal{B}_\sigma, \mu_\sigma)$ such that $T^g S = ST^g$ for all $g \in \mathbb{G}$. $C(\mathcal{T}_\sigma)$ contains a part coming directly from the Gaussian structure. It is the set of all $S \in C(\mathcal{T}_\sigma)$ which preserve the Gaussian space and we will denote it by $C^G(\mathcal{T}_\sigma)$. Let I be an isomorphism of Gaussian actions $\mathcal{T}_\sigma$, $\mathcal{T}_{\sigma'}$. I is called a Gaussian isomorphism if $U_I\mathcal{H}_{\sigma'} = \mathcal{H}_\sigma$. The existence of such an I is equivalent to saying that $\sigma \equiv \sigma'$ (see [5, Lemma 2]). In general, if Gaussian actions are isomorphic then the isomorphism need not be Gaussian. As an example, in the case of $\mathbb{Z}$-actions, let σ, σ' be symmetric Lebesgue measures on the circle restricted to disjoint sets.

Then T_σ, $T_{\sigma'}$ are Bernoulli automorphisms with infinity entropy (see e.g. [12]) and they have to be isomorphic by Ornstein's isomorphism theorem.

Let T_σ be a Gaussian automorphism with simple spectrum and denote by $\mathcal{H}^{(m)}$ its mth chaos. Then every Gaussian space of T_σ is equal to $\mathcal{H}^{(1)}$. This assertion has been proved in much stronger version, for instances, in [5]. However, one can obtain our statement with no effort. Indeed, let $\mathcal{H}$ be a Gaussian space of T_σ and let $h \in \mathcal{H}$. If $h = \sum_{m=1}^{\infty} h_m$, $h_m \in \mathcal{H}^{(m)}$ then $\sigma_{h_i} \perp \sigma_{h_j}$. Hence $h_m \in \bigoplus_{i=1}^{\infty} \mathbb{Z}(h_i) = \mathbb{Z}(h) \subset \mathcal{H}$, and h_m is a Gaussian variable. Therefore $h_m = 0$ for all $m \geq 2$, and $h \in \mathcal{H}^{(1)}$. Since $\mathcal{B}(\mathcal{H}) = \mathcal{B}$, we have $\mathcal{H} = \mathcal{H}^{(1)}$.

Assume now that $\mathcal{T}_\sigma$, $\mathcal{T}_{\sigma'}$ are isomorphic Gaussian $\mathbb{G}$-actions. Let I be an isomorphism between them. Assume that $C(\mathcal{T}_\sigma) = C^G(\mathcal{T}_\sigma)$. We can find in $C^G(\mathcal{T}_{\sigma'})$ some Gaussian automorphism S' with simple spectrum (see [5, Lemma 5]). Let $S = I^{-1}S'I$. Then S is a Gaussian automorphism with simple spectrum and $U_I\mathcal{H}_{\sigma'}$ is a Gaussian space of S. Since $S \in C(\mathcal{T}_\sigma) = C^G(\mathcal{T}_\sigma)$, $\mathcal{H}_\sigma$ is also a Gaussian space of S. Hence $U_I\mathcal{H}_{\sigma'} = \mathcal{H}_\sigma$. This proves:

PROPOSITION 4.1. *If $C(\mathcal{T}_\sigma) = C^G(\mathcal{T}_\sigma)$ then every isomorphism between $\mathcal{T}_\sigma$ and another Gaussian $\mathbb{G}$-action is Gaussian.*

PROPOSITION 4.2. *If every spectral type absolutely continuous with respect to σ has a finite multiplicity then $C(\mathcal{T}_\sigma) = C^G(\mathcal{T}_\sigma)$.*

PROOF. Let $S \in C(\mathcal{T}_\sigma)$ and assume that $S \notin C^G(\mathcal{T}_\sigma)$. There exists $h \in \mathcal{H}^{(1)}$ such that $U_S h \notin \mathcal{H}^{(1)}$. We have $h = \sum_{n=1}^{\infty} h_n$, $h_n \in H_n$, where H_n are spaces of the constant uniform multiplicity n from the spectral theorem ($n \neq \infty$). Since $\sigma_{h_i} \perp \sigma_{h_j}$ for $i \neq j$, it follows that $\bigoplus_{n=1}^{\infty} \mathbb{G}(h_n) = \mathbb{G}(h) \subset \mathcal{H}^{(1)}$. Hence $h_n \in \mathcal{H}^{(1)}$ for all $n \in \mathbb{N}$. Since $U_S h = \sum_{n=1}^{\infty} U_S h_n$, there exists $N \in \mathbb{N}$ such that $U_S h_N \notin \mathcal{H}^{(1)}$. But $U_S h_N$ is a Gaussian variable, therefore $U_S h_N = \sum_{m=1}^{\infty} h_N^{(m)}$, $h_N^{(m)} \in \mathcal{H}^{(m)}$ with infinitely many $h_N^{(m)}$ different from zero. Write $\sigma_m = \sigma_{h_N^{(m)}}$ for each $m \in \mathbb{N}$. Then infinitely many of $\sigma_1, \sigma_2, \ldots$ are different from zero.

We will find finite symmetric Borel measures $\sigma_1', \sigma_2', \ldots$ on $\widehat{\mathbb{G}}$ such that for all $m \in \mathbb{N}$:

(a) $\sigma_m' \ll \sigma_m$,
(b) $\sigma_m' \perp \sigma_l$ for all $l > m$,
(c) $\sum_{m=1}^{\infty} \sigma_m' \equiv \sum_{m=1}^{\infty} \sigma_m$.

Then obviously

(d) $\sigma_i' \perp \sigma_j'$ for all $i \neq j$.

For every $m \in \mathbb{N}$ we take decomposition of σ_m with respect to $\sum_{l=m+1}^{\infty} \sigma_l$

$$\sigma_m = \sigma_m' + \sigma_m'', \quad \sigma_m' \perp \sum_{l=m+1}^{\infty} \sigma_l, \quad \sigma_m'' \ll \sum_{l=m+1}^{\infty} \sigma_l.$$

Clearly (a) and (b) hold. Assume that $\sum_{m=1}^\infty \sigma_m$ are not absolutely continuous with respect to $\sum_{m=1}^\infty \sigma'_m$. Then

$$\sum_{m=1}^\infty \sigma_m = \gamma' + \gamma'', \quad \text{where } \gamma' \perp \sum_{m=1}^\infty \sigma'_m, \quad \gamma'' \ll \sum_{m=1}^\infty \sigma'_m, \quad \gamma' \neq 0.$$

There exists $m_1 \in \mathbb{N}$ such that γ' and σ_{m_1} are not mutually singular. Let us take decomposition

$$\sigma_{m_1} = \gamma'_1 + \gamma''_1, \quad \gamma'_1 \perp \gamma', \quad \gamma''_1 \ll \gamma', \quad \gamma''_1 \neq 0.$$

We have $\gamma''_1 \ll \sigma_{m_1} = \sigma'_{m_1} + \sigma''_{m_1}$, and $\gamma''_1 \ll \gamma' \perp \sigma'_{m_1}$. It follows that $\gamma''_1 \ll \sigma''_{m_1} \ll \sum_{l=m_1+1}^\infty \sigma_l$. Thus there exists $m_2 > m_1$ such that γ''_1 and σ_{m_2} are not mutually singular, and we repeat the above procedure. By $N+1$ steps, we get a nonzero measure γ''_{N+1} which is absolutely continuous with respect to $\sigma_{m_1}, \dots, \sigma_{m_N}$ and $\sum_{l=m_N+1}^\infty \sigma_l$. Hence the multiplicity of γ''_{N+1} is at least $N+1$, a contradiction.

Since $\sigma'_m \ll \sigma_{h_N}$, there exists $h'_m \in \mathbb{G}(h_N)$ such that $\sigma_{h'_m} \equiv \sigma'_m$ and $\sum_{m=1}^\infty h'_m$ is convergent. Let $h' = \sum_{m=1}^\infty h'_m \in \mathbb{G}(h_N)$. Since $\mathbb{G}(h_N^{(i)}) \subset \mathcal{H}^{(i)} \perp \mathcal{H}^{(j)} \supset \mathbb{G}(h_N^{(j)})$ for all $i \neq j$, we have

$$\sigma_{h_N} = \sigma_{U_S h_N} = \sum_{m=1}^\infty \sigma_m \equiv \sum_{m=1}^\infty \sigma'_m = \sigma_{h'}.$$

Therefore $\mathbb{G}(h') = \mathbb{G}(h_N)$. From (d) we obtain

$$U_S h'_m \in \mathbb{G}(U_S h') = \mathbb{G}(U_S h_N) \subset \bigoplus_{i=1}^\infty \mathbb{G}(h_N^{(i)})$$

for all $m \in \mathbb{N}$. Consequently, since $U_S h'_m$ is a Gaussian variable and the maximal spectral type on $\mathbb{G}(h_N^{(i)}) \subset \mathcal{H}^{(i)}$ is σ_i, from (b), we conclude that $U_S h'_m \in \mathcal{H}^{(1)}$ for all $m \in \mathbb{N}$. Thus $U_S h' \in \mathcal{H}^{(1)}$, and $U_S h_N \in \mathcal{H}^{(1)}$, a contradiction. □

Corollary 4.3. *If every spectral type absolutely continuous with respect to σ has a finite multiplicity then every isomorphism between $\mathcal{T}_\sigma$ and another Gaussian $\mathbb{G}$-action is Gaussian.*

This generalizes a fact for $\mathbb{Z}$-actions from [14] (given without proof), where author assumed that infinity is not an essential value of the multiplicity function of one automorphism. We proved a stronger version even in the case of $\mathbb{Z}$-action. There exists a measure σ on the circle singular to the Lebesgue measure λ such that $\sigma * \sigma \equiv \lambda$ (see [8]). It follows that $\sigma^{(m)} \equiv \lambda$ for each $m \geq 2$, and the multiplicity of λ is equal to infinity. Hence infinity is an essential value of the multiplicity function of T_σ. But every spectral type absolutely continuous with respect to σ has the multiplicity one.

Remark 4.4. In [5], there are considered Gaussian automorphisms whose ergodic self-joinings are Gaussian (GAG automorphisms). There is proved that Corollary 4.3 holds for GAG (see Corollary 5 in [5]). Every GAG satisfies the assumption of Corollary 4.3, but there is a Gaussian automorphism for which infinity is not an essential value of the multiplicity function, and it is not a GAG (Example 2 in [5]).

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