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MULTIPLICITY OF SOLUTIONS FOR SOME ELLIPTIC EQUATIONS INVOLVING CRITICAL AND SUPERCRITICAL SOBOLEV EXPONENTS

Shujie Li — Zhaoli Liu

Abstract. We study multiplicity of solutions of the following elliptic problems in which critical and supercritical Sobolev exponents are involved:

$$\begin{aligned} -\Delta u &= g(x,u) + \lambda h(x,u) \quad \text{in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \\ -\operatorname{div}(|\nabla u|^{p-2}\nabla u) &= g(x,u) + \lambda h(x,u) \quad \text{in } \Omega \text{ and } u = 0 \text{ on } \partial\Omega, \end{aligned}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $p > 1$, λ is a parameter, and $\lambda h(x,u)$ is regarded as a perturbation term of the problems. Except oddness with respect to u in some cases, we do not assume any condition on h. For the first problem, we get a result on existence of three nontrivial solutions for $|\lambda|$ small in the case where g is superlinear and $\limsup_{|t|\to\infty} g(x,t)/|t|^{2^*-1}$ is suitably small. We also prove that the first problem has $2k$ distinct solutions for $|\lambda|$ small when g and h are odd and there are k eigenvalues between $\lim_{t\to 0} g(x,t)/t$ and $\lim_{|t|\to\infty} g(x,t)/t$. For the second problem, we prove that it has more and more distinct solutions as λ tends to 0 assuming that g and h are odd and g is superlinear and $\lim_{|t|\to\infty} g(x,t)/|t|^{p^*-1} = 0$.

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1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\partial\Omega$, say, in $C^{2,\beta}$ for some $0 < \beta < 1$. We will consider in this paper the semilinear elliptic boundary value problem

$$(\mathrm{P1})_\lambda \qquad \begin{cases} -\Delta u = g(x,u) + \lambda h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the quasilinear elliptic boundary value problem

$$(\mathrm{P2})_\lambda \qquad \begin{cases} -\Delta_p u = g(x,u) + \lambda h(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $1 < p < +\infty$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, g and $h\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous, λ is a parameter and $\lambda h(x,t)$ is regarded as a perturbation term. By a solution u of $(\mathrm{P1})_\lambda$ we mean a classical solution, that is, $u \in C^2(\overline{\Omega})$ and u satisfies $(\mathrm{P1})_\lambda$ pointwise. By a solution u of $(\mathrm{P2})_\lambda$ we mean a weak solution, that is, $u \in W_0^{1,p}(\Omega)$ and u satisfies $(\mathrm{P2})_\lambda$ in the distribution sense. We will see that all weak solutions of $(\mathrm{P2})_\lambda$ obtained in this paper are in $C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.

The problem $(\mathrm{P1})_\lambda$ has a wide background in concrete problems from other branches of science such as mathematical biology, chemistry and physics (see [12], [21], [22], [24]). The problem $(\mathrm{P2})_\lambda$ arise also naturally in geometry and mechanics; it has geometrical interest for $p \geq 2$ and arise in the theory of non-Newtonian fluids both for $p > 2$ and $1 < p < 2$ (see [11], [19], [32] and the references cited therein).

The problem $(\mathrm{P1})_\lambda$ with $\lambda = 1$ was studied as a perturbation problem from symmetry in [5]–[7], [23], [39], [41]. In these papers the term $h(x,u)$ is considered to be a perturbation term and is assumed, as a nonlinear function of u, to be very small compared with the term $g(x,u)$. Under the oddness condition $g(x,-t) = -g(x,t)$ imposed only on g infinitely many solutions of the problem $(\mathrm{P1})_\lambda$ were then proved to exist. Similar results were obtained for $(\mathrm{P2})_\lambda$ in [30]. In the present paper, we consider the problems $(\mathrm{P1})_\lambda$ and $(\mathrm{P2})_\lambda$ also as perturbation problems and from a different point of view. Without assuming any conditions on h (except continuity and oddness in some cases) we shall prove multiplicity results on solutions of these problems with $|\lambda|$ sufficiently small.

Critical and supercritical Sobolev exponents are involved in the following sense. In our first result, $g(x,t)$ is allowed to be $\alpha|t|^{4/(N-2)}t$ for $|t|$ large and $h(x,t)$ can be any Lipschitz continuous function. In our second and third results, $h(x,t)$ is only assumed to be Lipschitz continuous and odd in t.

Denote $p^* = Np/(N-p)$ if $1 < p < N$ and $p^* = +\infty$ if $p \geq N$. Our first result is about $(\mathrm{P1})_\lambda$. For stating it, we formulate some conditions as below.

(g_1) $(\mathrm{P1})_0$ has a strict sub-solution $\phi \in C_0^2(\overline{\Omega})$ and a strict super-solution $\psi \in C_0^2(\overline{\Omega})$ with $\phi < 0 < \psi$.

(g_2) There exist constants $M > 0$ and $\mu > 2$ such that

$$0 < \mu G(x,t) \leq t g(x,t), \quad x \in \overline{\Omega},\ |t| \geq M,$$

where $G(x,t) = \int_0^t g(x,s)\,ds$.

(g_3) $\beta_\infty \triangleq \limsup_{|t|\to\infty} |g(x,t)|/|t|^{q-1} < \infty$ uniformly in $x \in \overline{\Omega}$, where $q = 2^*$ if $N > 2$ and q is any positive number if $N = 2$.

Denote by $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ all the eigenvalues of $-\Delta$ with 0-Dirichlet boundary condition and by $e_1, e_2, \ldots$ the corresponding eigenfunctions, with the explicit meaning that each λ_i is counted as many times as its multiplicity. We also denote $\lambda_0 = -\infty$. Note that $\lambda_1 > 0$ is simple and e_1 does not change sign. We assume that $e_1 > 0$ in Ω and $\|e_j\|_{1,2} = 1$ for $j \in \mathbb{N}$. From (g_2), it is easy to see that there exists $M_1 > 0$ such that

$$G(x,t) \geq \frac{1}{2}\lambda_2 t^2, \quad x \in \overline{\Omega},\ |t| \geq M_1. \tag{1.1}$$

Now, we define two numbers as

$$m_1 = \max_{x\in\overline{\Omega},|t|\leq M_1} \left\{\frac{1}{2}\lambda_2 t^2 - G(x,t)\right\}, \tag{1.2}$$

$$m_2 = \max_{x\in\overline{\Omega},|t|\leq M} \left\{G(x,t) - \frac{1}{\mu} t g(x,t)\right\}. \tag{1.3}$$

It is obvious that, $m_1 \geq 0$, $m_2 \geq 0$, and

$$G(x,t) \geq \frac{1}{2}\lambda_2 t^2 - m_1, \qquad x \in \overline{\Omega},\ t \in \mathbb{R}, \tag{1.4}$$

$$G(x,t) \leq \frac{1}{\mu} t g(x,t) + m_2, \quad x \in \overline{\Omega},\ t \in \mathbb{R}. \tag{1.5}$$

Denote by S the best constant of the critical Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$. That is,

$$S = \inf_{\phi\in H_0^1(\Omega),\|\phi\|_{2^*}=1} \|\nabla\phi\|_2^2. \tag{1.6}$$

It is well known that S depends on N and is independent of Ω. See, for example, [46] for discussions on such numbers. Our first main result is that

Theorem 1.1. *Suppose that* (g_1)–(g_3) *are satisfied and, in the case $N > 2$,*

$$\left(\frac{m_1 + m_2}{2^{-1} - \mu^{-1}}\right)^{2/(N-2)} \beta_\infty < \frac{8NS^{N/(N-2)}}{(N+2)^2|\Omega|^{2/(N-2)}}.$$

Then, for any h, there exists $\overline{\lambda} = \overline{\lambda}(h) > 0$ such that, for all $|\lambda| \le \overline{\lambda}$. Problem $(\mathrm{P1})_\lambda$ *has at least three classical solutions u_1, u_2 and u_3 satisfying $u_1 > \phi$, $u_1 \not\le \psi$, $u_2 < \psi$, $u_2 \not\ge \phi$, $u_3 \not\le \psi$, and $u_3 \not\ge \phi$.*

In some cases, we can get additional information about the solutions. For example, we have the following corollary from Theorem 1.1.

Corollary 1.2. *Instead of assuming* (g_1), *we assume that*

$$\limsup_{t\to 0} \frac{g(x,t)}{t} < \lambda_1$$

uniformly in $x \in \overline{\Omega}$. Then the three solutions obtained in Theorem 1.1 *are such that u_1 is positive, u_2 is negative, and u_3 is sign-changing provided that $h(x,0) = 0$.*

Remark 1.3. In Theorem 1.1, we do not need any assumption except Lipschitz continuity of $h(x,t)$, so critical Sobolev exponent may be involved if $\lambda \ne 0$. This is also the case even if $\lambda = 0$ since β_∞ may be positive (see Example 1.4 below). The existence of a positive solution and a negative solution was obtained by Ambrosetti and Rabinowitz [3] with a Mountain Pass argument (see also [40]). The existence of a third nontrivial solution was first proved by Wang [44]. Later, many authors proved that the third nontrivial solution is a sign-changing one (see, for example, [8], [9], [16], [21], [22], [33], [34], [37]). We should emphasize that the nonlinearities in those papers were always assumed to be subcritical. Even in the special case $\lambda = 0$, Theorem 1.1 generalizes the results mentioned above. But Theorem 1.1 says much more.

The following two examples shows that the critical Sobolev exponent is involved even in the case $\lambda = 0$.

Example 1.4. Consider

$$g(x,t) = \begin{cases} \alpha_1 t^{(N+2)/(N-2)} + \gamma_1 t^{p_1} & \text{for } t \ge 0, \\ \alpha_2 |t|^{4/(N-2)} t + \gamma_2 |t|^{p_2-1} t & \text{for } t < 0, \end{cases}$$

where $\alpha_i > 0, \gamma_i > 0$ and $1 < p_i < (N+2)/(N-2)$ $(i = 1,2)$. It is easy to see that, for all p_1, p_2, γ_1, γ_2, there exists a number $\overline{\alpha} = \overline{\alpha}(p_1,p_2,\gamma_1,\gamma_2,N,\Omega) > 0$ such that if $\max\{\alpha_1,\alpha_2\} \le \overline{\alpha}$ then all conditions in Theorem 1.1 are satisfied by $g(x,t)$.

Example 1.5. Consider

$$g(x,t) = \begin{cases} \alpha_1 t^{(N+2)/(N-2)} / \ln(2+t) + \gamma_1 t^{p_1} & \text{for } t \ge 0, \\ \alpha_2 |t|^{4/(N-2)} t / \ln(2+|t|) + \gamma_2 |t|^{p_2-1} t & \text{for } t < 0. \end{cases}$$

In this case, for all $\alpha_i > 0, \gamma_i > 0$ and $1 < p_i < (N+2)/(N-2)$ $(i = 1,2)$, $g(x,t)$ satisfies the conditions in Theorem 1.1.

Next, we consider Problem $(\mathrm{P}1)_\lambda$ in the case where $g(x,t)$ is asymptotically linear at $t=0$ and $t=\infty$. The following conditions will be used.

- (g_4) $\lim_{t\to 0} g(x,t)/t=\alpha_0$ and $\lim_{|t|\to\infty} g(x,t)/t=\alpha_\infty$ uniformly in $x\in\overline{\Omega}$, where α_0 and α_∞ are real numbers.
- (g_5) $g(x,-t)=-g(x,t)$ for all $x\in\overline{\Omega}$ and $t\in\mathbb{R}$.
- (h) $h(x,-t)=-h(x,t)$ for all $x\in\overline{\Omega}$ and $t\in\mathbb{R}$.

Our second result reads as follows.

Theorem 1.6. *Assume* (g_4) *and* (g_5).

- (a) *If* $\lambda_i<\alpha_0<\lambda_{i+1}\le\lambda_{i+2}\le\ldots\le\lambda_{i+k}<\alpha_\infty<\lambda_{i+k+1}$, *for some* $i\in\mathbb{N}\cup\{0\}$ *and* $k\in\mathbb{N}$ *then, for any* h *satisfying* (h), *there exists* $\overline{\lambda}=\overline{\lambda}(h)>0$ *such that, for* $|\lambda|\le\overline{\lambda}$, $(\mathrm{P}1)_\lambda$ *possesses at least* k *pairs of distinct classical solutions with positive energy.*
- (b) *If* $\lambda_i<\alpha_\infty<\lambda_{i+1}\le\lambda_{i+2}\le\ldots\le\lambda_{i+k}<\alpha_0<\lambda_{i+k+1}$, *for some* $i\in\mathbb{N}\cup\{0\}$ *and* $k\in\mathbb{N}$ *then, for any* h *satisfying* (h), *there exists* $\overline{\lambda}=\overline{\lambda}(h)>0$ *such that, for* $|\lambda|\le\overline{\lambda}$, $(\mathrm{P}1)_\lambda$ *possesses at least* k *pairs of distinct classical solutions with negative energy.*

Remark 1.7. In the case $\lambda=0$, results in Theorem 1.6 are well known and was proved with index theory (see e.g. [3], [18], [20], [40]). Since h can be any Lipschitz continuous function satisfying (h), $\lim_{t\to 0} h(x,t)/t$ and $\lim_{|t|\to\infty} h(x,t)/t$ may not exist. But when $|\lambda|$ is small enough, $|\lambda h(x,t)/t|$ is very small in a suitably large interval of t and this is sufficient for $(\mathrm{P}1)_\lambda$ to have k pairs of distinct classical solutions. Therefore, Theorem 1.6 shows that, in the case $\lambda=0$, the assumption (g_4) is stronger than what needed to guarantee existence of k pairs of distinct classical solutions of $(\mathrm{P}1)_0$. Conditions may be imposed on g only for t in a suitably large interval.

Our third result is about $(\mathrm{P}2)_\lambda$ in which g and h are odd in u and g is superlinear. To state such a result, we need the following assumptions.

- (g_6) There exist constants $M>0$ and $\mu>p$ such that
$$0<\mu G(x,t)\le tg(x,t),\quad x\in\overline{\Omega},|t|\ge M.$$
- (g_7) $\lim_{|t|\to\infty} g(x,t)/|t|^{q-1}=0$ uniformly in $x\in\overline{\Omega}$, where $q=p^*$ if $1<p<N$ and q is any positive number if $p\ge N$.

Theorem 1.8. *Suppose that* (g_5)–(g_7) *are satisfied. Then, for any* h *satisfying* (h) *and any* $j\in\mathbb{N}$, *there exists* $\overline{\lambda}_j=\overline{\lambda}_j(h)>0$ *such that, when* $|\lambda|\le\overline{\lambda}_j$, *Problem* $(\mathrm{P}2)_\lambda$ *possesses at least* j *pairs of solutions with positive energy. Moreover, these solutions are in* $C^{1,\alpha}(\overline{\Omega})$ *for some* $0<\alpha<1$.

It is easy to see that, in the case $1<p<N$, $g(x,t)=\alpha|t|^{p^*-2}t/\ln(2+|t|)+\gamma|t|^{q-2}t$ with $\alpha>0$, $\gamma>0$ and $p<q<p^*$ satisfies (g_5)–(g_7). In Theorem 1.8,

the only assumption imposed on h is the oddness in t, so, for example, $h(x,t) = t^{2n-1}e^{t^{2m}+1}$ with n and m being positive integers is a legitimate function.

REMARK 1.9. Theorem 1.8 implies that, in the case $\lambda = 0$, there exist infinitely many solutions of $(\mathrm{P2})_0$ if (g_5)–(g_7) are satisfied. In the case $p = 2$, under (g_5)–(g_6) and a condition stronger than (g_7) (with $p = 2$), existence of infinitely many solutions of $(\mathrm{P1})_0$ was first proved in [3] (see also [40]).

REMARK 1.10. In a recent paper [17], Chabrowski and Yang studied problem $(\mathrm{P1})_\lambda$ with $\Omega = \mathbb{R}^N$, $g(x,u) = Q(x)|u|^{q-1} - u$ and $h(x,u) = R(x)|u|^{r-1}u$, where $2 < q < (N+2)/(N-2) \leq r$ and $N \geq 3$. They proved existence of one positive solution and variant multiplicity results when $\lambda > 0$ is small. They used truncation argument and their argument depends on the special feature of both the subcritical and the supercritical terms. We will also use truncation argument to get the results in the present paper. Our results exhibit such a phenomenon that for any function h, the number of solutions of $(\mathrm{P1})_\lambda$ and $(\mathrm{P2})_\lambda$ is closer and closer to that of $(\mathrm{P1})_0$ and $(\mathrm{P2})_0$ as λ goes to 0, respectively.

REMARK 1.11. In the case where g is sublinear and odd, $(\mathrm{P1})_1$ and $(\mathrm{P2})_1$ were studied by Wang [45] where h is also a perturbation term with respect to g. Under suitable conditions on g and h, he proved that $(\mathrm{P1})_1$ and $(\mathrm{P2})_1$ have infinitely many solutions with negative energy and converging to 0. In [45], some other kinds of sublinear problems were also studied.

In order to give further information on the comparison between Theorem 1.8 and known results in the literature, we state the following two corollaries as very special cases of Theorem 1.8.

COROLLARY 1.12. *Assume that $p < q < p^*$ and $r > p$. Then for any $j \in \mathbb{N}$, there exists $\overline{\lambda}_j > 0$ such that, when $\lambda \geq \overline{\lambda}_j$, the problem*

$$(\mathrm{P3})^\pm_\lambda \qquad \begin{cases} -\Delta_p u = \lambda|u|^{q-2}u \pm |u|^{r-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

possesses at least j pairs of solutions with positive energy.

REMARK 1.13. Problems like $(\mathrm{P3})^+_\lambda$ have been studied by many authors.

(a) For the p-Laplacian case, Garcia and Peral [30] proved that, among other results, if $1 < p < N$ and $p < q < r = p^*$ then there exists $\lambda_0 > 0$ such that $(P3)^+_\lambda$ has at least one nontrivial solution for $\lambda \geq \lambda_0$. Corollary 1.12 strengthens their result.
(b) In the case $p = 2$ and $2 \leq q < r = 2^*$, it was studied by Brezis and Nirenberg in their celebrated paper [15].
(c) In the case $p = 2$ and $1 < q < 2 < r \leq 2^*$, such a problem was studied by Ambrosetti, Brezis and Cerami in [2]. The results in [2] have been

extended and generalized later in e.g. [10] and [45]. For related results, see also [29] and [13].

(d) In the case $p = 2$ and $\max\{2, 2^* - 1\} < q < r = 2^*$ and Ω being the unit ball, existence of radially symmetric solutions of $(\mathrm{P3})^+_\lambda$ was studied in [4] and the Neumann problem was studied in [1].

COROLLARY 1.14. *Assume that $p < q < p^*$ and $r > q$. Then for any $j \in \mathbb{N}$, there exists $\overline{\lambda}_j > 0$ such that, when $\lambda \geq \overline{\lambda}_j$, the problem*

$$(\mathrm{P4})^\pm_\lambda \qquad \begin{cases} -\Delta_p u = \lambda(|u|^{q-2}u \pm |u|^{r-2}u) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases}$$

possesses at least j pairs of solutions with positive energy.

REMARK 1.15.

(a) Problems like $(\mathrm{P4})^\pm_\lambda$ in unbounded domains were studied in [25], [26] where the nonlinearity was assumed to be subcritical.

(b) In Corollaries 1.12 and 1.14, r can be any large numbers. Especially, the critical and supercritical cases $r = p^*$ and $r > p^*$ are included.

Roughly speaking, the above theorems give bifurcation phenomena for the solutions of elliptic problems. Nevertheless, it seems that they are not obtainable from known bifurcation theory as well as global theory for nonlinear eigenvalue problems (cf. [38]).

2. Proof of Theorem 1.1

We will use $\|\cdot\|_q$ to denote the norm of $L^q(\Omega)$ $(1 \leq q \leq \infty)$ and $\|\cdot\|_{m,q}$ to denote the norm of $W^{m,q}(\Omega)$ $(1 \leq q < \infty)$. We will only consider the case $N > 2$.

The proof is based on a truncation argument. This technique has been used successfully by many authors in dealing with superlinear elliptic problems (see, for example, [27], [28], [17], [36]). Different truncation functions should be adopted in dealing with different problems. The arguments in the papers just mentioned above do not suffice for the present purpose. We define the truncation functions as follows.

Let ϕ, ψ, μ, M and M_1 be as in (g_1), (g_2) and (1.1). Without loss of generality, we can assume that $2 < \mu < 2N/(N-2)$. Take an increasing sequence of numbers $\{t_n\}$ such that $t_n \uparrow \infty$ as $n \to \infty$ and $t_1 > \max\{\|\phi\|_{C(\overline{\Omega})}, \|\psi\|_{C(\overline{\Omega})}, M, M_1\}$. For $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, define

$$g_{n,\lambda}(x,t) = \begin{cases} g(x,t) + \lambda h(x,t) & \text{if } |t| \leq t_n, \\ g(x,t_n)(t/t_n)^{\mu-1} + \lambda h(x,t_n)\mu_n(t) & \text{if } t > t_n, \\ g(x,-t_n)(-t/t_n)^{\mu-1} + \lambda h(x,-t_n)\mu_n(-t) & \text{if } t < -t_n, \end{cases}$$

where

$$\mu_n(t) = \begin{cases} 1 + t_n - t & \text{if } t_n \le t \le t_n + 1, \\ 0 & \text{if } t > t_n + 1. \end{cases}$$

Define $G_{n,\lambda}(x,t) = \int_0^t g_{n,\lambda}(x,s)\,ds$. Then $G_{n,\lambda}(x,t)$ has the expression

$$G_{n,\lambda}(x,t) = \begin{cases} G(x,t) + \lambda H(x,t) & \text{if } |t| \le t_n, \\ G(x,t_n) + \lambda H(x,t_n) & \\ \qquad + g(x,t_n)a_n(t) + \lambda h(x,t_n)b_n(t) & \text{if } t > t_n, \\ G(x,-t_n) + \lambda H(x,-t_n) & \\ \qquad - g(x,-t_n)a_n(-t) - \lambda h(x,-t_n)b_n(-t) & \text{if } t < -t_n. \end{cases}$$

where $H(x,t) = \int_0^t h(x,s)\,ds$,

$$a_n(t) = \int_{t_n}^t \left(\frac{s}{t_n}\right)^{\mu-1} ds = \frac{t^\mu - t_n^\mu}{\mu t_n^{\mu-1}}, \quad t \ge t_n,$$

and

$$b_n(t) = \int_{t_n}^t \mu_n(s)\,ds = \begin{cases} (1 - (1 + t_n - t)^2)/2 & \text{if } t_n \le t \le t_n + 1, \\ 1/2 & \text{if } t > t_n + 1. \end{cases}$$

Take a number $\delta > 0$ such that

$$\left(\frac{m_1 + m_2 + 2\delta}{2^{-1} - \mu^{-1}}\right)^{2/(N-2)} (\beta_\infty + 2\delta) < \frac{8NS^{N/(N-2)}}{(N+2)^2|\Omega|^{2/(N-2)}}. \tag{2.1}$$

Now we give some lemmas which will be used in the sequel.

LEMMA 2.1. *There exists a constant $C > 0$ depending only on g with the property that, for any $n^* \in \mathbb{N}$, there exists a number $\lambda^* = \lambda^*(n^*, h) > 0$ such that, for all $n \in \{1, \ldots, n^*\}$, all $|\lambda| \le \lambda^*$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,*

$$|g_{n,\lambda}(x,t)| \le (\beta_\infty + \delta)|t|^{(N+2)/(N-2)} + C,$$

and

$$|g_{n,\lambda}(x,t)||t|^{N/2} \le (\beta_\infty + \delta)|t|^{(N+2)/(N-2)+N/2} + C.$$

PROOF. We only prove the second inequality since the proof for the first one is similar. By (g_3), there exists a constant $M_2 > 0$ such that, for all $x \in \overline{\Omega}$ and all $t \in \mathbb{R}$ with $|t| \ge M_2$,

$$|g(x,t)| \le (\beta_\infty + \delta)|t|^{(N+2)/(N-2)}.$$

Define a constant C_1 as

$$C_1 = \max_{x \in \overline{\Omega}, |t| \le M_2} |g(x,t)||t|^{N/2}.$$

Then, for all $x \in \overline{\Omega}$ and all $t \in \mathbb{R}$,

$$|g(x,t)||t|^{N/2} \le (\beta_\infty + \delta)|t|^{(N+2)/(N-2)+N/2} + C_1.$$

Now, we claim that, for all $n \in \mathbb{N}$, all $x \in \overline{\Omega}$ and all $t \in \mathbb{R}$,

$$|g_n(x,t)||t|^{N/2} \le (\beta_\infty + \delta)|t|^{(N+2)/(N-2)+N/2} + C_1, \tag{2.2}$$

where $g_n(x,t) = g_{n,0}(x,t)$. Clearly, we need only to prove (2.2) for $|t| > t_n$. Without loss of generality, we can assume that $t_1 \ge M_2$. Since $\mu - 1 < (N+2)/(N-2)$, for any $n \in \mathbb{N}$ and $x \in \overline{\Omega}$, if $|t| > t_n$ then

$$|g_n(x,t)| \le (\beta_\infty + \delta)t_n^{(N+2)/(N-2)}(|t|/t_n)^{\mu-1} \le (\beta_\infty + \delta)|t|^{(N+2)/(N-2)}.$$

Hence (2.2) is true. For any $n^* \in \mathbb{N}$, let $\lambda^* = \lambda^*(n^*, h) > 0$ be small such that

$$\lambda^*|h(x,t)||t|^{N/2} \le 1, \quad x \in \overline{\Omega},\ |t| \le t_{n^*}. \tag{2.3}$$

Then (2.2) and (2.3) together with the definition of $g_{n,\lambda}(x,t)$ yield, for all $n \in \{1, \ldots, n^*\}$, all $|\lambda| \le \lambda^*$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,

$$|g_{n,\lambda}(x,t)||t|^{N/2} \le (\beta_\infty + \delta)|t|^{(N+2)/(N-2)+N/2} + C_1 + 1.$$

Letting $C = C_1 + 1$, we complete the proof. □

LEMMA 2.2. *Let m_1 be defined as in* (1.2). *Then, for any $n^* \in \mathbb{N}$, there exists a number $\lambda^* = \lambda^*(n^*, h) > 0$ such that, for all $n \in \{1, \ldots, n^*\}$, all $|\lambda| \le \lambda^*$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,*

$$G_{n,\lambda}(x,t) \ge \frac{1}{2}\lambda_2 t^2 - m_1 - \delta.$$

PROOF. We first claim that, for all $n \in \mathbb{N}$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,

$$G_n(x,t) \ge \frac{1}{2}\lambda_2 t^2 - m_1, \tag{2.4}$$

where $G_n(x,t) = G_{n,0}(x,t)$. Indeed, if $|t| \le t_n$ then (1.4) yields (2.4). If $t > t_n$ then, from (g_2), (1.1) and the fact that $t_n \ge \max\{M, M_1\}$, we have

$$\begin{aligned} G_n(x,t) &= G(x,t_n) + \frac{t_n}{\mu} g(x,t_n)\left(\left(\frac{t}{t_n}\right)^{\mu} - 1\right) \\ &\ge G(x,t_n)\left(\frac{t}{t_n}\right)^{\mu} \ge \frac{1}{2}\lambda_2 t_n^2 \left(\frac{t}{t_n}\right)^{\mu} \ge \frac{1}{2}\lambda_2 t^2. \end{aligned}$$

Similarly, we have $G_n(x,t) \ge \lambda_2 t^2/2$ for $t < -t_n$. Therefore, 2.4 is valid. For any $n^* \in \mathbb{N}$, let $\lambda^* = \lambda^*(n^*, h) > 0$ be small such that

$$\lambda^*(|H(x,t)| + |h(x,t)|) \le \delta, \quad x \in \overline{\Omega},\ |t| \le t_{n^*}. \tag{2.5}$$

Combining (2.4) and (2.5), we see that, for any $n^* \in \mathbb{N}$, there exists a number $\lambda^* = \lambda^*(n^*, h) > 0$ such that, for all $n \in \{1, \ldots, n^*\}$, all $|\lambda| \leq \lambda^*$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,

$$G_{n,\lambda}(x,t) \geq \frac{1}{2}\lambda_2 t^2 - m_1 - \delta. \qquad \square$$

LEMMA 2.3. *Let m_2 be defined as in* (1.3)*. Then, for any $n^* \in \mathbb{N}$, there exists a number $\lambda^* = \lambda^*(n^*, h) > 0$ such that, for all $n \in \{1, \ldots, n^*\}$, all $|\lambda| \leq \lambda^*$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,*

$$G_{n,\lambda}(x,t) \leq \frac{1}{\mu} t g_{n,\lambda}(x,t) + m_2 + \delta.$$

PROOF. We first claim that, for all $n \in \mathbb{N}$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,

$$G_n(x,t) \leq \frac{1}{\mu} t g_n(x,t) + m_2. \tag{2.6}$$

In fact, if $|t| \leq t_n$ then (2.6) is just from (1.5). If $t > t_n$ then, in view of (g_2) and the fact that $t_n \geq M$, we have

$$\begin{aligned} G_n(x,t) &= G(x,t_n) + \frac{1}{\mu} t_n g(x,t_n)\bigg(\bigg(\frac{t}{t_n}\bigg)^{\mu} - 1\bigg) \\ &\leq \frac{1}{\mu} t_n g(x,t_n) + \frac{1}{\mu} t_n g(x,t_n)\bigg(\bigg(\frac{t}{t_n}\bigg)^{\mu} - 1\bigg) = \frac{1}{\mu} t g_n(x,t). \end{aligned}$$

If $t < -t_n$, we have $G_n(x,t) \leq t g_n(x,t)/\mu$ in a similar way. Hence (2.6) is true. For any $n^* \in \mathbb{N}$, let $\lambda^* = \lambda^*(n^*, h) > 0$ be small such that

$$\lambda^*\Big(|H(x,t)| + |h(x,t)| + \frac{1}{\mu}|t||h(x,t)|\Big) \leq \delta, \quad x \in \overline{\Omega},\ |t| \leq t_{n^*}. \tag{2.7}$$

Combining (2.6) and (2.7), we see that, for any $n^* \in \mathbb{N}$, there exists a number $\lambda^* = \lambda^*(n^*, h) > 0$ such that, for all $n \in \{1, \ldots, n^*\}$, all $|\lambda| \leq \lambda^*$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,

$$G_{n,\lambda}(x,t) \leq \frac{1}{\mu} t g_{n,\lambda}(x,t) + m_2 + \delta. \qquad \square$$

For proving Theorem 1.1, we still need to recall some notations. Denote $H = H_0^1(\Omega)$, $X = C_0^1(\overline{\Omega})$. For $u_1, u_2 \in X$, we denote $u_1 \ll u_2$ if $u_1(x) < u_2(x)$ for $x \in \Omega$ and $(\partial u_1/\partial \nu)(x) > (\partial u_2/\partial \nu)(x)$ for $x \in \partial\Omega$, where ν is the outward normal at $x \in \partial\Omega$.

Now, we are ready to prove Theorem 1.1.

PROOF OF THEOREM 1.1. For every $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, consider the boundary value problem

$$\begin{cases} -\Delta u = g_{n,\lambda}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8$_{n,\lambda}$}$$

whose solutions correspond to critical points of

$$J_{n,\lambda}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G_{n,\lambda}(x,u), \quad u \in H.$$

It is well known that, due to the definition of $g_{n,\lambda}(x,t)$, $J_{n,\lambda}$ satisfies (PS) condition on H for each $n \in \mathbb{N}$ and each $\lambda \in \mathbb{R}$. Choose a number $m = m(n,\lambda) > 0$ such that $g_{n,\lambda}(x,t) + mt$ is strictly increasing in $t \in \mathbb{R}$. Since ϕ and ψ are a strict sub- and a strict super-solution of $(\mathrm{P1})_0$ respectively, we have, by strong maximum principle,

$$\phi \ll (-\Delta + m)^{-1}(g(\,\cdot\,,\phi) + m\phi)$$

and

$$\psi \gg (-\Delta + m)^{-1}(g(\,\cdot\,,\psi) + m\psi).$$

Then it is easy to see that there exists $\widetilde{\lambda} = \widetilde{\lambda}(h) > 0$ such that, for all $|\lambda| \le \widetilde{\lambda}$,

$$\phi \ll (-\Delta + m)^{-1}(g(\,\cdot\,,\phi) + \lambda h(\,\cdot\,,\phi) + m\phi) \tag{2.9}$$

and

$$\psi \gg (-\Delta + m)^{-1}(g(\,\cdot\,,\psi) + \lambda h(\,\cdot\,,\psi) + m\psi). \tag{2.10}$$

Since $t_1 > \max\{\|\phi\|_{C(\overline{\Omega})}, \|\psi\|_{C(\overline{\Omega})}\}$, ϕ and ψ are a strict sub- and a strict super-solution of $(2.8)_{n,\lambda}$ for all $n \in \mathbb{N}$ and all $|\lambda| \le \widetilde{\lambda}$, respectively. Taking the following equivalent norm of H

$$\|u\|_{1,2}^2 = \int_\Omega (|\nabla u|^2 + mu^2),$$

we have by a direct computation

$$J'_{n,\lambda}(u) = u - (-\Delta + m)^{-1}(g_{n,\lambda}(\,\cdot\,,u) + mu), \quad u \in H.$$

Denote

$$A_{n,\lambda}u = (-\Delta + m)^{-1}(g_{n,\lambda}(\,\cdot\,,u) + mu), \quad u \in H.$$

Then $A_{n,\lambda}$ is Lipschitz continuous from H to H as well as from X to X. Let $u_0 \in X$ and consider the following initial value problem both in H and in X:

$$\begin{cases} \dfrac{du(t)}{dt} = -u(t) + A_{n,\lambda}u(t) & \text{for } t \ge 0, \\ u(0) = u_0. \end{cases}$$

Let $u(t,u_0)$ (resp. $\widetilde{u}(t,u_0)$) be the unique solution with maximal interval of existence $[0,\eta(u_0))$ (resp. $[0,\widetilde{\eta}(u_0))$) in H (resp. X). By [37, Lemma 4.2], $\widetilde{\eta}(u_0) = \eta(u_0)$, $\widetilde{u}(t,u_0) = u(t,u_0)$ for $0 \le t < \eta(u_0)$, and if $\lim_{t\to\eta(u_0)} u(t,u_0) = u^*$ in the H topology for some $u^* \in K_{n,\lambda}$, the critical sets of $J_{n,\lambda}$, then $\lim_{t\to\eta(u_0)} u(t,u_0) = u^*$ in the X topology. Define

$$D_1 = \{u \in X \mid u \gg \phi\}, \quad D_2 = \{u \in X \mid u \ll \psi\}.$$

Then D_1 and D_2 are open convex subsets of X and

$$D_1 \cap D_2 = \{u \in X \mid \phi \ll u \ll \psi\} \neq \emptyset.$$

As in [37], we use $\partial_X D$ and $\overline{D}^X$ to denote the boundary and the closure of D relative to X. From (2.9), (2.10) and the increasing of the operator $A_{n,\lambda}$, we get that

$$A_{n,\lambda}(\partial_X D_i) \subset D_i, \quad i = 1, 2.$$

By the proof of [37, Theorem 4.1], there exists a path $h\colon [0,1] \to X$ such that

$$h(0) \in D_1 \setminus D_2, \quad h(1) \in D_2 \setminus D_1,$$

and

$$\inf_{u \in \overline{D}_1^X \cap \overline{D}_2^X} J_{n,\lambda}(u) > \max_{t \in [0,1]} J_{n,\lambda}(h(t)).$$

According to [37, Theorem 3.3], $J_{n,\lambda}$ has three critical points $u_{n,\lambda,1} \in D_1 \setminus \overline{D}_2{}^X$, $u_{n,\lambda,2} \in D_2 \setminus \overline{D}_1{}^X$ and $u_{n,\lambda,3} \in X \setminus (\overline{D}_1{}^X \cup \overline{D}_2{}^X)$. So, for all $n \in \mathbb{N}$ and all $|\lambda| \leq \widetilde{\lambda}$, $(2.8)_{n,\lambda}$ has three solutions $u_{n,\lambda,1}, u_{n,\lambda,2}$ and $u_{n,\lambda,3}$ satisfying $u_{n,\lambda,1} > \phi, u_{n,\lambda,1} \not\leq \psi, u_{n,\lambda,2} < \psi, u_{n,\lambda,2} \not\geq \phi, u_{n,\lambda,3} \not\leq \psi$, and $u_{n,\lambda,3} \not\geq \phi$. By the proof of [37, Theorem 3.3], $u_{n,\lambda,1}, u_{n,\lambda,2}$ and $u_{n,\lambda,3}$ are minimizers of $J_{n,\lambda}$ on $\partial_X C_X(D_1 \cap D_2) \cap \overline{D}_1{}^X$, $\partial_X C_X(D_1 \cap D_2) \cap \overline{D}_2{}^X$ and $\partial_X C_X(D_1 \cap D_2) \setminus (C_X(D_1) \cup C_X(D_2))$, respectively, where $C_X(D_1 \cap D_2)$ is the set of points u_0 in X for which there exists $0 < t < \eta(u_0)$ such that $u(t, u_0) \in D_1 \cap D_2$. Since $\phi < 0 < \psi$, the maximum principle implies $\phi \ll 0 \ll \psi$. So,

$$0 \in D_1 \cap D_2 \subset C_X(D_1 \cap D_2).$$

Therefore, each of the three sets $\partial_X C_X(D_1 \cap D_2) \cap \overline{D}_1{}^X$, $\partial_X C_X(D_1 \cap D_2) \cap \overline{D}_2{}^X$ and $\partial_X C_X(D_1 \cap D_2) \setminus (C_X(D_1) \cup C_X(D_2))$ intersects with $\operatorname{span}\{e_1, e_2\}$ (see [37] for details). Then we have, for $i = 1, 2, 3$,

$$J_{n,\lambda}(u_{n,\lambda,i}) \leq \sup_{u \in \operatorname{span}\{e_1, e_2\}} J_{n,\lambda}(u).$$

By Lemma 2.2 for all $n \in \{1, \ldots, n^*\}$ and $|\lambda| \leq \lambda^*$, if $u \in \operatorname{span}\{e_1, e_2\}$ then

$$J_{n,\lambda}(u) \leq \int_\Omega \bigg(\frac{1}{2}|\nabla u|^2 - \frac{1}{2}\lambda_2 u^2 + m_1 + \delta\bigg) \leq (m_1 + \delta)|\Omega|.$$

So, we arrive at, for all $n \in \{1, \ldots, n^*\}$ and $|\lambda| \leq \lambda^* := \min\{\lambda^*(n^*, h), \widetilde{\lambda}(h)\}$,

$$J_{n,\lambda}(u_{n,\lambda,i}) \leq (m_1 + \delta)|\Omega|, \quad i = 1, 2, 3, \tag{2.11}$$

here and in the sequel, we use λ^* to represent variant constants depending only on n^* and h. We need to prove that each $u_{n,\lambda,i}$ $(i = 1, 2, 3)$ is a solution of $(\mathrm{P}1)_\lambda$ when n is large and λ is small. For the sake of brevity, we omit the subscript i and denote $u_{n,\lambda,i}$ by $u_{n,\lambda}$ for $i = 1, 2, 3$. Now, we prove the existence of a constant $\widetilde{C}$ with the property that, for any $n^* \in \mathbb{N}$, there exists $\lambda^* > 0$ such that, for all $n \in \{1, \dots, n^*\}$ and all $|\lambda| \le \lambda^*$,

$$\|u_{n,\lambda}\|_\infty \le \widetilde{C}. \tag{2.12}$$

By Lemma 2.3, we see that, for all $n^* \in \mathbb{N}$, if $n \in \{1, \dots, n^*\}$ and $|\lambda| \le \lambda^*$,

$$J_{n,\lambda}(u_{n,\lambda}) \ge \frac{1}{2}\int_\Omega |\nabla u_{n,\lambda}|^2 - \frac{1}{\mu}\int_\Omega u_{n,\lambda} g_{n,\lambda}(x, u_{n,\lambda}) - (m_2 + \delta)|\Omega|.$$

Since $u_{n,\lambda}$ is a solution of $(2.8)_{n,\lambda}$, it follows that, for all $n^* \in \mathbb{N}$, if $n \in \{1, \dots, n^*\}$ and $|\lambda| \le \lambda^*$,

$$J_{n,\lambda}(u_{n,\lambda}) \ge \left(\frac{1}{2} - \frac{1}{\mu}\right)\int_\Omega |\nabla u_{n,\lambda}|^2 - (m_2 + \delta)|\Omega|. \tag{2.13}$$

Combining (2.11) and (2.13), we have, for all $n^* \in \mathbb{N}$, if $n \in \{1, \dots, n^*\}$ and $|\lambda| \le \lambda^*$,

$$\|u_{n,\lambda}\|_{1,2}^2 \le (2^{-1} - \mu^{-1})^{-1}(m_1 + m_2 + 2\delta)|\Omega|. \tag{2.14}$$

To get a $L^\infty(\Omega)$ bound for $u_{n,\lambda}$, we use a technique from [14] (see also [28]). Multiplying $(2.8)_{n,\lambda}$ with $|u_{n,\lambda}|^{N/2}\operatorname{sign} u_{n,\lambda}$ and integrating by parts, we have, for all $n \in \mathbb{N}$ and all $|\lambda| \le \lambda^*$,

$$\int_\Omega |\nabla |u_{n,\lambda}|^{(N+2)/4}|^2 = \tau_N \int_\Omega g_{n,\lambda}(x, u_{n,\lambda})|u_{n,\lambda}|^{N/2}\operatorname{sign} u_{n,\lambda},$$

where $\tau_N = (N+2)^2/(8N)$. According to Lemma 2.1, for any $n^* \in \mathbb{N}$, there exists $\lambda^* > 0$ such that, for all $n \in \{1, \dots, n^*\}$ and all $|\lambda| \le \lambda^*$,

$$\int_\Omega |\nabla |u_{n,\lambda}|^{(N+2)/4}|^2 \le \tau_N(\beta_\infty + \delta)\int_\Omega |u_{n,\lambda}|^{4/(N-2)+(N+2)/2} + C|\Omega|\tau_N.$$

Using Hölder inequality, (1.6) and (2.14), we estimate as follows, for all $n \in \{1, \dots, n^*\}$ and all $|\lambda| \le \lambda^*$,

$$\begin{aligned}
\int_\Omega |\nabla |u_{n,\lambda}|^{(N+2)/4}|^2 &\le \tau_N(\beta_\infty + \delta)\bigg(\int_\Omega |u_{n,\lambda}|^{2N/(N-2)}\bigg)^{2/N} \\
&\quad \cdot \bigg(\int_\Omega |u_{n,\lambda}|^{N(N+2)/(2(N-2))}\bigg)^{(N-2)/N} + C|\Omega|\tau_N \\
&\le \tau_N(\beta_\infty + \delta)\frac{1}{S^{2/(N-2)+1}}\|u_{n,\lambda}\|_{1,2}^{4/(N-2)}\||u_{n,\lambda}|^{(N+2)/4}\|_{1,2}^2 + C|\Omega|\tau_N \\
&\le \tau_N(\beta_\infty + \delta)\frac{1}{S^{N/(N-2)}}\left(\frac{(m_1 + m_2 + 2\delta)|\Omega|}{2^{-1} - \mu^{-1}}\right)^{2/(N-2)} \\
&\quad \cdot \||u_{n,\lambda}|^{(N+2)/4}\|_{1,2}^2 + C|\Omega|\tau_N.
\end{aligned}$$

The last inequality combined with (2.1) yields, for all $n \in \{1, \ldots, n^*\}$ and all $|\lambda| \leq \lambda^*$,

$$\||u_{n,\lambda}|^{(N+2)/4}\|_{1,2}^2 \leq \frac{\beta_\infty + \delta}{\beta_\infty + 2\delta}\||u_{n,\lambda}|^{(N+2)/4}\|_{1,2}^2 + C|\Omega|\tau_N.$$

This implies that a constant $C_1 > 0$ exists with the property that, for all $n^* \in \mathbb{N}$, there exists $\lambda^* > 0$ such that, for all $n \in \{1, \ldots, n^*\}$ and all $|\lambda| \leq \lambda^*$,

$$\||u_{n,\lambda}|^{(N+2)/4}\|_{1,2} \leq C_1,$$

and hence, by Sobolev inequality, $\|u_{n,\lambda}\|_{N(N+2)/2(N-2)} \leq C_2$, for some constant $C_2 > 0$ independent of n and λ. By Lemma 2.1 again, a constant $C_3 > 0$ exists with the property that, for all $n^* \in \mathbb{N}$, there exists $\lambda^* > 0$ such that, for all $n \in \{1, \ldots, n^*\}$ and all $|\lambda| \leq \lambda^*$, $\|g_{n,\lambda}(x, u_{n,\lambda})\|_{N/2} \leq C_3$. According to L^p theory of linear elliptic equations, a constant $C_4 > 0$ exists with the property that, for all $n^* \in \mathbb{N}$, there exists $\lambda^* > 0$ such that, for all $n \in \{1, \ldots, n^*\}$ and all $|\lambda| \leq \lambda^*$, $\|u_{n,\lambda}\|_{2,N/2} \leq C_4$. Since $W^{2,N/2}(\Omega) \hookrightarrow L^\infty(\Omega)$, a constant $\widetilde{C} > 0$ exists with the property that, for any $n^* \in \mathbb{N}$, there exists $\lambda^* > 0$ such that (2.12) is valid for all $n \in \{1, \ldots, n^*\}$ and all $|\lambda| \leq \lambda^*$. Since $\widetilde{C}$ is independent of $n^* \in \mathbb{N}$ and λ, we can choose a number $n^* \in \mathbb{N}$ such that $t_{n^*} > \widetilde{C}$. For such an n^*, let $\overline{\lambda} = \overline{\lambda}(h) := \lambda^*(n^*, h)$. Then, for all $|\lambda| \leq \overline{\lambda}$,

$$\|u_{n^*,\lambda}\|_\infty \leq \widetilde{C} < t_{n^*}$$

and $u_{n^*,\lambda}$ is a solution of $(\mathrm{P}1)_\lambda$. The proof is finished. □

Proof of Corollary 1.2. Since $\limsup_{t\to 0} g(x,t)/t < \lambda_1$ uniformly in $x \in \overline{\Omega}$ and $h(x, 0) = 0$, there exist $\delta > 0$ and $\widetilde{\lambda} > 0$ such that, for all $x \in \overline{\Omega}$, $0 < |t| \leq \delta$, and $|\lambda| \leq \widetilde{\lambda}$,

$$t^{-1}(g(x,t) + \lambda h(x,t)) < \lambda_1.$$

This implies that $\phi = -\delta^* e_1$ and $\psi = \delta^* e_1$ are a strict sub- and a strict super-solution of $(\mathrm{P}1)_\lambda$ provided $|\lambda| \leq \widetilde{\lambda}$, where $\delta^* > 0$ is such that $\delta^*\|e_1\|_\infty < \delta$. Let u_1, u_2 and u_3 be the three solutions obtained in Theorem 1.1. Then $u_1 > \phi, u_1 \not\leq \psi$, $u_2 < \psi$, $u_2 \not\geq \phi$, $u_3 \not\leq \psi$, and $u_3 \not\geq \phi$. Clearly, u_3 is a sign-changing solution. Now we prove that u_1 is a positive solution. Denote $\Omega^* = \{x \in \Omega \mid u_1(x) < 0\}$. If $\Omega^* \neq \emptyset$ then

$$\begin{cases} -\Delta u_1(x) = g(x, u_1(x)) + \lambda h(x, u_1(x)) > \lambda_1 u_1(x) & \text{for } x \in \Omega^*, \\ u_1(x) = 0 & \text{for } x \in \partial\Omega^*. \end{cases}$$

Let λ_1^* be the first eigenvalue of

$$-\Delta u = \lambda u \quad \text{in } \Omega^*; \qquad u = 0 \quad \text{on } \partial\Omega^*$$

with the positive eigenfunction e_1^*. Then $\lambda_1^* \geq \lambda_1$. Multiplying the last inequality with e_1^* and taking integral, we have

$$\lambda_1^* \int_{\Omega_1^*} u_1 e_1^* > \lambda_1 \int_{\Omega_1^*} u_1 e_1^*,$$

which is a contradiction. So, u_1 is a positive solution. Similarly, u_2 is a negative solution. $\square$

3. Proof of Theorem 1.6

In this section, we prove Theorem 1.6. The two cases (a) and (b) in Theorem 1.6 will be handled separately. Let $\Sigma = \{A \subset H\backslash\{0\} \mid A \text{ is closed and } A = -A\}$. We use $\gamma(A)$ to denote the genus of A (see [40]). We define a new kind of truncation functions to fulfill our task. Choose a sequence of positive numbers $\{t_n\}$ with the property that $t_n \uparrow +\infty$ and define, for $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$,

$$g_{n,\lambda}(x,t) = \begin{cases} g(x,t) + \lambda h(x,t) & \text{for } |t| \leq t_n, \\ g(x,t) + \lambda h(x,t_n) & \text{for } t > t_n, \\ g(x,t) + \lambda h(x,-t_n) & \text{for } t < -t_n. \end{cases}$$

Lemma 3.1. *There are constants $C > 0$, $\delta > 0$ and $\overline{\delta} > 0$ with the property that, for any $n^* \in \mathbb{N}$, there exists $\lambda^* = \lambda^*(n^*, h) > 0$ such that, for all $n \in \{1, \ldots, n^*\}$, all $|\lambda| \leq \lambda^*$, and all $x \in \overline{\Omega}$,*

- *in case* (a),

$$\left| g_{n,\lambda}(x,t) - \frac{\lambda_{i+k} + \lambda_{i+k+1}}{2} t \right| \leq \left(\frac{\lambda_{i+k+1} - \lambda_{i+k}}{2} - \delta \right) |t| + C, \quad t \in \mathbb{R}, \tag{3.1}$$

$$\lambda_i + \delta < \frac{g_{n,\lambda}(x,t)}{t} < \lambda_{i+1} - \delta, \quad 0 < |t| \leq \overline{\delta}, \tag{3.2}$$

- *and, in case* (b),

$$\left| g_{n,\lambda}(x,t) - \frac{\lambda_i + \lambda_{i+1}}{2} t \right| \leq \left(\frac{\lambda_{i+1} - \lambda_i}{2} - \delta \right) |t| + C, \quad t \in \mathbb{R}, \tag{3.3}$$

$$\lambda_{i+k} + \delta < \frac{g_{n,\lambda}(x,t)}{t} < \lambda_{i+k+1} - \delta, \quad 0 < |t| \leq \overline{\delta}. \tag{3.4}$$

Proof. We give the proof only in case (a) since it is similar in case (b). Take a $\delta > 0$ such that

$$\lambda_{i+k} + 2\delta < \alpha_\infty < \lambda_{i+k+1} - 2\delta \quad \text{and} \quad \lambda_i + 2\delta < \alpha_0 < \lambda_{i+1} - 2\delta.$$

By (g_4), there exist $C_1 > 0$ and $\overline{\delta} > 0$ such that, for all $x \in \overline{\Omega}$ and all $t \in \mathbb{R}$,

$$\left| g(x,t) - \frac{\lambda_{i+k} + \lambda_{i+k+1}}{2} t \right| \leq \left(\frac{\lambda_{i+k+1} - \lambda_{i+k}}{2} - \delta \right) |t| + C_1$$

and that, for all $x \in \overline{\Omega}$ and all $0 < |t| \le \overline{\delta}$,

$$\lambda_i + 2\delta < g(x,t)/t < \lambda_{i+1} - 2\delta.$$

For any $n^* \in \mathbb{N}$, choose $\lambda^* = \lambda^*(n^*, h) > 0$ small enough such that

$$\begin{aligned} \lambda^*|h(x,t)| &< 1, \quad x \in \overline{\Omega},\ |t| \le t_{n^*}, \\ \lambda^*|t^{-1}h(x,t)| &< \delta, \quad x \in \overline{\Omega},\ 0 < |t| \le \overline{\delta}. \end{aligned}$$

Let $C = C_1 + 1$. Then the result follows easily from the definition of $g_{n,\lambda}$. □

PROOF OF THEOREM 1.6. For any $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, consider

$$(3.5)_{n,\lambda} \qquad \begin{cases} -\Delta u = g_{n,\lambda}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and the associated functional $J_{n,\lambda}$. Let us begin with case (a). For $n^* \in \mathbb{N}$, let $\lambda^* = \lambda^*(n^*, h) > 0$ be as in Lemma 3.1. From (3.1) and (3.2), it is well known that (see, for example, [18, Theorem 4.3]), for $n \in \{1, \ldots, n^*\}$ and $|\lambda| \le \lambda^*$, $(3.5)_{n,\lambda}$ has at least k pairs of classical solutions

$$(3.6) \qquad \pm u_{n,\lambda,1}, \pm u_{n,\lambda,2}, \ldots, \pm u_{n,\lambda,k}$$

with positive critical values. This fact can also be seen with the following standard argument. For $n \in \mathbb{N}$, denote $E_n = \{e_1, \ldots, e_n\}$ and by $E_n^\perp$ the orthogonal complement of E_n. By (3.1), $J_{\lambda,n}$ satisfies (PS) condition and there exists $R > 0$ such that

$$J_{n,\lambda}(u) \le 0, \quad u \in E_{i+k},\ \|u\|_{1,2} \ge R.$$

By (3.2), there exist $\alpha > 0$ and $r > 0$ such that

$$J_{n,\lambda}(u) \ge \alpha, \quad u \in E_i^\perp,\ \|u\|_{1,2} = r.$$

Denote $D = \{u \in E_{i+k} \mid \|u\|_{1,2} \le R\}$ and $S = \{u \in E_i^\perp \mid \|u\|_{1,2} = r\}$. Define

$$\Phi = \{h \in C(D,H) \mid h \text{ is odd and } h(u) = u \text{ for } u \in \partial D\}$$

and

$$\Gamma_j = \{h(\overline{D \setminus Y}) \mid h \in \Phi,\ Y \in \Sigma,\ \gamma(Y) \le i + k - j\}, \quad j = i+1, \ldots, i+k.$$

If $B \in \Gamma_j$ for some $i + 1 \le j \le i + k$ then $B \cap S \ne \emptyset$ (see [40, Proposition 9.23]). Define

$$c_j = \inf_{B \in \Gamma_j} \max_{u \in B} J_{n,\lambda}(u), \quad j = i+1, \ldots, i+k.$$

Then

$$0 < \alpha \le c_{i+1} \le c_{i+2} \le \ldots \le c_{i+k} < +\infty.$$

So, $J_{n,\lambda}$ has at least $2k$ critical points denoted as in (3.6) such that (see [40, Proposition 9.30])

$$J_{n,\lambda}(\pm u_{n,\lambda,j}) = c_{i+j} > 0, \quad j = 1, \dots, k.$$

For $n \in \{1, \dots, n^*\}$, $|\lambda| \le \lambda^*$, and $j \in \{1, \dots, k\}$, we denote $u_{n,\lambda} = u_{n,\lambda,j}$ and $\overline{\lambda}_{i+k} = (\lambda_{i+k} + \lambda_{i+k+1})/2$ and we have

$$u_{n,\lambda} = (-\Delta - \overline{\lambda}_{i+k})^{-1}(g_{n,\lambda}(x, u_{n,\lambda}) - \overline{\lambda}_{i+k} u_{n,\lambda}),$$

which together with (3.1) implies that

$$\begin{aligned}\|u_{n,\lambda}\|_2 \le& \|(-\Delta - \overline{\lambda}_{i+k})^{-1}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} \|g_{n,\lambda}(x, u_{n,\lambda}) - \overline{\lambda}_{i+k} u_{n,\lambda}\|_2 \\ \le& \frac{2}{\lambda_{i+k+1} - \lambda_{i+k}} \left[\left(\frac{\lambda_{i+k+1} - \lambda_{i+k}}{2} - \delta\right)\|u_{n,\lambda}\|_2 + C|\Omega|^{1/2}\right].\end{aligned}$$

Here we have used the fact that $\|(-\Delta - \overline{\lambda}_{i+k})^{-1}\|_{\mathcal{L}(L^2(\Omega),L^2(\Omega))} = 2/(\lambda_{i+k+1} - \lambda_{i+k})$. Then, for any $n^* \in \mathbb{N}$, there exists $\lambda^* > 0$ such that, for all $n \in \{1, \dots, n^*\}$ and all $|\lambda| \le \lambda^*$,

$$\|u_{n,\lambda}\|_2 \le C\delta^{-1}|\Omega|^{1/2},$$

which combined with (3.1) implies $\|g_{n,\lambda}(x, u_{n,\lambda})\|_2 \le C_1$ for some constant $C_1 > 0$ independent of n and λ. Hence $\|u_{n,\lambda}\|_{2,2} \le C_2$ for some constant $C_2 > 0$ independent of n and λ. Then we can use the argument as in the proof of Theorem 1.1 to get the result.

In case (b), (3.3) and (3.4) implies that for $n \in \{1, \dots, n^*\}$ and $|\lambda| \le \lambda^*$, $(3.5)_{n,\lambda}$ has at least k pairs of classical solutions

$$\pm u'_{n,\lambda,1}, \pm u'_{n,\lambda,2}, \dots, \pm u'_{n,\lambda,k} \tag{3.7}$$

with negative critical values. This fact can be quoted from, for example, [18, Theorem 4.1]. Or, one may have it from the following standard argument. Note that (3.3) implies $J_{n,\lambda}$ satisfies the (PS) condition and is bounded on $E_i^\perp$ from below, while (3.4) implies existence of $\alpha_1 > 0$ and $r_1 > 0$ such that

$$J_{n,\lambda}(u) \le -\alpha_1, \quad u \in E_{i+k}, \ \|u\|_{1,2} = r_1.$$

Define

$$c'_j = \inf_{A\in\Sigma, \gamma(A)\ge j} \sup_{u\in A} J_{n,\lambda}(u), \quad j = i+1, \dots, i+k.$$

Since $A \cap E_i^\perp \ne \emptyset$ for any $A \in \Sigma$ with $\gamma(A) \ge i+1$ (see [40]), we have

$$-\infty < \inf_{u\in E_i^\perp} J_{n,\lambda}(u) \le c'_{i+1} \le c'_{i+2} \le \dots \le c'_{i+k} \le -\alpha_1 < 0.$$

Therefore, $J_{n,\lambda}$ has at least $2k$ critical points denoted as in (3.7) such that (see [40, Proposition 8.5])

$$J_{n,\lambda}(\pm u'_{n,\lambda,j}) = c'_{i+j} < 0, \quad j = 1, \dots, k.$$

Then the same argument as in case (a) leads to the result. $\square$

4. Proof of Theorem 1.8

Take an increasing sequence $\{t_n\}$ such that $t_1 > M$ and $t_n \to \infty$ as $n \to \infty$. Define $g_{n,\lambda}$ as in Section 2. But in the present case μ is taken from (g_6), in which one may assume that $p < \mu < p^*$ without loss of generality.

LEMMA 4.1. *Let q be as in* (g_7).

(a) *There exists a constant $C > 0$ depending only on g with the property that, for any $n^* \in \mathbb{N}$, there exists a number $\lambda^* = \lambda^*(n^*, h) > 0$ such that, for all $n \in \{1, \ldots, n^*\}$, all $|\lambda| \le \lambda^*$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,*

$$|g_{n,\lambda}(x,t)| \le C(|t|^{q-1} + 1).$$

(b) *Let $\delta > 0$ be any positive number. There exists a constant $C(\delta) > 0$ depending only on g and δ with the property that, for any $n^* \in \mathbb{N}$, there exists a number $\lambda^* = \lambda^*(n^*, h) > 0$ such that, for all $n \in \{1, \ldots, n^*\}$, all $|\lambda| \le \lambda^*$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,*

$$|g_{n,\lambda}(x,t)||t|^{N+1} \le \delta |t|^{q+N} + C(\delta).$$

The proof of Lemma 4.1 is similar to that of Lemma 2.1 and therefore is omitted. Note that Lemma 2.3 is also valid and it will be used in the sequel, and the number $\delta > 0$ in Lemma 2.3 can be any fixed number in the present case.

LEMMA 4.2. *There are two constants $C_1 > 0$ and $C_2 > 0$ depending only on g with the property that, for any $n^* \in \mathbb{N}$, there exists a number $\lambda^* = \lambda^*(n^*, h) > 0$ such that, for all $n \in \{1, \ldots, n^*\}$, all $|\lambda| \le \lambda^*$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,*

$$G_{n,\lambda}(x,t) \ge C_1 |t|^{\mu} - C_2.$$

PROOF. From (g_6), we see that, for all $x \in \overline{\Omega}$, and all $|t| \ge M$,

$$\frac{d}{dt}\left(\frac{G(x,t)}{|t|^{\mu}}\right) \begin{cases} \ge 0 & \text{if } t \ge M, \\ \le 0 & \text{if } t \le -M. \end{cases}$$

Define

$$C_1 = \min_{x \in \overline{\Omega}} \left\{ \frac{G(x,M)}{M^{\mu}}, \frac{G(x,-M)}{M^{\mu}} \right\}$$

and

$$C_2' = \max_{x \in \overline{\Omega}, |t| \le M} \{C_1 |t|^{\mu} - G(x,t)\}.$$

It follows that, for all $x \in \overline{\Omega}$ and all $t \in \mathbb{R}$,

$$G(x,t) \ge C_1 |t|^{\mu} - C_2'.$$

Note that $t_1 > M$. For any $n \in \mathbb{N}$, $x \in \overline{\Omega}$, and $t \in \mathbb{R}$, if $t > t_n$ then (g_6) yields

$$\begin{aligned} G_n(x,t) =& G(x,t_n) + \frac{1}{\mu} t_n g(x,t_n)\bigg(\bigg(\frac{t}{t_n}\bigg)^{\mu} - 1\bigg) \\ \geq& G(x,t_n)\bigg(\frac{t}{t_n}\bigg)^{\mu} \geq C_1 t_n^{\mu}\bigg(\frac{t}{t_n}\bigg)^{\mu} = C_1 t^{\mu}. \end{aligned}$$

If $t < -t_n$, we get $G_n(x,t) \geq C_1|t|^{\mu}$ in the same way. Therefore, for all $n \in \mathbb{N}$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,

$$G_n(x,t) \geq C_1|t|^{\mu} - C_2'.$$

For any $n^* \in \mathbb{N}$, take a number $\lambda^* = \lambda^*(n^*, h) > 0$ small enough such that

$$\lambda^*(|H(x,t)| + |h(x,t)|) \leq 1, \quad x \in \overline{\Omega},\ |t| \leq t_{n^*}.$$

Then we get the result letting $C_2 = C_2' + 1$. $\square$

In view of the argument of [40, Proposition 9.23], we have the next lemma.

Lemma 4.3. *Let E be a Banach space, V an k dimensional subspace of E, and $0 < r_1 < r_2 < +\infty$. Denote $T = \{u \in E \mid \|u\| = r_1\}$ and $D = \{u \in V \mid \|u\| \leq r_2\}$. If $h \in C(D,E)$, h is odd, $h(u) = u$ for $u \in \partial D$, $k \geq j$, $Y \in \Sigma$, and $\gamma(Y) \leq k - j$, then*

$$\gamma(h(\overline{D \setminus Y}) \cap T) \geq j.$$

Denote $M = \{u \in W_0^{1,p}9\Omega) \mid \|u\|_p = 1\}$. Define $I(u) = \|\nabla u\|_p^p$ for $u \in W_0^{1,p}(\Omega)$ and

$$\lambda_k = \inf_{A \subset M, \gamma(A) \geq k} \sup_{u \in A} I(u), \quad k = 1, 2, \ldots .$$

The following lemma is taken from [42, Theorem 4.4] and can be proved with the argument of [40, Proposition 9.33].

Lemma 4.4. *$\lambda_k \to \infty$ as $k \to \infty$.*

Now we recall some important facts (the following two lemmas) about regularity of the solutions of p-Laplacian equations. The proofs of these two lemmas are included only for reasons of completeness and convenience.

Lemma 4.5. *Assume $f\colon \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is continuous and there exist $C > 0$ and $1 < r < p^* - 1$ such that, for all $x \in \overline{\Omega}$ and $t \in \mathbb{R}$, $|f(x,t)| \leq C(1 + |t|^r)$. If $u \in W_0^{1,p}(\Omega)$ is a weak solution of*

$$-\Delta_p u = f(x,u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

then $u \in L^q(\Omega)$ for all $q > 1$.

PROOF. The result for $p \geq N$ is trivial. Now, we assume $1 < p < N$. Let $s > 0$. Multiplying the equation with $|u|^s u$ and taking integral, we have

$$\int_\Omega |\nabla(|u|^{s/p}u)|^p = \frac{(s+p)^p}{(s+1)p^p}\int_\Omega f(x,u)|u|^s u.$$

So, there exists a constant $C = C(p,s) > 0$ such that

$$\int_\Omega |\nabla(|u|^{s/p}u)|^p \leq C\int_\Omega |u|^{s+r+1} + C.$$

By Sobolev inequality,

$$S_p\|u\|^{p+s}_{N(p+s)/(N-p)} \leq C\|u\|^{r+s+1}_{r+s+1} + C,$$

where

$$S_p = \inf_{\phi\in W_0^{1,p}(\Omega),\|\phi\|_{p^*}=1} \|\nabla\phi\|_p^p.$$

From the last inequality, we see that if $u \in L^{r+s+1}(\Omega)$ for some $s > 1$ then $u \in L^{N(p+s)/(N-p)}(\Omega)$. Denote $s_0 = p^* - (r+1) > 0$. Since $u \in L^{p^*}(\Omega)$, we have $u \in L^{q_1}(\Omega)$, where

$$q_1 = \frac{N(p+s_0)}{N-p} = p^* + \frac{p^*}{p}s_0.$$

Since $u \in L^{q_1}(\Omega)$, we have $u \in L^{q_2}(\Omega)$ where

$$q_2 = \frac{N}{N-p}(p + q_1 - (r+1)) = p^* + \frac{p^*}{p}s_0 + \left(\frac{p^*}{p}\right)^2 s_0.$$

Continuing this procedure of iteration, we have $u \in L^{q_n}(\Omega)$ for all $n \in \mathbb{N}$ where

$$q_n = p^* + s_0\sum_{i=1}^n \left(\frac{p^*}{p}\right)^i.$$

Therefore, $u \in L^q(\Omega)$ for all $q > 1$. □

LEMMA 4.6. *Assume there exists $q > N$ such that $g \in L^{q/p}(\Omega)$. If $u \in W_0^{1,p}(\Omega)$ is a weak solution of*

$$-\Delta_p u = g \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

then $u \in L^\infty(\Omega)$ and there exists a constant $C = C(p,q,N,\Omega) > 0$ such that

$$\|u\|_\infty \leq C\|g\|_{q/p}^{1/(p-1)}.$$

PROOF. The result for $p > N$ is trivial. We only consider $1 < p < N$ since it is similar for $p = N$. The proof is very similar to that of [31, Theorem 8.15], so we will be sketchy and indicate the differences. Denote $k = \|g\|_{q/p}^{1/(p-1)}$. For $\beta \geq 1$ and $M > k$, define

$$H(z) = \begin{cases} z^\beta - k^\beta & \text{for } k \leq z \leq M, \\ \beta M^{\beta-1}(z-M) + M^\beta - k^\beta & \text{for } z > M. \end{cases}$$

Denote $w = u^+ + k$ and define

$$v = G(w) = \int_k^w |H'(s)|^p \, ds.$$

Using v as a testing function, we obtain

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla G(w) = \int_\Omega g G(w),$$

which implies

$$\int_\Omega |\nabla H(w)|^p \le \int_\Omega |g| w G'(w) \le k^{-p+1} \int_\Omega |g| w^p |H'(w)|^p.$$

By Sobolev inequality and Hölder inequality,

$$S_p \|H(w)\|_{p^*}^p \le \|wH'(w)\|_{qp/(q-p)}^p.$$

Letting $M \to \infty$, we obtain

$$S_p^{1/p} \|w^\beta - k^\beta\|_{p^*} \le \beta \|w^\beta\|_{\overline{q}},$$

where $\overline{q} = qp/(q-p)$. So,

$$S_p^{1/p} \|w\|_{\beta p^*}^\beta \le S_p^{1/p} k^\beta |\Omega|^{1/p^*} + \beta \|w\|_{\beta \overline{q}}^\beta.$$

Since $k \le w$, we obtain

$$\|w\|_{\beta \chi \overline{q}} \le (C\beta)^{1/\beta} \|w\|_{\beta \overline{q}},$$

where $C = |\Omega|^{1/p^* - 1/\overline{q}} + S_p^{-1/p}$ and $\chi = N(q-p)/q(N-p) > 1$. Since $\beta \ge 1$ is arbitrary, we have $w \in L^r$ for all $r \ge 1$. Taking $\beta = \chi^m$ $(m = 0, 1, \dots)$ in the last inequality, we come to

$$\|w\|_{\chi^n \overline{q}} \le \prod_{m=0}^\infty (C\chi^m)^{\chi^{-m}} \|w\|_{\overline{q}} = C\|w\|_{\overline{q}},$$

where $C = C(p, q, N, \Omega)$. Letting $n \to \infty$, we arrive at

$$\|w\|_\infty \le C\|w\|_{\overline{q}}.$$

So,

$$\|u^+\|_\infty \le C(\|u\|_{p^*} + \|g\|_{q/p}^{1/(p-1)}).$$

But from the equation, the Sobolev inequality and Hölder inequality yield

$$S_p \|u\|_{p^*}^p \le \|u\|_{q/(q-p)} \|g\|_{q/p} \le C\|u\|_{p^*} \|g\|_{q/p}.$$

Then we have

$$\|u^+\|_\infty \le C\|g\|_{q/p}^{1/(p-1)}.$$

Similarly,

$$\|u^-\|_\infty \le C\|g\|_{q/p}^{1/(p-1)}.$$

Combining the last two inequalities gives the result. $\square$

Proof of Theorem 1.6. For every $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$, consider the boundary value problem

$$(4.1)_{n,\lambda} \qquad \begin{cases} -\Delta_p u = g_{n,\lambda}(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

whose solutions correspond to critical points of

$$J_{n,\lambda}(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - \int_\Omega G_{n,\lambda}(x,u), \quad u \in W_0^{1,p}(\Omega).$$

In view of the definition of $g_{n,\lambda}(x,t)$, (g_5) and (h), we see that $J_{n,\lambda}$ is an even functional. $J_{n,\lambda}$ satisfies (PS) condition on $W_0^{1,p}(\Omega)$ for each $n \in \mathbb{N}$ and each $\lambda \in \mathbb{R}$ since, in the definition of $g_{n,\lambda}$, $p < \mu < p^*$.

At the present stage, we fix $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}$. We prove that $J_{n,\lambda}$ has infinitely many critical points. Take a sequence of vectors $\{e_k\}_1^\infty \subset W_0^{1,p}(\Omega)$ such that the vectors are linearly independent and $\|e\|_{1,p} = 1$. Denote $E_k = \operatorname{span}\{e_1,\ldots,e_k\}$ $(k = 1,2,\ldots)$. For any k, in view of the definition of $g_{n,\lambda}$ and the fact that $\mu > p$, there exists $R_k > 1$ such that

$$J_{n,\lambda}(u) < 0, \quad u \in E_k,\ \|u\|_{1,p} \ge R_k.$$

Set $D_k = \{u \in E_k \mid \|u\|_{1,p} \le R_k\}$ and define

$$G_k = \{h \in C(D_k, W_0^{1,p}(\Omega)) \mid h \text{ is odd and } h(u) = u \text{ on } \partial D_k\}, \quad k = 1,2,\ldots,$$
$$\Gamma_j = \{h(\overline{D_k \setminus Y}) \mid h \in G_k,\ k \ge j,\ Y \in \Sigma, \text{ and } \gamma(Y) \le k - j\},$$

and

$$c_{n,\lambda,j} = \inf_{B \in \Gamma_j} \max_{u \in B} J_{n,\lambda}(u), \quad j = 1,2,\ldots.$$

Clearly, each $c_{n,\lambda,j}$ is finite and

$$c_{n,\lambda,1} \le c_{n,\lambda,2} \le \ldots \le c_{n,\lambda,j} \le \ldots.$$

We assert that there exists $j_0 \in \mathbb{N}$ such that for any $n^* \in \mathbb{N}$, there exists $\lambda^* = \lambda^*(n^*) > 0$ such that if $n \in \{1,\ldots,n^*\}$ and $|\lambda| \le \lambda^*$ then $c_{n,\lambda,j_0} \ge 1/2p$. For proving this, we only consider $1 < p < N$ since it is similar for $p \ge N$. For any $B \in \Gamma_j$, by Lemma 4.3, we see that $\gamma(B \cap T) \ge j$, where $T = \{u \in W_0^{1,p}(\Omega) \mid \|u\|_{1,p} = 1\}$. Define $g\colon B \cap T \to M$ as $g(u) = u/\|u\|_p$. Then, since $B \cap T$ is compact, g is continuous. So, $\gamma(g(B \cap T)) \ge j$ and, by the definition of λ_j, there exists $v_j \in g(B \cap T)$ such that $I(v_j) \ge \lambda_j$. Let $u_j \in B \cap T$ be such that $g(u_j) = v_j$. Then

$$1 = \int_\Omega |\nabla u_j|^p \ge \lambda_j \int_\Omega |u_j|^p.$$

By (g_5), (h), (g_7) and the definition of $g_{n,\lambda}$, there is a constant $c > 0$ depending only on g with the property that, for any $n^* \in \mathbb{N}$, there exists a number $\lambda^* =$

$\lambda^*(n^*,h) > 0$ such that, for all $n \in \{1,\dots,n^*\}$, all $|\lambda| \le \lambda^*$, all $x \in \overline{\Omega}$, and all $t \in \mathbb{R}$,

$$G_{n,\lambda}(x,t) \le \begin{cases} ct^2 & \text{for } |t| \le 1, \\ c|t|^p + (S_p^{p^*/p}/4p)|t|^{p^*} & \text{for } |t| \ge 1. \end{cases}$$

So, if $p \le 2$, then

$$J_{n,\lambda}(u_j) \ge \frac{1}{p} - c\int_\Omega |u_j|^p - \frac{S_p^{p^*/p}}{4p}\int_\Omega |u_j|^{p^*} \ge \frac{3}{4p} - \frac{c}{\lambda_j}.$$

If $p > 2$, then

$$J_{n,\lambda}(u_j) \ge \frac{1}{p} - c\int_\Omega (u_j^2 + |u_j|^p) - \frac{S_p^{p^*/p}}{4p}\int_\Omega |u_j|^{p^*} \ge \frac{3}{4p} - c_1\biggl(\biggl(\frac{1}{\lambda_j}\biggr)^{2/p} + \frac{1}{\lambda_j}\biggr).$$

By Lemma 4.4 there exists j_0 such that for any $n^* \in \mathbb{N}$, if $n \in \{1,\dots,n^*\}$ and $|\lambda| \le \lambda^*$ then $J_{n,\lambda}(u_{j_0}) \ge 1/2p$, which implies, since $B \in \Gamma_{j_0}$ is arbitrary, $c_{n,\lambda,j_0} \ge 1/2p$. According to [40, Propositions 9.30 and 9.33], for $n \in \{1,\dots,n^*\}$ and $|\lambda| \le \lambda^*$, $J_{n,\lambda}$ has an unbounded sequence of solutions

$$\pm u_{n,\lambda,j_0}, \pm u_{n,\lambda,j_0+1}, \dots, \pm u_{n,\lambda,j_0+j}, \dots$$

corresponding to an unbounded sequence of positive critical values

$$c_{n,\lambda,j_0}, c_{n,\lambda,j_0+1}, \dots, c_{n,\lambda,j_0+j}, \dots .$$

By Lemma 4.2, we have, for $n \in \{1,\dots,n^*\}$ and $|\lambda| \le \lambda^*$,

$$J_{n,\lambda}(u) \le J^*(u), \quad u \in W_0^{1,p}(\Omega),$$

where

$$J^*(u) = \frac{1}{p}\int_\Omega |\nabla u|^p - C_1\int_\Omega |u|^\mu + C_2|\Omega|, \quad u \in W_0^{1,p}(\Omega).$$

Therefore, for $n \in \{1,\dots,n^*\}$, $|\lambda| \le \lambda^*$, and $j \ge j_0$,

$$J_{n,\lambda}(u_{n,\lambda,j}) = c_{n,\lambda,j} \le c_j^* := \inf_{B\in\Gamma_j}\max_{u\in B} J^*(u).$$

Since $u_{n,\lambda,j}$ are solutions of $(4.1)_{n,\lambda}$, by Lemma 2.3 we see that and $j \ge j_0$,

$$J_{n,\lambda}(u_{n,\lambda,j}) \ge \biggl(\frac{1}{p} - \frac{1}{\mu}\biggr)\int_\Omega |\nabla u_{n,\lambda,j}|^p - (m_2+\delta)|\Omega|.$$

for $n \in \{1,\dots,n^*\}$, $|\lambda| \le \lambda^*$. Combining the last two inequalities, we obtain, for $n \in \{1,\dots,n^*\}$, $|\lambda| \le \lambda^*$, and $j \ge j_0$,

$$\|u_{n,\lambda,j}\|_{1,p}^p \le (p^{-1} - \mu^{-1})^{-1}(c_j^* + (m_2+\delta)|\Omega|). \tag{4.2}$$

If $p > N$ then $W_0^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$. So, for any $j \ge j_0$ there is a constant $C_j > 0$ with the property that, for any $n^* \in \mathbb{N}$, if $n \in \{1,\dots,n^*\}$ and $|\lambda| \le \lambda^*$ then

$$\|u_{n,\lambda,j}\|_\infty \le C_j. \tag{4.3}$$

We assert that (4.3) is also valid in the case $1 < p < N$. Multiplying $(4.1)_{n,\lambda}$ with $|u_{n,\lambda,j}|^N u_{n,\lambda,j}$ and taking integral, by Lemma 4.1(b), we have, for $n \in \{1,\dots,n^*\}$, $|\lambda| \le \lambda^*$, and $j \ge j_0$,

$$\int_\Omega |\nabla(|u_{n,\lambda,j}|^{N/p} u_{n,\lambda,j})|^p \le \delta \int_\Omega |u_{n,\lambda,j}|^{p^*+N} + C(\delta).$$

By Lemma 4.5, $u_{n,\lambda,j} \in L^q(\Omega)$ for any $q > 1$ and therefore $|u_{n,\lambda,j}|^{N/p} u_{n,\lambda,j} \in W_0^{1,p}9\Omega)$. Using (4.2), Sobolev inequality and Holder inequality, we have

$$\begin{aligned}
&S_p\bigg(\int_\Omega |u_{n,\lambda,j}|^{(N+p)p^*/p}\bigg)^{p/p^*} \\
&\le \delta\left(\int_\Omega |u_{n,\lambda,j}|^{p^*}\right)^{\alpha(p^*+N)/p^*}\bigg(\int_\Omega |u_{n,\lambda,j}|^{p^*(N+p)/p}\bigg)^{(1-\alpha)(p^*+N)p/((N+p)p^*)} + C(\delta) \\
&\le \delta C'_j\bigg(\int_\Omega |u_{n,\lambda,j}|^{p^*(N+p)/p}\bigg)^{(1-\alpha)(p^*+N)p/((N+p)p^*)} + C(\delta),
\end{aligned}$$

where C'_j depends only on j and $0 < \alpha < 1$ satisfies

$$\frac{\alpha}{p^*} + \frac{(1-\alpha)p}{(N+p)p^*} = \frac{1}{p^*+N}.$$

It is easy to see that $(1-\alpha)(p^*+N) = N+p$. Then, choosing $\delta = \delta(j)$ small enough, we have, for $n \in \{1,\dots,n^*\}$, $|\lambda| \le \lambda^*$, and $j \ge j_0$,

$$\|u_{n,\lambda,j}\|_{(N+p)p^*/p} \le C''_j,$$

for some constant C''_j depending only on j. By Lemma 4.1(a), a direct computation shows that, there exists a constant C'''_j depending only on j such that, for $n \in \{1,\dots,n^*\}$, $|\lambda| \le \lambda^*$, and $j \ge j_0$,

$$\|g_{n,\lambda}(u_{n,\lambda,j})\|_{(N+p)/p} \le C'''_j.$$

Then Lemma 4.6 implies, for $n \in \{1,\dots,n^*\}$, $|\lambda| \le \lambda^*$, and $j \ge j_0$, (4.3) is valid. If $p = N$, then we obtain (4.3) in a similar way by [31, Theorem 7.15]. For any $j \in \mathbb{N}$, choose $n^* = n^*(j) > 0$ such that $t_{n^*} > \max\{C_{j_0}, C_{j_0+1},\dots,C_{j_0+j-1}\}$. Denote $\overline{\lambda}_j = \lambda^*(n^*,h)$. If $|\lambda| \le \overline{\lambda}_j$ then, for $k = j_0,\dots,j_0+j-1$,

$$\|u_{n^*,\lambda,k}\|_\infty \le C_k \le t_{n^*}.$$

Hence, $(\mathrm{P2})_\lambda$ has at least j pairs of solutions $\pm u_{n^*,\lambda,j_0+k}$, $k = 0,\dots,j-1$. According to [11], [35], [43], these solutions are in $C^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$.□

PROOF OF COROLLARY 1.10. Let $v = \lambda^{1/(q-p)}u$. Then $(P3)^{\pm}_\lambda$ is transformed to

$$\begin{cases} -\Delta_p v = |v|^{q-2}v \pm \lambda^{-(r-p)/(q-p)}|v|^{r-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then one get the result easily from Theorem 1.6. □

Proof of Corollary 1.14. Let $v = \lambda^{1/(q-p)}u$. Then $(\mathrm{P4})^{\pm}_{\lambda}$ is converted to

$$\begin{cases} -\Delta_p v = |v|^{q-2}v \pm \lambda^{-(r-q)/(q-p)}|v|^{r-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The result comes also from Theorem 1.6 easily. □

References

[1] Adimurthi and S. L. Yadava, *Nonexistence of positive radial solutions of a quasilinear Neumann problem with a critical Sobolev exponent*, Arch. Rational Mech. Anal. **139** (1997), 239–253.

[2] A. Ambrosetti, H. Brézis and G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), 519–543.

[3] A. Ambrosetti and P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[4] F. V. Atkinson and L. A. Peletier, *Oscillation of solutions of perturbed autonomous equations with an application to nonlinear elliptic eigenvalue problems involving critical Sobolev exponents*, Differential Integral Equations **3** (1990), 401–433.

[5] A. Bahri, *Topological results on a certain class of functionals and applications*, J. Funct. Anal. **41** (1981), 397–427.

[6] A. Bahri and H. Berestycki, *A perturbation method in critical point theory and applications*, Trans. Amer. Math. Soc. **267** (1981), 1–32.

[7] A. Bahri and P. L. Lions, *Morse-index of some min-max critical points*, Comm. Pure Appl. Math. **41** (1988), 1027–1037.

[8] T. Bartsch, K. C. Chang and Z.-Q. Wang, *On the Morse indices of sign changing solutions of nonlinear elliptic problems*, Math. Z. **233** (2000), 655–677.

[9] T. Bartsch and Z.-Q. Wang, *On the existence of sign changing solutions for semilinear Dirichlet problem*, Topol. Methods Nonlinear Anal. **7** (1996), 115–131.

[10] T. Bartsch and M. Willem, *On an elliptic equation with concave and convex nonlinearities*, Proc. Amer. Math. Soc. **123** (1995), 3555–3561.

[11] E. di Benedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), 827–850.

[12] H. Berestycki and P. L. Lions, *Nonlinear scalar field equations, I: Existence of a ground state, II: Existence of infinitely many solutions*, Arch. Rational Mech. Anal. **82** (1983), 313–345, 347–375.

[13] H. Brézis, *Some variational problems with lack of compactness*, Nonlinear Functional Analysis and its Applications (F. E. Browder, ed.), Proceeding of Symposia in Pure Mathematics, vol. 45, 1983, pp. 165–201.

[14] H. Brézis and T. Kato, *Remarks on the Schrödinger operator with singular complex potentials*, J. Math. Pure Appl. **58** (1979), 137–151.

[15] H. Brézis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), 437–477.

[16] A. Castro, J. Cossio and M. Neuberger, *A sign-changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. Math. **27** (1997), 1041–1053.

[17] J. Chabrowski and J. F. Yang, *Existence theorems for elliptic equations involving supercritical Sobolev exponent*, Adv. Differential Equations **2** (1997), 231–256.

[18] K. C. Chang, *Infinite Dimensional Morse Theory and Multiple Solution Problems*, Birkhäuser, Boston, 1993.

[19] F.-C. St. Cirstea and V. D. Radulescu, *On a double bifurcation quasilinear problem arising in the study of anisotropic continuous media*, Proc. Edinburgh Math. Soc. **44** (2001), 527–548.

[20] D. C. Clark, *A variant of Lusternik–Schnirelman theory*, Indiana Univ. Math. J. **22** (1972), 65–74.

[21] E. N. Dancer and Y. Du, *Existence of changing sign solutions for some semilinear problems with jumping nonlinearities at zero*, Proc. Roy. Soc. Edinburgh Sect. A **124** (1994), 1165–1176.

[22] ______, *Multiple solutions of some semilinear elliptic equations via the generalized Conley index*, J. Math. Anal. Appl. **189** (1995), 848–871.

[23] G. C. Dong and S. J. Li, *On the existence of infinitely many solutions of the Dirichlet problem for some nonlinear elliptic equations*, Sci. Sinica Ser. A **25** (1982), 468–475.

[24] Y. Du, *Exact multiplicity and S-shaped bifurcation curve for some semilinear elliptic problems from combustion theory*, SIAM J. Math. Anal. **32** (2000), 707–733.

[25] P. Drábek, *Nonlinear eigenvalue problems for p-Laplacian in* $\mathbb{R}^N$, Math. Nachr. **173** (1995), 131–139.

[26] P. Drábek and C. Simader, *Nonlinear eigenvalue problems for quasilinear equations in unbounded domains*, Math. Nachr. **203** (1999), 5–30.

[27] D. G. de Figueiredo and P. L. Lions, *On pairs of positive solutions for a class of semilinear elliptic problems*, Indiana Univ. Math. J. **34** (1985), 591–606.

[28] D. G. de Figueiredo, P. L. Lions, and R. D. Nussbaum, *A priori estimates and existence results for positive solutions of semilinear elliptic equations*, J. Math. Pure Appl. **61** (1982), 41–63.

[29] J. Garcia Azorero and I. Peral Alonso, *Existence and nonuniqueness for the p-Laplacian: nonlinear eigenvalues*, Comm. Partial Differential Equations **12** (1987), 1389–1430.

[30] ______, *Multiplicity of solutions for elliptic problems with critical exponent or with a nonsymmetric term*, Trans. Amer. Math. Soc. **323** (1991), 877–895.

[31] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, second edition, Spinger–Verlag, Berlin, 1998.

[32] Z. Guo and J. R. L. Webb, *Large and small solutions of a class of quasilinear elliptic eigenvalue problems*, J. Differential Equations **180** (2002), 1–50.

[33] S. J. Li and Z.-Q. Wang, *Mountain pass theorem in order interval and multiple solutions for semilinear elliptic Dirichlet problems*, J. Anal. Math. **81** (2000), 373–396.

[34] S. J. Li and Z. T. Zhang, *Sign-changing and multiple solutions theorems for semilinear elliptic boundary value problems with jumping nonlinearities*, Acta Math. Sinica Engl. Ser. **16** (2000), 113–122.

[35] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), 1203–1219.

[36] Z. L. Liu, *Positive solutions of superlinear elliptic equations*, J. Funct. Anal. **167** (1999), 370–398.

[37] Z. L. Liu and J. X. Sun, *Invariant sets of descending flow in critical point theory with applications to nonlinear differential equations*, J. Differential Equations **172** (2001), 257–299.

[38] P. H. Rabinowitz, *Some global results for nonlinear eigenvalue problems*, J. Funct. Anal. **7** (1971), 183–200.

[39] ______, *Multiple critical points of perturbed symmetric functionals*, Trans. Amer. Math. Soc. **272** (1982), 753–769.

[40] ———, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS, No. 65,, Amer. Math. Soc., Providence, 1985.

[41] M. Struwe, *Infinitely many critical points for functionals which are not even and applications to superlinear boundary value problems*, Manuscripta Math. **32** (1980), 335–364.

[42] A. Szulkin and M. Willem, *Eigenvalue problems with indefinite weight*, Studia Math. **135** (1999), 191–201.

[43] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150.

[44] Z.-Q. Wang, *On a semilinear elliptic equation*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **8** (1991), 43–57.

[45] ———, *Nonlinear boundary value problems with concave nonlinearities near the origion*, NoDEA Nonlinear Differential Equations and Appl. **8** (2001), 15–33.

[46] M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

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Shujie Li
Academy of Mathematics and System Sciences
Academia Sinica
Beijing 100080, P.R. CHINA
and
Yuan-Rong Zeng Functional Analysis Research Center
Harbin Normal University
Harbin 150025, P.R. China

E-mail address: lisj@math.ac.cn

Zhaoli Liu
Department of Mathematics
Capital Normal University
Beijing 100037, P.R. CHINA

E-mail address: zliu@mail.cnu.edu.cn