Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 30, 2007, 235–249

EXISTENCE OF SOLUTIONS FOR $p(x)$-LAPLACIAN PROBLEM ON AN UNBOUNDED DOMAIN

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Abstract. In this paper we study the following $p(x)$-Laplacian problem:

$$\begin{aligned} -\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u)+b(x)|u|^{p(x)-2}u &= f(x,u) \quad x\in\Omega,\\ u&=0 \qquad\quad \text{on } \partial\Omega, \end{aligned}$$

where $1<p_1\le p(x)\le p_2<n$, $\Omega\subset\mathbb{R}^n$ is an exterior domain. Applying Mountain Pass Theorem we obtain the existence of solutions in $W_0^{1,p(x)}(\Omega)$ for the $p(x)$-Laplacian problem in the superlinear case.

1. Introduction

After Kovacik and Rakosnik first discussed the $L^{p(x)}$ space and $W^{k,p(x)}$ space in [20], a lot of research have been done concerning this kind of variable exponent spaces, see for example [1]–[3], [6], [7], [11]–[13] and [16]–[18] and the references therein. We don't want to list all the works in this field here. In [22] Ruzicka presented the mathematical theory for the application of variable exponent spaces in electro-rheological fluids.

Inspired by their works, we want to study the $p(x)$-Laplacian problem:

$$\begin{aligned} -\operatorname{div}(a(x)|\nabla u|^{p(x)-2}\nabla u)+b(x)|u|^{p(x)-2}u &= f(x,u), \quad x\in\Omega,\\ u&=0 \qquad\qquad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

2000 *Mathematics Subject Classification.* Primary 35J05; Secondary 35D05, 46N20.
Key words and phrases. Existence, $p(x)$-Laplacian problem, unbounded domain.

where Ω is an exterior domain in $\mathbb{R}^n$, i.e. Ω is the complement of a bounded domain, $0 < a_0 \le a(x) \in L^\infty(\Omega)$, $0 < b_0 \le b(x) \in L^\infty(\Omega)$, p is Lipschitz continuous on $\overline{\Omega}$ and satisfies

$$1 < p_1 \le p(x) \le p_2 < n. \tag{1.2}$$

Our object is to obtain sufficient conditions on f for (1.1) to admit nontrivial and nonnegative solutions in the general case of the following prototype:

$$f(x,u) = g(x)u^{\alpha(x)}, \quad p(x) - 1 < \alpha(x) < p^*(x) - 1 \tag{1.3}$$

where $p^*(x) = np(x)/(n - p(x))$.

When $p(x)$ is a constant function, there are a lot of studies. For the case of bounded domains, see for example [4], [9], [10], [14] and [19] and the references therein. For the case of unbounded domains, there are also many studies, see for example [5], [8], [21], [24] and [25]. It is beyond our ability to write out all the works in this direction here. When $p(x)$ is a variable function, Fan and Zhang [17] studied the $p(x)$-Laplacian problems on bounded domains. Under some conditions, they established some results on the existence of solutions. For unbounded domains, Fan and Han [15] investigated the existence of solutions for $p(x)$-Laplacian equations. In this paper we discuss the $p(x)$-Laplacian problem in the case of unbounded domain. Our method is a bit different from that in [15] and [17] and in some sense we discuss the $p(x)$-Laplacian problem in a more general setting than that in [15] and [17] as well.

2. Preliminaries

In this section we first recall some facts on variable exponent spaces $L^{p(x)}(\Omega)$ and $W^{k,p(x)}(\Omega)$. For the details see [20] and [16].

Let $\mathbf{P}(\Omega)$ be the set of all Lebesgue measurable functions $p\colon \Omega \to [1,\infty]$.

$$\rho_p(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + \inf_{\Omega_\infty} |f(x)|, \tag{2.1}$$

$$\|f\|_p = \inf\{\lambda > 0 : \rho_p(f/\lambda) \le 1\}, \tag{2.2}$$

where $\Omega_\infty = \{x \in \Omega : p(x) = \infty\}$. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is the class of all functions f such that $\rho_p(\lambda f) < \infty$ for some $\lambda = \lambda(f) > 0$. $L^{p(x)}(\Omega)$ is a Banach space endowed with the norm (2.2). $\rho_p(f)$ is called the modular of f in $L^{p(x)}(\Omega)$.

For a given $p(x) \in \mathbf{P}(\Omega)$ we define the conjugate function $p'(x)$ as:

$$p'(x) = \begin{cases} \infty & \text{if } x \in \Omega_1 = \{x \in \Omega : p(x) = 1\}, \\ 1 & \text{if } x \in \Omega_\infty, \\ \dfrac{p(x)}{p(x)-1} & \text{for other } x \in \Omega. \end{cases}$$

THEOREM 2.1. *Let $p \in \mathbf{P}(\Omega)$. Then the inequality*

$$\int_\Omega |f(x)g(x)|\,dx \le r_p \|f\|_p \|g\|_{p'}$$

holds for every $f \in L^{p(x)}(\Omega)$ and $g \in L^{p'(x)}(\Omega)$ with the constant r_p depending on $p(x)$ and Ω only.

THEOREM 2.2. *The topology of the Banach space $L^{p(x)}(\Omega)$ endowed by the norm* (2.2) *coincides with the topology of modular convergence if and only if $p \in L^\infty(\Omega)$.*

THEOREM 2.3. *The dual space to $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$ if and only if $p \in L^\infty(\Omega)$. The space $L^{p(x)}(\Omega)$ is reflexive if and only if*

$$1 < \inf_\Omega p(x) \le \sup_\Omega p(x) < \infty. \tag{2.3}$$

Next we assume that $\Omega \subset \mathbb{R}^n$ is a nonempty open set, $p \in \mathbf{P}(\Omega)$ and k is a given natural number.

Given a multiindex $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we set $|\alpha| = \alpha_1 + \ldots + \alpha_n$ and $D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}$, where $D_i = \partial/\partial x_i$ is the generalized derivative operator.

The generalized Sobolev space $W^{k,p(x)}(\Omega)$ is the class of all functions f on Ω such that $D^\alpha f \in L^{p(x)}(\Omega)$ for every multiindex α with $|\alpha| \le k$, endowed with the norm

$$\|f\|_{k,p} = \sum_{|\alpha| \le k} \|D^\alpha f\|_p. \tag{2.4}$$

By $W_0^{k,p(x)}(\Omega)$ we denote the subspace of $W^{k,p(x)}(\Omega)$ which is the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.4).

THEOREM 2.4. *The space $W^{k,p(x)}(\Omega)$ and $W_0^{k,p(x)}(\Omega)$ are Banach spaces, which are reflexive if p satisfies* (2.3).

We denote the dual space of $W_0^{k,p(x)}(\Omega)$ by $W^{-k,p'(x)}(\Omega)$, then we have

THEOREM 2.5. *Let $p \in \mathbf{P}(\Omega) \cap L^\infty(\Omega)$. Then for every $G \in W^{-k,p'(x)}(\Omega)$ there exists a unique system of functions $\{g_\alpha \in L^{p'(x)}(\Omega) : |\alpha| \le k\}$ such that*

$$G(f) = \sum_{|\alpha| \le k} \int_\Omega D^\alpha f(x) g_\alpha(x)\,dx, \quad f \in W_0^{k,p(x)}(\Omega).$$

The norm of $W_0^{-k,p'(x)}(\Omega)$ is defined as

$$\|G\|_{-k,p'} = \sup\left\{\frac{|G(f)|}{\|f\|_{k,p}} : f \in W_0^{k,p(x)}(\Omega)\right\}.$$

Theorem 2.6. *If Ω is a bounded domain with cone property, $p(x) \in C(\overline{\Omega})$ satisfies* (1.2) *and $q(x)$ is any Lebesgue measurable function defined on Ω with $p(x) \leq q(x)$ a.e. on $\overline{\Omega}$ and $\inf_{x\in\Omega}\{p^*(x) - q(x)\} > 0$, then there is a compact embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.*

Theorem 2.7. *Let Ω be a domain with cone property. If $p:\overline{\Omega} \to \mathbb{R}$ is Lipschitz continuous and satisfies* (1.2), *and $q(x) \in \mathbf{P}(\Omega)$ satisfies $p(x) \leq q(x) \leq p^*(x)$ a.e. on $\overline{\Omega}$, then there is a continuous embedding $W^{1,p(x)}(\Omega) \to L^{q(x)}(\Omega)$.*

For the $p(x)$-Laplacian problem (1.1) we define two functionals $K(u)$ and $J(u)$ on Ω:

$$K(u) = \int_\Omega F(x,u)\,dx, \quad J(u) = \int_\Omega \frac{1}{p(x)}(a(x)|\nabla u|^{p(x)} + b(x)|u|^{p(x)})\,dx - K(u)$$

where $F(x,t) = \int_0^t f(x,s)\,ds$.

Next we discuss the properties of $K(u)$ in the case (1.3). We assume that f satisfies the following conditions:

- (H1) $f \in C(\overline{\Omega} \times \mathbb{R})$, $f(x,t) > 0$ in $\Omega_0 \times (0,\infty)$ for some nonempty open set $\Omega_0 \subseteq \Omega$ and $f(x,t) = 0$ for all $x \in \Omega$ and $t \leq 0$.
- (H2) $|f(x,t)| \leq g(x)|t|^{\alpha(x)}$, $\alpha + 1 \in \mathbf{P}(\Omega)$ is uniformly continuous on $\overline{\Omega}$ with $\widehat{a} = \inf_{x\in\Omega}\{\alpha(x) - p(x) + 1\} > 0$ and $a = \inf_{x\in\Omega}\{p^*(x) - \alpha(x) - 1\} > 0$, $0 \not\equiv g \in L^\infty(\Omega) \cap L^{p_0}(\Omega)$ where

$$p_0(x) = \frac{np(x)}{np(x) - (\alpha(x)+1)(n-p(x))}.$$

- (H3) There exists $\mu > p(x)$ with $\inf_{x\in\Omega}\{\mu - p(x)\} > 0$ such that $\mu F(x,t) \leq tf(x,t)$ for $(x,t) \in \Omega \times \mathbb{R}$.

Lemma 2.8. *Suppose that $\alpha(x)$ satisfies the conditions in* (H2). *Let r be a positive constant. Then, if $|u(x)| \geq r$,*

$$\lim_{\|u\|_{1,p}\to 0} \frac{\|u\|_{\alpha+1}}{\|u\|_{p^*}} = 0.$$

Proof. For any $0 < \varepsilon < 1$, we have

$$\begin{aligned}\int_\Omega \bigg(\frac{|u|}{\varepsilon\|u\|_{p^*}}\bigg)^{\alpha(x)+1} dx &= \int_\Omega \bigg(\frac{|u|}{\varepsilon\|u\|_{p^*}}\bigg)^{p^*(x)} \bigg(\frac{\varepsilon\|u\|_{p^*}}{|u|}\bigg)^{p^*(x)-\alpha(x)-1} dx \\ &\leq \int_\Omega \bigg(\frac{|u|}{\|u\|_{p^*}}\bigg)^{p^*(x)} \bigg(\frac{\|u\|_{p^*}}{r}\bigg)^{p^*(x)-\alpha(x)-1} \bigg(\frac{1}{\varepsilon}\bigg)^{\alpha(x)+1} dx.\end{aligned}$$

As $a = \inf_{x\in\Omega}\{p^*(x) - \alpha(x) - 1\} > 0$, we can choose $\|u\|_{p^*}$ sufficiently small such that

$$\int_\Omega \bigg(\frac{|u|}{\varepsilon\|u\|_{p^*}}\bigg)^{\alpha(x)+1} dx \leq \int_\Omega \bigg(\frac{|u|}{\|u\|_{p^*}}\bigg)^{p^*(x)} dx \leq 1$$

and further

$$\|u\|_{\alpha+1} \le \varepsilon \|u\|_{p^*}.$$

By Theorem 2.7 we know $\|u\|_{p^*} \to 0$ as $\|u\|_{1,p} \to 0$. □

Theorem 2.9. *Suppose that f satisfies* (H1) *and* (H2), *then $K(u)$ is weakly continuous on $W_0^{1,p(x)}(\Omega)$.*

Proof. Let $\Omega_k = \{x \in \Omega : |x| \le k\}$ where k is a natural number. Let $u_j \to u$ weakly in $W_0^{1,p(x)}(\Omega)$. We have

$$\begin{aligned} |K(u_j) - K(u)| \le \int_{\Omega_k} & |F(x,u_j) - F(x,u)|\,dx \\ & + C\|g\|_{p_0,\Omega\setminus\Omega_k}(\||u_j|^{\alpha+1}\|_{(\alpha+1)^{-1}p^*} + \||u_j|^{\alpha+1}\|_{(\alpha+1)^{-1}p^*}). \end{aligned}$$

As $\{u_j\}$ is bounded in $W_0^{1,p(x)}(\Omega)$, $\{u_j\}$ is bounded in $W^{1,p(x)}(\Omega_k)$ for fixed k as well. By Theorem 2.6 there is a compact embedding $W^{1,p(x)}(\Omega_k) \to L^{\alpha(x)+1}(\Omega_k)$ and further there exists a subsequence of $\{u_j\}$ (still denote the subsequence by $\{u_j\}$) such that $u_j \to u$ in $L^{\alpha(x)+1}(\Omega)$ and by Theorem 2.2 $u_j \to u$ in modular as well. From (H2) we get

$$|F(x,t)| \le \frac{1}{\alpha(x)+1} g(x)|t|^{\alpha(x)+1}.$$

Then by Vitali Theorem, after subtracting a subsequence if necessary, for fixed k we have

$$\int_{\Omega_k} F(x,u_j)\,dx \to \int_{\Omega_k} F(x,u)\,dx \quad \text{as } j \to \infty.$$

Let $\chi_{\Omega\setminus\Omega_k}$ be the characteristic function of $\Omega\setminus\Omega_k$. Denote $\overline{\alpha} = \sup_\Omega \alpha(x)$. From

$$\begin{aligned} &\int_\Omega \bigg(\frac{|u_j|^{\alpha(x)+1}}{(1+\|u_j\|_{p^*})^{\overline{\alpha}+1}}\bigg)^{(\alpha(x)+1)^{-1}p^*(x)} dx \\ &= \int_\Omega \bigg(\frac{|u_j|}{((1+\|u\|_{p^*})^{\overline{\alpha}+1})^{(\alpha(x)+1)^{-1}}}\bigg)^{p^*(x)} dx \\ &\le \int_\Omega \bigg(\frac{|u_j|}{((1+\|u\|_{p^*})^{\overline{\alpha}+1})^{(\overline{\alpha}+1)^{-1}}}\bigg)^{p^*(x)} dx = \int_\Omega \bigg(\frac{|u_j|}{1+\|u\|_{p^*}}\bigg)^{p^*(x)} dx, \end{aligned}$$

we have

$$\||u_j|^{\alpha+1}\|_{(\alpha+1)^{-1}p^*} \le (1+\|u_j\|_{p^*})^{\overline{\alpha}+1}.$$

Furthermore, by Theorem 2.7,

$$\||u_j|^{\alpha+1}\|_{(\alpha+1)^{-1}p^*} \le C(1+\|u_j\|_{1,p})^{\overline{\alpha}+1}.$$

Similarly

$$\||u|^{\alpha+1}\|_{(\alpha+1)^{-1}p^*} \le C(1+\|u\|_{1,p})^{\overline{\alpha}+1}.$$

As $g \in L^{p_0(x)}(\Omega)$, we know

$$\int_\Omega g^{p_0(x)}\,dx < \infty \quad \text{and} \quad \int_{\Omega\setminus\Omega_k} g^{p_0(x)}\,dx = \int_\Omega (g\chi_{\Omega\setminus\Omega_k})^{p_0(x)}\,dx \to 0$$

as $k \to \infty$. By Theorem 2.2 $\|g\chi_{\Omega\setminus\Omega_k}\|_{p_0} = \|g\|_{p_0,\Omega\setminus\Omega_k} \to 0$ as $k \to \infty$. $\square$

THEOREM 2.10. *Suppose that f satisfies* (H1) *and* (H2), *then $K(u)$ is differentiable on $W_0^{1,p(x)}(\Omega)$ with*

$$K'(u)\phi = \int_\Omega f(x,u)\phi\,dx, \quad \text{for all } \phi \in W_0^{1,p(x)}(\Omega)$$

and $K'(u)$ is a continuous and compact mapping from $W_0^{1,p(x)}(\Omega)$ to $W^{-1,p'(x)}(\Omega)$.

PROOF. For differentiability of K, we will show that for any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon,u) > 0$ such that

$$\begin{aligned}\Big|K(u+\phi) - K(u) - \int_\Omega f(x,u)\phi\,dx\Big|\\ = \Big|\int_\Omega F(x,u+\phi) - F(x,u) - f(x,u)\phi\,dx\Big| < \varepsilon\|\phi\|_{1,p}\end{aligned}$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$ with $\|\phi\|_{1,p} < \delta$.

Let $\Omega_k = \{x \in \Omega : |x| \le k\}$, $\Omega_{k1} = \{x \in \Omega_k : |u(x)| \ge \beta\}$, $\Omega_{k2} = \{x \in \Omega_k : |\phi(x)| \ge r\}$, $\Omega_{k3} = \{x \in \Omega_k : |u(x)| < \beta \text{ and } |\phi(x)| < r\}$ where k, β, r are constant which will be determined later. First on $\Omega\setminus\Omega_k$ we have

$$\begin{aligned}\Big|\int_{\Omega\setminus\Omega_k} F(x,u+\phi) - F(x,u) - f(x,u)\phi\,dx\Big|\\ \le \int_{\Omega\setminus\Omega_k} g((|u|+|\phi|)^{\alpha(x)}|\phi| + |u|^{\alpha(x)}|\phi|)\,dx\\ \le C\int_{\Omega\setminus\Omega_k} g(|u|^{\alpha(x)}|\phi| + |\phi|^{\alpha(x)+1})\,dx\end{aligned}$$

since

$$(|u|+|\phi|)^{\alpha(x)} \le 2^{\alpha(x)}(|u|^{\alpha(x)} + |\phi|^{\alpha(x)}) \le 2^{\overline{\alpha}}(|u|^{\alpha(x)} + |\phi|^{\alpha(x)}).$$

Observe that

$$\int_{\Omega\setminus\Omega_k} g|u|^{\alpha(x)}|\phi|\,dx \le C\|g|u|^{\alpha(x)}\|_{(p^*)',\Omega\setminus\Omega_k}\|\phi\|_{p^*} \le C\|g|u|^{\alpha(x)}\|_{(p^*)',\Omega\setminus\Omega_k}\|\phi\|_{1,p}.$$

As

$$(p^*(x))' = \frac{np(x)}{np(x) - (n-p(x))} < \frac{np(x)}{np(x) - (\alpha(x)+1)(n-p(x))} = p_0(x),$$

we have

$$\int_{\Omega\setminus\Omega_k} (g|u|^{\alpha(x)})^{(p^*(x))'}\,dx \le C\|g^{(p^*)'}\|_{p_0/(p^*)',\Omega\setminus\Omega_k}\||u|^{\alpha(p^*)'}\|_{(p_0/(p^*)')'}.$$

Since, by Theorems 2.2 and 2.8,

$$\int_\Omega (|u|^{\alpha(x)(p^*(x))'})^{(p_0(x)/(p^*(x))')'}\,dx = \int_\Omega |u|^{p^*(x)}\,dx < \infty,$$

we get $\||u|^{\alpha(p^*)'}\|_{(p_0/(p^*)')'} < \infty$ by applying Theorem 2.2 once more. In view of

$$\int_{\Omega\setminus\Omega_k} (g^{(p^*(x))'})^{p_0(x)/(p^*(x))'}\,dx = \int_\Omega (g\chi_{\Omega\setminus\Omega_k})^{p_0(x)}\,dx \to 0$$

as $k \to \infty$, from Theorem 2.2 we obtain

$$\|g^{(p^*)'}\|_{p_0/(p^*)',\Omega\setminus\Omega_k} \to 0 \quad \text{as } k \to \infty.$$

Similarly we can deal with the term $\int_{\Omega\setminus\Omega_k} g|\phi|^{\alpha(x)+1}\,dx$. Therefore we conclude

$$\left|\int_{\Omega\setminus\Omega_k} F(x,u+\phi) - F(x,u) - f(x,u)\phi\,dx\right| < \frac{\varepsilon}{2}\|\phi\|_{1,p} \tag{2.5}$$

for sufficiently large k and $\|\phi\|_{1,p} \le 1$.

On Ω_k we have

$$\left|\int_{\Omega_k} F(x,u+\phi) - F(x,u) - f(x,u)\phi\,dx\right| \le \sum_{i=1}^{3}\int_{\Omega_{ki}} |F(x,u+\phi) - F(x,u) - f(x,u)\phi|\,dx.$$

Second similar to the above

$$\begin{aligned}\left|\int_{\Omega_{k1}} F(x,u+\phi) - F(x,u) - f(x,u)\phi\,dx\right| &\le C\int_{\Omega_{k1}} g(|u|^{\alpha(x)}|\phi| + |\phi|^{\alpha(x)+1})\,dx\\ &\le C\int_{\Omega_{k1}} (|u|^{\alpha(x)}|\phi| + |\phi|^{\alpha(x)+1})\,dx = I_1 + I_2.\end{aligned}$$

For I_1 we have

$$I_1 \le C\||u|^\alpha\|_{(p^*)',\Omega_{k1}}\|\phi\|_{p^*} \le C\||u|^\alpha\|_{(p^*)',\Omega_{k1}}\|\phi\|_{1,p}.$$

As $\alpha(x)(p^*(x))' = \alpha(x)p^*(x)/(p^*(x)-1) < \alpha(x)+1$, we have

$$\int_{\Omega_{k1}} |u|^{\alpha(x)(p^*(x))'}\,dx \le \||u|^{\alpha(p^*)'}\|_{(\alpha+1)/\alpha(p^*)',\Omega_{k1}}\|\chi_{\Omega_{k1}}\|_{((\alpha+1)/\alpha(p^*)')'}.$$

In view of

$$\int_\Omega (\chi_{\Omega_{k1}})^{((\alpha(x)+1)/\alpha(x)(p^*(x))')'}\,dx = \operatorname{meas}\Omega_{k1} \le \operatorname{meas}\Omega_k < \infty,$$

by Theorem 2.2

$$\|\chi_{\Omega_{k1}}\|_{((\alpha+1)/\alpha(p^*)')'} < \infty.$$

Because $u \in W^{1,p(x)}(\Omega)$, we can get

$$\infty > \int_{\Omega_{k1}} |u|^{p(x)}\,dx \ge \int_{\Omega_{k1}} \beta^{p(x)}\,dx \ge \min\{\beta^{p_1},\beta^{p_2}\}\operatorname{meas}\Omega_{k1}. \tag{2.6}$$

From (2.6), $\operatorname{meas}\Omega_{k1} \to 0$ as $\beta \to \infty$. In view of

$$\int_{\Omega_{k1}} |u|^{\alpha(x)(p^*(x))'((\alpha(x)+1)/\alpha(x)(p^*(x))')}\,dx = \int_{\Omega_{k1}} |u|^{\alpha(x)+1}\,dx$$

and by Theorem 2.6 we conclude

$$\int_{\Omega_{k1}} |u|^{\alpha(x)+1}\,dx = \int_{\Omega} |u\chi_{\omega_{k1}}|^{\alpha(x)+1}\,dx \to 0$$

as $\beta \to \infty$ and $\|u^{\alpha(p^*)'}\|_{(\alpha+1)/\alpha(p^*)',\Omega_{k1}} \to 0$ as $\beta \to \infty$. Therefore

$$\int_{\Omega_{k1}} |u|^{\alpha(x)(p^*(x))'}\,dx \to 0 \quad \text{as } \beta \to \infty$$

and, by Theorem 2.2, we can choose β so large that

$$I_1 \le \frac{\varepsilon}{12}\|\phi\|_{1,p}.$$

Similarly for I_2 we can also show for sufficiently large β

$$I_2 \le \frac{\varepsilon}{12}\|\phi\|_{1,p}$$

and therefore

$$\left|\int_{\Omega_{k1}} F(x,u+\phi) - F(x,u) - f(x,u)\phi\,dx\right| < \frac{\varepsilon}{6}\|\phi\|_{1,p}. \tag{2.7}$$

Third from $f \in C(\overline{\Omega}\times\mathbb{R})$ we have $F \in C^1(\overline{\Omega}\times\mathbb{R})$. For any $\varepsilon_1, \beta > 0$, there exists $r > 0$ such that

$$|F(x,\xi+h) - F(x,\xi) - f(x,\xi)h| < \varepsilon_1|h| \tag{2.8}$$

whenever $x \in \overline{\Omega_k}$, $|\xi| \le \beta$ and $|h| < r$. From (2.8) we have

$$\int_{\Omega_{k3}} |F(x,u+\phi) - F(x,u) - f(x,u)\phi|\,dx \le \varepsilon_1\|\phi\|_p\|\chi_{\Omega_k}\|_{p'}.$$

Choose ε_1 such that $\varepsilon_1\|\chi_{\Omega_k}\|_{p'} < \varepsilon/6$, then

$$\int_{\Omega_{k3}} |F(x,u+\phi) - F(x,u) - f(x,u)\phi|\,dx \le \frac{\varepsilon}{6}\|\phi\|_{1,p}. \tag{2.9}$$

Here $\|\chi_{\Omega_k}\|_{p'} < \infty$ because $\int_\Omega (\chi_{\Omega_k})^{p'(x)}\,dx = \operatorname{meas}\Omega_k < \infty$.

Fourth similar to the above we have

$$\begin{aligned}\left|\int_{\Omega_{k2}} F(x,u+\phi) - F(x,u) - f(x,u)\phi\,dx\right| &\le C\int_{\Omega_{k2}} |u|^{\alpha(x)}|\phi| + |\phi|^{\alpha(x)+1}\,dx\\ &\le C(\||u|^{\alpha}\|_{(\alpha+1)/\alpha,\Omega_{k2}} + \||\phi|^{\alpha}\|_{(\alpha+1)/\alpha,\Omega_{k2}})\|\phi\|_{\alpha+1,\Omega_{k2}}.\end{aligned}$$

By Theorem 2.8 $\|\phi\|_{\alpha+1,\Omega_{k2}} \le \varepsilon_2\|\phi\|_{p^*} \le C\varepsilon_2\|\phi\|_{1,p}$ for sufficiently small $\|\phi\|_{1,p}$. From $u \in W^{1,p(x)}(\Omega)$ and Theorem 2.6, $u \in L^{\alpha(x)+1}(\Omega_k)$, so

$$\int_{\Omega_{k2}} (|u|^{\alpha(x)})^{(\alpha(x)+1)/\alpha(x)}\,dx < \infty$$

and further $\||u|^\alpha\|_{(\alpha+1)/\alpha,\Omega_{k2}} < \infty$. Similarly $\||\phi|^\alpha\|_{(\alpha+1)/\alpha,\Omega_{k2}} < \infty$ if $\|\phi\|_{1,p} \le 1$. Choose ε_2 such that

$$\int_{\Omega_{k2}} |F(x,u+\phi) - F(x,u) - f(x,u)\phi|\,dx \le \frac{\varepsilon}{6}\|\phi\|_{1,p}. \tag{2.10}$$

From (2.5), (2.7), (2.9) and (2.10) we conclude that $K(u)$ is differentiable on $W_0^{1,p(x)}(\Omega)$ with

$$K'(u)\phi = \int_\Omega f(x,u)\phi\,dx \quad \text{for all } \phi \in W_0^{1,p(x)}(\Omega).$$

Next we consider the continuity of $K'(u)$. From

$$\begin{aligned}
&|K'(u_j)\phi - K'(u)\phi| \\
&\le \int_{\Omega_k} |f(x,u_j) - f(x,u)||\phi|\,dx + \int_{\Omega\setminus\Omega_k} |f(x,u_j) - f(x,u)||\phi|\,dx \\
&\le C(\|f(x,u_j) - f(x,u)\|_{(p^*)',\Omega_k}\|\phi\|_{p^*} + \|g(|u_j|^\alpha + |u|^\alpha)\|_{(p^*)',\Omega\setminus\Omega_k}\|\phi\|_{p^*}) \\
&\le C(\|f(x,u_j) - f(x,u)\|_{(p^*)',\Omega_k} + \|g(|u_j|^\alpha + |u|^\alpha)\|_{(p^*)',\Omega\setminus\Omega_k})\|\phi\|_{1,p},
\end{aligned}$$

we have

$$\begin{aligned}
&\|K'(u_j) - K'(u)\|_{-1,p'} \\
&\qquad \le C(\|f(x,u_j) - f(x,u)\|_{(p^*)',\Omega_k} + \|g(|u_j|^\alpha + |u|^\alpha)\|_{(p^*)',\Omega\setminus\Omega_k}).
\end{aligned}$$

Similarly to the differentiability of $K(u)$ we get the result.

At last we show the compactness of $K'(u)$ by the diagonal method. Let $\{u_j\}$ be a bounded sequence in $W_0^{1,p(x)}(\Omega)$. For each k the compactness of the embedding $W^{1,p(x)}(\Omega_k) \to L^{q(x)}(\Omega_k)$, where $q(x)$ satisfies the conditions in Theorem 2.6, and the boundedness of $\{u_j\}$ in $W^{1,p(x)}(\Omega_k)$ imply that $\{u_j\}$ has a Cauchy subsequence $\{u_{jk}\}$ in $L^{q(x)}(\Omega_k)$. By taking $q(x) = \alpha(x) + 1$, similar to the above we can choose j and k sufficiently large such that

$$\|f(x,u_{jj}) - f(x,u_{ii})\|_{(p^*)',\Omega_k} + \|g(|u_{jj}|^\alpha + |u_{ii}|^\alpha)\|_{(p^*)',\Omega\setminus\Omega_k} < \varepsilon.$$

Then $\{K'(u_{jj})\}$ is a Cauchy sequence in $W^{-1,p'(x)}(\Omega)$ and the compactness of K' follows immediately. □

3. Existence of solutions

The critical points u of $J(u)$, i.e.

$$J'(u)(\phi) = \int_\Omega a(x)|\nabla u|^{p(x)-2}\nabla u\nabla\phi + b(x)|u|^{p(x)-2}u\phi - f(x,u)\phi\,dx = 0 \tag{3.1}$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$ are weak solutions of

$$-\mathrm{div}(a(x)|\nabla u|^{p(x)-2}\nabla u) + b(x)|u|^{p(x)-2}u = f(x,u).$$

So next we need only to consider the existence of nontrivial critical points of $J(u)$.

In the following we study the general case of the prototype (1.3).

THEOREM 3.1. *Under conditions* (H1)–(H3) *the* $p(x)$*-Laplacian problem* (1.1) *has a nontrivial and nonnegative solution* $u \in W_0^{1,p(x)}(\Omega)$.

PROOF. By condition (H2),

$$J(u) \geq \int_\Omega \frac{a_0}{p(x)}|\nabla u|^{p(x)} + \frac{b_0}{p(x)}|u|^{p(x)}\,dx - \int_\Omega \frac{1}{\alpha(x)+1} g(x)|u|^{\alpha(x)+1}\,dx$$
$$\geq \frac{1}{p_2}\int_\Omega a_0|\nabla u|^{p(x)} + b_0|u|^{p(x)} - C|u|^{\alpha(x)+1}\,dx.$$

By Theorem 2.7 we have $\|u\|_{\alpha+1} \leq C\|u\|_{1,p}$. If $\|u\|_{1,p} < 1$ is sufficiently small such that $C\|u\|_{1,p} < 1$, then $\|u\|_{\alpha+1} < 1$. As $\alpha(x)$ and $p(x)$ are uniformly continuous on $\overline{\Omega}$, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|p(x) - p(y)| < \varepsilon \quad \text{and} \quad |\alpha(x) - \alpha(y)| < \varepsilon$$

whenever $x = (x^1, \dots, x^n), y = (y^1, \dots, y^n) \in \overline{\Omega}$ satisfy $|y^i - x^i| < \delta$, $i = 1, \dots, n$. Take $\varepsilon = \widehat{a}/4$ and define $u(x) = 0$ on $\mathbb{R}^n \setminus \Omega$. Divide $\mathbb{R}^n$ into countable open hypercubes $\{Q_j\}_{j=1}^\infty$ with edges parallel to the coordinate axes, the length of each edge is $\delta/2$, $\{Q_j\}_{j=1}^\infty$ mutually have no common points and $\mathbb{R}^n = \bigcup_{j=1}^\infty \overline{Q}_j$. It is obvious that

$$\alpha_{j1} + 1 - p_{j2} > \frac{\widehat{a}}{2}$$

where $p_{j2} = \sup_{x\in Q_j\cap\Omega}\{p(x)\}$ and $\alpha_{j1} = \inf_{x\in Q_j\cap\Omega}\{\alpha(x)\}$. By [11]

$$\int_{Q_j\cap\Omega} |u|^{\alpha(x)+1}\,dx \leq (C\|u\|_{1,p,Q_j\cap\Omega})^{\alpha_{j1}+1}. \tag{3.2}$$

As $\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p < 1$, we have

$$\int_{Q_j\cap\Omega} (|u|^{p(x)} + |\nabla u|^{p(x)})\,dx \geq \|u\|_{p,Q_j\cap\Omega}^{p_{j2}} + \|\nabla u\|_{p,Q_j\cap\Omega}^{p_{j2}} \geq C\|u\|_{1,p,Q_j\cap\Omega}^{p_{j2}}. \tag{3.3}$$

From (3.2) and (3.3) we have

$$\begin{aligned}\frac{1}{p_2}\int_{Q_j\cap\Omega} & a_0|\nabla u|^{p(x)} + b_0|u|^{p(x)} - C|u|^{\alpha(x)+1}\,dx \\ &\geq C_2\|u\|_{1,p,Q_j\cap\Omega}^{p_{j2}} - C_1\|u\|_{1,p,Q_j\cap\Omega}^{\alpha_{j1}+1} \\ &= C_2\|u\|_{1,p,Q_j\cap\Omega}^{p_{j2}}\left(1 - \frac{C_1}{C_2}\|u\|_{1,p,Q_j\cap\Omega}^{\alpha_{j1}+1-p_{j2}}\right) \\ &\geq C_2\|u\|_{1,p,Q_j\cap\Omega}^{p_2}\left(1 - \frac{C_1}{C_2}\|u\|_{1,p,Q_j\cap\Omega}^{\widehat{a}/2}\right) > 0\end{aligned}$$

if $\|u\|_{1,p,Q_j\cap\Omega} < (C_2/C_1)^{2/\widehat{a}}$. So, if $u \neq 0$ and $\|u\|_{1,p} \leq d = \min\{1/2, (C_2/2C_1)^{2/\widehat{a}}\}$, then

$$J(u) \geq \sum_{j=1}^\infty C_2\|u\|_{1,p,Q_j\cap\Omega}^{p_2}\left(1 - \frac{C_1}{C_2}\|u\|_{1,p,Q_j\cap\Omega}^{\widehat{a}/2}\right) > 0.$$

Set $S_d = \{u \in W_0^{1,p(x)}(\Omega) : \|u\|_{1,p} = d\}$, $B_d = \{u \in W_0^{1,p(x)}(\Omega) : \|u\|_{1,p} \le d\}$. Next we show $\inf_{u\in S_d} J(u) > 0$. Otherwise $\inf_{u\in S_d} J(u) = 0$ and there exists $\{u_n\} \subset S_d$ such that $J(u_n) \to 0$. As B_d is weakly compact, there exist a subsequence of $\{u_n\}$ (still denote it by $\{u_n\}$) and $u \in B_d$ such that $u_n \to u$ weakly in $W_0^{1,p(x)}(\Omega)$. As $J(u) + K(u)$ is convex and differentiable, it is weakly semicontinuous and then $J(u) \le \liminf_{n\to\infty} J(u_n) = 0$ in view of Theorem 2.9. If $u \ne 0$, we have $J(u) > 0$ and so $u = 0$. But similar to (3.3)

$$J(u_n) + K(u_n) = \int_\Omega a(x)|\nabla u_n|^{p(x)} + b(x)|u_n|^{p(x)} dx \ge C\|u\|_{1,p}^{p_2} = Cd^{p_2} > 0,$$

we know $J(u_n) \not\to 0$ as $K(u_n) \to 0$, which is a contradiction. By (H1) and (H3) we have $F(x,t) \ge a_1 t^\mu - a_2$ where $(x,t) \in \Omega_0 \times \mathbb{R}$ and $a_1, a_2 > 0$ are constant.

Pick $x_0 \in \Omega_0$ and $B_{2R}(x_0) = \{x : |x - x_0| < 2R\} \subset \Omega_0$ with $2R < 1$. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ such that $\phi \equiv 1$, $x \in B_R(x_0)$; $0 \le \phi(x) \le 1$ and $|\nabla\phi| \le 1/R$. Denote $l = \inf_{x\in\Omega}\{\mu - p(x)\}$. Then, for $s > 1$,

$$\begin{aligned}
J(s\phi) \le & \int_{B_{2R}(x_0)} \frac{s^{p(x)}}{p(x)} (a(x)|\nabla\phi|^{p(x)} + b(x)|\phi|^{p(x)})\, dx \\
& - \int_{B_{2R}(x_0)} s^\mu a_1 |\phi|^\mu \, dx + a_2 \mathrm{meas}\, B_{2R}(x_0) \\
\le & C\left(\frac{1}{R^{p_2}} + 1\right) \int_{B_{2R}(x_0)} s^{p(x)}\, dx \\
& - s^\mu a_1 \int_{B_{2R}(x_0)} |\phi|^\mu \, dx + a_2 \mathrm{meas}\, B_{2R}(x_0) \\
= & \int_{B_{2R}(x_0)} s^{p(x)} \left(\frac{C}{R^{p_2}} + C - \overline{C} s^{\mu - p(x)}\right) dx + a_2 \mathrm{meas}\, B_{2R}(x_0) \\
\le & \int_{B_{2R}(x_0)} s^{p(x)} \left(\frac{C}{R^{p_2}} + C - \overline{C} s^{l}\right) dx + a_2 \mathrm{meas}\, B_{2R}(x_0) < 0
\end{aligned}$$

if s is sufficiently large. Here $\overline{C} = (\int_{B_{2R}(x_0)} a_1 |\phi|^\mu \, dx)/\mathrm{meas}\, B_{2R}(x_0)$.

Next we show that the (PS) condition holds. Suppose that $\{u_i\} \subset W_0^{1,p(x)}(\Omega)$ is a sequence such that $J(u_i) \le C$ and $J'(u_i) \to 0$ in $W^{-1,p'(x)}(\Omega)$. By (H3) we have

$$\begin{aligned}
J(u_i) \ge & \int_\Omega \frac{a(x)}{p(x)} |\nabla u_i|^{p(x)} + \frac{b(x)}{p(x)} |u_i|^{p(x)} \, dx - \int_\Omega \frac{1}{\mu} f(x, u_i) u_i \, dx \\
= & \int_\Omega \left(\frac{1}{p(x)} - \frac{1}{\mu}\right) (a(x)|\nabla u_i|^{p(x)} + b(x)|u_i|^{p(x)})\, dx \\
& + \frac{1}{\mu} \int_\Omega (a(x)|\nabla u_i|^{p(x)} + b(x)|u_i|^{p(x)} - f(x, u_i) u_i)\, dx \\
\ge & \frac{l}{\mu p_2} \int_\Omega a_0 |\nabla u_i|^{p(x)} + b_0 |u_i|^{p(x)} \, dx - \frac{1}{\mu} \|J'(u_i)\|_{-1,p'} \|u_i\|_{1,p}.
\end{aligned}$$

We consider four cases to show that $\{u_i\}$ is bounded in $W^{1,p(x)}(\Omega)$.

Case 1. If $\|u_i\|_p \le 1$ and $\|\nabla u_i\|_p \le 1$, it is immediate that $\|u_i\|_{1,p} \le C$.

Case 2. If $\|u_i\|_p > 1$ and $\|\nabla u_i\|_p > 1$, then

$$\|u_i\|_{1,p} \le \int_\Omega |u_i|^{p(x)} + |\nabla u_i|^{p(x)}\, dx.$$

For i sufficiently large we have

$$\frac{1}{\mu}\|J'(u_i)\|_{-1,p'} < \frac{l}{2\mu p_2}\min\{a_0, b_0\}$$

and then

$$\int_\Omega |\nabla u_i|^{p(x)}\, dx \le C \quad \text{and} \quad \int_\Omega |u_i|^{p(x)}\, dx \le C,$$

and furthermore, by Theorem 2.2, $\|u_i\|_{1,p} \le C$.

Case 3. If $\|u_i\|_p > 1$ and $\|\nabla u_i\|_p \le 1$, then

$$J(u_i) \ge \frac{lb_0}{\mu p_2}\int_\Omega |u_i|^{p(x)}\, dx - \frac{1}{\mu}\|J'(u_i)\|_{-1,p'}\|u_i\|_p - \frac{1}{\mu}\|J'(u_i)\|_{-1,p'}.$$

If i is sufficiently large

$$\frac{1}{\mu}\|J'(u_i)\|_{-1,p'} < \frac{lb_0}{2\mu p_2},$$

by $\|u_i\|_p \le \int_\Omega |u_i|^{p(x)}\, dx$ we know $\int_\Omega |u_i|^{p(x)}\, dx \le C$ and $\|u_i\|_{1,p} \le C$.

Case 4. If $\|u_i\|_p \le 1$ and $\|\nabla u_i\|_p > 1$, we can get $\|u_i\|_{1,p} \le C$ similar to Case 3.

From Cases 1–4 we conclude that $\{u_i\}$ is bounded in $W^{1,p(x)}(\Omega)$ and by Theorem 2.10 there exists a subsequence of $\{u_i\}$ (we still denote it by $\{u_i\}$) such that $K'(u_i)$ is a Cauchy sequence in $W^{-1,p'(x)}(\Omega)$.

Divide Ω into two parts: $\Omega_1 = \{x \in \Omega : p(x) < 2\}$, $\Omega_2 = \{x \in \Omega : p(x) \ge 2\}$. From (3.1) it is easy to get

$$\begin{aligned}(3.4)\quad \int_\Omega a(x)(|\nabla u_i|^{p(x)-2}\nabla u_i - |\nabla u_j|^{p(x)-2}\nabla u_j)(\nabla u_i - \nabla u_j) \\ + b(x)(|u_i|^{p(x)-2}u_i - |u_j|^{p(x)-2}u_j)(u_i - u_j)\, dx \\ \le |J'(u_i)(u_i - u_j)| + |J'(u_j)(u_i - u_j)| \\ + \left|\int_\Omega (f(x,u_i) - f(x,u_j))(u_i - u_j)\, dx\right| \\ \le C(\|J'(u_i)\|_{-1,p'} + \|J'(u_j)\|_{-1,p'} + \|K'(u_i) - K'(u_j)\|_{-1,p'}) \to 0.\end{aligned}$$

On Ω_1 we have

$$\begin{aligned}
&\int_{\Omega_1} |\nabla u_i - \nabla u_j|^{p(x)} + |u_i - u_j|^{p(x)}\,dx \\
&\le \int_{\Omega_1} ((|\nabla u_i|^{p(x)-2}\nabla u_i - |\nabla u_j|^{p(x)-2}\nabla u_j)(\nabla u_i - \nabla u_j))^{p(x)/2} \\
&\quad \times (|\nabla u_i|^{p(x)} + |\nabla u_j|^{p(x)})^{(2-p(x))/2}\,dx \\
&\quad + \int_{\Omega_1} ((|u_i|^{p(x)-2}u_i - |u_j|^{p(x)-2}u_j) \\
&\quad \times (u_i - u_j))^{p(x)/2}(|u_i|^{p(x)} + |u_j|^{p(x)})^{(2-p(x))/2}\,dx \\
&\le \|((|\nabla u_i|^{p(x)-2}\nabla u_i - |\nabla u_j|^{p(x)-2}\nabla u_j)(\nabla u_i - \nabla u_j))^{p(x)/2}\|_{2/p,\Omega_1} \\
&\quad \times \|(|\nabla u_i|^{p(x)} + |\nabla u_j|^{p(x)})^{(2-p(x))/2}\|_{2/(2-p),\Omega_1} \\
&\quad + \|((|u_i|^{p(x)-2}u_i - |u_j|^{p(x)-2}u_j) \\
&\quad \times (u_i - u_j))^{p(x)/2}\|_{2/p,\Omega_1}\|(|u_i|^{p(x)} + |u_j|^{p(x)})^{(2-p(x))/2}\|_{2/(2-p),\Omega_1}.
\end{aligned}$$

From (3.4) and Theorem 2.2 we get

$$\|((|\nabla u_i|^{p(x)-2}\nabla u_i - |\nabla u_j|^{p(x)-2}\nabla u_j)(\nabla u_i - \nabla u_j))^{p(x)/2}\|_{2/p,\Omega_1} \to 0, \tag{3.5}$$

$$\|((|u_i|^{p(x)-2}u_i - |u_j|^{p(x)-2}u_j)(u_i - u_j))^{p(x)/2}\|_{2/p,\Omega_1} \to 0. \tag{3.6}$$

As

$$\int_{\Omega_1} (|\nabla u_i|^{p(x)} + |\nabla u_j|^{p(x)})^{((2-p(x))/2)\cdot(2/(2-p(x)))}\,dx$$

and

$$\int_{\Omega_1} (|u_i|^{p(x)} + |u_j|^{p(x)})^{((2-p(x))/2)\cdot(2/(2-p(x)))}\,dx$$

are all bounded, by Theorem 2.2, (3.5) and (3.6), we have

$$\int_{\Omega_1} |\nabla u_i - \nabla u_j|^{p(x)} + |u_i - u_j|^{p(x)}\,dx \to 0. \tag{3.7}$$

On Ω_2, by (3.4) we have

$$\begin{aligned}
&\int_{\Omega_2} |\nabla u_i - \nabla u_j|^{p(x)} + |u_i - u_j|^{p(x)}\,dx \\
&\le C\int_{\Omega_2} (|\nabla u_i|^{p(x)-2}\nabla u_i - |\nabla u_j|^{p(x)-2}\nabla u_j)(\nabla u_i - \nabla u_j) \\
&\qquad + (|u_i|^{p(x)-2}u_i - |u_j|^{p(x)-2}u_j)(u_i - u_j)\,dx \to 0.
\end{aligned} \tag{3.8}$$

Combining (3.7) with (3.8) and by Theorem 2.2 we conclude $\|u_i - u_j\|_{1,p} \to 0$. Thus the (PS) condition holds.

The Mountain Pass Theorem guarantees that J has a nontrivial critical point u. Let $\phi = \max\{-u(x), 0\}$ in (3.1) we arrive at the conclusion that $u \ge 0$ in Ω. □

Acknowledgements. Part of this work was done while the author was visiting Department of mathematics, The University of Queensland, Australia. He would like to express his sincere thanks to all members of Department of Mathematics, The University of Queensland, for their hospitality, especially to Dr. Jan Chabrowski, who gave a lot of helps there. Part of this work was supported by Natural Science Foundation of Heilongjiang Province.

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Manuscript received September 14, 2006

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