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MIN-MAX LEVELS ON THE DOUBLE NATURAL CONSTRAINT

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Abstract. A question about the possibility of using min-max methods on the double natural constraint, in spite of its lack of regularity, has been raised in some recent papers. In this note we give an answer by topological arguments which show the equivalence between constrained and unconstrained min-max classes, avoiding in this way any regularity problem.

1. Introduction

In studying problems like

$$\text{(P)}\qquad \begin{cases} -\Delta u = |u|^{p-2}u + \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ and $p \leq 2^*$, through variational methods we deal with the associated functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_\Omega |u|^2 - \frac{1}{p}\int_\Omega |u|^p. \tag{1.1}$$

Introducing suitable constraints turns out to be an useful tool (see for instance [9]) in order to prove the existence of solutions to (P) and a classical approach

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relies in searching constrained critical points for the functional I on the so called *natural constraint* manifold

$$\mathcal{V} = \{u \mid u \neq 0,\ \nabla I(u) \cdot u = 0\}. \tag{1.2}$$

More recently ([2]) a similar approach was carried out on the *double natural constraint* manifold

$$\mathcal{W} = \{u \mid u^{\pm} \neq 0,\ \nabla I(u) \cdot u^{\pm} = 0\} \subset \mathcal{V}. \tag{1.3}$$

In [2] the existence of minimum points is proved, whereas in other works (see for instance [5]) the goal relies in detecting min-max levels. The proof of the critical character of such levels does not appear sufficiently clear for the poor regularity properties of the manifold $\mathcal{W}$ (see [3], [1]). Hence, also to avoid any doubt, it seems to be quite useful to spend a few pages in order to show how a suitable use of the genus allows to find infinitely many critical min-max levels. To this aim two approaches appear reasonable, namely (1) to prove a suitable deformation lemma which uses less regularity of the classical ones and works on $\mathcal{W}$, (2) to prove the equivalence between the variational characterization of the min-max levels on the constraint and an unconstrained one through topological methods. We shall pursue the approach (2) which deserves an interest which goes beyond this particular context. Indeed, even in the case of the natural constraint, where no regularity problem arises, it is worth to know that the critical points corresponding to constrained classes also admit a variational characterization as unconstrained min-max points.

The reader is supposed to be familiar with the use of variational methods in nonlinear analysis and, in particular, with the connections, which we will not deepen here, between the topological properties of Krasnosel'skiĭ Genus and the estimates on the corresponding min-max levels. The paper is organized as follows. In Section 2 we introduce a class of test maps and consequently a notion of genus which is a slight variant of the well known Krasnosel'skiĭ Genus. In Section 3 we introduce the constrained min-max classes and the corresponding admissible unconstrained classes in the case of the double natural constraint. In Section 4 we show the variants needed to deal with the natural constraint and, finally, Section 5 is devoted to the proof of some topological results which are needed in order to compute the genus of a sphere.

2. Genus of a symmetric set

For any $k \in \mathbb{N}$, we adopt the following notation: $Q_k = [-1,1]^k$, $F^i_{\pm} = \{x \in Q_k \mid x_i = \pm 1\}$. Given $n, k \in \mathbb{N}$, for every $x \in \mathbb{R}^{n+k}$ we shall split x as $x = (x_0, \ldots, x_k)$ with $x_0 \in \mathbb{R}^n$ and $x_i \in \mathbb{R}$ for $i = 1, \ldots, k$. Analogously, if $\varphi(x) \in \mathbb{R}^{n+k}$, we shall write $\varphi(x) = (\varphi_0(x), \ldots, \varphi_k(x))$ with $\varphi_0(x) \in \mathbb{R}^n$ and

$\varphi_i(x) \in \mathbb{R}$ for $i = 1, \dots, k$. Sometimes, we shall also use the notation $x = (x_0, x')$, $\varphi(x) = (\varphi_0(x), \varphi'(x))$ instead of the previous one, by assuming $x', \varphi'(x) \in \mathbb{R}^k$. We shall set $B_n = \{x \in \mathbb{R}^n \mid |x| \le 1\}$, $S^n = \partial B_{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| = 1\}$ and we shall denote by E any given Banach space.

For any given $k \in \mathbb{N}$, we set $I_k = \{1, \dots, k\}$ and we denote by $\mathcal{P}_k$ the set of all the involutive permutations on I_k, i.e. $\pi \in \mathcal{P}_k$ if and only if $\pi: I_k \to I_k$ and $\pi \circ \pi = \mathrm{id}$. Given any $\pi \in \mathcal{P}_k$, we introduce the map $\widehat{\pi}: \mathbb{R}^k \to \mathbb{R}^k$ so defined:

$$\widehat{\pi}(x_1, \dots, x_k) = (x_{\pi(1)}, \dots, x_{\pi(k)}).$$

We have

$$\widehat{\pi}(\widehat{\pi}(x)) = \widehat{\pi}(x_{\pi(1)}, \dots, x_{\pi(k)}) = (x_{\pi(\pi(1))}, \dots, x_{\pi(\pi(k))}) = x,$$

which means $\widehat{\pi} \circ \widehat{\pi} = \mathrm{id}$. Furthermore, given any Banach space E, let us define the map $\pi_E: E \times \mathbb{R}^k \to E \times \mathbb{R}^k$ such that $\pi_E: (x_0, x') \mapsto (-x_0, \widehat{\pi}(x'))$. It is easily seen that π_E is involutive too, indeed

$$\pi_E(\pi_E(x)) = \pi_E(-x_0, \widehat{\pi}(x')) = (x_0, \widehat{\pi}(\widehat{\pi}(x'))) = (x_0, x') = x$$

and so we also have $\pi_E \circ \pi_E = \mathrm{id}$. We shall set $\widetilde{\pi} = \pi_E$ if $E = \mathbb{R}^n$. Now, let $A \subset E$ be any symmetric subset and let $\varphi: A \times Q_k \to \mathbb{R}^{n+k}$ and $\pi \in \mathcal{P}_k$ be given, we set

$$\varphi_\pi = \widetilde{\pi} \circ \varphi \circ \pi_A: x \mapsto \widetilde{\pi}(\varphi(\pi_E(x))),$$

where $\pi_A: A \times Q_k \to A \times Q_k$ is the restriction of π_E. We shall refer to φ_π as to the π-*symmetric* of φ. We shall say that φ is π-*symmetric* if $\varphi = \varphi_\pi$ and that φ is *symmetric* if there exists π such that $\varphi = \varphi_\pi$. We note that $(\varphi_\pi)_\pi = \widetilde{\pi} \circ (\widetilde{\pi} \circ \varphi \circ \pi_A) \circ \pi_A = \varphi$.

Remrk 2.1. We observe that if $\pi = \mathrm{id}$ φ is π-symmetric if

(S1) $\varphi_0(-x_0, x') = -\varphi_0(x_0, x')$, for all $x \in A \times Q_k$,
(S2) $\varphi'(-x_0, x') = \varphi'(x_0, x')$, for all $x \in A \times Q_k$.

Definition 2.2. Let $A \subset E$ be any given symmetric subset, for any $k \in \mathbb{N}$ we shall say that a symmetric function $\varphi \in C(A \times Q_k, \mathbb{R}^{n+k})$ is a *test function of dimension* n for A if the following condition holds

(T) $\pm\varphi_i(x) \ge 0$, for all $x \in A \times F^i_\pm$ and all $i = 1, \dots, k$.

We shall denote by $\Lambda^*_n(A)$ the set of the n dimensional test functions for A, obtained for any value of k. Then we are ready to define the *genus* of a set A as the number

$$\gamma^*(A) = \inf\{n \in \mathbb{N} \mid \text{there exists } \varphi \in \Lambda^*_n(A) \text{ such that } 0 \notin \varphi(A \times Q_k)\}.$$

Firstly, we remark that, since $\Lambda^*_n(A)$ is a larger set than the set $\Lambda_n(A)$ of the test maps related to Krasnosel'skiĭ Genus $\gamma(A)$ (see [6], [9]) (indeed the last one coincides with the subset of the former one which contains the test functions constructed by taking $k = 0$), we have that, in general, $\gamma^*(A) \le \gamma(A)$. By definition, given any $\varphi \in \Lambda_n^*(A)$ with $n < \gamma^*(A)$, $0 \in \varphi(A)$.

In view of proving the next propositions concerning the genus of a symmetric set, we prove some useful lemmas stating some properties of the test functions.

Lemma 2.3. *Let $\varphi: A \times Q_k \to \mathbb{R}^{n+k}$ satisfy* (T) *and let $\pi \in \mathcal{P}_k$ be given, then φ_π also satisfies* (T).

Proof. Let $x = (x_0, x') \in A \times F_i^{\pm}$, for $i = 1, \dots, k$. Then $\widehat{\pi}(x') \in F_{\pi(i)}^{\pm}$ and, since φ satisfies (T), we have

$$\pm\varphi_{\pi(i)}(\pi_E(x_0, x')) = \pm\varphi_{\pi(i)}(-x_0, \widehat{\pi}(x')) \ge 0.$$

By the previous definitions we know that $\varphi_{\pi(i)} = (\widetilde{\pi}\circ\varphi)_i$ and so $\pm(\widetilde{\pi}\circ\varphi)_i(\pi_E(x)) \ge 0$, that is $\pm(\varphi_\pi)_i(x) \ge 0$, as stated in the thesis. □

Lemma 2.4. *Let $A \subset E$ be any given symmetric subset. Then every test function $\varphi \in \Lambda_n^*(A)$ can be extended to a map in $\Lambda_n^*(E)$.*

Proof. Firstly, by virtue of Tietze–Dugundji Theorem (see [4]) we take an extension of the components φ_i on $E \times F_i^{\pm}$ valued in $\mathbb{R}_\pm$ keeping the sign property (T), then we extend them and φ_0 continuously on all of $E \times Q_k$ so getting a map $\overline{\varphi}: E \times Q_k \to \mathbb{R}^{n+k}$ which, obviously, satisfies (T). By Lemma 2.3 the map $\overline{\varphi}_\pi$, which is an extension of φ_π, satisfies (T) and the same happens for $\varphi_s = \overline{\varphi}/2 + \overline{\varphi}_\pi/2: E \times Q_k \to \mathbb{R}^{n+k}$, where $\pi \in \mathcal{P}_k$ is such that φ is π-symmetric. Obviously, φ_s is symmetric since

$$(\varphi_s)_\pi = \frac{1}{2}\overline{\varphi}_\pi + \frac{1}{2}(\overline{\varphi}_\pi)_\pi = \frac{1}{2}\overline{\varphi}_\pi + \frac{1}{2}\overline{\varphi} = \varphi_s,$$

so $\varphi_s \in \Lambda_n^*(E)$. If $x \in A \times Q_k$ then $\varphi(x) = \varphi_\pi(x)$, hence $\varphi(x) = \overline{\varphi}(x) = \overline{\varphi}_\pi(x)$ and so $\varphi(x) = \varphi_s(x)$. Therefore φ_s extends φ to all of E. □

The following statement shows how the lower estimates on the genus are preserved by the gradient flow.

Proposition 2.5. *Let $A \subset E$ be any given symmetric subset and let $\eta: A \to E$ be a given odd continuous map. Then $\gamma^*(\eta(A)) \ge \gamma^*(A)$.*

Proof. Let $n < \gamma^*(A)$, $\varphi \in \Lambda_n(\eta(A))$ and let $\overline{\eta}(x_0, x') = (\eta(x_0), x')$, $\overline{\eta}: A \times Q_k \to \eta(A) \times Q_k$. Given $\pi \in \mathcal{P}_k$, we have

$$\pi_E(\overline{\eta}(x_0, x')) = (-\eta(x_0), \widehat{\pi}(x')) = (\eta(-x_0), \widehat{\pi}(x')) = \overline{\eta}(-x_0, \widehat{\pi}(x')) = \overline{\eta}(\pi_E(x'))$$

and so $\pi_{\eta(A)} \circ \overline{\eta} = \overline{\eta} \circ \pi_A$. We take $\overline{\varphi} = \varphi \circ \overline{\eta}: A \times Q_k \to \mathbb{R}^{n+k}$, then

$$\overline{\varphi}_\pi = \widetilde{\pi} \circ \overline{\varphi} \circ \pi_A = \widetilde{\pi} \circ \varphi \circ \overline{\eta} \circ \pi_A = \widetilde{\pi} \circ \varphi \circ \pi_{\eta(A)} \circ \overline{\eta} = \varphi_\pi \circ \overline{\eta}.$$

Therefore, if we fix π such that φ is π-symmetric, that is $\varphi = \varphi_\pi$, then $\overline{\varphi}_\pi = \varphi \circ \overline{\eta} = \overline{\varphi}$, hence also $\overline{\varphi}$ is symmetric. Since $\overline{\eta}(A \times F_i^{\pm}) = \eta(A) \times F_i^{\pm}$, we deduce that $\overline{\varphi}$ satisfies (T). Thus $\overline{\varphi} \in \Lambda_n^*(A)$, hence

$$0 \in \overline{\varphi}(A \times Q_k) = \varphi(\overline{\eta}(A \times Q_k)) = \varphi(\eta(A) \times Q_k).$$

It follows that $n < \gamma^*(\eta(A))$ by the arbitrariness of φ and, consequently, $\gamma^*(A) \leq \gamma^*(\eta(A))$ by the arbitrariness of n. $\square$

We state the following result, which will be proved in Section 5 after some topological lemmas and shows the existence of sets of large genus.

Proposition 2.6. $\gamma^*(S^n) = n + 1$.

Analogously to the case of Krasnosel'skiĭ Genus, we can state a trace property for the genus $\gamma^*(A)$ of any subset $A \subset E$ which allows to deduce easily that if k min-max levels coincide, in a case of compactness, the set of the critical points at such level has a Krasnosel'skiĭ Genus bigger or equal to $k + 1$.

Theorem 2.7. *Let $A \subset E$ be any given symmetric subset and let $\varphi \in \Lambda_k(A)$, with $k < \gamma^*(A)$. Then $\gamma^*(\varphi^{-1}(0)) \geq \gamma^*(A) - k$.*

Proof. Let $k \leq n < \gamma^*(A)$ and $\psi \in \Lambda^*_{n-k}(\varphi^{-1}(0))$, $\psi : \varphi^{-1}(0) \times Q_h \to \mathbb{R}^{n-k+h}$. By Lemma 2.4 ψ can be extended to $\overline{\psi}$ defined on all of E (and so on A) and such that $\overline{\psi} \in \Lambda^*_{n-k}(A)$. Let us consider the function $\varphi \times \overline{\psi}\colon A \times Q_h \to \mathbb{R}^{n+h}$ defined below

$$(\varphi \times \overline{\psi})(x) = (\varphi(x_0), \overline{\psi}(x)) = ((\varphi(x_0), \overline{\psi}_0(x)), \overline{\psi}'(x)).$$

It is easily seen that $\varphi \times \overline{\psi}$ satisfies (T), since the components involved in such condition only belong to $\overline{\psi}$. Moreover, if $\pi \in \mathcal{P}_h$ then

$$\begin{aligned}
\widetilde{\pi}((\varphi \times \overline{\psi})(\pi_E(x))) &= \widetilde{\pi}((\varphi(-x_0), \overline{\psi}_0(\pi_E(x)), \overline{\psi}'(\pi_E(x)) \\
&= ((-\varphi(-x_0), -\overline{\psi}_0(\pi_E(x)), \widehat{\pi}(\overline{\psi}'(\pi_E(x))) \\
&= ((\varphi(x_0), (\widetilde{\pi} \circ \overline{\psi})_0(\pi_E(x)), (\widetilde{\pi} \circ \overline{\psi})'\pi_E(x))) = (\varphi \times \overline{\psi}_\pi)(x).
\end{aligned}$$

Therefore $(\varphi \times \overline{\psi})_\pi = \varphi \times \overline{\psi}_\pi$, hence since $\overline{\psi}$ is symmetric, then also $\varphi \times \overline{\psi}$ satisfies the same condition. Thus $\varphi \times \overline{\psi} \in \Lambda_n(A)$ and then $0 \in (\varphi \times \overline{\psi})(A \times Q_h)$, that is $0 \in \overline{\psi}(\varphi^{-1}(0) \times Q_h)$. By the arbitrariness of $\overline{\psi}$ we get $n - k < \gamma^*(\varphi^{-1}(0))$ and by the arbitrariness of n the thesis follows. $\square$

A different trace property, which actually is the main motivation for passing to the present variant of the notion of genus, can be now also proved.

Theorem 2.8. *Let $A \subset E$ be any symmetric subset and let $\sigma: A \times Q_k \to E$ and $\varphi: E \to \mathbb{R}^k$ be given. If there exists $\pi \in \mathcal{P}_k$ such that*

(a) $\sigma(\pi_E(x)) = -\sigma(x)$ *for all* $x \in A \times Q_k$*;*
(b) $\varphi(-u) = \widehat{\pi}(\varphi(u))$ *for all* $u \in E$*;*
(c) $\pm\varphi_i(\sigma(x)) \geq 0$, *for all* $x \in A \times F_i^{\pm}$ *and all* $i = 1, \dots, k$,

then $\gamma^*(\sigma(A \times Q_k) \cap \varphi^{-1}(0)) \geq \gamma^*(A)$.

Proof. Firstly we observe that conditions (a), (b) allow to state respectively that $\sigma(A \times Q_k)$ and $\varphi^{-1}(0)$ are symmetric subsets of E and so the assertion makes sense. We fix $n < \gamma^*(A)$ and a test function ψ of dimension n defined on $(\sigma(A \times Q_k) \cap \varphi^{-1}(0))$ and extended by Lemma 2.4 on E. Then $\psi : E \times Q_h \to \mathbb{R}^{n+h}$ satisfies (T) and is symmetric for some $\pi_1 \in \mathcal{P}_h$.

Let us define $\rho \in \mathcal{P}_{k+h}$ such that $\rho(i) = \pi_1(i)$ if $i \leq h$ and $\rho(i) = h + \pi(i-h)$ if $i > h$. In this way, if $x' \in \mathbb{R}^{h+k}$ is decomposed as $x' = (z', y')$ with $z' \in \mathbb{R}^h$ and $y' \in \mathbb{R}^k$, we have $\widehat{\rho}(x') = (\widehat{\pi}_1(z'), \widehat{\pi}(y'))$.

We define $\vartheta: A \times Q_{h+k} \to \mathbb{R}^{n+h+k}$ by setting

$$\vartheta(x_0, x') = \vartheta(x_0, z', y') = (\psi(\sigma(x_0, y'), z'), \varphi(\sigma(x_0, y'))),$$

that is $\vartheta_0 = \psi_0$ and $\vartheta' = (\psi', \varphi)$.

We are going to prove that $\vartheta \in \Lambda_n^*(A)$. Since ψ satisfies (T) we have that, for $i \leq h$, $\pm\vartheta_i = \pm\psi_i \geq 0$ if $z' \in F_i^{\pm}$, whereas condition (c) states that $\pm\vartheta_{h+i} = \pm\varphi_i \geq 0$ if $y' \in F_i^{\pm}$ and so ϑ satisfies (T). Moreover, by (a), (b) and the symmetry of ψ,

$$\begin{aligned}
\vartheta(\rho_E(x)) &= \vartheta(-x_0, \widehat{\pi}_1(z'), \widehat{\pi}(y')) \\
&= (\psi(\sigma(-x_0, \widehat{\pi}(y')), \widehat{\pi}_1(z')), \varphi(\sigma(-x_0, \widehat{\pi}(y')))) \\
&= (\psi(\sigma(\pi_E(x_0, y')), \widehat{\pi}_1(z')), \varphi(\sigma(\pi_E(x_0, y')))) \\
&= (\psi(-\sigma(x_0, y'), \widehat{\pi}_1(z')), \varphi(-\sigma(x_0, y'))) \\
&= (\psi((\pi_1)_E(\sigma(x_0, y'), z')), \widehat{\pi}(\varphi(\sigma(x_0, y')))) \\
&= (\widetilde{\pi}_1(\psi(\sigma(x_0, y'), z')), \widehat{\pi}(\varphi(\sigma(x_0, y')))) \\
&= (-\psi_0(\sigma(x_0, y'), z'), \widehat{\pi}_1(\psi'(\sigma(x_0, y'), z')), \widehat{\pi}(\varphi(\sigma(x_0, y')))) \\
&= (-\psi_0(\sigma(x_0, y'), z'), \widehat{\rho}(\psi'(\sigma(x_0, y'), z')), \varphi(\sigma(x_0, y'))) \\
&= (-\vartheta_0(x), \widehat{\rho}(\vartheta'(x))) = \widetilde{\rho}(\vartheta(x)).
\end{aligned}$$

Then ϑ is symmetric and so $\vartheta \in \Lambda_n^*(A)$ as claimed above and, since $n < \gamma^*(A)$, $0 \in \vartheta(A \times Q_{h+k})$. This means that there exist $x_0 \in A$, $y' \in Q_k$ and $z' \in Q_h$ such that $\psi(\sigma(x_0, y'), z') = 0$ and $\varphi(\sigma(x_0, y')) = 0$. So $0 \in \psi(((\sigma(A \times Q_k) \cap \varphi^{-1}(0))) \times Q_h)$. By the arbitrariness of ψ one gets $n \leq \gamma^*(\sigma(A \times Q_k) \cap \varphi^{-1}(0))$ and by the arbitrariness of n one gets the thesis. $\square$

3. Min-max classes on the double natural constraint

In this section we shall apply the above introduced notion of genus to the study of the min-max classes on the double natural constraint. To this aim we fix $\pi \in \mathcal{P}_2$, $\pi \neq \mathrm{id}$, so that we simply have $\pi(1) = 2$, $\pi(2) = 1$, and consequently we fix the maps $\widehat{\pi}$ and π_E as defined in the previous section. Let $\Omega \subset \mathbb{R}^N$ be given and let $I\colon H \to \mathbb{R}$ be the functional defined in (1.1), related to problem (P), with $H = H_0^1(\Omega)$. For every $x \in \Omega$, we set, as usual, $u^+(x) = \max(u(x), 0)$, $u^-(x) = \max(-u(x), 0)$. Let λ_1 be the first eigenvalue of $-\Delta$ on Ω, we suppose $\lambda < \lambda_1$. Let $\mathcal{W}$ be the double natural constraint given by (1.3), we set for $n \geq 1$

$$\Gamma_n^{\mathcal{W}} = \{A \subset \mathcal{W} \mid A \text{ is compact}, \ \gamma^*(A) \geq n\} \quad \text{and} \quad c_n = \inf_{A \in \Gamma_n^{\mathcal{W}}} \sup_{u \in A} I(u).$$

Since $\lambda < \lambda_1$ then there exists a ground state level $\overline{c} > 0$ so, for every n, $c_n > 2\overline{c}$ since, for every $u \in \mathcal{W}$, $I(u^\pm) \geq \overline{c}$ (see [2]). Let us introduce the sets

$$\begin{aligned} C^- &= \{u \in H \setminus \{0\} \mid I(u) \leq \overline{c}/3, \ \nabla I(u) \cdot u \geq 0\}, \\ C^+ &= \{u \in H \setminus \{0\} \mid I(u) \leq 0, \ (\text{and so } \nabla I(u) \cdot u \leq 0)\}, \\ L_n &= \{u \in H \setminus \{0\} \mid I(u) \leq c_n - \overline{c}/3\}. \end{aligned}$$

Let us observe that, being $\lambda < \lambda_1$, given any $u \in H \setminus \{0\}$, the function $\alpha \mapsto I(\alpha u)$ grows with a positive derivative for α small as far as it reaches its maximum, then it has a negative derivative and tends to $-\infty$ as $\alpha \to +\infty$. So two constants $\varepsilon, c > 0$ such that $\varepsilon \leq 1 \leq c$, $\varepsilon u \in C^-$ and $cu \in C^+$ always exist. Moreover, if $A \subset H \setminus \{0\}$ is any compact set, ε and c can be uniformly fixed for $u \in A$, as well as, if $A \subset \mathcal{W}$, for $u \in A^\pm = \{u^\pm \mid u \in A\}$, since $A^\pm$ turn out to be also compact sets which do not contain 0. Furthermore, ε can be fixed in a maximal way and c can be fixed in a minimal way so that both the two values can be considered as functions of A. We introduce the function $\sigma_A : A \times Q_2 \to H$, defined as

$$\sigma_A(u, \alpha, \beta) = \left(\varepsilon + (\alpha + 1)\frac{c - \varepsilon}{2}\right)u^+ - \left(\varepsilon + (\beta + 1)\frac{c - \varepsilon}{2}\right)u^-$$

and the sets

$$C_A = \sigma_A(A \times \partial Q_2) \quad \text{and} \quad \Sigma_A = \sigma_A(A \times Q_2).$$

We shall refer to Σ_A as to the *double homothetic expansion of* A. Let us notice that if $A \in \Gamma_n^{\mathcal{W}}$ and $\sup_A I < c_n + \overline{c}/3$, then σ_A belongs (see Lemma 3.3 below) to the class of continuous functions defined by

$$\mathcal{F}_n = \{\sigma\colon A \times Q_2 \to H \mid A \in \Gamma_n^{\mathcal{W}} \text{ and (1)–(3) hold}\},$$

where

(1) $\sigma(\pi_H(x)) = -\sigma(x)$ for all $x \in A \times Q_2$;
(2) $\sigma(u, \alpha, \beta) \in L_n$ and $\sigma(u, \alpha, \beta)^+ \in C^\pm$ if $\alpha = \pm 1$;

(3) $\sigma(u,\alpha,\beta) \in L_n$ and $\sigma(u,\alpha,\beta)^- \in C^\pm$ if $\beta = \pm 1$.

Therefore Σ_A belongs to the class of sets defined by

$$\Gamma_n = \{X \subset H\setminus\{0\} \mid \text{there exists } A \in \Gamma_n^{\mathcal{W}}, \text{ and exists } \sigma\colon A\times Q_2 \to H,\ \sigma \in \mathcal{F}_n, \text{ such that } X = \sigma(A\times Q_2)\}.$$

Let us introduce two further classes of functions and sets, namely

$$\mathcal{F}^* = \{\varphi\colon H \to \mathbb{R}^2 \mid (1^*)\text{–}(3^*) \text{ hold}\},$$

where

(1*) $\varphi(-u) = \widehat{\pi}(\varphi(u))$;
(2*) $\pm\varphi_1(u) \geq 0$ if $u \in L_n$ and $u^+ \in C^\pm$;
(3*) $\pm\varphi_2(u) \geq 0$ if $u \in L_n$ and $u^- \in C^\pm$,

and

$$\Gamma_n^* = \{X \subset H \setminus \{0\} \mid X \text{ is compact},\ X = -X \text{ and } \gamma^*(X \cap \varphi^{-1}(0)) \geq n \text{ for all } \varphi \in \mathcal{F}_n^*\}.$$

Lemma 3.1. $\Gamma_n \subset \Gamma_n^*$.

Proof. Let $\varphi \in \mathcal{F}^*$, $A \in \Gamma_n^{\mathcal{W}}$, $\sigma\colon A \times Q_2 \to H$, $\sigma \in \mathcal{F}_n$ be fixed. By virtue of Theorem 2.8 we have $\gamma^*(\sigma(A\times Q_2) \cap \varphi^{-1}(0)) \geq n$. By the arbitrariness of $\varphi \in \mathcal{F}^*$, we get $\sigma(A \times Q_2) \in \Gamma_n^*$. □

Lemma 3.2. *For every $A \in \Gamma_n^*$ we have $A \cap \mathcal{W} \in \Gamma_n^{\mathcal{W}}$.*

Proof. Let $g(u) = \nabla I(u)\cdot u$ cut off by a positive constant near 0 so that $g(0) > 0$. To prove the statement it suffices to note that the function $\varphi_\pm\colon u \mapsto -g(u^\pm) \in \mathbb{R}^2$ belongs to $\mathcal{F}^*$ and hence by the definition of Γ_n^* the thesis follows. □

Lemma 3.3. *If $A \in \Gamma_n^{\mathcal{W}}$ then $\sup_{\Sigma_A} I = \sup_A I$ and $\sup_{C_A} I \leq \sup_A I - 2\overline{c}/3$.*

Proof. Of course $A \subset \Sigma_A$, so $\sup_A I \leq \sup_{\Sigma_A} I$. Conversely, if $u \in \Sigma_A$, u has the form $u = \mu\bar{u}^+ + \nu\bar{u}^-$, with $\bar{u} \in A$. Since $\bar{u}^\pm \in \mathcal{V}$, we have

$$I(u) = I(\mu\overline{u}^+) + I(\nu\overline{u}^-) \leq I(\overline{u}^+) + I(\overline{u}^-) = I(\overline{u}) \leq \sup_A I.$$

If $u \in C_A$, then $\{\mu,\nu\} \cap \{\varepsilon, c\} \neq \emptyset$ so $I(\overline{u}^\pm) - I(u^\pm) \geq \overline{c} - \overline{c}/3$ for at least one of the $+$ and $-$ sign. □

Lemma 3.4. *The min-max levels of the above defined classes are the same, i.e.*

$$\inf_{A\in\Gamma_n} \sup_{u\in A} I(u) = \inf_{A\in\Gamma_n^*} \sup_{u\in A} I(u) = c_n.$$

Proof. By virtue of the inclusion $\Gamma_n \subset \Gamma_n^*$ we have

$$\inf_{A\in\Gamma_n^*} \sup_{u\in A} I(u) \leq \inf_{A\in\Gamma_n} \sup_{u\in A} I(u).$$

Moreover, for every $X \in \Gamma_n^*$, from Lemma 3.2 we have

$$c_n \leq \sup_{u \in X \cap \mathcal{W}} I(u) \leq \sup_{u \in X} I(u)$$

and so, by the arbitrariness of X, $c_n \leq \inf_{A \in \Gamma_n^*} \sup_{u \in A} I(u)$. Finally, by Lemma 3.3 we have that, for every $X \in \Gamma_n^{\mathcal{W}}$, since $\Sigma_X \in \Gamma_n$,

$$\inf_{A \in \Gamma_n} \sup_{u \in A} I(u) \leq \sup_{u \in \Sigma_X} I(u) = \sup_{u \in X} I(u).$$

By the arbitrariness of X in $\Gamma_n^{\mathcal{W}}$ we have

$$\inf_{A \in \Gamma_n} \sup_{u \in A} I(u) \leq \inf_{A \in \Gamma_n^{\mathcal{W}}} \sup_{u \in A} I(u) = c_n. \qquad \square$$

Proposition 3.5. *Γ_n and Γ_n^* are two admissible min-max classes.*

Proof. Since $c_n > 0$, we can find a cut-off function φ such that $\varphi(c_n) = 1$ and $\varphi(s) = 0$ for $s \leq c_n - \overline{c}/3$ (see [9], [8]). By multiplying the gradient $\nabla I(u)$ by $\varphi(I(u))$ we obtain a cut-off gradient flow $\eta: \mathbb{R} \times H \to H$ which lets the points at level c_n move along the reverse gradient direction and leaves the points in L_n fixed. It follows that, setting $\eta_t: x \mapsto \eta(t, x)$, η_t is odd and if $\sigma \in \mathcal{F}_n$ and $\varphi \in \mathcal{F}_n^*$ then $\eta_t \circ \sigma \in \mathcal{F}_n$ and $\varphi \circ \eta_t \in \mathcal{F}_n^*$. So if $A \in \Gamma_n$ then $\eta_t(A) \in \Gamma_n$ and if $A \in \Gamma_n^*$ then $\eta_t(A) \in \Gamma_n^*$. $\square$

By the above results we can state that, for every n, the min-max levels of the classes in $\Gamma_n^{\mathcal{W}}$ are all critical levels provided they satisfy the Palais–Smale condition, regardless any regularity property of the constraint since they are levels of the unconstrained admissible min-max classes Γ_n and Γ_n^*. The corresponding critical points also have a characterization of unconstrained min-max points with the relative properties like, for instance, the direct estimates on the Morse index in the whole space. Moreover, the three classes are also equivalent from the point of view of the localization of the critical points near the maximum points of the terms of a minimizing sequence of sets, as stated in the next proposition.

Proposition 3.6. *If $(A_i)_{i \in \mathbb{N}}$ is any minimizing sequence in $\Gamma_n^{\mathcal{W}}$, i.e. for every $i \in \mathbb{N}$ $A_i \in \Gamma_n^{\mathcal{W}}$ and $\lim_i(\sup_{u \in A_i} I(u)) = c_n$, then the sequence of the double homothetic expansions $(\Sigma_{A_i})_{i \in \mathbb{N}}$ is a minimizing sequence in Γ_n (and so, in particular, in Γ_n^*) for i large.*

Proof. The assertion easily follows from Lemmas 3.3 and 3.4. $\square$

Proposition 3.7. *If $(A_i)_{i \in \mathbb{N}}$ is any minimizing sequence in Γ_n^* (and so, in particular, in Γ_n) then the sequence of the traces on $\mathcal{W}$ $(A_i \cap \mathcal{W})_{i \in \mathbb{N}}$ is a minimizing sequence in $\Gamma_n^{\mathcal{W}}$.*

Proof. The assertion easily follows from Lemmas 3.2 and 3.4. $\square$

It is worth to remark that the two previous propositions imply, in particular, that the admissible min-max classes Γ_n and Γ_n^* produce Palais–Smale sequences in $\mathcal{W}$.

4. Min-max classes on the natural constraint

As stated in the introduction, a similar (but surely simpler) construction can be also made for the natural constraint $\mathcal{V}$ defined in (1.2). In such a case, we do not need to assume $\lambda < \lambda_1$ but we only need to restrict our attention to the values of n such that the min-max level c_n, which we are going to define, is not zero. For any fixed n we introduce the class of sets

$$\Gamma_n^{\mathcal{V}} = \{A \subset \mathcal{V} \mid A \text{ is compact}, \ \gamma^*(A) \geq n\}$$

and denote by c_n the corresponding min-max level, that is

$$c_n = \inf_{A \in \Gamma_n^{\mathcal{V}}} \sup_{u \in A} I(u).$$

Let $k = \min\{n \in \mathbb{N} \mid c_n > 0\}$, we introduce the sets

$$\begin{aligned} C^- &= \{u \in H \setminus \{0\} \mid I(u) \leq c_k/2, \ \nabla I(u) \cdot u \geq 0\}, \\ C^+ &= \{u \in H \setminus \{0\} \mid I(u) \leq 0, \ \nabla I(u) \cdot u \leq 0\}. \end{aligned}$$

Now two constant ε and c as above such that $\varepsilon u \in C^-$ and $cu \in C^+$ can be fixed when $u \in \mathcal{V}$ (not for all u if $\lambda > \lambda_1$) and can be fixed uniformly for a compact set $A \subset \mathcal{V}$ and considered as a function of A. Given A, we introduce the function $\sigma_A \colon A \times Q_1 \to H$, defined as

$$\sigma_A(u, \alpha) = \left(\varepsilon + (\alpha + 1)\frac{c - \varepsilon}{2}\right)u,$$

satisfying the conditions

$$\sigma_A(u, -1) = \varepsilon u, \quad \sigma_A(u, 1) = cu, \quad \text{for all } u \in A.$$

Then the set $\Sigma_A = \sigma_A(A \times Q_1)$ will be named *simple homothetic expansion of* A. Let us notice that if $A \in \Gamma_n^{\mathcal{V}}$ then σ_A belongs to the class of continuous functions defined by

$$\begin{aligned} \mathcal{F}_n = \{\sigma \colon A \times Q_1 \to H \mid A \in \Gamma_n^{\mathcal{V}}, \ &\text{for all } u \in A \\ &\text{such that } \sigma(-u, \alpha) = -\sigma(u, \alpha), \ \sigma(u, -1) \in C^-, \ \sigma(u, 1) \in C^+\} \end{aligned}$$

and Σ_A belongs to the class of sets defined by

$$\begin{aligned} \Gamma_n = \{X \subset H \setminus \{0\} \mid \text{there exists } A \in \Gamma_n^{\mathcal{V}}, \ &\text{and exists } \sigma \colon A \times Q_1 \to H, \sigma \in \mathcal{F}_n \\ &\text{such that } X = \sigma(A \times Q_1)\}. \end{aligned}$$

As in the previous case, we set

$$\begin{aligned}\mathcal{F}^* = &\{\varphi: H \to \mathbb{R} \mid \varphi(-u) = \varphi(u), \varphi(u) \le 0 \text{ for all } u \in C^-,\\ &\qquad\qquad\qquad\qquad\qquad \varphi(u) \ge 0 \text{ for all } u \in C^+\},\\ \Gamma_n^* = &\{X \subset H \setminus \{0\} \mid X \text{ is compact},\\ &\qquad\qquad X = -X,\ \gamma^*(X \cap \varphi^{-1}(0)) \ge n \text{ for all } \varphi \in \mathcal{F}^*\}.\end{aligned}$$

Thus, by using the above variants of the previous notation, we can state, also for the case of the natural constraint, the analogous results as in the previous section which can be formally stated and proved in the same way and allow to characterize the critical points arising from min-max levels of the classes in $\Gamma_n^{\mathcal{V}}$ as unconstrained min-max points.

5. A topological lemma

THEOREM 5.1. *Let $f: B_n \times Q_k \to \mathbb{R}^{n+k}$ be a continuous map symmetric on $\partial B_n \times Q_k$ (i.e. such that, for some $\pi \in \mathcal{P}_k$, $f(x) = f_\pi(x)$, for all $x \in S^{n-1} \times Q_k$) and assume that* (T) *holds for $\varphi = f$ and $A = B_n$. Then there exists $x \in B_n \times Q_k$ such that $f(x) = 0$.*

PROOF. The first step consists in introducing a suitable change of variables which will allow to deal with the symmetry properties involved in the most convenient way.

For given $k \in \mathbb{N}$ and $\pi \in \mathcal{P}_k$, let us introduce the sets

$$I_0 = \{i \in I_k \mid i = \pi(i)\}, \quad I_1 = \{i \in I_k \mid i < \pi(i)\}.$$

We note that, if $k_i = \sharp I_i$, $k_0 + 2k_1 = k$. Let us introduce the functions p_1 and p_2, both defined on $\mathbb{R}^k$, as

$$p_1(x) = (x_i + x_{\pi(i)})_{i \in I_1} + (x_i)_{i \in I_0}, \quad p_2(x) = (x_i - x_{\pi(i)})_{i \in I_1}.$$

We have $\mathbb{R}^k \cong p_1(\mathbb{R}^k) \oplus p_2(\mathbb{R}^k)$. Then, for every $x \in B_n \times Q_k$, we set

$$\mathbf{x}_1 = (x_0, p_2(x')), \quad \mathbf{x}_2 = p_1(x')$$

and so we have $x = (x_0, x') \cong (\mathbf{x}_1, \mathbf{x}_2)$. Analogously, for any $f = (f_0, f') \in \mathbb{R}^{n+k}$, we define the functions

$$\mathbf{f}_1 = (f_0, p_2 \circ f'), \quad \mathbf{f}_2 = p_1 \circ f'$$

and so $f \cong (\mathbf{f}_1, \mathbf{f}_2)$. Now we observe that $\mathbf{x}_1 = 0$ if and only if $x_0 = 0$ and, for every $i \in I_k$, $x_i = x_{\pi(i)}$, that $(-\mathbf{x}_1, \mathbf{x}_2) \cong (-x_0, \widehat{\pi}(x')) = \widetilde{\pi}(x)$ and that, consequently, f π-symmetric means

$$f(-\mathbf{x}_1, \mathbf{x}_2) = \widetilde{\pi}(\varphi(\mathbf{x}_1, \mathbf{x}_2)) = (-\mathbf{f}_1(\mathbf{x}_1, \mathbf{x}_2), \mathbf{f}_2(\mathbf{x}_1, \mathbf{x}_2)). \tag{5.1}$$

We assume by contradiction that

$$0 \notin f(\partial(B_n \times Q_k)) = f\bigg((S^{n-1} \times Q_k) \cup \bigcup_{i=1}^{k}(B_n \times (F_i^+ \cup F_i^-))\bigg).$$

In such a case, we shall show that the topological degree of f in zero is different from zero, that is $\deg(B_n \times Q_k, f, 0) \neq 0$. The first step in this direction consists in forcing the assumption (T) to be satisfied with strict inequalities, by adding to f' the function $\varepsilon x'$ with $\varepsilon > 0$ suitably small in order to keep the value of the topological degree. We just remark that the function $g\colon (x_0, x') \mapsto (0, x')$ is π-symmetric, as it can be easily seen. Then, through a linear homotopy, we can pass from f to $(f + f_\pi)/2$, which we will continue to denote by f. By Lemma 2.3 we know that the symmetrization does not change the degree since f is not modified on $\partial B_n \times Q_k$ and f_i and $(f_\pi)_i$ have a fixed sign on $B_n \times F_i^{\pm}$.

By a standard perturbation argument we can also assume $f \in C^1(B_n \times Q_k, \mathbb{R}^{n+k})$. We set

$$X = \{x \in B_n \times Q_k \mid \mathbf{x}_1 = 0\}.$$

We shall introduce further modifications of f which make $f^{-1}(0) \cap X$ *contain only regular points*. Firstly we know that, by the oddness of $\mathbf{f}_1$ with respect to the variable $\mathbf{x}_1$ stated in (5.1), $\mathbf{f}_1 = 0$ on X and so the zeroes of f on X are the zeros of $\mathbf{f}_2$. Then we observe that, since after the previous symmetrization $\mathbf{f}_2$ is even with respect to the variable $\mathbf{x}_1$, so the partial Jacobian matrix $J_{\mathbf{x}_1}\mathbf{f}_2(x)$ is identically zero for every $x \in X$. Then, for any $x \in X$, by Laplace Rule we get

$$|Jf(x)| = |J_{\mathbf{x}_1}\mathbf{f}_1(x)||J_{\mathbf{x}_2}\mathbf{f}_2(x)|.$$

We can force $|J_{\mathbf{x}_2}\mathbf{f}_2| \neq 0$ on every $x \in X$ such that $\mathbf{f}_2(x) = 0$ by subtracting from $\mathbf{f}_2$ a small regular value $h \in p_1(\mathbb{R}^k)$, given by Sard Theorem. Note that this perturbation does not affect the symmetry properties of f and, if h is taken sufficiently small, does not change the value of the topological degree. In particular, we get that $f^{-1}(0) \cap X$ is a finite set and therefore the set L which contains all the eigenvalues of $J_{\mathbf{x}_1}\mathbf{f}_1(x)$ in the points of $f^{-1}(0) \cap X$ is also finite. Then we can force the determinant $|J_{\mathbf{x}_1}\mathbf{f}_1(x)|$ to be different from zero in such points by adding to $\mathbf{f}_1$ the function $-\lambda\mathbf{x}_1$ with $\lambda \in \mathbb{R} \setminus L$. Again, this new perturbation preserves the symmetry properties of f and, if λ is sufficiently small, does not change the value of $\deg(B_n \times Q_k, f, 0)$ and the zeros of $\mathbf{f}_2$ remain of course the same. So we can be sure that f has only regular zeros on X.

Now, since $\mathbf{f}_2$ satisfies the hypotheses of Miranda Theorem (see [7]) on $X \cong Q_{k_0+k_1}$, we can state that $f^{-1}(0) \cap X$ is composed by an odd number of regular zeroes. By continuity, we have that there exists a small $\varepsilon > 0$ such that the closed tubolar neighbourhood $X_\varepsilon = \{x \in B_n \times Q_k \mid d(x, X) \leq \varepsilon\}$ contains only regular zeros. In order to force 0 to be a regular value for f we have to deal with the set $(B_n \times Q_k) \setminus X_\varepsilon$ and, to this aim, we argue as follows.

Let $A_i^{\pm} = \{x \in B_n \times Q_k \mid \pm \mathbf{x}_1^i \geq \varepsilon/n\}$, where, for $i = 1, \ldots, n + k_1$, $\mathbf{x}_1^i$ is the component of $\mathbf{x}_1$ of index i. One can easily see that

$$(B_n \times Q_k) \setminus X_\varepsilon \subset \bigcup_{i=1}^{n+k_1} (A_i^+ \cup A_i^-).$$

We are going to perform a new perturbation of f, which keeps the symmetry properties and is too small to change the topological degree or to introduce singular zeroes in X_ε, in order to exclude the presence of singular zeroes of f also on $A_1^{\pm}$. Let S be the set of singular values of f. By Sard Theorem $S \cup \widetilde{\pi}(S)$ is a negligible set, so we can take $h_1 \in \mathbb{R}^{n+k} \setminus (S \cup \widetilde{\pi}(S))$ arbitrarily small. Let $g\colon \mathbb{R} \to [0, 1]$ be a smooth function such that $g(x) = 0$ for $x \leq 0$ and $g(x) = 1$ for $x \geq 1$. Let $\psi\colon \mathbb{R}^n \to \mathbb{R}^{n+k}$ be defined as

$$\psi(x) = g\bigg(\frac{n}{\varepsilon}\mathbf{x}_1^1\bigg)h_1 + g\bigg(-\frac{n}{\varepsilon}\mathbf{x}_1^1\bigg)\widetilde{\pi}(h_1).$$

One easily sees that $\psi(x) = h_1$ if $x \in A_1^+$ and $\psi(x) = \widetilde{\pi}(h_1)$ if $x \in A_1^-$. So the function $\overline{f}\colon x \mapsto f(x) - \psi(x)$ has no singular zeroes on $A_1^+ \cup A_1^-$. Moreover, $\overline{f}$ keeps the symmetry properties of f. The value of the degree and the regularity of the zeroes in X_ε are preserved by stability, provided h_1 is taken suitably small.

We proceed in this construction through $n+k_1$ steps, from $i = 1$ to $i = n+k_1$. Thanks to the stability property of the regular points, we are sure that at each step the regularity gained at the previous step on $X_\varepsilon \cup \bigcup_{j=1}^{i-1}(A_j^+ \cup A_j^-)$ is kept, provided the perturbation term h_i given by Sard Theorem is chosen sufficiently small. Therefore, we can conclude that we have regularity everywhere on $B_n \times Q_k$ and so 0 is a regular value for f. Finally, we know that $f^{-1}(0) \cap X$ is made by an odd number of points and, since $f^{-1}(0) = \widetilde{\pi}(f^{-1}(0))$, $f^{-1}(0) \setminus X$ is made by an even number of points, indeed $\widetilde{\pi}(x) \neq x$ for $x \notin X$. Then we can conclude that $\deg(B_n \times Q_k, f, 0)$ is odd and so we get the thesis. □

REMARK 5.2. It is worth to notice that the previous theorem reduces to Borsuk Theorem when $k = 0$ and to Miranda Theorem when $n = 0$ and $\pi = id$. If it is easy to see that Miranda Theorem can be deduced from Borsuk Theorem, nevertheless reconducing the above statement to Borsuk Theorem does not seem to be an obvious task.

REMARK 5.3. In the case $\pi = \mathrm{id}$ the above theorem takes a simpler form and the assumption f π-symmetric reduces to ask (S1) and (S2) for $x_0 \in \partial B_n$. The proof also becomes easier since $\mathbf{x}_1 = x_0$ and $\mathbf{x}_2 = x'$ in such a case. One may also wonder if (S2) is essential or if the thesis holds true under the only assumptions (S1) and (T). To this aim one can consider the following counterexample.

Let $n = k = 1$, we take $f(x)$ such that $f_0(x_0, x') = |x_0 + x'| - 1$, $f'(x_0, x') = x_0 + x'$. If $x_0 = \pm 1$ and $|x'| \leq 1$ we have $|x_0 + x'| = 1 \pm x'$ and so $f_0(\pm 1, x') = \pm x'$ as asked by (S1) and $\pm f'(x_0, x') \geq 0$ for $x' = \pm 1$ so (T) holds. Clearly f does not satisfy the thesis of the theorem.

Theorem 5.1 can be also stated in the following two forms.

COROLLARY 5.4. *Let $\varphi: S^n \times Q_k \to \mathbb{R}^{n+k}$ be a continuous mapping such that $\pm\varphi_i(x) \geq 0$ for all $x \in S^n \times F_\pm^i$ and for all $i = 1, \ldots, k$. Then for every $\pi \in \mathcal{P}_k$ there exists $x \in S^n \times Q_k$ such that $\varphi(x) = -\varphi_\pi(x)$.*

PROOF. Let us define $\psi: x \mapsto \varphi(x) + \varphi_\pi(x)$ for some π. We have that ψ is π-symmetric on $S^n \times Q_k$ and, by virtue of Lemma 2.3, ψ satisfies (T). Let $S_+^n = \{(x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} \geq 0\}$. S_+^n is homeomorphic to B_n and so, by Theorem 5.1, there exists $x \in S_+^n \times Q_k$ such that $\psi(x) = 0$, that is $\varphi(x) = -\varphi_\pi(x)$. □

COROLLARY 5.5. *Let $\varphi: S^n \times Q_k \to \mathbb{R}^{n+k}$, $\varphi \in \Lambda_n^*(S^n)$ be given. Then there exists $x \in S^n \times Q_k$ such that $\varphi(x) = 0$.*

PROOF. By applying Corollary 5.4 we have the existence of $x \in S^n \times Q_k$ such that $\varphi(x) = -\varphi_\pi(x) = -\varphi(x)$, that is $\varphi(x) = 0$. □

We can finally fill the gap in Section 2 by deducing Proposition 2.6 from the above statements.

PROOF OF PROPOSITION 2.6. We know that $\gamma^*(S^n) \leq \gamma(S^n) \leq n + 1$. On the other hand, if $\varphi \in \Lambda_n^*(S^n)$ then by virtue of Corollary 5.5 we have $0 \in \varphi(S^n \times Q_k)$ and so $n < \gamma^*(S^n)$. □

Corollary 5.5 is clearly equivalent to Proposition 2.6. The proof that it also implies Theorem 5.1 is trivial and is left to the reader.

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