Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 29, 2007, 119–161

GRAPH-APPROXIMATION OF MULTIVALUED WEIGHTED MAPS

Robert Skiba

(*Submitted by L. Górniewicz*)

Abstract. In this paper we study the existence of weighted graph-approximations of w-carriers whose values satisfy a certain w-UV-property. In particular, we prove that any upper semicontinuous set-valued map with compact and acyclic values (with respect to the Čech homology with rational coefficients) from a compact ANR to an ANR admits arbitrarily close weighted graph-approximations.

1. Introduction

The approximations methods provide a powerful tool to study multivalued maps, which were initiated in 1935 by J. von Neumann. The further development of his idea is related to the names A. Cellina, A. Granas, L. Górniewicz, W. Kryszewski, W. Lasonde, and many others; for more historical remarks and the related references, see the survey paper [21].

It is well-known that a multivalued upper semicontinuous map with not necessarily connected values may not admit (sufficiently close) single-valued continuous approximations. Therefore we study the finite-valued version of this problem. The study of such approximations was initiated by J. Pejsachowicz and G. Conti (see [6], [25]). In this article we are going to develop the approximation methods introduced in [6] and [25]. We hope our approximation

2000 *Mathematics Subject Classification.* 54C60, 55N35, 55N05, 55N20.
Key words and phrases. Weighted maps, graph-approximation, UV-property.
The author was partially supported by KBN Grant 2/PO3A/015/25.

techniques being very simple and entirely elementary may be useful in a further development. Let us note that some applications of these approximations can be found in [6], [27], [31].

The paper is organized as follows. After this Introduction it consists of five sections. In Section 2 we give preliminary notations. Moreover, we define w-maps and we give some of their elementary properties. Section 3 is devoted to the study of topological properties of the class of multivalued maps discovered by Gabriele Darbo in 1950 under the name of weighted carriers (w-carriers). In Section 4 we study sets having various w-UV-properties. In particular, we shall compare w-UV-notions with acyclicity with respect to the Čech homology. In Section 5 we improve an approximability theorem for weighted carriers defined on compact polyhedra due to G. Conti and J. Pejsachowicz (see Theorem 4.1 in [6]). Next we state our main result (see Theorem 5.17). In Appendix we give a necessary and sufficient condition for a given compact subset A of an ANR X to be k-acyclic in the sense of Čech homology with the coefficients in the field of rational numbers $\mathbb{Q}$.

The author express his gratitude to L. Górniewicz, W. Kryszewski, J. Pejsachowicz for many valuable comments, suggestions and remarks.

Finally, let us notice that this paper is the refined version of the part of Ph. D. thesis of the author (see [31]).

2. Preliminaries and definitions

By a space we always mean a metric space. If (X, d_X) is a metric space, $\varepsilon > 0$ and $A \subset X$, then by the ε-neighbourhood of A in X we mean the set $O_\varepsilon(A) := \{x \in X \mid d_X(x, A) < \varepsilon\}$, where $d_X(x, A) = \inf_{a \in A} d_X(x, a)$ is the distance of a point $x \in X$ from the set A. In addition, $B(x, \varepsilon) = \{y \in X \mid d_X(y, x) < \varepsilon\}$ (resp. $D(x, \varepsilon) = \{y \in X \mid d_X(y, x) \leq \varepsilon\}$) is the open ball (closed disk) of radius ε centered at $x \in X$. In what follows by a map we understand a single-valued continuous transformation of spaces and by a multivalued map φ of a space X into a space Y we mean a correspondence which associates to every $x \in X$ a non-empty and compact subset $\varphi(x) \subset Y$, and we write $\varphi\colon X \multimap Y$. In the sequel, the symbol $f\colon X \to Y$ is reserved for single-valued mappings. A multivalued map $\varphi\colon X \multimap Y$ is upper semicontinuous (u.s.c.) if for any open subset U of Y the set $\varphi^{-1}(U) := \{x \in X \mid \varphi(x) \subset U\}$ is open in X. Moreover, we associate with φ the graph Γ_φ of φ by putting: $\Gamma_\varphi := \{(x, y) \in X \times Y \mid y \in \varphi(x)\}$.

Given a space X, a piece of X is any open and closed subset of X. Throughout this paper, $\#X$ denotes a power of a set X. By a pair of spaces we understand a pair (X, A) where X is a space and A is a subset of X. Given pairs (X, A), (Y, B), we write $\varphi\colon (X, A) \multimap (Y, B)$ if $\varphi\colon X \multimap Y$ and $\varphi(A) \subset B$. Additionally, by $\varphi_A\colon A \multimap B$ we denote the multivalued map determined by $\varphi\colon (X, A) \multimap (Y, B)$.

If $\varphi\colon X \multimap Y$ and $A \subset X$, then the composite of the inclusion $i\colon A \to X$ with $\varphi\colon X \multimap Y$ is denoted by $\varphi|A\colon A \multimap Y$.

If $A \subset X$, then $\operatorname{cl} A$, $\operatorname{int} A$ and ∂A denote the closure, the interior and the boundary of A, respectively. By D^{n+1} we shall understand the unit closed disk in $\mathbb{R}^{n+1}$ and $\partial D^{n+1} = S^n$. Here and in what follows we shall denote by Δ_n the n-dimensional standard simplex.

By $\check{H}_*(X,G)$ we denote the Čech homology (graded) of a space X with coefficients in a group G ([9]). A space X will be called positively acyclic (resp. k-acyclic, $k \geq 1$) if $\check{H}_n(X,\mathbb{Q}) = 0$ for $n \geq 1$ (resp. $\check{H}_i(X,\mathbb{Q}) = 0$ for $1 \leq i \leq k$). The following nontrivial theorem will be useful for our present purposes.

Theorem 2.1 ([28]). *There exists a transformation T form the arbitrarily homology theory with compact supports over a coefficient group G to the Čech homology over the same coefficient group G such that*

(a) *to each metric space X assigns a homomorphism $T(X)\colon H(X,G) \to \check{H}(X,G)$,*
(b) *for any single-valued map $f\colon X \to Y$ the diagram*

$$\begin{array}{ccc} H(X,G) & \xrightarrow{f_*} & H(Y,G) \\ {\scriptstyle T(X)}\downarrow & & \downarrow{\scriptstyle T(Y)} \\ \check{H}(X,G) & \xrightarrow[f_*]{} & \check{H}(Y,G) \end{array}$$

commutes. Moreover, if X is a metric absolute neighbourhood retract, then $T(X)\colon H(X,G) \to \check{H}(X,G)$ is an isomorphism.

Now we shall gather the basic properties of multivalued weighted maps which are needed in the sequel.

Definition 2.2. A *weighted mapping* from X to Y with coefficients in $\mathbb{Q}$ (or simply a w-map) is a pair $\psi = (\sigma_\psi, w_\psi)$ satisfying the following conditions:

(a) $\sigma_\psi\colon X \multimap Y$ is a multivalued upper semicontinuous mapping such that $\sigma_\psi(x)$ is a finite subset of Y for any $x \in X$;
(b) $w_\psi\colon X \times Y \to \mathbb{Q}$ is a function with the following properties:
   - $w_\psi(x,y) = 0$ for any $y \notin \sigma_\psi(x)$;
   - for any open subset U of Y and $x \in X$ such that $\sigma_\psi(x) \cap \partial U = \emptyset$ there exists an open neighbourhood V of the point x such that:

$$\sum_{y \in U} w_\psi(x,y) = \sum_{y \in U} w_\psi(z,y)$$

for every $z \in V$.

For simplicity of notation, we shall denote a multivalued weighted mapping from X to Y briefly by $\psi\colon X \multimap Y$. So, by $\psi(x)$ we shall mean $\sigma_\psi(x)$ for all $x \in X$. The mapping σ_ψ from the above definition will be called the support of ψ and w_ψ the weight of ψ. The class of weighted maps was introduced in 1958 by G. Darbo and independently by R. Jerrard. Let us notice that in this paper the notion of weighted map is introduced with very little change from the original definition, but all the results of [16], [24]–[26] are also true for weighted maps defined above. Moreover, the above definition seems to be more convenient in our considerations.

Now we give some examples of weighted maps (see also [24], [34]).

Example 2.3. Each continuous map $f\colon X \to Y$ can be considered as a weighted one by assigning the coefficient 1 to each $f(x)$.

Example 2.4. Let $\psi\colon X \multimap Y$ be a continuous map such that for all $x \in X$, $\psi(x)$ consists of 1 or exactly n points (with n fixed). A weight $w_\psi\colon X \times Y \to \mathbb{Q}$ we can define by the formula:

$$w_\psi(x,y) = \begin{cases} 0 & \text{if } y \notin \psi(x), \\ n & \text{if } \{y\} = \psi(x), \\ 1 & \text{otherwise.} \end{cases}$$

It is not difficult to verify that $\psi = (\psi, w_\psi)$ is a weighted map.

Example 2.5. Let $f\colon X \to SP^nY$ be a continuous single-valued map and let $\Pi\colon SP^nY \multimap Y$ be a multivalued map which is defined by

$$\Pi(x_1^{k_1} \ldots x_s^{k_s}) = \{x_1, \ldots , x_s\},$$

where SP^nY denotes the n-th symmetric product of Y and $x_1^{k_1} \ldots x_s^{k_s}$ denotes an equivalence class in SP^nY. Then f induces a w-map $\varphi = (\sigma_\varphi, w_\varphi)$, where $\sigma_\varphi\colon X \multimap Y$ *and* $w_\varphi\colon X \times Y \to \mathbb{Q}$ are defined as follows:

$$\sigma_\varphi(x) = \Pi \circ f(x)$$

and

$$w_\varphi(x,y) = \begin{cases} k_i & \text{if } y \in \sigma_\varphi(x), \\ 0 & \text{if } y \notin \sigma_\varphi(x). \end{cases}$$

Now we shall gather the basic properties of weighted maps which are needed in the sequel (see [10], [16]).

Proposition 2.6. *If $\psi, \varphi\colon X \multimap Y$ are w-maps, then $\psi \cup \varphi = (\sigma_{\psi\cup\varphi}, w_{\psi\cup\varphi})$ is also one, where $\sigma_{\psi\cup\varphi}\colon X \multimap Y$ and $w_{\psi\cup\varphi}\colon X \times Y \to \mathbb{Q}$ are defined by the* *formulas:*

$$\begin{aligned}\sigma_{\psi\cup\varphi}(x) &= \sigma_\psi(x)\cup\sigma_\varphi(x),\\ w_{\psi\cup\varphi}(x,y) &= w_\psi(x,y)+w_\varphi(x,y),\end{aligned}$$

for every $x\in X$ and $y\in Y$.

PROPOSITION 2.7. *If $\psi\colon X\multimap Y$ is a w-map and $\alpha\in\mathbb{Q}$, then $\alpha\cdot\psi=(\sigma_{\alpha\cdot\psi},w_{\alpha\cdot\psi})$ is also one, where $\sigma_{\alpha\cdot\psi}\colon X\multimap Y$ and $w_{\alpha\cdot\psi}\colon X\times Y\to\mathbb{Q}$ are defined as follows: $\sigma_{\alpha\cdot\psi}(x)=\sigma_\psi(x)$ and $w_{\alpha\cdot\psi}(x,y)=\alpha\cdot w_\psi(x,y)$ for every $x\in X$ and $y\in Y$.*

PROPOSITION 2.8. *If $\psi\colon X\multimap Y$ and $\varphi\colon Y\multimap Z$ are w-maps, then $\varphi\circ\psi\colon X\multimap Z$ is a w-map, where its support $\sigma_{\varphi\circ\psi}$ is the composition of σ_φ and σ_ψ and a weight $w_{\varphi\circ\psi}\colon X\times Z\to\mathbb{Q}$ is defined by the formula:*

$$w_{\varphi\circ\psi}(x,z)=\sum_{y\in Y}w_\psi(x,y)\cdot w_\varphi(y,z),$$

for every $x\in X$ and $z\in Z$.

DEFINITION 2.9. Given two w-maps ψ and φ from X to Y, we say that ψ is *w-homotopic to φ* ($\psi\sim_w\varphi$) if there exists a w-map $\theta\colon X\times[0,1]\multimap Y$ such that:

$$w_\theta((x,0),y)=w_\psi(x,y)\quad\text{and}\quad w_\theta((x,1),y)=w_\varphi(x,y),\tag{2.1}$$

$$\sigma_\theta(x,0)=\sigma_\psi(x)\quad\text{and}\quad\sigma_\theta(x,1)=\sigma_\varphi(x),\tag{2.2}$$

for any $x\in X$, $y\in Y$.

DEFINITION 2.10. Let $\varphi\colon X\multimap Y$ be a w-map and let X be a connected space. Then the sum

$$\sum_{y\in Y}w_\varphi(x,y)$$

is called the *weighted index of* φ, where $x\in X$. We shall denote it by $I_w(\varphi)$.

The above definition is correct because the sum $\sum_y w_\varphi(x,y)$ does not depend on the choice of $x\in X$ if the space X is connected ([16]).

PROPOSITION 2.11. *The above index has the following properties:*

(a) *If φ, $\psi\colon X\multimap Y$ are w-homotopic, then $I_w(\varphi)=I_w(\psi)$.*
(b) *If $\varphi\colon X\multimap Y$ and $\psi\colon Y\multimap Z$ are w-maps, then $I_w(\psi\circ\varphi)=I_w(\psi)\cdot I_w(\varphi)$.*
(c) *If $f\colon X\to Y$ is a continuous map, then $I_w(f)=1$.*

The proof is straightforward.

Proposition 2.12 ([16]). *Let $\varphi\colon X \multimap Y$ be a weighted map such that $\varphi(X) \subset \bigcup_{i=1}^{s} V_i$, where V_i, $i = 1, \dots, s$, are open subsets of Y with $V_i \cap V_j = \emptyset$ for $i \neq j$. Assume also that the following condition is satisfied: $\varphi(x) \cap V_i \neq \emptyset$ for all $x \in X$ and $i = 1, \dots, s$. Then there exist w-maps $\varphi_i\colon X \multimap Y$ with $\varphi_i(X) \subset V_i$, $1 \leq i \leq s$, such that $\varphi = \bigcup_{i=1}^{s} \varphi_i$.*

Lemma 2.13. *Let $\psi, \varphi\colon X \multimap Y$ be two w-maps such that*

$$w_\psi(x, y) = w_\psi(x, y),$$

for each $x \in X, y \in Y$. Then there exists a weighted map $\theta\colon X \times [0, 1] \multimap Y$ such that

$$\begin{aligned} \theta(x, 0) &= \varphi(x), \quad & w_\theta((x, 0), y) &= w_\varphi(x, y), \\ \theta(x, 1) &= \psi(x), \quad & w_\theta((x, 1), y) &= w_\psi(x, y), \end{aligned}$$

for $x \in X, y \in Y$.

Proof. It is enough to define a w-map $\theta\colon X \times [0, 1] \multimap Y$ as follows:

$$\theta(x, t) = \begin{cases} \varphi(x) & \text{if } t \in [0, 1/3), \\ \varphi(x) \cup \psi(x) & \text{if } t \in [1/3, 2/3], \\ \psi(x) & \text{if } t \in (2/3, 1] \end{cases}$$

and

$$w_\theta((x, t), y) = w_\varphi(x, y), \quad \text{for } x \in X,\ y \in Y,\ t \in [0, 1]. \qquad \square$$

Lemma 2.14. *Let Y be a path-connected space. Then for any w-map $\varphi\colon \{0, 1\} \multimap Y$ satisfying condition*

$$\sum_{y \in Y} w_\varphi(0, y) = \sum_{y \in Y} w_\varphi(1, y)$$

there exists a w-map $\widetilde{\varphi}\colon [0, 1] \multimap Y$ such that

$$\widetilde{\varphi} = \bigcup_{i=1}^{s} \lambda_i f_i \quad \text{and} \quad \widetilde{\varphi}|\{0, 1\} = \varphi,$$

where $\lambda_i \in \mathbb{Q}$ and $f_i\colon [0, 1] \to Y$ are single-valued continuous functions, for $1 \leq i \leq s$.

Proof. The proof may be found in [6]. But we provide a simple direct proof. Let $\varphi(0) = \{x_1, \dots, x_n\}$ and $\varphi(1) = \{y_1, \dots, y_m\}$. The proof will be divided into two steps.

Step 1. We assume that $n \geq m$. Let

$$\begin{aligned} \alpha_i &:= w_\varphi(0, x_i) \quad \text{for } 1 \leq i \leq n, \\ \beta_j &:= w_\varphi(1, y_j) \quad \text{for } 1 \leq j \leq m. \end{aligned}$$

First, we shall consider the case $m = 1$. Then by the connectedness of Y there exist continuous functions $h_i\colon [0,1] \to Y$ such that

$$h_i(0) = x_i \quad \text{and} \quad h_i(1) = y_1,$$

for $i = 1, \dots, n$. Consequently, $\widetilde{\varphi}\colon [0,1] \multimap Y$ it is enough to define as follows

$$\widetilde{\varphi} := \bigcup_{i=1}^{n} \alpha_i h_i.$$

Let $m > 1$. We put

$$\begin{gathered} \gamma_{x_1} = \alpha_1, \\ \gamma_{y_i} = \beta_i - \gamma_{x_i}, \quad \gamma_{x_{i+1}} = \alpha_{i+1} - \gamma_{y_i}, \quad \text{for } i = 1, \dots, m-1, \\ \text{if } n > m, \text{ then we put } \gamma_{x_{m+l}} = \alpha_{m+l}, \text{ for } l = 1, \dots, n-m. \end{gathered}$$

Since Y is path-connected, there exist continuous functions

$$h_{x_1}, \dots, h_{x_n}\colon [0,1] \to Y \quad \text{and} \quad h_{y_1}, \dots, h_{y_{m-1}}\colon [0,1] \to Y$$

such that

$$\begin{aligned} h_{x_i}(0) &= x_i, & h_{x_i}(1) &= y_i, & &\text{for } i = 1, \dots, m-1, \\ h_{y_i}(0) &= y_i, & h_{y_i}(1) &= x_{i+1}, & &\text{for } i = 1, \dots, m-1, \\ h_{x_{m+l}}(0) &= x_{m+l}, & h_{x_{m+l}}(1) &= y_m, & &\text{for } l = 0, \dots, n-m. \end{aligned}$$

Now it is enough to define $\widetilde{\varphi}\colon [0,1] \multimap Y$ as follows

$$\widetilde{\varphi} = \bigg(\bigcup_{i=1}^{n} \gamma_{x_i} h_{x_i} \bigg) \cup \bigg(\bigcup_{j=1}^{m-1} \gamma_{y_j} h_{y_j}^{-1} \bigg),$$

where $h_{y_j}^{-1}(t) := h_{y_j}(1-t)$ for all $t \in [0,1]$ and $j = 1, \dots, m-1$.

Step 2. We assume that $m \geq n$. Let us define a w-map $\psi\colon \{0,1\} \multimap Y$ by $\psi(t) = \varphi(1-t)$ for $t \in \{0,1\}$. Then by Step 1 there exists a w-map $\widetilde{\psi}\colon [0,1] \multimap Y$ such that

$$\widetilde{\psi} = \bigcup_{i=1}^{s} \lambda_i f_i \quad \text{and} \quad \widetilde{\psi}|\{0,1\} = \psi,$$

where $\lambda_i \in \mathbb{Q}$ and $f_i\colon [0,1] \to Y$ are continuous functions for $i = 1, \dots, s$. Consequently, a w-map $\widetilde{\varphi}\colon [0,1] \multimap Y$ defined by the formula

$$\widetilde{\varphi} = \bigcup_{i=1}^{s} \lambda_i f_i^{-1}$$

is the desired extension of $\varphi\colon \{0,1\} \multimap Y$, where $f_i^{-1}(t) := f_i(1-t)$ for $t \in [0,1]$ and $i = 1, \dots, s$. This completes the proof of the lemma. $\square$

Lemma 2.15 (Gluing lemma). *Assume that a space X is a union of two closed subsets $X = A_1 \cup A_2$ and $A_1 \cap A_2 \neq \emptyset$. If there are two weighted maps $\varphi_1 \colon A_1 \multimap Y, \varphi_2 \colon A_2 \multimap Y$ such that*

$$\begin{aligned} \sigma_{\varphi_1}(x) &= \sigma_{\varphi_2}(x) && \text{for all } x \in A_1 \cap A_2, \\ w_{\varphi_1}(x,y) &= w_{\varphi_2}(x,y) && \text{for all } x \in A_1 \cap A_2,\ y \in Y, \end{aligned}$$

then a pair $\varphi = (\sigma_\varphi, w_\varphi)$ given by

$$\sigma_\varphi(x) := \begin{cases} \sigma_{\varphi_1}(x) & \text{if } x \in A_1, \\ \sigma_{\varphi_2}(x) & \text{if } x \in A_2, \end{cases}$$

and

$$w_\varphi(x,y) := \begin{cases} w_{\varphi_1}(x,y) & \text{if } x \in A_1,\ y \in Y, \\ w_{\varphi_2}(x,y) & \text{if } x \in A_2,\ y \in Y, \end{cases}$$

is a weighted map.

The simple proof of the above lemma is left to the reader.

By $\mathcal{W}$ we shall denote the category of metric spaces and weighted maps with coefficients in $\mathbb{Q}$. In particular, by $\mathcal{W}(X,Y)$ we shall understand the class of all w-maps from X to Y. Let us define an equivalence relation $\sim$ on $\mathcal{W}(X,Y)$ as follows: $\psi \sim \varphi$ if and only if $w_\psi = w_\varphi$. The class of equivalence classes we shall denote by $(X,Y) := \mathcal{W}(X,Y)/\sim$.

Darbo constructed a homology theory for weighted maps by adopting the usual construction of the singular homology functor. In what follows we briefly describe his construction. Let Δ_k be the geometrical k-simplex. For any $0 \leq i \leq k$ consider the map $d_k^i \colon \Delta_{k-1} \to \Delta_k$ given by the inclusion of Δ_{k-1} as the face opposite to the i-th vertex of Δ_k. Given a space X we shall consider the graded vector space $\mathbb{C}(X,\mathbb{Q}) = \{\mathbb{C}_k(X,\mathbb{Q})\}_{k\geq 0}$, where $\mathbb{C}_k(X,\mathbb{Q}) := (\Delta_k, X)$. So, we can define a boundary operator ∂ in $\mathbb{C}(X)$ as follows:

$$\partial_k s = \bigcup_{i=0}^{k} (-1)^i s \circ d_k^i \in \mathbb{C}_{k-1}(X,\mathbb{Q}),$$

for any $s \in \mathbb{C}_k(X,\mathbb{Q})$. The graded vector space $\mathbb{H}(X,\mathbb{Q}) = \{\mathbb{H}_k(X,\mathbb{Q})\}_{k\geq 0}$ of the complex $(\mathbb{C}(X,\mathbb{Q}),\partial)$ will be called the Darbo homology of the space X over $\mathbb{Q}$. Any weighted map $\varphi \colon X \multimap Y$ induces in a functorial way a linear map $\varphi_* \colon \mathbb{H}_*(X,\mathbb{Q}) \to \mathbb{H}_*(Y,\mathbb{Q})$ (of degree zero). Let us note that two w-homotopic w-maps induce the same linear map in Darbo homology. With this $\mathbb{H}_*$ becomes additive functor from $\mathcal{W}$ to the category of graded vector spaces which is invariant under the w-homotopy. Darbo (and Jerrard) showed that the functor $\mathbb{H}_*$ satisfies the Eilenberg–Steenrod axioms for a homology theory with compact carriers. For more details concerning the notion of Darbo Homology, we recommend [7], [16].

A w-map $\varphi\colon (X, x_0) \multimap (Y, y_0)$ between pointed spaces will be called a *pointed w-map* if $\varphi(x_0) = y_0$. Let $\mathcal{W}_0$ be the category of pointed spaces and pointed w-maps with the weighted index equal to 0. Given two weighted maps φ_0 and φ_1 from (X, x_0) to (Y, y_0), we say that φ_0 is w-homotopic relative to x_0 to φ_1 (written $\varphi_0 \sim_w \varphi_1$ rel. x_0) if there exists a weighted map $\theta\colon X \times [0,1] \multimap Y$ satisfying two conditions of Definition 2.9 and $\theta(x_0, t) = y_0$ for any $t \in [0,1]$. This θ is called the *pointed w-homotopy* between φ_0 and φ_1. It is easy to see that the pointed w-homotopy is an equivalence relation on $\mathcal{W}_0$. For a space X with a basepoint $x_0 \in X$, define $\pi_n^w(X, x_0)$ to be the set of the pointed classes of w-maps $\varphi\colon (S^n, s_0) \multimap (X, x_0)$ having the weighted index $I_w(\varphi) = 0$, where s_0 is a base point of the n-sphere S^n. Notice that $\pi_n^w(X, x_0)$ admits a natural structure of $\mathbb{Q}$-module under the following operations:

$$[\varphi] + [\psi] := [\varphi \cup \psi], \qquad \lambda[\varphi] = [\lambda\varphi],$$

where $[\varphi], [\psi] \in \pi_n^w(X, x_0)$, $\lambda \in \mathbb{Q}$. For any pointed space X, and $n \geq 0$, the $\mathbb{Q}$-module $\pi_n^w(X, x_0)$ is called the *n-th w-homotopy $\mathbb{Q}$-module of X*. It is easy to see that we can replace in the definition of $\pi_n^w(X, x_0)$ the unit sphere S^n by $\partial\Delta_{n+1}$. Notice that the concept of w-homotopy was systematically studied in [18], [24], [26].

The Hurewicz map $h_n\colon \pi_n^w(X, x_0) \to \widetilde{\mathbb{H}}_n(X, \mathbb{Q})$ is defined in the usual way. Namely, $h_n(\alpha) = \alpha_*(1_n)$, where $\widetilde{\mathbb{H}}$ denotes the reduced (Darbo) homology and 1_n is a generator of $\widetilde{\mathbb{H}}_n(S^n, \mathbb{Q})$. In the sequel we shall use the following result:

Theorem 2.16 ([26]). *If X is an absolute neighbourhood retract, then the Hurewicz map $h_n\colon \pi_n^w(X, x_0) \to \widetilde{\mathbb{H}}_n(X, \mathbb{Q})$ is an isomorphism for every $n \geq 0$ and any $x_0 \in X$. Moreover, we have the following commutative diagram:*

$$\begin{array}{ccc} \pi_n^w(X, x_0) & \xrightarrow{\varphi_{*n}} & \pi_n^w(Y, y_0) \\ {\scriptstyle h_n}\downarrow & & \downarrow{\scriptstyle h_n} \\ \widetilde{\mathbb{H}}_n(X, \mathbb{Q}) & \xrightarrow[\varphi_{*n}]{} & \widetilde{\mathbb{H}}_n(Y, \mathbb{Q}) \end{array}$$

for any weighted map $\varphi\colon X \multimap Y$ and $n \geq 0$.

We will also make use of the following lemma.

Lemma 2.17. *Let $\varphi\colon S^n \multimap Y$ be a weighted map. In addition, let us assume that there exists a point $x_0 \in S^n$ such that $\varphi(x_0)$ consists of one point. If φ can be extended over D^{n+1}, then φ is w-homotopic relative to x_0 to $I_w(\varphi)k$, where $k\colon S^n \to Y$ is the constant map at $\varphi(x_0)$ (1).*

(1) By the constant map at $y_0 \in Y$ we shall understand the function $k\colon X \to Y$ with $k(x) = y_0$ for all $x \in X$.

Proof. Let $\widetilde{\varphi}: D^{n+1} \multimap Y$ be an extension of φ and let $c: S^n \to D^{n+1}$ be defined by $c(x) = x_0$ for all $x \in S^n$. Since the inclusion $i: S^n \to D^{n+1}$ and $c: S^n \to D^{n+1}$ are w-homotopic relative to x_0, it follows that $\widetilde{\varphi} \circ i$ and $\widetilde{\varphi} \circ c$ are also w-homotopic relative to x_0. Let $k: S^n \to Y$ be defined to be $\widetilde{\varphi} \circ c$. Consequently, $\varphi \sim_w I_w(\varphi)k$, because $\widetilde{\varphi} \circ i = \varphi$ and $\widetilde{\varphi} \circ c = I_w(\varphi)k$, which completes the proof. $\square$

3. Weighted carriers

Given any multivalued map $\Phi: X \multimap Y$ we put

$$D(\Phi) = \{(V, x) \mid V \text{ is an open subset of Y and } \Phi(x) \cap \partial V = \emptyset\}.$$

Definition 3.1. A multivalued u.s.c. map $\Phi: X \multimap Y$ with compact values is said to be a *weighted carrier* if there exists a function $I_{w\mathrm{loc}}: D(\Phi) \to \mathbb{Q}$ satisfying the following conditions:

(a) (Existence) If $I_{w\mathrm{loc}}(\Phi, V, x) \neq 0$, then $\Phi(x) \cap V \neq \emptyset$.

(b) (Local invariance) For every $(V, x) \in D(\Phi)$ there exists an open neighbourhood U_x of a point x such that for each $x' \in U_x$ we have

$$I_{w\mathrm{loc}}(\Phi, V, x) = I_{w\mathrm{loc}}(\Phi, V, x').$$

(c) (Additivity) If $\Phi(x) \cap V \subset \bigcup_{j=1}^{k} V_i$, where V_i, $1 \leq i \leq k$, are open disjoint subsets of V, then

$$I_{w\mathrm{loc}}(\Phi, V, x) = \sum_{i=1}^{k} I_{w\mathrm{loc}}(\Phi, V_i, x).$$

A function $I_{w\mathrm{loc}}: D(\Phi) \to \mathbb{Q}$ verifying the above conditions will be called the local weighted index of Φ.

Remark 3.2. Let us notice that Definition 3.1 is equivalent to that of [6], but our definition of weighted carriers is much more useful to our work.

Remark 3.3. The additivity property in the case of $k = 1$ will be called the excision property.

Below we shall present a number of examples.

Example 3.4. It is easy to see that if $\Phi: X \multimap Y$ is an upper semicontinuous map with connected values, then Φ is a w-carrier. Indeed, a function $I_{w\mathrm{loc}}: D(\Phi) \to \mathbb{Q}$ it is enough to define as follows

$$I_{w\mathrm{loc}}(\Phi, U, x) := \begin{cases} 1 & \text{if } \Phi(x) \cap U \neq \emptyset, \\ 0 & \text{if } \Phi(x) \cap U = \emptyset, \end{cases}$$

for any $(U, x) \in D(\Phi)$.

Example 3.5. If $\varphi: X \multimap Y$ is a weighted map, then $I_{w\mathrm{loc}}: D(\varphi) \to \mathbb{Q}$ we define by $I_{w\mathrm{loc}}(\varphi, U, x) := \sum_{y \in U} w_\varphi(x, y)$ for any $(U, x) \in D(\varphi)$.

Example 3.6. Let X be a compact ANR and let $f: X \times [0,1] \to X$ be a continuous function with the Lefschetz number $\lambda(f_0) \neq 0$ of f_0, where $f_0(x) = f(x, 0)$ for all $x \in X$. Then a multivalued (u.s.c.) map $\Phi: [0,1] \multimap X$ defined by $\Phi(t) = \{x \mid f_t(x) := f(x,t) = x\}$ for all $t \in [0,1]$ is a weighted carrier, because a function $I_{w\mathrm{loc}}: D(\Phi) \to \mathbb{Q}$ given by

$$I_{w\mathrm{loc}}(\Phi, U, t) := \mathrm{ind}(f_t, U, X)$$

verifies all the conditions of Definition 3.1, where $\mathrm{ind}(f_t, U, X)$ denotes the fixed point index for single-valued maps (for more information on the fixed point index for single-valued maps see [14]).

For other examples see [6]. Now we shall prove a lemma which will be used repeatedly throughout this paper.

Lemma 3.7. *Let $\Psi: X \multimap Y$ be a weighted carrier and let U be an open subset of Y. In addition, let X_0 be a connected subset of X such that $\Psi(x) \cap \partial U = \emptyset$ for each $x \in X_0$. Then*

$$I_{w\mathrm{loc}}(\Psi, U, x) = I_{w\mathrm{loc}}(\Psi, U, y)$$

for any $x, y \in X_0$.

Proof. Let us define a map $I: X_0 \to \mathbb{Q}$ by $I(y) := I_{w\mathrm{loc}}(\Psi, U, y)$, where the set $\mathbb{Q}$ of rational numbers is endowed with the discrete topology. Then from the local invariance of $I_{w\mathrm{loc}}$ we infer that the above function I is locally constant. Therefore, by the connectedness of X_0, I is constant, which completes the proof. □

Definition 3.8. Let U be an open subset of Y and let $\Psi: X \multimap Y$ be a weighted carrier. Let C be a connected subset of X satisfying the following condition: $\Psi(x) \cap \partial U = \emptyset$. Define $I_{w\mathrm{loc}}(\Psi|(C, U))$ to be $I_{w\mathrm{loc}}(\Psi|(C, U)) := I_{w\mathrm{loc}}(\Psi, U, c_0)$, where $c_0 \in C$ is an arbitrary fixed point.

Let $\Psi: Y \multimap Z$ and $\Phi: X \multimap Y$ be two weighted carriers. Assume also that the sets $\Phi(x)$ have finitely many connected components $C_1^x, \ldots, C_s^x$, for all $x \in X$. Now let us fix a point $x \in X$. Since C_i^x, $i = 1, \ldots, s_x$, are compact disjoint subsets of $\Phi(x)$, there exist open subsets V_i^x of Z such that

$$C_i^x \subset V_i^x \quad \text{and} \quad V_i^x \cap V_j^x = \emptyset, \tag{3.1}$$

for $i \neq j$ and $i, j = 1, \ldots, s_x$. Let U be an open subset of Z such that $\Psi \circ \Phi(x) \cap \partial U = \emptyset$.

Definition 3.9. Under the above assumptions we let

$$I_{w\mathrm{loc}}(\Psi\circ\Phi,U,x)=\sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi,V_i^x,x)\cdot I_{w\mathrm{loc}}(\Psi|(C_i^x,U)),$$

where $I_{w\mathrm{loc}}(\Psi|(C_i^x,U))$ is defined as in Definition 3.8.

Let us observe that from the localization property of $I_{w\mathrm{loc}}$ for Φ it follows that $I_{w\mathrm{loc}}(\Phi,V_i^x,x)$ does not depend on the choice of V_i^x, and hence the above definition is correct.

Proposition 3.10. *Let $\Psi\colon Y\multimap Z$ and $\Phi\colon X\multimap Y$ be as above. Then a function $I_{w\mathrm{loc}}\colon D(\Psi\circ\Phi)\to\mathbb{Q}$ defined as in Definition* 3.9 *satisfies the properties of existence, local invariance and additivity* (*and hence $\Psi\circ\Phi$ is a weighted carrier*).

Proof. Let us fix $x\in X$. Let $\Phi(x)=C_1^x\cup\ldots\cup C_{s_x}^x$, where C_i^x are components of $\Phi(x)$. Moreover, let U be an open subset of Z such that $\Psi\circ\Phi(x)\cap\partial U=\emptyset$.

(Existence) Let $I_{w\mathrm{loc}}(\Psi\circ\Phi,U,x)\neq 0$. Then there exists $1\le i_0\le s_x$ such that

$$I_{w\mathrm{loc}}(\Phi,V_{i_0}^x,x)\cdot I_{w\mathrm{loc}}(\Psi|(C_{i_0}^x,U))\neq 0.$$

Since $I_{w\mathrm{loc}}(\Psi|(C_{i_0}^x,U))=I_{w\mathrm{loc}}(\Psi,U,c_{i_0})$ for any point $c_{i_0}\in C_{i_0}^x$, it follows that $I_{w\mathrm{loc}}(\Psi,U,c_{i_0})\neq 0$. Consequently, $\Psi(c_{i_0})\cap U\neq\emptyset$ and hence $\Psi(\Phi(x))\cap U\neq\emptyset$, because $c_{i_0}\in C_{i_0}^x\subset\Phi(x)$.

(Local invariance) First, we shall show that for any C_i^x, $i=1,\ldots,s_x$, there exists an open neighbourhood W_i^x of C_i^x in Y such that

$$I_{w\mathrm{loc}}(\Psi|(C_i^x,U))=I_{w\mathrm{loc}}(\Psi,U,y)\quad\text{for all } y\in W_i^x \tag{3.2}$$

and $W_i^x\cap W_j^x=\emptyset$ for $i\neq j$. For this purpose, we fix C_j^x. By the local invariance of $I_{w\mathrm{loc}}$ for Ψ we infer that for any $y\in C_j^x$ there exists an open neighbourhood O'_y of y such that for each $y'\in O'_y$ the following equalities hold

$$I_{w\mathrm{loc}}(\Psi,U,y')=I_{w\mathrm{loc}}(\Psi,U,y)=I_{w\mathrm{loc}}(\Psi|(C_j^x,U)).$$

Since Ψ is u.s.c. and $\Psi(y)\cap\partial U=\emptyset$ for $y\in C_j^x$, it follows that for any $y\in C_j^x$ there exists an open neighbourhood O''_y of y such that $\Psi(y')\cap\partial U=\emptyset$ for each $y'\in O''_y$. Let $O_y:=O'_y\cap O''_y$ for $y\in C_j^x$. Moreover, let $\widetilde{W}_j^x:=\bigcup_{y\in C_j^x}O_y$. Then

$$I_{w\mathrm{loc}}(\Psi,U,y)=I_{w\mathrm{loc}}(\Psi|(C_j^x,U))\quad\text{for } y\in\widetilde{W}_j^x.$$

It is easy to see that there exist open sets $\widehat{W}_i^x$, $i=1,\ldots,s_x$, such that

$$C_i^x\subset\widehat{W}_i^x\quad\text{and}\quad\widehat{W}_i^x\cap\widehat{W}_j^x=\emptyset\quad\text{for } i\neq j.$$

Obviously, if we put $W_i^x := \widetilde{W}_i^x \cap \widehat{W}_i^x$, then $W_i^x \cap W_j^x = \emptyset$ for $i \neq j$ and $i, j = 1, \ldots, s_x$; which completes the proof of (3.2). Now let us put $I_{w\text{loc}}(\Psi|(W_i^x, U)) := I_{w\text{loc}}(\Psi, U, y)$, where $y \in W_i^x$ is an arbitrary fixed point. Hence

$$I_{w\text{loc}}(\Psi|(W_i^x, U)) = I_{w\text{loc}}(\Psi|(C_i^x, U)) \tag{3.3}$$

for all $1 \leq i \leq s_x$. Consequently, from the local invariance of $I_{w\text{loc}}$ for Φ we infer that for each $1 \leq i \leq s_x$ there exists an open neighbourhood O_i^x of the point x such that

$$I_{w\text{loc}}(\Phi, W_i^x, x) = I_{w\text{loc}}(\Phi, W_i^x, x') \tag{3.4}$$

for all $x' \in O_i^x$. Since Φ is u.s.c. we can deduce that there exists an open neighbourhood $\widetilde{O}_x$ of the point x such that $\Phi(\widetilde{O}_x) \subset \bigcup_{i=1}^{s_x} W_i^x$. Let $O_x := \widetilde{O}_x \cap (\bigcap_{i=1}^{s_x} O_i^x)$. Since the sets W_i^x, $1 \leq i \leq s_x$, satisfy the condition (3.1), we have

$$I_{w\text{loc}}(\Psi \circ \Phi, U, x) = \sum_{j=1}^{s_x} I_{w\text{loc}}(\Phi, W_j^x, x) \cdot I_{w\text{loc}}(\Psi|(C_j^x, U)).$$

Now we shall show that for any $x' \in O_x$ the following equality holds

$$I_{w\text{loc}}(\Psi \circ \Phi, U, x) = I_{w\text{loc}}(\Psi \circ \Phi, U, x'),$$

where

$$I_{w\text{loc}}(\Psi \circ \Phi, U, x') = \sum_{j=1}^{s_{x'}} I_{w\text{loc}}(\Phi, V_j^{x'}, x') \cdot I_{w\text{loc}}(\Psi|(C_j^{x'}, U)),$$

$C_j^{x'}$ are components of $\Phi(x')$, $1 \leq j \leq s_{x'}$, and $V_j^{x'}$ are open subsets in Y such that

$$C_j^{x'} \subset V_j^{x'} \quad \text{and} \quad V_i^{x'} \cap V_j^{x'} = \emptyset \quad \text{for } i \neq j.$$

For this purpose, let us fix $x' \in O_x$. Let $I_i^{x'} := \{1 \leq k \leq s_{x'} \mid C_k^{x'} \subset W_i^x\}$ for $1 \leq i \leq s_x$ (2). Then

$$\begin{aligned} &\sum_{j=1}^{s_{x'}} I_{w\text{loc}}(\Phi, V_j^{x'}, x') \cdot I_{w\text{loc}}(\Psi|(C_j^{x'}, U)) \\ &\quad = \sum_{i=1}^{s_x} \sum_{j \in I_i^{x'}} I_{w\text{loc}}(\Phi, V_j^{x'}, x') \cdot I_{w\text{loc}}(\Psi|(C_j^{x'}, U)) \end{aligned} \tag{3.5}$$

(2) Let us note that the set $I_i^{x'}$ defined above can be empty, but it holds only in case $I_{w\text{loc}}(\Phi, W_i^x, x) = 0$. Moreover, if $I_i^{x'} = \emptyset$, then we put

$$\sum_{j \in I_i^{x'}} I_{w\text{loc}}(\Phi, V_j^{x'}, x') \cdot I_{w\text{loc}}(\Psi|(C_j^{x'}, U)) = 0.$$

$$= \sum_{i=1}^{s_x} \sum_{j \in I_i^{x'}} I_{w\text{loc}}(\Phi, V_j^{x'}, x') \cdot I_{w\text{loc}}(\Psi|(C_i^x, U)),$$

where the last equality follows from the fact that for any $j \in I_i^{x'}$ and any $y \in C_j^{x'} \subset W_i^x$ we have

$$I_{w\text{loc}}(\Psi|(C_j^{x'}, U)) = I_{w\text{loc}}(\Psi, U, y) = I_{w\text{loc}}(\Psi|(W_i^x, U)) \overset{(3.3)}{=} I_{w\text{loc}}(\Psi|(C_i^x, U)).$$

Consequently, we have

$$(3.5) = \sum_{i=1}^{s_x} I_{w\text{loc}}(\Psi|(C_i^x, U)) \cdot \bigg(\sum_{j \in I_i^{x'}} I_{w\text{loc}}(\Phi, V_j^{x'}, x') \bigg).$$

Now let us observe that if we show that

$$\sum_{j \in I_i^{x'}} I_{w\text{loc}}(\Phi, V_j^{x'}, x') = I_{w\text{loc}}(\Phi, W_i^x, x), \tag{3.6}$$

then the proof of the local invariance of $I_{w\text{loc}}$ will be completed, because

$$\sum_{i=1}^{s_x} I_{w\text{loc}}(\Psi|(C_i^x, U)) \cdot I_{w\text{loc}}(\Phi, W_i^x, x) = I_{w\text{loc}}(\Psi \circ \Phi, U, x).$$

Now let us fix $i \in \{1, \ldots, s_x\}$. Since

$$\Phi(x') \cap V_j^{x'} = C_j^{x'} \subset W_i^x \textit{ and } \Phi(x') \cap V_j^{x'} \subset V_j^{x'} \text{for any } j \in I_i^{x'},$$

we deduce from the excision property of $I_{w\text{loc}}$ for Φ that

$$I_{w\text{loc}}(\Phi, V_j^{x'}, x') = I_{w\text{loc}}(\Phi, V_j^{x'} \cap W_i^x, x')$$

for all $j \in I_i^{x'}$. Hence

$$\begin{aligned}
\sum_{j \in I_i^{x'}} I_{w\text{loc}}(\Phi, V_j^{x'}, x') &= \sum_{j \in I_i^{x'}} I_{w\text{loc}}(\Phi, V_j^{x'} \cap W_i^x, x') \\
&\overset{(*)}{=} I_{w\text{loc}}\bigg(\Phi, \bigg(\bigcup_{j \in I_i^{x'}} V_j^{x'} \cap W_i^x \bigg), x' \bigg) \\
&= I_{w\text{loc}}\bigg(\Phi, \bigg(\bigcup_{j \in I_i^{x'}} V_j^{x'} \bigg) \cap W_i^x, x' \bigg),
\end{aligned}$$

where the equality $(*)$ holds true by the additivity property of $I_{w\text{loc}}$ for Φ. Consequently, applying the excision property of $I_{w\text{loc}}$, we obtain

$$I_{w\text{loc}}\bigg(\Phi, \bigg(\bigcup_{j \in I_i^{x'}} V_j^{x'} \bigg) \cap W_i^x, x' \bigg) = I_{w\text{loc}}(\Phi, W_i^x, x') \overset{(3.4)}{=} I_{w\text{loc}}(\Phi, W_i^x, x),$$

which completes the proof of (3.6).

(Additivity) Let $\Psi\circ\Phi(x)\cap U\subset\bigcup_{j=1}^k U_j$, $U_m\cap U_n=\emptyset$ for $m\neq n$, $U_j\subset U$ for $1\le j\le k$. First, we shall show that

$$I_{w\mathrm{loc}}(\Psi|(C_i^x,U))=\sum_{j=1}^k I_{w\mathrm{loc}}(\Psi|(C_i^x,U_j)) \tag{3.7}$$

for $1\le i\le s_x$. For this purpose, let us fix $1\le i_0\le s_x$ and $c_{i_0}^x\in C_{i_0}^x$. Since $\Psi(c_{i_0}^x)\cap U\subset\bigcup_{j=1}^k U_j$, we deduce from the additivity property of $I_{w\mathrm{loc}}$ for Ψ that

$$I_{w\mathrm{loc}}(\Psi|(C_i^x,U))=I_{w\mathrm{loc}}(\Psi,U,c_{i_0}^x)=\sum_{j=1}^k I_{w\mathrm{loc}}(\Psi,U_j,c_{i_0}^x),$$

and taking into account the following equality

$$I_{w\mathrm{loc}}(\Psi|(C_i^x,U_j))=I_{w\mathrm{loc}}(\Psi,U_j,c_{i_0}^x),$$

we obtain (3.7). Consequently

$$\begin{aligned}
I_{w\mathrm{loc}}(\Psi\circ\Phi,U,x)&=\sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi,V_i^x,x)\cdot I_{w\mathrm{loc}}(\Psi|(C_i^x,U))\\
&=\sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi,V_i^x,x)\cdot\bigg(\sum_{j=1}^k I_{w\mathrm{loc}}(\Psi|(C_i^x,U_j)\bigg)\\
&=\sum_{j=1}^k\sum_{i=1}^{s_x} I_{w\mathrm{loc}}(\Phi,V_i^x,x)\cdot I_{w\mathrm{loc}}(\Psi|(C_i^x,U_j))\\
&=\sum_{j=1}^k I_{w\mathrm{loc}}(\Psi\circ\Phi,U_j,x),
\end{aligned}$$

which completes the proof of the additivity property of $I_{w\mathrm{loc}}$ for $\Psi\circ\Phi$. □

As an easy consequence of Proposition 3.10 we obtain the following corollary:

Corollary 3.11. *Let $f\colon Y\to Z$ be a single-valued map and let $\Psi\colon X\multimap Y$ be a weighted carrier. Then*

$$I_{w\mathrm{loc}}(\Psi\circ f,U,x)=I_{w\mathrm{loc}}(\Psi,U,f(x)).$$

Definition 3.12. Let $\Psi\colon X\multimap Y$ be a weighted carrier and let $f\colon Y\to Z$ be a single-valued map. Then $I_{w\mathrm{loc}}\colon D(f\circ\Psi)\to\mathbb{Q}$ is defined by

$$I_{w\mathrm{loc}}(f\circ\Psi,U,x):=I_{w\mathrm{loc}}(\Psi,f^{-1}(U),x),$$

for any $(U,x)\in D(f\circ\Psi)$.

It is easy to see that if $(U,x)\in D(f\circ\Psi)$ then $(f^{-1}(U),x)\in D(\Psi)$ and therefore the above definition is correct.

Proposition 3.13. *Let X be an* ANR*, let A be a closed* ANR *subspace of X and let Y be an arbitrary metric space. If $\Psi\colon A\times[0,1]\multimap Y$ is a weighted carrier such that $\Psi_0\colon A\multimap Y$ is extendable to a w-carrier $\widetilde{\Psi_0}\colon X\multimap Y$, then there is a w-carrier $\overline{\Psi}\colon X\times[0,1]\multimap Y$ such that*

(a) $\overline{\Psi}|X\times\{0\}=\widetilde{\Psi_0}$,

(b) *for every $t\in[0,1]$, $\overline{\Psi_t}|A=\Psi_t$,*

where $\Psi_t(x):=\Psi(t,x)$ and $\overline{\Psi_t}(x):=\overline{\Psi}(t,x)$ for all $t\in[0,1]$ and $x\in A$.

The proof of Proposition 3.13 proceeds along the same line as in the case of single valued maps in [15] and therefore we omit further details. Now we are able to prove:

Corollary 3.14. *Let $X,A\subset X,Y$ be as in Proposition 3.13 and let $V\subset U$ be subsets of Y. In addition, let $\varphi\colon A\multimap V$ be a w-map. Then φ can be extended to a w-map $\widetilde{\varphi}\colon X\multimap U$ if and only if $\varphi\cup(-I_w(\varphi))y_0\colon A\multimap V$ can be extended to a w-map $\overline{\varphi}\colon X\multimap U$, where $y_0\in Y$ is any fixed point.*

Proof. This implication $\Rightarrow$ is obvious. For $\Leftarrow$, let $\varphi\cup(-I_w(\varphi))y_0\colon A\multimap V$ be a weighted map and let $\overline{\varphi}\colon X\multimap U$ be an extension of $\varphi\cup(-I_w(\varphi))y_0$ over X. Then a weighted map $\overline{\varphi}\cup I_w(\varphi)y_0\colon X\multimap U$ satisfies the following condition $w_{\overline{\varphi}\cup I_w(\varphi)y_0}(x,y)=w_{i\circ\varphi}(x,y)$ for all $x\in A,y\in U$, where $i\colon V\hookrightarrow U$ is the inclusion. Hence, in view of Lemma 2.13, a w-map $\overline{\varphi}\cup I_w(\varphi)y_0|A\colon A\multimap U$ is w-homotopic to $i\circ\varphi\colon A\multimap U$. Consequently, by Proposition 3.13, it follows that there exists a w-map $\widetilde{\varphi}\colon X\multimap U$ with $\widetilde{\varphi}(x)=\varphi(x)$ for all $x\in A$, which completes the proof. □

4. w-UV-sets

Following [20] we propose the following definitions, which will play a crucial role in the sequel.

Definition 4.1. Let $V\subset U$ be subsets of a space Y. We say that the inclusion $V\hookrightarrow U$ is *w-homotopy 0-trivial* if for any connected component C of V and for any weighted map $\varphi\colon\partial\Delta_1\multimap C$ satisfying the condition

$$\sum_{y\in C}w_\varphi(0,y)=\sum_{y\in C}w_\varphi(1,y)$$

there exists a weighted map $\widetilde{\varphi}\colon\Delta_1\multimap U$ such that $\widetilde{\varphi}(x)=\varphi(x)$ for every $x\in\partial\Delta_1$.

Definition 4.2. Let $V\subset U$ be subsets of a space Y and let $n\geq 1$ be an integer. The inclusion $V\hookrightarrow U$ is said to be *w-homotopy n-trivial* if it is w-homotopy 0-trivial and for any integer $1<k\leq n+1$ and for every weighted map $\varphi\colon\partial\Delta_k\multimap V$ there exists a w-map $\widetilde{\varphi}\colon\Delta_k\multimap U$ such that $\widetilde{\varphi}(x)=\varphi(x)$ for every $x\in\partial\Delta_k$.

Definition 4.3. Let K be a compact subset of a space X. We say that the inclusion $A \hookrightarrow X$ has:

(a) *w-UV^n-property* ($n \geq 0$) if for every $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that the inclusion $O_\delta(K) \to O_\varepsilon(K)$ is w-homotopy n-trivial;

(b) *w-UV^ω-property* if it has w-UV^n-property for each $n \geq 0$.

Now we are going to show some facts concerning the above notions. In particular, we will prove that the class of sets satisfying some w-UV-properties is quite large.

Proposition 4.4. *Let X be a locally connected space* (3)*, let K be a compact subset of X and let $n \geq 1$. If for any $\varepsilon > 0$ there exists δ, $0 < \delta < \varepsilon$, such that:*

(a) *$O_\delta(K) \hookrightarrow O_\varepsilon(K)$ is w-homotopy 0-trivial,*

(b) *for each positive integer $1 \leq k \leq n$ and $x_0 \in O_\delta(K)$, the inclusion $O_\delta(K) \hookrightarrow O_\varepsilon(K)$ induces the trivial homomorphism*

$$\pi_k^w(O_\delta(K), x_0) \to \pi_k^w(O_\varepsilon(K), x_0),$$

then the inclusion $K \hookrightarrow X$ has a w-UV^n-property.

Proof. The proof will be divided into a number of steps. (We proceed by proving successively more general cases.)

Step 1. Fix $\varepsilon > 0$ and let $\delta > 0$ be such that the induced homomorphism

$$\pi_k^w(O_\delta(K), x_0) \to \pi_k^w(O_\varepsilon(K), x_0) \tag{4.1}$$

is trivial for $1 \leq k \leq n$ and for all $x_0 \in O_\delta(A)$. We divide Step 1 into a sequence of cases.

Case A. Let $\varphi: S^n \multimap O_\delta(A)$ be a w-map with $I_w(\varphi) = 0$ and $\varphi(s_0) = x_0$, where $s_0 \in S^n$ is a fixed point. Since the homomorphism (4.1) is trivial, it follows that a w-map $i \circ \varphi: S^n \multimap O_\varepsilon(A)$ is w-homotopic to the constant map at x_0 (with the weighted index equals 0), where $i: O_\delta(A) \to O_\varepsilon(A)$ is the inclusion. Hence, in view of Proposition 3.13, we conclude that $i \circ \varphi$ can be extended to a w-map $\widetilde{\varphi}: D^{n+1} \multimap O_\varepsilon(A)$.

Case B. Let $\varphi: S^n \multimap O_\delta(A)$ be a w-map with $I_w(\varphi) \neq 0$ and $\varphi(s_0) = x_0$ (s_0 as in Case 1). Let $\widetilde{\psi}: S^n \multimap O_\delta(A)$ be given by $\psi = \varphi \cup (-I_w(\varphi))y_0$, where y_0 is an arbitrary fixed point of A. Since $I_w(\psi) = 0$, we conclude by Case 1 that there exists a weighted map $\widetilde{\psi}: D^{n+1} \multimap O_\varepsilon(A)$ such that $\widetilde{\psi}(x) = \psi(x)$ for each $x \in S^n$. Therefore by Corollary 3.14 we obtain a w-map $\widetilde{\varphi}: D^{n+1} \multimap O_\varepsilon(A)$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in S^n$.

---

(3) If a space X is locally connected and V is an open subset of X, then V is locally connected. Hence any connected component C of V is open in X. This observation will be of use in the proof of Proposition 4.4 and later.

Case C. Let $\varphi\colon S^n \multimap O_\delta(A)$ be a w-map and let us assume that $\#\varphi(s_0) \neq 1$. Assume also that there exists a w-map $\alpha\colon [0,1] \multimap O_\delta(A)$ such that

$$\alpha(0) = \varphi(s_0), \quad \alpha(1) = x_0,$$
$$w_\alpha(0,y) = w_\varphi(s_0,y), \quad \text{for all } y \in O_\delta(A).$$

Now let us define $\Upsilon\colon (S^n \times \{0\}) \cup (\{s_0\} \times [0,1]) \multimap O_\delta(A)$ by

$$\Upsilon(x,t) = \begin{cases} \varphi(x) & \text{if } t = 0, \\ \alpha(t) & \text{if } x = s_0. \end{cases}$$

Then in view of Proposition 3.13 there exists a w-map $\widetilde{\Upsilon}\colon S^n \times [0,1] \multimap O_\delta(A)$ such that $\widetilde{\Upsilon}|(S^n \times \{0\}) \cup (\{s_0\} \times [0,1]) = \Upsilon$. Now, applying Case A or Case B to $\widetilde{\Upsilon}(\,\cdot\,,1)\colon S^n \multimap O_\delta(A)$ (4), we obtain an extension $\overline{\Upsilon}\colon D^{n+1} \multimap O_\varepsilon(A)$ of $\widetilde{\Upsilon}(\,\cdot\,,1)$. Let

$$\Upsilon_0\colon (S^n \times \{0\}) \cup (\{s_0\} \times [0,1]) \cup (D^{n+1} \times \{1\}) \multimap O_\varepsilon(A)$$

be defined as follows

$$\Upsilon_0(x,t) = \begin{cases} \varphi(x) & \text{if } t = 0, \\ \alpha(t) & \text{if } x = s_0, \\ \overline{\Upsilon}(x) & \text{if } t = 1. \end{cases}$$

Since $(S^n \times \{0\}) \cup (\{s_0\} \times [0,1]) \cup (D^{n+1} \times \{1\})$ is an ANR and is closed in $D^{n+1} \times [0,1]$, we infer that due to Proposition 3.13 we can extend Υ_0 to a weighted map $\widehat{\Upsilon}\colon D^{n+1} \times [0,1] \multimap O_\varepsilon(A)$. Finally, let us observe that $\widehat{\Upsilon}(\,\cdot\,,0)\colon D^{n+1} \multimap O_\varepsilon(A)$ satisfies the following condition $\widehat{\Upsilon}(x,0) = \varphi(x)$, for every $x \in S^n$.

Step 2. Let $\varepsilon > 0$. Then under the assumptions of Proposition 4.4 it follows that there exists $\delta < \varepsilon$ such that

(a) $O_\delta(K) \hookrightarrow O_\varepsilon(K)$ is w-homotopy 0-trivial,

(b) for each positive integer $1 \le k \le n$ and $x_0 \in O_\delta(K)$, the inclusion $O_\delta(K) \hookrightarrow O_\varepsilon(K)$ induces the trivial homomorphism

$$\pi_k^w(O_\delta(K), x_0) \to \pi_k^w(O_\varepsilon(K), x_0).$$

Moreover, for δ there exists $\eta < \delta$ such that

(c) $O_\eta(K) \hookrightarrow O_\delta(K)$ is w-homotopy 0-trivial.

Let us fix $1 \le k \le n$ and let s_0 be the base point of S^k. Now we shall show that for any w-map $\varphi\colon S^k \multimap O_\eta(A)$ there exists a w-map $\widetilde{\varphi}\colon D^{k+1} \multimap O_\varepsilon(A)$ with $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in S^k$. To see this, let us fix a w-map $\varphi\colon S^k \multimap O_\eta(A)$. Let us observe that if for a given w-map $\varphi\colon S^k \multimap O_\eta(A)$ there exists a w-map $\alpha\colon [0,1] \multimap O_\delta(A)$ such that

$$\alpha(0) = \varphi(s_0),\ w_\alpha(0,y) = w_\varphi(s_0,y) \quad \text{for all } y \in O_\eta(A),\ \#\alpha(1) = 1, \tag{4.2}$$

(4) If $I_w(\widetilde{\Upsilon}(\,\cdot\,,1)) = 0$, then we apply Case A, otherwise we apply Case B.

then by Step 1 we infer that there exists a w-map $\widetilde{\varphi}: D^{k+1} \multimap O_\varepsilon(A)$ with $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in S^k$. Therefore we can assume that for $\varphi: S^k \multimap O_\eta(A)$ there is no $\alpha: [0,1] \multimap O_\delta(A)$ satisfying (4.2). Let $O_\eta(A) = \bigcup_{j\in I} O_j^A$, where O_j^A is the connected component of $O_\eta(A)$. Since $O_\eta(A)$ is locally connected, it follows that the connected components of $O_\eta(A)$ are open in $O_\eta(A)$. Hence, by the compactness of $\varphi(S^k)$, we obtain

$$\#I' := \{j \in I \mid \varphi(S^k) \cap O_j^A \neq \emptyset\} < \infty.$$

Obviously, $\varphi(S^k) \subset \bigcup_{j_m \in I'} O_{j_m}^A$. Let us choose a point y_{j_m} in each component $O_{j_m}^A$ and let us define a w-map $\alpha: S^k \multimap O_\eta(A)$ as follows

$$\alpha(x) = \{y_{j_1}, \dots, y_{j_s}\}, \quad w_\alpha(x,y) = 0,$$

for all $x \in S^k, y \in O_\eta(A)$, where $s := \#I'$. Let $\varphi^\alpha: S^k \multimap O_\delta(A)$ be defined by $\varphi^\alpha := \varphi \cup \alpha$. Then, by Proposition 2.12, a w-map φ^α has the following decomposition $\varphi^\alpha = \varphi_1^\alpha \cup \ldots \cup \varphi_s^\alpha$, where any w-map φ_m^α satisfies the following condition $\varphi_m^\alpha(S^k) \subset O_{j_m}^A$. Let $\beta_m: \{0,1\} \multimap O_{j_m}^A$, $m = 1, \dots, s$, be defined as follows

$$\begin{gathered}
\beta_m(0) = \varphi_m^\alpha(s_0), \quad \beta_m(1) = y_{j_m}, \\
w_{\beta_m}(0,y) := w_{\varphi_m^\alpha}(s_0,y), \quad \text{for all } y \in O_{j_m}^A, \\
w_{\beta_m}(1,y_{j_m}) := I_w(\varphi_m^\alpha), \quad w_{\beta_m}(1,y) := 0 \quad \text{for } y \neq y_{j_m},
\end{gathered}$$

where $1 \leq m \leq s$. Since the inclusion $O_\eta(A) \hookrightarrow O_\delta(A)$ is w-homotopy 0-trivial, it follows that for any β_m there exists a w-map $\widetilde{\beta_m}: [0,1] \multimap O_\delta(A)$ with $\widetilde{\beta_m}|\{0,1\} = \beta_m$. Hence, by Step 1, for any w-map $\varphi_m^\alpha: S^k \multimap O_\eta(A)$ there exists a w-map $\widetilde{\varphi}_m^\alpha: D^{k+1} \multimap O_\varepsilon(A)$ such that $\widetilde{\varphi}_m^\alpha(x) = \varphi_m^\alpha(x)$ for all $x \in S^k$. Consequently, a w-map $\widetilde{\varphi}: D^{k+1} \multimap O_\varepsilon(A)$ given by

$$\widetilde{\varphi}^\alpha = \widetilde{\varphi}_1^\alpha \cup \ldots \cup \widetilde{\varphi}_m^\alpha$$

is an extension of $\varphi^\alpha: S^k \multimap O_\eta(A)$. Since w-maps φ and φ^α satisfy the following condition

$$w_\varphi(x,y) = w_{\varphi^\alpha}(x,y),$$

for all $x \in S^k$ and $y \in O_\varepsilon(A)$, Lemma 2.13 implies that φ is w-homotopic to φ^α; and hence, by Proposition 3.13, we infer that there exists a w-map $\widetilde{\varphi}: D^{k+1} \multimap O_\varepsilon(A)$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in S^k$. This completes the proof. $\square$

Now we will prove that the converse of the last statement is also true.

Proposition 4.5. *Let X be a space and let K be a compact subset of X. If the inclusion $K \hookrightarrow X$ has a w-UV^n-property ($n \geq 1$), then for each $\varepsilon > 0$, there exists δ, $0 < \delta < \varepsilon$, such that the homomorphism*

$$h_k: \pi_k^w(O_\delta(K), x_0) \to \pi_k^w(O_\varepsilon(K), x_0)$$

induced by the inclusion $i\colon O_\delta(A) \to O_\varepsilon(A)$ *is trivial for* $1 \le k \le n$ *and for all* $x_0 \in O_\delta(A)$.

Proof. Let us fix $\varepsilon > 0$. Let $\delta > 0$ be such that for any $1 \le k \le n$ and any w-map $\varphi\colon S^k \multimap O_\delta(A)$ there exists a w-map $\widetilde{\varphi}\colon D^{k+1} \multimap O_\varepsilon(A)$ with $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in S^k$. Now we are going to show that the induced homomorphism $h_k\colon \pi_k^w(O_\delta(K), x_0) \to \pi_k^w(O_\varepsilon(K), x_0)$ is trivial for any $1 \le k \le n$ and for each $x_0 \in O_\delta(A)$. To see this, let us fix $1 \le k \le n$ and $x_0 \in O_\delta(A)$. Let $\varphi\colon (S^k, s_0) \multimap (O_\delta(A), x_0)$ be a pointed w-map with $I_w(\varphi) = 0$. Hence, by the definition of δ, we infer that there exists a w-map $\widetilde{\varphi}\colon D^{k+1} \multimap O_\varepsilon(A)$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for $x \in S^k$. Then, in view of Lemma 2.17, $i \circ \varphi$ is w-homotopic relative to s_0 to the constant map at x_0 (with the weighted index equal to 0), which proves that the homomorphism h_k is trivial. □

As an immediate consequence of the above propositions we obtain:

Corollary 4.6. *Let X be a locally connected space and let K be a compact subset of X. Then the inclusion $K \hookrightarrow X$ has a w-UV^n-property $(n \ge 1)$ if and only if, for any $\varepsilon > 0$, there exists δ, $0 < \delta < \varepsilon$, such that:*

(a) *$O_\delta(K) \hookrightarrow O_\varepsilon(K)$ is w-homotopy 0-trivial,*

(b) *for each positive integer $1 \le k \le n$ and $x_0 \in O_\delta(K)$, the inclusion $O_\delta(K) \hookrightarrow O_\varepsilon(K)$ induces the trivial homomorphism*

$$\pi_k^w(O_\delta(K), x_0) \to \pi_k^w(O_\varepsilon(K), x_0).$$

Proposition 4.7. *Let X be a locally path-connected space and let $A \subset X$ be a compact subspace. Then for any open subsets U and V satisfying condition $A \subset V \subset U \subset X$ the inclusion $V \hookrightarrow U$ is a w-homotopy 0-trivial.*

Proof. Let $V \subset U$ be open subsets of X and let C be a connected component of V. Due to our assumptions C is locally path-connected and connected. Hence C is path-connected. Let $\varphi\colon S^0 \multimap C$ be a weighted map with $\sum_{y\in C} w_\varphi(0, y) = \sum_{y\in C} w_\varphi(1, y)$. Then in view of Lemma 2.14 there exists a weighted map $\widetilde{\varphi}\colon D^1 \multimap C$ with $\widetilde{\varphi}|S^0 = \varphi$. This completes the proof. □

Consequently, combining Corollary 4.6 and Proposition 4.7, we obtain the following corollary.

Corollary 4.8. *Let A be a compact subset of a locally path-connected space X, $n \ge 1$. Then A has a w-UV^n-property if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that the inclusion $O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces the trivial homomorphism $h_k\colon \pi_k^w(O_\delta(A), x_0) \to \pi_k^w(O_\varepsilon(A), x_0)$ for any $1 \le k \le n$ and for all $x_0 \in O_\delta(A)$.*

Taking into account Corollary 4.8, Theorems 2.1 and 2.16 we get the following theorem.

THEOREM 4.9. *Let X be an* ANR *and let A be a compact subset of X, $k \geq 1$. Then the inclusion $j: A \hookrightarrow X$ has a w-UV^k-property if and only if for each $\varepsilon > 0$ there exists $0 < \delta < \varepsilon$ such that the induced homomorphism $j_i: \check{H}_i(O_\delta(A), \mathbb{Q}) \to \check{H}_i(O_\varepsilon(A), \mathbb{Q})$ is trivial for each $1 \leq i \leq k$.*

Now let us observe that Theorem 4.9 together with Theorem 6.1 (in Appendix) implies the following theorem.

THEOREM 4.10. *Let X nad $A \subset X$ be as in Theorem* 4.9 *and $k \geq 1$. Then A is k-acyclic if and only if the inclusion $A \hookrightarrow X$ has a w-UV^k-property.*

Since a subset A of a space X is positively acyclic if and only if it is k-acyclic for all $k \geq 1$, we obtain, by Theorem 4.10, the following corollary.

COROLLARY 4.11. *Let X be an* ANR *and let A be a compact subset of X. Then A is positively acyclic if and only if the inclusion $j: A \hookrightarrow X$ has a w-UV^ω-property.*

We shall conclude this section by introducing the following notion, which will be used in what follows.

DEFINITION 4.12. Let $0 \leq n < \infty$ or $n = \omega$. A weighted carrier $\Psi: X \multimap Y$ is said to be a *w-UV^n-valued carrier* if, for each $x \in X$, the inclusion $\Psi(x) \hookrightarrow Y$ has w-UV^n-property.

5. Approximation of w-carrier by w-maps

DEFINITION 5.1 ([25]). Let $\Psi: X \multimap Y$ be a weighted carrier and $X_0 \subset X$, and let $\varepsilon > 0$. A weighted map $\psi: X_0 \multimap Y$ is said to be an *ε-approximation of* $\Psi: X \multimap Y$ if

(a) $\psi(x) \subset O_\varepsilon(\Psi(O_\varepsilon(x)))$ for all $x \in X_0$,
(b) $I_{w\mathrm{loc}}(\psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, x)$ for any piece C of $O_\varepsilon(\Psi(O_\varepsilon(x)))$ ([5]) and $x \in X_0$.

REMARK 5.2. The above definition is correct, i.e. $(C, x) \in D(\Psi)$ and $(C, x) \in D(\psi)$, because the following simple lemma holds true.

LEMMA 5.3. *Let U be an open subset of X and let C be a piece of U. If K is a subset of U, then $K \cap \partial C = \emptyset$ (where ∂C denotes the boundary of C with respect to X).*

PROOF. It is enough to show that $\partial C \cap U = \emptyset$. For this purpose, let us observe that C and $U \setminus C$ are open in X. Consequently, $\partial C \cap (U \setminus C) = \emptyset$ and $\partial C \cap C = \emptyset$; and hence $\partial C \cap U = \emptyset$. □

Moreover, we have the following result:

([5]) Recall that by a piece C of $O_\varepsilon(\Psi(O_\varepsilon(x)))$ we understand a subset C of $O_\varepsilon(\Psi(O_\varepsilon(x)))$ which is open and closed in $O_\varepsilon(\Psi(O_\varepsilon(x)))$.

Proposition 5.4. *Let* $\Psi: X \multimap Y$ *be a* w*-carrier and let* $\varphi: X \multimap Y$ *be a* w*-map. In addition, let* $0 < \varepsilon_1 < \varepsilon_2$*. If* φ *is an* ε_1*-approximation of* Ψ*, then* φ *is also an* ε_2*-approximation of* Ψ*.*

Proof. The first condition of Definition 5.1 is obviously satisfied, only the second condition needs to be proved. For this purpose, let us fix $x \in X$ and let C be a piece of $O_{\varepsilon_2}(\Psi(O_{\varepsilon_2}(x)))$. Then $\widetilde{C} := C \cap O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x)))$ is a piece of $O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x)))$. Since φ is an ε_1-approximation of Ψ, it follows that

$$I_{w\mathrm{loc}}(\varphi, \widetilde{C}, x) = I_{w\mathrm{loc}}(\Psi, \widetilde{C}, x). \tag{5.1}$$

Consequently, by the excision property of $I_{w\mathrm{loc}}$, we obtain

$$I_{w\mathrm{loc}}(\varphi, \widetilde{C}, x) = I_{w\mathrm{loc}}(\varphi, C, x) \tag{5.2}$$

and

$$I_{w\mathrm{loc}}(\Psi, \widetilde{C}, x) = I_{w\mathrm{loc}}(\Psi, C, x). \tag{5.3}$$

Now, taking into account (5.1)–(5.3), we have

$$I_{w\mathrm{loc}}(\varphi, C, x) = I_{w\mathrm{loc}}(\Psi, C, x),$$

which completes the proof. □

We need the following lemma.

Lemma 5.5 ([13]). *Let* $\psi: X \multimap Y$ *and* $\varphi: Y \multimap Z$ *be two upper semicontinuous multivalued maps. If* X *is a compact space, then for every* $\varepsilon > 0$ *there is* $\delta > 0$ *such that* $O_\delta(\varphi)O_\delta(\psi)(x) \subset O_\varepsilon(\varphi \circ \psi(O_\varepsilon(x)))$ *for any* $x \in X$*, where* $O_\delta(\varphi)O_\delta(\psi)(x) := O_\delta(\varphi(O_\delta(O_\delta(\psi(O_\delta(x))))))$*.*

Now we use the above lemma to obtain the following proposition which will needed in the sequel.

Proposition 5.6. *Let* X *be a compact space,* $\varphi: X \multimap Y$ *a weighted map and* $\Phi: Y \multimap Z$ *a weighted carrier. Then for each* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that if* $\psi: Y \multimap Z$ *is a* δ*-approximation of* Φ*, then* $\psi \circ \varphi$ *is an* ε*-approximation of* $\Phi \circ \varphi$*.*

Proof. Let $\varepsilon > 0$. From Lemma 5.5 it follows that there exists $\delta > 0$ such that

$$O_\delta(\Phi)O_\delta(\varphi)(x) \subset O_\varepsilon(\Phi \circ \varphi)(x),$$

for all $x \in X$. Let $\psi: Y \multimap Z$ be a δ-approximation of $\Phi: Y \multimap Z$. Let us fix $x \in X$. Then

$$\psi(\varphi(x)) \subset O_\delta(\Phi(O_\delta(\varphi(x)))) \subset O_\delta(\Phi)O_\delta(\varphi)(x) \subset O_\varepsilon(\Phi \circ \varphi)(x).$$

What is left is to show that

$$I_{w\text{loc}}(\psi\circ\varphi, C, x) = I_{w\text{loc}}(\Phi\circ\varphi, C, x)$$

for any piece C of $O_\varepsilon(\Phi\circ\varphi(O_\varepsilon(x)))$. Let $\varphi(x) = \{y_1, \ldots, y_{n_x}\}$. Now let us observe (see Definition 3.9) that

$$\begin{aligned} I_{w\text{loc}}(\psi\circ\varphi, C, x) &= \sum_{i=1}^{n_x} I_{w\text{loc}}(\varphi, V_i^x, x)\cdot I_{w\text{loc}}(\psi, C, y_i),\\ I_{w\text{loc}}(\Phi\circ\varphi, C, x) &= \sum_{i=1}^{n_x} I_{w\text{loc}}(\varphi, V_i^x, x)\cdot I_{w\text{loc}}(\Phi, C, y_i), \end{aligned}$$

where $V_1^x, \ldots, V_{n_x}^x$ satisfy the following conditions

$$y_i \in V_i^x \quad \text{and} \quad V_i^x \cap V_j^x = \emptyset \quad \text{for } i \neq j.$$

Consequently, it is enough to show that the following equality holds

$$I_{w\text{loc}}(\varphi, C, y_i) = I_{w\text{loc}}(\Phi, C, y_i).$$

For this purpose, let us observe that

$$\begin{aligned} I_{w\text{loc}}(\psi, C, y_i) &= I_{w\text{loc}}(\psi, C\cap O_\delta(\Phi(O_\delta(y_i))), y_i)\\ &= I_{w\text{loc}}(\Phi, C\cap O_\delta(\Phi(O_\delta(y_i))), y_i) = I_{w\text{loc}}(\Phi, C, y_i), \end{aligned} \tag{5.4}$$

where the first equality and the last one follow from the excision property of $I_{w\text{loc}}$, because

$$\begin{aligned} \psi(y_i)\cap C &\subset C\cap O_\delta(\Phi(O_\delta(y_i))) \subset C,\\ \Phi(y_i)\cap C &\subset C\cap O_\delta(\Phi(O_\delta(y_i))) \subset C. \end{aligned}$$

Moreover, since $C\cap O_\delta(\Phi(O_\delta(y_i)))$ is a piece of $O_\delta(\Phi(O_\delta(y_i)))$, we deduce that the second equality in (5.4) follows from the fact that ψ is a δ-approximation of Φ.□

Before proceeding further, we prove some necessary lemmas.

Lemma 5.7. *Let X be a locally connected space and let $\Psi: X \multimap Y$ be a weighted carrier. Then for every $\varepsilon > 0$ and $x \in X$ there exists $\delta_x > 0$ such that for any $y \in B(x, \delta_x)$ and any piece C of $O_\varepsilon(\Psi(O_\varepsilon(x)))$ the following equation holds:*

$$I_{w\text{loc}}(\Psi, C, x) = I_{w\text{loc}}(\Psi, C, y).$$

Proof. Let us fix $\varepsilon > 0$ and $x \in X$. Since Ψ is a weighted carrier, it follows that there exists $\eta_x > 0$ such that

$$\Psi(B(x, \eta_x)) \subset O_\varepsilon(\Psi(x)) \tag{5.5}$$

and

$$I_{w\text{loc}}(\Psi, O_\varepsilon(\Psi(x)), x) = I_{w\text{loc}}(\Psi, O_\varepsilon(\Psi(x)), y)$$

for all $y \in B(x, \eta_x)$. Additionally, since X is locally connected, it follows that there exists a connected neighbourhood [6] V_x of x and $\delta_x > 0$ such that $B(x, \delta_x) \subset V_x \subset B(x, \eta_x)$. Now let us observe that for all $y \in B(x, \eta_x)$ and for any piece C of $O_\varepsilon(\Psi(x))$ we have $\Psi(y) \cap \partial C = \emptyset$ (where ∂C denotes the boundary of C with respect to Y), by (5.5) and Lemma 5.3. Consequently, in view of Lemma 3.7, we obtain

$$I_{w\mathrm{loc}}(\Psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, y)$$

for all $y \in V_x$; and hence for all $y \in B(x, \delta_x)$. This completes the proof. □

Lemma 5.8. *Let X be a compact space and let $\Psi: X \multimap Y$ be a weighted carrier. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that if two points $x, y \in X$ satisfy the following condition $d_X(x, y) < \delta$, then there exists a point $z \in X$ such that*

$$\Psi(x) \subset O_\varepsilon(\Psi(z)) \quad \textit{and} \quad \Psi(y) \subset O_\varepsilon(\Psi(z)), \tag{5.6}$$

$$z \in O_\varepsilon(x) \quad \textit{and} \quad z \in O_\varepsilon(y), \tag{5.7}$$

$$I_{w\mathrm{loc}}(\Psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, z) = I_{w\mathrm{loc}}(\Psi, C, y), \tag{5.8}$$

for any piece C of $O_\varepsilon(\Psi(z))$.

Proof. Let us fix $\varepsilon > 0$. Since Ψ is an upper semicontinuous multivalued map with compact values, it follows that for any $x \in X$ there exists $0 < \delta'_x < \varepsilon$ such that $\Psi(B(x, \delta'_x)) \subset O_\varepsilon(\Psi(x))$. Moreover, in view of Lemma 5.7, there exists $\delta''_x > 0$ such that for any piece of C of $O_\varepsilon(\Psi(x))$ and any $y \in B(x, \delta''_x)$ we have the following equality

$$I_{w\mathrm{loc}}(\Psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, y). \tag{5.9}$$

Let $\delta_x := (1/2)\min\{\delta'_x, \delta''_x\}$ and let $\{B(x, \delta_x)\}_{x \in X}$ be the open covering of X. Since X is compact, there exists a finite subcovering $B(x_1, \delta_{x_1}), \dots, B(x_k, \delta_{x_k})$ of this covering. Let us put $\delta := (1/2)\min\{\delta_{x_1}, \dots, \delta_{x_k}\}$. Now we shall show that such a δ satisfies the conclusion of Lemma 5.8. Indeed, let us take two points x and y with $d_X(x, y) < \delta$. Then for a point x there exists $1 \le i_0 \le k$ such that $x \in B(x_{i_0}, \delta_{x_{i_0}})$. Let $z := x_{i_0}$. Then $\Psi(x) \subset O_\varepsilon(\Psi(z))$. Since $d_X(y, z) \le d_X(y, x) + d_X(x, z) < \delta + \delta_z < \varepsilon/2 + \varepsilon/2 = \varepsilon$, we deduce that $\Psi(y) \subset O_\varepsilon(\Psi(z))$ and $y \in O_\varepsilon(z)$; and hence (5.6) and (5.7) are satisfied. Finally, (5.8) follows from (5.9) and the fact that $d_X(x, z) < \delta''_z$ and $d_X(y, z) < \delta''_z$, which completes the proof. □

Now we are able to prove the following corollary.

---

[6] Recall that by neighbourhood of x in X we mean always a set containing x in its interior; thus a neighbourhood need not be open.

Corollary 5.9. *Let X be a compact space and let $\Psi: X \multimap Y$ be a weighted carrier. Then for every $\varepsilon > 0$ there exists $\delta_\Psi > 0$ such that for every $x \in X$ and every piece C of $O_\varepsilon(\Psi(O_\varepsilon(x)))$ we have*

$$I_{w\mathrm{loc}}(\Psi, C, x) = I_{w\mathrm{loc}}(\Psi, C, y)$$

for any $y \in B(x, \delta_\Psi)$.

Proof. Let us fix $\varepsilon > 0$ and let $\delta > 0$ be as in Lemma 5.8 according to Ψ and ε. We shall show that such a δ satisfies the conclusion of the above corollary. For this purpose, let us choose a point y such that $d_X(x,y) < \delta$. Then, by Lemma 5.8, we deduce that there exists a point z such that

$$O_\varepsilon(\Psi(z)) \subset O_\varepsilon(\Psi(O_\varepsilon(x))),$$
$$\Psi(x) \subset O_\varepsilon(\Psi(z)) \quad \text{and} \quad \Psi(y) \subset O_\varepsilon(\Psi(z)).$$

Let C be a piece of $O_\varepsilon(\Psi(O_\varepsilon(x)))$. Since $C \cap O_\varepsilon(\Psi(z))$ is open and closed in $O_\varepsilon(\Psi(z))$, it follows by Lemma 5.8 and the excision property of $I_{w\mathrm{loc}}$ that

$$\begin{aligned} I_{w\mathrm{loc}}(\Psi, C, x) = I_{w\mathrm{loc}}(\Psi, C \cap O_\varepsilon(\Psi(z)), x) &\overset{\text{Lemma 5.8}}{=} I_{w\mathrm{loc}}(\Psi, C \cap O_\varepsilon(\Psi(z)), z) \\ &\overset{\text{Lemma 5.8}}{=} I_{w\mathrm{loc}}(\Psi, C \cap O_\varepsilon(\Psi(z)), y) = I_{w\mathrm{loc}}(\Psi, C, y), \end{aligned}$$

which completes the proof. □

The following lemma is crucial in what follows.

Lemma 5.10. *Let X be a compact space and let Y be a space, $n \geq 0$. If $\Psi: X \multimap Y$ is a w-UV^n-valued carrier, then for each $\varepsilon > 0$ there exists a δ, $0 < \delta < \varepsilon$, such that for each $x \in X$ two properties hold:*

(a) *for any connected component C of $O_\delta(\Psi(O_\delta(x)))$ and for every w-map $\varphi: \partial\Delta_1 \multimap C$ with $\sum_{y \in C} w_\varphi(0,y) = \sum_{y \in C} w_\varphi(1,y)$ there exists a w-map $\widetilde{\varphi}: \Delta_1 \multimap O_\varepsilon(\Psi(O_\varepsilon(x)))$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in \partial\Delta_1$;*

(b) *if $n > 0$, then for each k, $1 < k \leq n+1$, and any weighted map $\varphi: \partial\Delta_k \multimap O_\delta(\Psi(O_\delta(x)))$ there exists a w-map $\widetilde{\varphi}: \Delta_k \multimap O_\varepsilon(\Psi(O_\varepsilon(x)))$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in \partial\Delta_k$.*

The proof of the above lemma is similar in spirit to that of [12, Lemma 5.8], so the details are left to the reader.

We are now going to establish the first approximation result of this section.

Theorem 5.11. *Let X be a compact polyhedron and let A be a subpolyhedron of X. Let Y be a locally connected space. If $dim(X \setminus A) \leq n+1$ and $\Psi: X \multimap Y$ is a w-UV^n-valued carrier, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that* *if* $\varphi_0: A \multimap Y$ *is a* δ*-approximation of* $\Psi: X \multimap Y$*, then there exists a* w*-map* $\varphi: X \multimap Y$ *being an* ε*-approximation of* Ψ *with* $\varphi|A = \varphi_0$*.*

Proof. The main idea of our proof follows from [6], [12]. Let us fix $\varepsilon > 0$ and let $\dim(X \setminus A) = n_0$. Using Lemma 5.10 we can construct a sequence $\{\varepsilon_i\}_{i=0}^{n_0}$ (7) such that

(1) $\varepsilon_{n_0} := \varepsilon$,
(2) $4\varepsilon_i < \varepsilon_{i+1}$ for $0 \leq i \leq n_0 - 1$,
(3) for any $x \in X$, any connected component C of $O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x)))$, and any weighted map $\varphi: \partial\Delta_1 \multimap C$ with $\sum_{y\in C} w_\varphi(0,y) = \sum_{y\in C} w_\varphi(1,y)$ there exists a weighted map $\widetilde{\varphi}: \Delta_1 \multimap O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(x)))$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in \partial\Delta_1$,
(4) for any $x \in X$, any positive k, $1 \leq k \leq n_0 - 1$, and any w-map $\varphi: \partial\Delta_{k+1} \multimap O_{2\varepsilon_k}(\Psi(O_{2\varepsilon_k}(x)))$ there exists a weighted map $\widetilde{\varphi}: \Delta_{k+1} \multimap O_{\varepsilon_{k+1}/2}(\Psi(O_{\varepsilon_{k+1}/2}(x)))$ such that $\widetilde{\varphi}(x) = \varphi(x)$ for all $x \in \partial\Delta_{k+1}$.

Let $\delta := \varepsilon_0$ and let $\varphi_0: A \multimap Y$ be a δ-approximation of $\Psi: X \multimap Y$. Let (K, L) be a triangulation of (X, A) finer than the covering $\{O_{\varepsilon_0/2}(x)\}_{x\in X}$ of X, i.e. $|K| = X$, $|L| = A$ and L is a subcomplex of K. We shall prove now that $\varphi: A \multimap Y$ can be extended to an ε-approximation of $\Psi: X \multimap Y$. For this purpose, choose for each simplex σ of $K \setminus L$ a point x_σ such that $|\sigma| \subset O_{\varepsilon_0/2}(x_\sigma)$. Let us notice that if σ is a vertex v of $K \setminus L$, then we can take $x_\sigma = v$. Let v be a vertex of K such that $v \notin L$. Since Y is locally connected, it follows that the open set $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v)))$ is also locally connected and hence the connected components of $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v)))$ are open in $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v)))$. Consequently, by the compactness of $\Psi(v)$, we infer that it meets only a finite number of connected components of $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v)))$, say $C_1^v, \ldots, C_{r_v}^v$. Let us choose a point y_i^v in each C_i^v. We define a weighted map $\varphi^0: |K^{(0)}| \cup |L| \multimap Y$ (8) by the formula

$$\varphi^0(x) = \begin{cases} \varphi_0(x) & \text{if } x \in |L|, \\ \displaystyle\sum_{i=1}^{r_v} I_{w\mathrm{loc}}(\Psi, C_i^v, v) y_i^v & \text{if } x = v \in |K^{(0)}| \setminus |L|. \end{cases}$$

Obviously, φ^0 is an ε_0-approximation of $\Psi: |K| \multimap Y$. Now we extend φ^0 to $|K^{(1)}| \cup |L|$. For this purpose, let us fix a 1-dimensional simplex $\sigma = \langle v_0, v_1\rangle$ such that $\sigma \notin L$. Since $|\sigma| \subset O_{\varepsilon_0/2}(x_\sigma)$, we have

$$O_{\varepsilon_0}(v_i) \subset O_{\varepsilon_0}(|\sigma|) \subset O_{\varepsilon_0}(O_{\varepsilon_0/2}(x_\sigma)) \subset O_{2\varepsilon_0}(x_\sigma), \quad \text{for } i = 0, 1.$$

(7) During construction we can assume that $n_0 \geq 1$ because otherwise $n_0 = 0$ and then we put $\{\varepsilon_i\}_{i=0}^{n_0} := \varepsilon$.

(8) Given simplicial complex we shall denote by $K^{(i)}$ the simplex of K consisting of all simplexes $\sigma \in K$ with $dim(\sigma) \leq i$.

Moreover, since φ^0 is an ε_0-approximation of $\Psi\colon |K| \multimap Y$, we infer that

$$\varphi^0(v_i) \subset O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_i))) \subset O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x_\sigma))), \quad \text{for } i = 1, 2.$$

Now we shall show that for each piece C of $O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x_\sigma)))$ the following condition holds:

$$I_{w\mathrm{loc}}(\varphi^0, C, v_0) = I_{w\mathrm{loc}}(\varphi^0, C, v_1).$$

Indeed, let us fix a piece C of $O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x_\sigma)))$. Let

$$C_i := C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_i))),$$

for $i = 1, 2$. Then C_i is a piece of $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_i)))$, for $1 \le i \le 2$. Since φ^0 is an ε_0-approximation of $\Psi\colon |K| \multimap Y$, we obtain

$$I_{w\mathrm{loc}}(\varphi^0, C, v_i) = I_{w\mathrm{loc}}(\varphi^0, C_i, v_i) = I_{w\mathrm{loc}}(\Psi, C_i, v_i) = I_{w\mathrm{loc}}(\Psi, C, v_i), \tag{5.10}$$

for $i = 1, 2$; where the first equality and the last one above follow form the excision property of $I_{w\mathrm{loc}}$. Consequently, since $|\sigma| = |\langle v_0, v_1\rangle|$ is connected and

$$\Psi(|\sigma|) \subset \Psi(O_{\varepsilon_0/2}(x_\delta)) \subset O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x_\delta))),$$

we deduce from Lemmas 3.7 and 5.3 that

$$I_{w\mathrm{loc}}(\Psi, C, v_0) = I_{w\mathrm{loc}}(\Psi, C, v_1).$$

Hence, taking into account (5.10) and (5.11), we obtain

$$I_{w\mathrm{loc}}(\varphi^0, C, v_0) = I_{w\mathrm{loc}}(\varphi^0, C, v_1).$$

Thus, by the definition of ε_0, we can extend $\varphi^0||\partial\sigma|\colon |\partial\sigma| \multimap O_{2\varepsilon_0}(\Psi(O_{2\varepsilon_0}(x_\sigma)))$ to

$$\varphi_\sigma\colon |\sigma| \multimap O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(x_\sigma))).$$

Now we are going to show that φ_σ is an ε_1-approximation of $\Psi\colon |K| \multimap Y$. First, let us observe that for each $x \in |\sigma|$ we have $x_\sigma \in O_{\varepsilon_0/2}(x)$, since $|\sigma| \subset O_{\varepsilon_0/2}(x_\sigma)$. Thus

$$O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(x_\sigma))) \subset O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(O_{\varepsilon_0/2}(x)))) \subset O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x))).$$

This shows that $\varphi_\sigma(x) \subset O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x)))$, for each $x \in |\sigma| = |\langle v_0, v_1\rangle|$. So, it is enough to show that if C is any piece of $O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x)))$, then

$$I_{w\mathrm{loc}}(\varphi_\sigma, C, x) = I_{w\mathrm{loc}}(\Psi, C, x).$$

For this purpose, let us fix $x \in |\sigma|$ and let C be a piece of $O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x)))$. Since

$$\varphi_\sigma(v_0) \subset \varphi_\sigma(|\sigma|) \subset O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(x_\sigma))) \subset O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x))),$$
$$\Psi(v_0) \subset \Psi(|\sigma|) \subset O_{\varepsilon_1/2}(\Psi(O_{\varepsilon_1/2}(x_\sigma))) \subset O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x))),$$

it follows, in view of Lemmas 5.3 and 3.7, that

$$I_{w\mathrm{loc}}(\varphi_\sigma, C, v_0) = I_{w\mathrm{loc}}(\varphi_\sigma, C, x), \tag{5.12}$$

$$I_{w\mathrm{loc}}(\Psi, C, v_0) = I_{w\mathrm{loc}}(\Psi, C, x). \tag{5.13}$$

Since $\varphi_\sigma||\partial\sigma|$ is an ε_0-approximation of $\Psi\colon |K| \multimap Y$, we conclude that

$$\varphi_\sigma(v_0) \subset O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0))) \subset O_{\varepsilon_1}(\Psi(O_{\varepsilon_1}(x))).$$

Then, by the excision property of $I_{w\mathrm{loc}}$, we have

$$I_{w\mathrm{loc}}(\varphi_\sigma, C, v_0) = I_{w\mathrm{loc}}(\varphi_\sigma, C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0))), v_0), \tag{5.14}$$

$$I_{w\mathrm{loc}}(\Psi, C, v_0) = I_{w\mathrm{loc}}(\Psi, C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0))), v_0). \tag{5.15}$$

Now let us observe that $C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0)))$ is a piece of $O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0)))$. Hence, taking into account the fact that $\varphi_\sigma||\partial\sigma|$ is an ε_0-approximation of $\Psi\colon |K| \multimap Y$, we obtain

$$I_{w\mathrm{loc}}(\varphi_\sigma, C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0))), v_0) = I_{w\mathrm{loc}}(\Psi, C \cap O_{\varepsilon_0}(\Psi(O_{\varepsilon_0}(v_0))), v_0). \tag{5.16}$$

Consequently, from (5.12)–(5.16) we get

$$I_{w\mathrm{loc}}(\varphi_\sigma, C, x) = I_{w\mathrm{loc}}(\Psi, C, x),$$

which proves that φ_σ satisfies the second condition of Definition 5.1. Now using the gluing lemma we obtain a weighted map $\varphi^1\colon |K^{(1)}| \cup |L| \multimap Y$ being an ε_1-approximation of $\Psi\colon |K| \multimap Y$ with $\varphi^1||K^{(0)}| \cup |L| = \varphi^0$. Suppose now that $\varphi^r\colon |K^{(r)}| \cup |L| \multimap Y$ is an ε_r-approximation of $\Psi\colon |K| \multimap Y$, $r < n_0$. Let τ be an $(r+1)$-dimensional simplex such that $\tau \notin L$. Then $|\tau| \subset O_{\varepsilon_0/2}(x_\tau) \subset O_{\varepsilon_r/2}(x_\tau)$ and $\varphi^r(x) \subset O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(x)))$, for all $x \in |\partial\tau|$. Consequently,

$$\varphi^r(|\partial\tau|) \subset O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(|\partial\tau|))) \subset O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(O_{\varepsilon_r}(x_\tau)))) \subset O_{2\varepsilon_r}(\Psi(O_{2\varepsilon_r}(x_\tau))).$$

Thus, by the definition of ε_r, a w-map $\varphi^r||\partial\tau|\colon |\partial\tau| \multimap O_{2\varepsilon_r}(\Psi(O_{2\varepsilon_r}(x_\tau)))$ admits an extension to

$$\varphi^{r+1}_\tau\colon |\tau| \multimap O_{\varepsilon_{r+1}/2}(\Psi(O_{\varepsilon_{r+1}/2}(x_\tau))).$$

Let us observe now that for each $x \in |\tau|$ we have

$$O_{\varepsilon_{r+1}/2}(\Psi(O_{\varepsilon_{r+1}/2}(x_\tau))) \subset O_{\varepsilon_{r+1}}(\Psi(O_{\varepsilon_{r+1}}(x))),$$

because $|\tau| \subset O_{\varepsilon_0/2}(x_\tau) \subset O_{\varepsilon_{r+1}/2}(x_\tau)$ and hence

$$\varphi^{r+1}_\tau(x) \subset O_{\varepsilon_{r+1}}(\Psi(O_{\varepsilon_{r+1}}(x))),$$

for each $x \in |\tau|$. This implies that φ^{r+1}_τ satisfies the first condition of Definition 5.1. Let us fix $x_0 \in |\partial\tau|$. Now we shall prove that φ^{r+1}_τ verifies also the second condition of Definition 5.1. For this end, let us fix $x \in |\tau|$ and let C be a piece of $O_{\varepsilon_{r+1}}(\Psi(O_{\varepsilon_{r+1}}(x)))$. Then by Lemmas 5.3 and Lemma 3.7 we get

$$I_{w\mathrm{loc}}(\varphi_\tau^{r+1}, C, x_0) = I_{w\mathrm{loc}}(\varphi_\tau^{r+1}, C, x), \tag{5.17}$$

$$I_{w\mathrm{loc}}(\Psi, C, x_0) = I_{w\mathrm{loc}}(\Psi, C, x). \tag{5.18}$$

Moreover,

$$I_{w\mathrm{loc}}(\varphi_\tau^{r+1}, C \cap O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(x_0))), x_0) = I_{w\mathrm{loc}}(\Psi, C \cap O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(x_0))), x_0), \tag{5.19}$$

because $\varphi_\tau^{r+1}||\partial\tau|$ is an ε_r-approximation of Ψ. Next, by the excision property of $I_{w\mathrm{loc}}$, we infer that

$$I_{w\mathrm{loc}}(\varphi_\tau^{r+1}, C \cap O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(x_0))), x_0) = I_{w\mathrm{loc}}(\varphi_\tau^{r+1}, C, x_0) \tag{5.20}$$

and

$$I_{w\mathrm{loc}}(\Psi, C \cap O_{\varepsilon_r}(\Psi(O_{\varepsilon_r}(x_0))), x_0) = I_{w\mathrm{loc}}(\Psi, C, x_0). \tag{5.21}$$

Therefore, taking into account (5.17)–(5.21), we obtain

$$I_{w\mathrm{loc}}(\varphi_\tau^{r+1}||\partial\tau|, C, x) = I_{w\mathrm{loc}}(\Psi, C, x),$$

which ends the proof that φ_τ^{r+1} is an ε_{r+1}-approximation of Ψ. Now using the gluing lemma we obtain a weighted map $\varphi^{r+1}: |K^{(r+1)}| \cup |L| \multimap Y$ being ε_{r+1}-approximation of $\Psi: |K| \multimap Y$ with $\varphi^{r+1}||K^{(r)}| \cup |L| = \varphi^r$. This completes the inductive step. Thus, after n_0 steps, we arrive to an ε-approximation $\varphi = \varphi^{n_0}$ of Ψ. The theorem is proved. □

Let us notice that the following theorem was proved in [6].

Theorem 5.12. *Let $X_0 \subset X$ be a finite polyhedral pair, let Y be a metric* ANR *and let $\Phi: X \multimap Y$ be a weighted carrier with positively acyclic values. Given any $\varepsilon > 0$ there exists $\delta > 0$ such that any δ-approximation $\varphi: X_0 \multimap Y$ of $\Phi|X_0: X_0 \multimap Y$ can be extended to an ε-approximation $\widetilde{\varphi}: X \multimap Y$ of Φ.*

It should be noted that Theorem 5.11 was proved under the weaker assumptions than Theorem 5.12 but with a slight different conclusion, which will be much more convenient in applications.

The following three lemmas will be used in the proof of the main result of this section.

Lemma 5.13 ([10]). *Let K be a compact subset of X and let U be an open neighbourhood of K in X. Then for any retraction $r: U \to X$ and any $\varepsilon > 0$ there exists $\delta > 0$ such that $O_\delta(K) \subset U$ and $d_X(r(x), x) < \varepsilon$ for each $x \in O_\delta(K)$.*

Lemma 5.14 ([23]). *Let (X,A) be a pair of compact* ANR*'s and let $\eta > 0$. Then there is a finite polyhedral pair (P,P_0) and maps of pairs $i:(P,P_0)\to(X,A)$ and $q:(X,A)\to(P,P_0)$ such that $d_X(i\circ q(x),x)<\eta$ for each $x\in X$.*

Lemma 5.15 ([5]). *Let X be a compact* ANR *and let $\varepsilon>0$ be given. Then there exists $\delta>0$ such that if $f_0,f_1:X\to X$ are δ-close* (9)*, then there exists a map $h:X\times[0,1]\to X$ such that*

(a) $h(x,0)=f_0(x)$ *for all* $x\in X$,
(b) $h(x,1)=f_1(x)$ *for all* $x\in X$,
(c) $\operatorname{diam}(h(\{x\}\times[0,1]))<\varepsilon$ *for any* $x\in X$,

where $\operatorname{diam}(h(\{x\}\times[0,1])):=\sup\{d_X(h(x,t_1),h(x,t_2)),\ t_1,t_2\in[0,1]\}$.

We shall now prove the following lemma that will play a central role in the sequel.

Lemma 5.16. *Let (X,A) be a pair of compact* ANR*'s and let $\Psi:X\multimap Y$ be a weighted carrier, Y a space. Let $\varepsilon>0$. Then there exists $\gamma>0$ such that if a weighted map $\psi_0:(X\times\{0\})\cup(A\times[0,1])\multimap Y$ has the property that the weighted maps $\psi_0(\,\cdot\,,0):X\multimap Y$ and $\psi_0(\,\cdot\,,t):A\multimap Y$, for each $t\in[0,1]$, are γ-approximations of Ψ, then there exists a weighted map $\psi:X\times[0,1]\multimap Y$ such that for each $t\in[0,1]$ the weighted map $\psi(\,\cdot\,,t):X\multimap Y$ is an ε-approximation of Ψ and $\psi|(X\times\{0\})\cup(A\times[0,1])=\psi_0$.*

Proof. The basic idea of the proof follows from [2]. Let $M:=(X\times\{0\})\cup(A\times[0,1])$. Since $X\times\{0\}$, $A\times[0,1]$, $(X\times\{0\})\cap(A\times[0,1])=A\times\{0\}$ are ANR-spaces, we infer from [4, Theorem 6.1, p. 90] that M is also an ANR. Hence there exists an open neighbourhood $U\subset X\times[0,1]$ of M and a retraction $r:U\to M$. Let δ_Ψ be as in Corollary 5.9 for Ψ and $\varepsilon/2$. From Lemma 5.13 it follows that there exists $0<\gamma<\min\{\varepsilon/2,\delta_\Psi\}$ such that

$$O_\gamma(M)\subset U \quad\text{and}\quad d_{X\times[0,1]}(r(z),z)<\min\{\varepsilon/2,\delta\} \tag{5.22}$$

for every $z\in O_\gamma(M)$. Now take a w-map ψ_0 as in the formulation of Lemma 5.16 according to γ. Define a w-map $\overline{\psi}:O_\gamma(M)\multimap Y$ by the formula: $\psi_0\circ r$. Let us observe that for each $(x,t)\in O_\gamma(M)$ we have

$$\overline{\psi}(x,t)\subset O_\varepsilon(\Psi(O_\varepsilon(x))). \tag{5.23}$$

Indeed, let $(x',t'):=r(x,t)$. Then by (5.22) we get

$$d_X(x',x)\le d_{X\times[0,1]}(r(x,t),(x,t))<\min\{\varepsilon/2,\delta_\Psi\}. \tag{5.24}$$

(9) Let $f,g:Y\to X$ be two mappings and let d_Y be a metric in Y, $\varepsilon>0$. We shall say that f and g are ε-close provided for every $y\in Y$ we have $d_Y(f(y),g(y))<\varepsilon$.

Therefore

$$\overline{\psi}(x,t) = \psi_0(x',t') \subset O_\gamma(\Psi(O_\gamma(x'))) \subset O_\varepsilon(\Psi(O_\varepsilon(x))), \tag{5.25}$$

which verifies (5.23). Let V be an open neighbourhood of A in X such that $V \times [0,1] \subset O_\gamma(M)$. Since A and $X \setminus V$ are disjoint subsets of X, there exists an Urysohn function, i.e. there is a map $u\colon X \to [0,1]$ such that $u(x) = 1$, for every $x \in A$ and $u(x) = 0$, for every $X \setminus V$. Define a w-map $\psi\colon X \times [0,1] \multimap Y$ by

$$\psi(x,t) = \overline{\psi}(x, u(x)t).$$

Now, let us observe that from (5.23) we get

$$\psi(x,t) \subset O_\varepsilon(\Psi(O_\varepsilon(x)))$$

for all $(x,t) \in X \times [0,1]$. Therefore the proof will be completed, if we show that for each $t \in [0,1]$ a w-map $\psi(\,\cdot\,,t)\colon X \multimap Y$ satisfies the second condition of Definition 5.1. To this end we need consider 3 cases.

Case 1. Let $x_0 \in A$ and let C be a piece of $O_\varepsilon(\Psi(O_\varepsilon(x_0)))$. In addition, let us fix $t_0 \in [0,1]$. Then we have

$$\psi(x_0,t_0) = \overline{\psi}(x_0, u(x_0)t_0) = \overline{\psi}(x_0,t_0) = \psi_0 \circ r(x_0,t_0) = \psi_0(x_0,t_0).$$

Hence

$$I_{w\mathrm{loc}}(\psi(\,\cdot\,,t_0), C, x_0) = I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,t_0), C, x_0).$$

Since $O_\gamma(\Psi(O_\gamma(x_0))) \subset O_\varepsilon(\Psi(O_\varepsilon(x_0)))$ and $C \subset O_\varepsilon(\Psi(O_\varepsilon(x_0)))$, we infer that $C \cap O_\gamma(\Psi(O_\gamma(x_0)))$ is a piece of $O_\gamma(\Psi(O_\gamma(x_0)))$. Therefore

$$\begin{aligned} I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,t_0), C, x_0) &= I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,t_0), C \cap O_\gamma(\Psi(O_\gamma(x_0))), x_0) \\ &= I_{w\mathrm{loc}}(\Psi, C \cap O_\gamma(\Psi(O_\gamma(x_0))), x_0), \end{aligned}$$

where the first equality follows from the excision property of $I_{w\mathrm{loc}}$ for $\psi_0(\,\cdot\,,t_0)$, but the second one follows from the fact that $\psi_0(\,\cdot\,,t_0)\colon A \multimap Y$ is a γ-approximation of Ψ. Using once again the excision property of $I_{w\mathrm{loc}}$, we get

$$I_{w\mathrm{loc}}(\Psi, C \cap O_\gamma(\Psi(O_\gamma(x_0))), x_0) = I_{w\mathrm{loc}}(\Psi, C, x_0),$$

which proves that

$$I_{w\mathrm{loc}}(\psi(\,\cdot\,,t_0), C, x_0) = I_{w\mathrm{loc}}(\Psi, C, x_0).$$

The proof of Case 1 is complete.

Case 2. Let $x_0 \in X \setminus V$ and let $C \subset O_\varepsilon(\Psi(O_\varepsilon(x_0))$ be as above. In addition, let us fix $t_0 \in [0,1]$. Then

$$\psi(x_0,t_0) = \overline{\psi}(x_0, u(x_0)t_0) = \overline{\psi}(x_0,0) = \psi_0 \circ r(x_0,0) = \psi_0(x_0,0).$$

Hence

$$I_{w\mathrm{loc}}(\psi(\,\cdot\,,t_0), C, x_0) = I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,0), C, x_0).$$

Since $\psi_0(\,\cdot\,,0)$ is a γ-approximation of Ψ we have

$$\begin{aligned} I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,0),C,x_0) &= I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,0),C\cap O_\gamma(\Psi(O_\gamma(x_0))),x_0) \qquad (5.26)\\ &= I_{w\mathrm{loc}}(\Psi,C\cap O_\gamma(\Psi(O_\gamma(x_0))),x_0)\\ &= I_{w\mathrm{loc}}(\Psi,C,x_0) \qquad (5.27)\end{aligned}$$

where the equalities in (5.26) and (5.27) follow from the excision property of $I_{w\mathrm{loc}}$. Therefore

$$I_{w\mathrm{loc}}(\psi(\,\cdot\,,t_0),C,x_0) = I_{w\mathrm{loc}}(\Psi,C,x_0),$$

which completes the proof of Case 2.

Case 3. (In this case Corollary 5.9 plays a crucial role) Let $x_0 \in V\setminus A$ and let $C\subset O_\varepsilon(\Psi(O_\varepsilon(x_0)))$ be a piece of $O_\varepsilon(\Psi(O_\varepsilon(x_0)))$. In addition, let us fix $t_0\in[0,1]$. Let $(x',t') := r(x_0,t_0)$. Since

$$\psi(x_0,t_0) = \psi_0\circ r(x_0,t_0) = \psi_0(x',t'),$$

we have

$$I_{w\mathrm{loc}}(\psi(\,\cdot\,,t_0),C,x_0) = I_{w\mathrm{loc}}(\psi_0\circ r,C,(x_0,t_0)) = I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,t'),C,x'). \qquad (5.28)$$

Moreover,

$$\psi_0(x',t')\subset O_\gamma(\Psi(O_\gamma(x')))\subset O_\varepsilon(\Psi(O_\varepsilon(x_0))),$$

because $\psi_0(\,\cdot\,,t')$ is a γ-approximation of Ψ and $d_X(x_0,x') < \varepsilon/2$ by (5.24). Consequently, the same reasoning as in Case 1 establishes that

$$I_{w\mathrm{loc}}(\psi_0(\,\cdot\,,t'),C,x') = I_{w\mathrm{loc}}(\Psi,C,x'). \qquad (5.29)$$

Since $d_X(x_0,x') < \delta_\Psi$, Corollary 5.9 implies that

$$I_{w\mathrm{loc}}(\Psi,C,x') = I_{w\mathrm{loc}}(\Psi,C,x_0). \qquad (5.30)$$

Therefore, taking into account (5.28)–(5.30) we get

$$I_{w\mathrm{loc}}(\psi(\,\cdot\,,t_0),C,x_0) = I_{w\mathrm{loc}}(\Psi,C,x_0),$$

which completes the proof of Case 3, and hence the lemma follows. □

We now prove the main result of this section.

THEOREM 5.17. *Let X be a compact* ANR*, let $A\subset X$ be a closed* ANR*, and let Y be a locally connected space. In addition, let $\Psi\colon X\multimap Y$ be a w-UV^ω-valued carrier. Then for each $\varepsilon>0$ there exists $\delta>0$ such that if $\varphi_0\colon A\multimap Y$ is a δ-approximation of $\Psi\colon X\multimap Y$, then there exists a weighted map $\varphi\colon X\multimap Y$ being an ε-approximation of Ψ with $\varphi|A=\varphi_0$.*

PROOF. The proof is based on [2]. Let $\varepsilon>0$ be fixed. Let γ be as in Lemma 5.16 according to X, A, Ψ and ε. Let δ_Ψ be as in Corollary 5.9 according to Ψ and ε. In addition, let $\eta > 0$ be as in Lemma 5.15 for X and $\min\{\gamma/2, \delta_\Psi\}$. We can assume that $\eta \le \min\{\gamma/2, \delta_\Psi\}$. Then for a pair of compact ANR's and η by Lemma 5.14 there is a polyhedral pair (P, P_0) and maps of pairs

$$p\colon (P, P_0) \to (X, A) \quad \text{and} \quad q\colon (X, A) \to (P, P_0) \tag{5.31}$$

such that for each $x \in X$ we have $d_X(p \circ q(x), x) < \eta$. Since P is compact space, we infer that there exists $0 < \mu \le \gamma$ such that if $x, y \in P$, then $d_X(p(x), q(x)) < \gamma/2$ provided that $d_P(x, y) < \mu$. Thus $p(O_\mu(q(x))) \subset O_\gamma(x)$ for each $x \in X$. Let $\Phi\colon P \multimap Y$ be a weighted carrier given by $\Phi = \Psi \circ p$. It is easy to see that Φ is a w-UV^ω-valued carrier. In view of Theorem 5.11, there exists $\nu > 0$ such that if $\theta_0\colon P_0 \multimap Y$ is a ν-approximation of $\Phi\colon P \multimap Y$, then there exists a μ-approximation $\theta\colon P \multimap Y$ of Φ with $\theta | P_0 = \theta_0$. Next, in view of Proposition 5.6, there exists $0 < \delta \le \gamma/2$ such that if $\psi\colon A \multimap Y$ is a δ-approximation of $\Psi\colon X \multimap Y$, then $\psi \circ p | P_0\colon P_0 \multimap Y$ is a ν-approximation of $\Phi\colon P \multimap Y$. Now, let ψ_0 be a δ-approximation of Ψ. We shall prove that there exists a weighted map $\psi\colon X \multimap Y$ being an δ-approximation of Ψ with $\psi | A = \psi_0$. Let us observe that by the choice of δ the composition $\psi_0 \circ p | P_0$ is a ν-approximation of $\Phi\colon P \multimap Y$. Therefore, from Theorem 5.11 it follows that a weighted map $\psi_0 \circ p | P_0\colon P_0 \multimap Y$ admits an extension $\widetilde{\psi}\colon P \multimap Y$ being a μ-approximation of Φ. Let us define now a weighted map $\overline{\psi}\colon X \multimap Y$ by $\overline{\psi} = \widetilde{\psi} \circ q$. We shall show now that $\overline{\psi}$ is a γ-approximation of $\Psi\colon X \multimap Y$. Since $\widetilde{\psi}$ is a γ-approximation of Φ, we have

$$\overline{\psi}(x) = \widetilde{\psi}(q(x)) \subset O_\mu(\Phi(O_\mu(q(x))))$$

for each $x \in X$. Hence, taking into account a definition of Φ and (5.31), we get

$$O_\mu(\Phi(O_\mu(q(x)))) = O_\mu(\Psi \circ p(O_\mu(q(x)))) \subset O_\mu(\Psi(O_\gamma(x))) \subset O_\gamma(\Psi(O_\gamma(x))),$$

for each $x \in X$. Now, we are going to show that for any $x \in X$ and for any piece C of $O_\gamma(\Psi(O_\gamma(x)))$ the following condition is satisfied:

$$I_{w\text{loc}}(\overline{\psi}, C, x) = I_{w\text{loc}}(\Psi, C, x).$$

To this end, let us fix $x \in X$ and $C \subset O_\gamma(\Psi(O_\gamma(x)))$ and let us observe that

$$\begin{aligned}
I_{w\text{loc}}(\overline{\psi}, C, x) &= I_{w\text{loc}}(\overline{\psi}, C \cap O_\mu(\Psi(O_\mu(q(x)))), x) && (\text{excision of } I_{w\text{loc}}) \\
&= I_{w\text{loc}}(\widetilde{\psi} \circ q, C \cap O_\mu(\Psi(O_\mu(q(x)))), x) && (\overline{\psi} = \widetilde{\psi} \circ q) \\
&= I_{w\text{loc}}(\widetilde{\psi}, C \cap O_\mu(\Psi(O_\mu(q(x)))), q(x)) && (\text{Corollary 3.11}) \\
&= I_{w\text{loc}}(\Phi, C \cap O_\mu(\Psi(O_\mu(q(x)))), q(x)),
\end{aligned}$$

where the last equality follows from the fact that $\widetilde{\psi}$ is a μ-approximation of Φ. Consequently, we get

$$\begin{aligned} I_{w\mathrm{loc}}(\Phi, C\cap O_\mu(\Psi(O_\mu(q(x)))), q(x)) &= I_{w\mathrm{loc}}(\Phi, C, q(x)) && (\text{excision of } I_{w\mathrm{loc}}) \\ &= I_{w\mathrm{loc}}(\Psi\circ p, C, q(x)) && (\Phi = \Psi\circ p) \\ &= I_{w\mathrm{loc}}(\Psi, C, p\circ q(x)). && (\text{Corollary 3.11}) \end{aligned}$$

Since $d_X(p\circ q(x), x) < \eta \leq \delta_\Psi$, due to Corollary 5.9, we get

$$I_{w\mathrm{loc}}(\Psi, C, p\circ q(x)) = I_{w\mathrm{loc}}(\Psi, C, x).$$

Summing up, we have showed that

$$I_{w\mathrm{loc}}(\overline{\psi}, C, x) = I_{w\mathrm{loc}}(\Psi, C, x),$$

which proves that $\overline{\psi}$ is a γ-approximation of Ψ. Now we shall use Lemma 5.16 to modify a weighted map $\overline{\psi}$ because $\overline{\psi}$ is not yet the required approximation of Ψ. For this purpose, let us recall that for each $a\in A$ we have

$$\overline{\psi}(a) = \widetilde{\psi}\circ q(a) = \psi_0(p\circ q(a)).$$

Moreover, $id_A\colon A\to A$ and $p\circ q\colon A\to A$ are η-close. Therefore, in view of Lemma 5.15, there exists a map $h\colon A\times[0,1]\to A$ such that

$$\begin{gathered} h(\,\cdot\,,0) = p\circ q|A \quad \text{and} \quad h(\,\cdot\,,1) = \mathrm{id}_A, \\ \operatorname{diam}(h(\{a\}\times[0,1])) < \min\{\gamma/2, \delta_\Psi\}. \end{gathered} \tag{5.32}$$

Let $\phi_0\colon (X\times\{0\})\cup(A\times[0,1]) \multimap Y$ be given by

$$\phi_0(x,t) = \begin{cases} \overline{\psi}(x) & \text{if } (x,t)\in X\times\{0\}, \\ \psi_0\circ h(x,t) & \text{if } (x,t)\in A\times[0,1]. \end{cases}$$

Since

$$\psi_0\circ h(z,0) = (\psi_0\circ p)\circ q(a) = \overline{\psi},$$

in view of Corollary 3.11, we get

$$\begin{aligned} I_{w\mathrm{loc}}(\overline{\psi}, U, a) &= I_{w\mathrm{loc}}(\psi_0\circ(p\circ q), U, a) = I_{w\mathrm{loc}}(\psi_0, U, p\circ q(a)) \\ &= I_{w\mathrm{loc}}(\psi_0, U, h(a,0)) = I_{w\mathrm{loc}}(\psi_0\circ h, U, (a,0)), \end{aligned}$$

where $a\in A$ and U is an open subset of Y such that $\overline{\psi}(a)\cap\partial U = \emptyset$. Hence, by the gluing lemma, ϕ_0 is a weighted map. We shall show now that for all $t\in[0,1]$ a w-map $\phi_0(\,\cdot\,,t)$ is a γ-approximation of Ψ (the case $t=0$ has already been proved). Let us fix $t\in(0,1]$ and $a\in A$. Then

$$\phi_0(a,t) = \psi_0\circ h(a,t) \subset O_\delta(\Psi(O_\delta(h(a,t)))),$$

because ψ_0 is a δ-approximation of Ψ. Moreover, since $h(a,t) \in O_{\gamma/2}(a)$, we have $O_\delta(h(a,t)) \subset O_{\gamma/2+\delta}(a)$. Thus

$$\phi_0(a,t) \subset O_\gamma(\Psi(O_\gamma(a))), \tag{5.33}$$

since $\delta \leq \gamma/2$. Let C be a piece of $O_\gamma(\Psi(O_\gamma(a)))$. Now we are going to prove that

$$I_{w\mathrm{loc}}(\phi_0(\,\cdot\,,t),C,a) = I_{w\mathrm{loc}}(\Psi,C,a).$$

First, note that

$$\begin{aligned} I_{w\mathrm{loc}}(\phi_0(\,\cdot\,,t),C,a) &= I_{w\mathrm{loc}}(\psi_0 \circ h(\,\cdot\,,t),C,a) \quad (\phi_0(\,\cdot\,,t) = \psi_0 \circ h(\,\cdot\,,t)) \\ &= I_{w\mathrm{loc}}(\psi_0,C,h(a,t)). \qquad \text{(Corollary 3.11)} \end{aligned}$$

Additionally,

$$\begin{aligned} \psi_0 \circ h(a,t) &\subset O_\delta(\Psi(O_\delta(h(a,t)))) \subset O_\delta(\Psi(O_\delta(O_{\gamma/2}(a)))) \\ &\subset O_{\gamma/2}(\Psi(O_\gamma(a))) \subset O_\gamma(\Psi(O_\gamma(a))), \end{aligned}$$

and hence $C \cap O_\delta(\Psi(O_\delta(h(a,t))))$ is a piece of $O_\delta(\Psi(O_\delta(h(a,t))))$. Consequently, by the excision property of $I_{w\mathrm{loc}}$, we get

$$I_{w\mathrm{loc}}(\psi_0,C,h(a,t)) = I_{w\mathrm{loc}}(\psi_0,C \cap O_\delta(\Psi(O_\delta(h(a,t)))),h(a,t))$$

and

$$\begin{aligned} &I_{w\mathrm{loc}}(\psi_0,C \cap O_\delta(\Psi(O_\delta(h(a,t)))),h(a,t)) \\ &\qquad = I_{w\mathrm{loc}}(\Psi,C \cap O_\delta(\Psi(O_\delta(h(a,t)))),h(a,t)), \end{aligned}$$

because ψ_0 is a δ-approximation of Ψ. Thus

$$I_{w\mathrm{loc}}(\Psi,C \cap O_\delta(\Psi(O_\delta(h(a,t)))),h(a,t)) = I_{w\mathrm{loc}}(\Psi,C,h(a,t)) = I_{w\mathrm{loc}}(\Psi,C,a),$$

where the first equality follows form the excision property of $I_{w\mathrm{loc}}$, and the second one holds by Corollary 5.9, because, by (5.32), $d_X(h(a,t),a) < \delta_\Psi$ (let us recall that δ_Ψ was defined at the beginning of our proof). Consequently, we have showed that

$$I_{w\mathrm{loc}}(\phi_0(\,\cdot\,,t),C,a) = I_{w\mathrm{loc}}(\Psi,C,a). \tag{5.34}$$

From (5.33) and (5.34) we infer that the assumptions of Lemma 5.16 are satisfied, and hence ϕ_0 admits an extension $\phi\colon X \times [0,1] \multimap Y$ such that for each $t \in [0,1]$ the w-map $\phi(\,\cdot\,,t)\colon X \multimap Y$ is an ε-approximation of Ψ. Finally, to complete the proof, we define $\varphi\colon X \multimap Y$ by putting $\varphi := \phi(\,\cdot\,,1)$. □

In particular, we obtain the following corollary.

Corollary 5.18. *Let X be a compact* ANR*, let $A \subset X$ be a closed* ANR*, and let Y be an* ANR*. In addition, let $\Psi: X \multimap Y$ be upper semicontinuous multivalued map with acyclic values. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi_0: A \multimap Y$ is a δ-approximation of $\Psi: X \multimap Y$, then there exists a weighted map $\varphi: X \multimap Y$ being an ε-approximation of Ψ with $\varphi|A = \varphi_0$.*

Moreover, from the above considerations we obtain the following corollaries.

Corollary 5.19. *Let $\Psi: X \multimap Y$ be a w-UV^ω-valued carrier, let X be a compact* ANR *and let Y be a locally connected space. Then for each $\varepsilon > 0$ there exists an ε-approximation $\varphi: X \multimap Y$ of Ψ.*

Proof. It is enough to take $A = \emptyset$ in Theorem 5.17. □

Corollary 5.20. *Let $\Theta: X \times [0,1] \multimap Y$ be a w-UV^ω-valued carrier and let X, Y be as above. Then for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $\varphi_i: X \multimap Y$ is a δ-approximation of Θ, then there exists an ε-approximation $\psi: X \times [0,1] \multimap Y$ of Θ such that $\psi|X \times \{i\} = \varphi_i$ for $i = 0, 1$.*

Proof. Let us take $A = X \times \{0\} \cup X \times \{1\}$ and let $\varphi: A \multimap Y$ be defined as follows

$$\varphi(x,t) = \begin{cases} \varphi_0(x) & \text{if } (x,t) \in X \times \{0\}, \\ \varphi_1(x) & \text{if } (x,t) \in X \times \{1\}. \end{cases}$$

This completes the proof if we invoke Theorem 5.17. □

Corollary 5.21. *Let $\Psi: X \multimap Y$ be a w-UV^ω-valued carrier. Then for each $\varepsilon > 0$ there is a $\delta > 0$ such that for any two δ-approximations $\varphi_0, \varphi_1: X \multimap Y$ of Ψ there exists a w-homotopy $\psi: X \times [0,1] \multimap Y$ such that*

(a) $\psi(\,\cdot\,, 0) = \varphi_0$ *and* $\psi(\,\cdot\,, 1) = \varphi_1$,
(b) $\psi(\,\cdot\,, t)$ *is an ε-approximation of Ψ for any $t \in [0,1]$.*

Proof. Let $\Theta: X \times [0,1] \multimap Y$ be a w-UV^ω-valued carrier defined by $\Theta(x,t) = \Psi(x)$ for all $x \in X$ and $t \in [0,1]$. Then in view of Corollary 5.20 there exists the required weighted homotopy ψ. □

In particular, we get:

Corollary 5.22. *Let $\Psi: X \multimap Y$ and X, Y be as above. Then there is $\delta > 0$ such that any two δ-approximations $\psi_1, \psi_2: X \multimap Y$ of Ψ are w-homotopic.*

We shall end this section by proving approximation results for w-carriers defined on pairs of compact ANR's.

Let us recall that $\Psi: (X, A) \multimap (Y, B)$ is a w-carrier if $\Psi: X \multimap Y$ is a w-carrier and $\Psi(A) \subset B$. It is easy to see that if $\Psi: (X, A) \multimap (Y, B)$ is a w-carrier, then $\Psi_A: A \multimap B$ is also a w-carrier.

Definition 5.23. Let $\Psi\colon (X,A) \multimap (Y,B)$ be a w-carrier and let $\varepsilon > 0$. We say that a w-map $\varphi\colon (X,A) \multimap (Y,B)$ is an *ε-approximation of* Ψ if $\varphi_A\colon A \multimap B$ is an ε-approximation of Ψ_A and $\varphi\colon X \multimap Y$ is an ε-approximation of $\Psi\colon X \multimap Y$.

Definition 5.24. We say that $\Psi\colon (X,A) \multimap (Y,B)$ is a *w-UV^ω-valued carrier* if Ψ_A and Ψ_X are w-UV^ω-valued carriers.

Theorem 5.25. *Let (X,A) be pair of compact* ANR*'s, (Y,B) a pair of locally connected spaces and $\Phi\colon (X,A) \multimap (Y,B)$ a w-UV^ω-valued carrier. Then for each $\varepsilon > 0$ there is a w-map $\varphi\colon (X,A) \multimap (Y,B)$ such that φ is an ε-approximation of Φ.*

Proof. The proof is similar to that of [2, Theorem 3.1(i)], but for the sake of completeness we give details. Let us take $\varepsilon > 0$ and let $0 < \delta < \varepsilon$ be as in Theorem 5.17. Since $\Phi_A\colon A \multimap B$ is a w-UV^ω-valued carrier, we conclude, using Corollary 5.19, that there is a δ-approximation $\varphi_0\colon A \multimap B$ of Φ_A. Consequently, in view of Theorem 5.17, there exists a weighted map $\varphi\colon X \multimap Y$ such that φ is an ε-approximation of Φ and $\varphi|A = \varphi_0$. Hence we obtain a weighted map $\varphi\colon (X,A) \multimap (Y,B)$ being an ε-approximation of $\Phi\colon (X,A) \multimap (Y,B)$. $\square$

Similarly to [2, Theorem 3.1(ii)], we get the following theorem.

Theorem 5.26. *Let (X,A), (Y,B) and $\Phi\colon (X,A) \multimap (Y,B)$ be as in the formulation of Theorem* 5.25. *Then for each $\varepsilon > 0$ there is $\delta > 0$ such that if $\varphi_1, \varphi_2\colon (X,A) \multimap (Y,B)$ are δ-approximations of Φ, then there exists a w-homotopy $\theta\colon (X \times [0,1], A \times [0,1]) \multimap (Y,B)$ such that $\theta(\,\cdot\,,t)\colon (X,A) \multimap (Y,B)$ is an ε-approximation of Φ for each $t \in [0,1]$.*

Proof. Let $\varepsilon > 0$ and let $\overline{\Phi}\colon (X \times [0,1], A \times [0,1]) \multimap (Y,B)$ be defined by $\overline{\Phi}(x,t) := \Phi(x)$ for all $t \in [0,1]$, $x \in X$. It is easy to see that $\overline{\Phi}$ is a w-UV^ω-valued carrier. Additionally, let us define $M := (X \times \{0\}) \cup (A \times [0,1]) \cup (X \times \{1\})$. Using the same arguments as in the proof of Lemma 5.16, we see that M is an absolute neighbourhood retract. Let $0 < \gamma < \varepsilon$ be as in the formulation of Theorem 5.17 for $X \times [0,1]$, M, Φ and ε. Moreover, Corollary 5.21 provides $0 < \delta < \gamma$ according to $\Phi_A\colon A \multimap B$ and γ. Then, by Corollary 5.21, there is a w-homotopy $\overline{\theta}\colon A \times [0,1] \multimap B$ such that

(1) $\overline{\theta}(\,\cdot\,,t)$ is a γ-approximation of Φ_A for all $t \in [0,1]$,
(2) $\overline{\theta}(\,\cdot\,,0) = \varphi_A(\,\cdot\,)$ and $\overline{\theta}(\,\cdot\,,1) = \psi_A(\,\cdot\,)$.

Now let us define a w-map $\widetilde{\theta}\colon M \multimap Y$ as follows

$$\widetilde{\theta}(x,t) = \begin{cases} \varphi_X(x) & \text{if } (x,t) \in X \times \{0\}, \\ \overline{\theta}(x,t) & \text{if } (x,t) \in A \times [0,1], \\ \psi_X(x) & \text{if } (x,t) \in X \times \{1\}. \end{cases}$$

Since $\widetilde{\theta}$ is a γ-approximation of $\Phi: X \times [0,1] \multimap Y$, by Theorem 5.17, there exists an extension $\theta: X \times [0,1] \multimap Y$ of $\widetilde{\theta}$ over $X \times [0,1]$ such that θ is an ε-approximation of $\Phi: X \multimap Y$, which implies that there is a weighted map $\theta: (X, A) \multimap (Y, B)$ satisfying all requirements of the assertion. □

Corollary 5.27. *Let (X, A), (Y, B) and $\Phi: (X, A) \multimap (Y, B)$ be as above. Then there is $\delta > 0$ such that if $\varphi, \psi: (X, A) \multimap (Y, B)$ are δ-approximations of Φ, then there is a w-homotopy $\theta: (X \times [0,1], A \times [0,1]) \multimap (Y, B)$ such that $\theta(\,\cdot\,, 0) = \varphi(\,\cdot\,)$ and $\theta(\,\cdot\,, 1) = \psi(\,\cdot\,)$.*

Remark 5.28. In the forthcoming paper of the author ([32]) it will be presented how the approximability theorems obtained above can be used to show new fixed point results for weighted carriers (see also [6], [27], [31], [33]).

6. Appendix

In Appendix we will use the Čech homology with the coefficients in the field of rational numbers $\mathbb{Q}$, and for simplicity we will omit $\mathbb{Q}$ from the notation. The aim of this section is to prove the following theorem.

Theorem 6.1. *Let X be an ANR and let $A \subset X$ be a compact subset, $k \geq 1$. Then A is k-acyclic if and only if for each $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that the inclusion $j: O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces a trivial homomorphism $j_{*i}: \check{H}_i(O_\delta(A)) \to \check{H}_i(O_\varepsilon(A))$ for all $1 \leq i \leq k$.*

Before we give the actual proof of the main result in this section we derive a few propositions. For this purpose we shall need a number of lemmas.

Lemma 6.2 ([15]). *Let $Y \in$ ANR, X be an arbitrary space and $A \subset X$ be a closed subset. Assume that $f, g: X \to Y$ are such that there is a homotopy $h: A \times [0,1] \to Y$ with $h(x, 0) = f(x)$, $h(x, 1) = g(x)$ for every $x \in A$. Then there exists a neighbourhood U of A in X and a homotopy $H: U \times [0,1] \to Y$ such that $H|A \times [0,1] = h$, $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for every $x \in U$.*

Lemma 6.3 ([4]). *Let Q^ω be the Hilbert cube and let $A \subset Q^\omega$ be a compact subset. Then there exists a sequence $\{Z_i\}_{i=1}^\infty$ of compact ANR-spaces such that*

$$Z_1 = Q^\omega, \quad Z_{i+1} \subset \operatorname{int} Z_i, \quad A = \bigcap_{i=1}^{\infty} Z_i.$$

Lemma 6.4 ([14]). *Let X be a compact space and let $X_1 \supset X_2 \supset \ldots$ be a descending sequence of compact spaces with $X = \bigcap_{i=1}^\infty X_i$. Then*

$$\check{H}_*(X) = \varprojlim \check{H}_*(X_i).$$

Lemma 6.5 ([14]). *Let $(X_1, A_1) \supset (X_2, A_2) \supset \dots$ be a descending sequence of compact pairs with $(X, A) = \bigcap_{i=1}^{\infty}(X_i, A_i)$. In addition, fix n and let $\{z_1, \dots, z_s\}$ be a linearly independent set in $\check{H}_n(X, A)$. Then there exists an index k_n such that:*

(a) *$\{j_k(z_1), \dots, j_k(z_s)\}$ is linearly independent in $\check{H}_n(X_k, A_k)$ for all $k \geq k_n$, where $j_k \colon \check{H}_n(X, A) \to \check{H}_n(X_k, A_k)$ are induced by $(X, A) \subset (X_k, A_k)$,*

(b) *in particular, for all $k \geq k_n$, $j_k \colon \check{H}_n(X, A) \to \check{H}_n(X_k, A_k)$ is a monomorphism on the space $E_s := \langle z_1, \dots, z_s \rangle$ generated by $\{z_1, \dots, z_s\}$.*

Lemma 6.6. *Let X, Y be ANR's and let $X_0 \subset X$ and $Y_0 \subset Y$ be compact subsets. In addition, assume that the inclusion $X_0 \hookrightarrow X$ has a w-UV^n-property, where $n \geq 1$. If Y_0 is homeomorphic to X_0, then the inclusion $Y_0 \hookrightarrow Y$ has also a w-UV^n-property.*

The proof of the above lemma is the same as in [3]. The only difference is using the w-homotopy functor instead of the homotopy functor.

Proposition 6.7. *Let X be the Hilbert cube and let $A \subset X$ be a compact subset. Assume, furthermore, that A is k-acyclic, $k \geq 1$. Then for each $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that the inclusion $j \colon O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces a trivial homomorphism $j_{*l} \colon \check{H}_l(O_\delta(A)) \to \check{H}_l(O_\varepsilon(A))$ for each $1 \leq l \leq k$.*

Proof. Our proof is based upon ideas found in [14]. Since X is the Hilbert cube there exists, in view of Lemma 6.3, a sequence $\{Z_i\}_{i=1}^{\infty}$ of compact ANR-spaces such that $Z_{i+1} \subset Z_i$, for $i \geq 1$, and $\bigcap_{i=1}^{\infty} Z_i = A$. Let us fix Z_{i_0}. Now consider the diagram

$$
\begin{array}{ccccc}
 & & \check{H}_l(Z_s) & & \\
 & & \big\downarrow{\scriptstyle p_l^s} & & \\
\check{H}_l(A) & \xrightarrow{\ j_l\ } & \check{H}_l(Z_{i_0}) & \xrightarrow{\ w_l\ } & \check{H}_l(Z_{i_0}, A) \\
 & & \big\downarrow{\scriptstyle \lambda_l^s} & \swarrow{\scriptstyle \mu_l^s} & \\
 & & \check{H}_l(Z_{i_0}, Z_s), & &
\end{array}
$$

where all homomorphisms are induced by inclusions, the triangle is commutative; and both the horizontal and vertical lines are exact, $s \geq i_0$, $l \geq 1$. Let us observe that if $\check{H}_l(A) = 0$ for some $l \geq 1$, then from the above diagram we deduce that $\ker w_l = 0$. Additionally, $\dim \operatorname{Im} w_l < \infty$, because compact ANR's have the Čech homology of finite type. Now we shall show that there exists an index N_{i_0} such that the homomorphisms $p_l^s \colon \check{H}_l(Z_s) \to \check{H}_l(Z_{i_0})$ are trivial for $s \geq N_{i_0}$ and $1 \leq l \leq k$. Let $1 \leq l_0 \leq k$ be fixed and let $z_1^{l_0}, \dots, z_{s_l}^{l_0}$ be a basis for

$w_{l_0}(\check{H}_{l_0}(Z_{i_0})) \subset \check{H}_{l_0}(Z_{i_0}, A)$. Now, by applying Lemma 6.5 to

$$(Z_{i_0}, Z_s) \supset (Z_{i_0}, Z_{s+1}) \supset (Z_{i_0}, Z_{s+2}) \supset \dots$$

and $\mu^s_{l_0}: \check{H}_{l_0}(Z_{i_0}, A) \to \check{H}_{l_0}(Z_{i_0}, Z_s)$ for $s \geq i_0$, we obtain $N^{i_0}_{l_0} \geq i_0$ such that the homomorphism $\mu^s_{l_0}|\langle z^{l_0}_1, \dots, z^{l_0}_{s_{l_0}}\rangle: \langle z^{l_0}_1, \dots, z^{l_0}_{s_{l_0}}\rangle \to \check{H}_{l_0}(Z_{i_0}, Z_s)$ is a monomorphism for $s \geq N^{i_0}_{l_0}$. Moreover, since $\ker w_{l_0} = 0$ and $\lambda^s_{l_0} = \mu^s_{l_0} \circ w_{l_0}$, we deduce that the homomorphism $\lambda^s_{l_0}: \check{H}_{l_0}(Z_{i_0}) \to \check{H}_{l_0}(Z_{i_0}, Z_s)$ is a monomorphism for all $s \geq N^{i_0}_{l_0}$. Thus, from the exactness of the vertical sequence in the above diagram, we infer that $\operatorname{Im} p^s_{l_0} = 0$ for $s \geq N^{i_0}_{l_0}$. Let $N_{i_0} := max\{N^{i_0}_1, \dots, N^{i_0}_k\}$. Then for $1 \leq l \leq k$ and $s \geq N_{i_0}$ the homomorphism

$$p^s_l: \check{H}_l(Z_s) \to \check{H}_l(Z_{i_0}) \tag{6.1}$$

is trivial. Let $\varepsilon > 0$. Now, let us observe that there exists i_0 such that $Z_s \subset O_\varepsilon(A)$ for $s \geq i_0$, because $A = \bigcap_{i=1}^\infty Z_i$ and $Z_{i+1} \subset Z_i$. Let us fix $s \geq N_{i_0}$ ($N_{i_0} \geq i_0$). Since Z_s is a compact ANR, there exists an open subset $U \subset X$ with $Z_s \subset U \subset O_\varepsilon(A)$ and a retraction $r_s: U \to Z_s$. Let $f: U \to O_\varepsilon(A)$ be factored as

$$U \xrightarrow{r_s} Z_s \xrightarrow{j_s} Z_{i_0} \xrightarrow{i_s} O_\varepsilon(A),$$

where j_s and i_s are the inclusions. Then by the compactness of A and Lemma 6.2 we infer that there exists $\delta < \varepsilon$ with $O_\delta(A) \subset U$ and such that $f|O_\delta(A)$ is homotopic to the inclusion $i: O_\delta(A) \to O_\varepsilon(A)$. Hence

$$(f|O_\delta(A))_{*l} = i_{*l}: \check{H}_l(O_\delta(A)) \to \check{H}_l(O_\varepsilon(A)).$$

But $(f|O_\delta(A))_{*l} = (i_s)_{*l} \circ (j_s)_{*l} \circ (r_s|O_\delta(A))_{*l}$ and $(j_s)_{*l} = p^s_l$, so, in view of (6.1), the homomorphism $(f|O_\delta(A))_{*l}$ is trivial for $1 \leq l \leq k$. Consequently, $i_{*l}: \check{H}_l(O_\delta(A)) \to \check{H}_l(O_\varepsilon(A))$ is the trivial homomorphism for $1 \leq l \leq k$, which completes the proof. □

Corollary 6.8. *Let X be an* ANR *and let $A \subset X$ be a compact subset. Assume, furthermore, that A is k-acyclic, $k \geq 1$. Then for each $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that the inclusion $j: O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces a trivial homomorphism $j_{*l}: \check{H}_l(O_\delta(A)) \to \check{H}_l(O_\varepsilon(A))$ for $1 \leq l \leq k$.*

Proof. Since any compact metric space admits an embedding into the Hilbert cube Q^ω, it follows that there exists a compact subset B of Q^ω which is homeomorphic to A. Moreover, since A is k-acyclic and since A is homeomorphic to B, we deduce that B is also k-acyclic. Now, in view of Proposition 6.7 and Theorem 4.9, we infer that the inclusion $B \hookrightarrow Q^\omega$ has a w-UV^k-property. Hence, by Lemma 6.6, the inclusion $A \hookrightarrow X$ has a w-UV^k-property. Consequently, by Theorem 4.9, the assertion follows. □

Proposition 6.9. *Let X and $A \subset X$ be as above. If for each $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that the inclusion $j\colon O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces a trivial homomorphism $j_{*l}\colon \check{H}_l(O_\delta(A)) \to \check{H}_l(O_\varepsilon(A))$ for any $1 \le l \le k$, then A is k-acyclic.*

Proof. Let $\{Z_n\}_{n=1}^\infty$ be sequence as in Lemma 6.3. Then, under our assumption, there exist two sequences $\{\varepsilon_m\}_{m=1}^\infty$ and $\{i_m\}_{m=1}^\infty$ such that

(1) $Z_{i_1} = Q^\omega$, $\varepsilon_1 = 1$, $i_1 = 1$, $\varepsilon_{m+1} < \varepsilon_m$;
(2) $Z_{i_{m+1}} \subset O_{\varepsilon_m}(A) \subset Z_{i_m}$ for $m \ge 1$;
(3) the inclusion $j\colon O_{\varepsilon_{m+1}}(A) \to O_{\varepsilon_m}(A)$ induces the trivial homomorphism $j_{*l}\colon \check{H}_l(O_{\varepsilon_{m+1}}(A)) \to \check{H}_l(O_{\varepsilon_m}(A))$ for any $m \ge 1$, $1 \le l \le k$.

Since the inclusion $J_{i_m}\colon Z_{i_{m+2}} \hookrightarrow Z_{i_m}$ can be factored as

$$Z_{i_{m+2}} \hookrightarrow O_{\varepsilon_{m+1}}(A) \hookrightarrow O_{\varepsilon_m}(A) \hookrightarrow Z_{i_m},$$

the induced homomorphism $J_{i_m}\colon \check{H}_l(Z_{i_{m+2}}) \hookrightarrow \check{H}_l(Z_{i_m})$ is trivial for any $m \ge 1$ and $1 \le l \le k$. Therefore

$$\varprojlim_m \check{H}_l(Z_{i_{(2m-1)}}) = 0, \tag{6.2}$$

and since $A = \bigcap_{m=1}^\infty Z_{i_{(2m-1)}}$, so by Lemma 6.4 and (6.2) we infer that $\check{H}_l(A) = 0$ for $1 \le l \le k$, which completes the proof. □

Corollary 6.10. *Let X be an* ANR *and let $A \subset X$ be a compact subset. If for each $\varepsilon > 0$ there exists $\delta < \varepsilon$ such that the inclusion $j\colon O_\delta(A) \hookrightarrow O_\varepsilon(A)$ induces a trivial homomorphism $j_{*l}\colon \check{H}_l(O_\delta(A)) \to \check{H}_l(O_\varepsilon(A))$ for any $1 \le l \le k$, then A is k-acyclic.*

Proof. By the same argument as in the proof of Corollary 6.8, there exists a compact subset B of the Hilbert cube Q^ω which is homeomorphic to A. Under our assumptions Theorem 4.9 implies that the inclusion $A \hookrightarrow X$ has a w-UV^k-property. Hence by Lemma 6.6 it follows that the inclusion $B \hookrightarrow Q^\omega$ has a w-UV^k-property. Consequently, by Theorem 4.9 and Proposition 6.9, we infer that B is k-acyclic. Since A is homeomorphic to B, we deduce that A is also k-acyclic. □

Finally, from Corollaries 6.8 and 6.10 we obtain Theorem 6.1.

References

[1] J. Andres and L. Górniewicz, *Topological Fixed Point Principles for Boundary Value Problems*, Topological Fixed Point Theory and Its Applications, vol. 1, Kluwer Academic Publishers, Dordrecht, 2003.

[2] R. Bader, G. Gabor and W. Kryszewski, *On the extension of approximations for set-valued maps and the repulsive fixed points*, Boll. Un. Mat. Ital. B (7) **10** (1996), no. 2, 399–416.

[3] R. Bader and W. Kryszewski, *Fixed point index for compositions of set-valued with proximally ∞-connected values on arbitrary* ANR*s repulsive fixed points*, Set-Valued Anal. **2** (1994), 459–480.

[4] K. Borsuk, *Theory of Retracts*, Monografie Matematyczne, PWN, Warszawa, 1967.

[5] R. Brown, *The Lefschetz Fixed Point Theorem*, Scott, Foresman and Co., Glenview III, London, 1971.

[6] G. Conti and J. Pejsachowicz, *Fixed point theorems for multivalued weighted maps*, Ann. Mat. Pura Appl. (4) **126** (1981), 319–341.

[7] G. Darbo, *Teoria dell'omologia in una categoria di mappe plurivalenti ponderate*, Rend. Sem. Mat. Univ. Padova **28** (1958), 188–220.

[8] ______, *Estensione alle mappe ponderate del teorema di Lefschetz sui punti fissi*, Rend. Sem. Mat. Univ. Padova **31** (1961), 46–57.

[9] S. Eilenberg and N. Steenrod, *Foundations of Algebraic Topology*, Princeton Univerity Press, Princeton, New Jersey, 1952.

[10] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Second edition, Topological Fixed Point Theory and Its Applications, vol. 4, Springer, Dordrecht, 2006.

[11] L. Górniewicz, A. Granas and W. Kryszewski, *Sur la méthode de l'homotopie dans la théorie des points fixes pour les applications multivoques.* II. *L'indice dans les* ANR*s compacts*, C. R. Acad. Sci. Paris Sér. I Math. **308** (1989), no. 14, 449–452.

[12] ______, *On the homotopy method in the fixed point index theory of multi-valued mappings of compact absolute neighborhood retracts*, J. Math. Anal. Appl. **161** (1991), no. 2, 457–473.

[13] L. Górniewicz and M. Lassonde, *Approximation and fixed points for compositions of R_δ-maps*, Topology Appl. **55** (1994), no. 3, 239–250.

[14] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer Monographs in Mathematics, Springer–Verlag, New York, 2003.

[15] Sze-tsen Hu, *Theory of Retracts*, Wayne State University Press, Detroit, 1965.

[16] R. Jerrard, *Homology with multiple-valued functions applied to fixed points*, Trans. Amer. Math. Soc. **213** (1975), 407–427.

[17] ______, *Fixed points and product spaces*, Houston J. Math. **11** (1985), no. 2, 191–198.

[18] R. Jerrard and M. D. Meyerson, *Homotopy with m-functions*, Pacific J. Math. **84** (1979), no. 2, 305–318.

[19] W. Kryszewski, *The fixed-point index for the class of compositions of acyclic set-valued maps on* ANR*s*, Bull. Sci. Math. **120** (1996), no. 2, 129–151.

[20] ______, *Homotopy properties of set-valued mappings* (1997), Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Toruń, (Habilitation dissertation).

[21] ______, *Graph-approximation of set valued maps. A survey.*, Differential Inclusions and Optimal Control (J. Andres, L. Górniewicz and P. Nistri, eds.), Lecture Notes in Nonlinear Analysis, vol. 2, Juliusz Schauder Center for Nonlinear Studies, Nicolaus Copernicus University, Toruń, 1998, pp. 223–235.

[22] ______, *Graph-approximation of set-valued maps on noncompact domains*, Topology Appl. **83** (1998), no. 1, 1–21.

[23] S. Mardešić and J. Segal, *Shape Theory*, North-Holland Mathematical Library, vol. 26, North-Holland Publishing Co., Amsterdam, 1982.

[24] J. Pejsachowicz, *The homotopy theory of weighted mappings*, Boll. Un. Mat. Ital. B (5) **14** (1977), no. 3, 702–721.

[25] ______, *A Lefschetz fixed point theorem for multivalued weighted mappings*, Boll. Un. Mat. Ital. A (5) **14** (1977), no. 2, 391–397.

[26] ———, *Relation between the homotopy and the homology theory of weighted mappings*, Boll. Un. Mat. Ital. B (5) **15** (1978), no. 1, 285–302.

[27] J. Pejsachowicz and R. Skiba, *Fixed point theory of multivalued weighted maps*, Handbook of Topological Fixed Point Theory, Springer, Dordrecht, 2005, pp. 217–263.

[28] C. B. Petkova, *Coincidence of homologies on homologically locally connected spaces*, C. R. Acad. Bulgare Sci. 35 **4** (1978), no. 1, 427–430.

[29] R. Skiba, *On the Lefschetz fixed point theorem for multivalued weighted mappings*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **40** (2001), 201–214.

[30] ———, *Topological essentiality for multivalued weighted mappings*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **41** (2002), 131–145.

[31] ———, *Fixed points of multivalued weighted maps*, Ph. D. Thesis (2005), Toruń, Poland. (in Polish)

[32] ———, *Fixed points of multivalued weighted maps*, the revised version of Ph. D. thesis, in preperation. (in English)

[33] F. von Haeseler and H.-O. Peitgen and G. Skordev, *Lefschetz fixed point theorem for acyclic maps with multiplicity*, Topol. Methods Nonlinear Anal. **19** (2002), no. 2, 339–374.

[34] F. von Haeseler and G. Skordev, *Borsuk–Ulam theorem, fixed point index and chain approximations for maps with multiplicity*, Pacific J. Math. **153** (1992), no. 2, 369–396.

Manuscript received July 1, 2006

Robert Skiba
Faculty of Mathematics and Computer Sciences
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, POLAND

E-mail address: robo@mat.uni.torun.pl