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EXISTENCE OF MINIMIZER OF SOME FUNCTIONALS INVOLVING HARDY-TYPE INEQUALITIES

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(*Submitted by M. Willem*)

Abstract. We study a class of p-laplacian-type problems with various unbounded weights and a forcing term on open subsets of $\mathbb{R}^N$ or on the positive real axis. To prove the existence of solution, we use variational methods involving concentration-compactness technique and Hardy-type inequalities.

1. Introduction

This paper is devoted to the existence of solution of the problem

$$\begin{cases} -\Delta_p u - \lambda \dfrac{|u|^{p-2}u}{|x|^p} = \dfrac{f(x)}{p} & \text{in } \Omega, \\ u = 0 & \text{in } \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{1.1}$$

This is a p-laplacian-like equation with an unbounded potential and a forcing term. We will study the corresponding energy functional to obtain a solution of (1.1). The classical minimization method cannot be used here because the Lagrangian

$$L(x,u,\nabla u) = |\nabla u|^p - \lambda \frac{|u|^p}{|x|^p} - fu$$

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is, in general, not bounded from below and Ω is, in general, unbounded. To overcome this difficulty, we use a concentration-compactness argument (see Lions [5]) or more precisely the decomposition technique (see Smets [10] or Willem [11]). Contrary to the usual case, we consider here a free minimization problem. Another key ingredient to prove the result is the Hardy inequality (see e.g. [4], [12]). Let us remark that a lack of compactness is due to the invariance by dilation of the quotient

$$\frac{\int_{\mathbb{R}^N} |\nabla u|^p \, dx}{\int_{\mathbb{R}^N} (|u|^p/|x|^p) \, dx}.$$

The idea of the proof is to consider a minimizing sequence (u_n) weakly converging to u and Ω_R, a region of the space designed to control the lack of compactness, i.e. the components of Ω included in the union of $B(0,1/R)$ and $B^c(0,R)$. We decompose the functional in two components

$$J(u_n) = \int_{\Omega \setminus \Omega_R} L(x, u_n, \nabla u_n) \, dx + \int_{\Omega_R} L(x, u_n, \nabla u_n) \, dx$$

and we take the limit of each part

$$\begin{aligned} \lim_{R\to\infty} \underline{\lim}_{n\to\infty} \int_{\Omega\setminus\Omega_R} L(x, u_n, \nabla u_n) \, dx &\geq J(u), \\ \lim_{R\to\infty} \overline{\lim}_{n\to\infty} \int_{\Omega_R} L(x, u_n, \nabla u_n) \, dx &\geq 0. \end{aligned}$$

These inequalities are obtained using a decomposition lemma. We deduce from this that u is a minimizer of the functional.

Problem (1.1) for bounded domains was studied by García and Peral in [3]. They proved the existence of a minimizer using an Hardy-type inequality, Ekeland principle and a convergence theorem. An improvement obtained here is the validity of the result for unbounded domains Ω.

Moreover, our method is applicable for a large class of problems, provided the existence of an adapted Hardy-type inequality. As Secchi, Smets and Willem proved a cylindrical Hardy inequality in [9], we can study the corresponding equation involving the operator

$$-\mathrm{div}\left(\frac{|\nabla u|^{p-2}\nabla u}{|y|^{ap}}\right) - \lambda \frac{|u|^{p-2}u}{|y|^{(a+1)p}},$$

where $x = (y,z) \in \mathbb{R}^k \times \mathbb{R}^{N-k}$. This is done in Section 3. The particular case $k = N$, corresponding to a radial singularity, is also considered.

In Section 4, we use the same technique to study a corresponding ordinary differential equation using the one dimensional Hardy inequality.

As Marcus, Mizel and Pinchover [6], Colin [2] and Chabrowski and Willem [1] developped an inequality on exterior domains with a singularity located on the boundary, we can also obtain the existence of solution of associated problems. This is done in Section 5.

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Remark 1.1. After the completion of this work, we received a manuscript by X. Zhong [13] and B. Pellacci sended us [7] and [8], papers concerning the study of problem (1.1). In [13] the case of a bounded Ω is treated by an elementary argument. In [7] and [8] the classical concentration-compactness principle from [5] is used.

2. The p-laplacian case

To obtain solutions of problem (1.1), we study the associated energy functional

$$J(u) := \int_\Omega |\nabla u|^p\,dx - \lambda \int_\Omega \frac{|u|^p}{|x|^p}\,dx - \langle f, u\rangle$$

defined on $\mathcal{D}_0^{1,p}(\Omega)$, the closure of $\mathcal{D}(\Omega)$ with respect to the norm $||\nabla u||_p$ and where Ω is a smooth open subset of $\mathbb{R}^N$ with compact boundary, $0 \notin \partial\Omega$ and

$$\text{if } 0 \notin \Omega \text{ then } 1 < p \neq N \text{ and if } 0 \in \Omega \text{ then } 1 < p < N, \qquad f \in (\mathcal{D}_0^{1,p}(\Omega))'. \tag{2.1}$$

We consider $\inf J := \inf_{u\in\mathcal{D}_0^{1,p}(\Omega)} J(u)$.

The aim of this section is to prove the following result.

Theorem 2.1. *Assume that* (2.1) *is satisfied. If* $\lambda < \lambda_p := |(N-p)/p|^p$, *then* $\inf J$ *is achieved and problem* (1.1) *has a solution.*

We first recall the classical Hardy inequality (see e.g. [4,12]).

Lemma 2.2. *Assume that* (2.1) *is satisfied. If* $u \in \mathcal{D}_0^{1,p}(\Omega)$, *then*

$$\lambda_p \int_\Omega \frac{|u|^p}{|x|^p}\,dx \le \int_\Omega |\nabla u|^p\,dx.$$

This inequality implies that the functional J is continuous, G-differentiable and coercive if $\lambda < \lambda_p$. The key ingredient proving Theorem 2.1 is the following lemma quantifying the loss of compactness.

Lemma 2.3. *Let* $(u_n) \subset \mathcal{D}_0^{1,p}(\Omega)$ *be such that* $u_n \rightharpoonup u$ *in* $\mathcal{D}_0^{1,p}(\Omega)$. *We define*

$$\Omega_R := \{x \in \Omega : |x| < R^{-1} \text{ or } |x| > R\},$$

$$\mu_0 := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} |\nabla u_n|^p\,dx,$$

$$\nu_0 := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{|x|^p}\,dx,$$

then we have

$$\overline{\lim_{n\to\infty}} \int_\Omega \frac{|u_n|^p}{|x|^p}\,dx = \int_\Omega \frac{|u|^p}{|x|^p}\,dx + \nu_0, \tag{2.2}$$

$$\overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p\,dx \geq \int_\Omega |\nabla u|^p\,dx + \mu_0, \tag{2.3}$$

$$\lambda_p\,\nu_0 \leq \mu_0. \tag{2.4}$$

Proof. We have, for every $R > 0$,

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_\Omega \frac{|u_n|^p}{|x|^p}\,dx &= \lim_{n\to\infty} \int_{\Omega\setminus\Omega_R} \frac{|u_n|^p}{|x|^p}\,dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{|x|^p}\,dx,\\ &= \int_{\Omega\setminus\Omega_R} \frac{|u|^p}{|x|^p}\,dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{|x|^p}\,dx,\end{aligned}$$

as $u_n \to u$ in $L^p_{loc}(\Omega)$, by Rellich's theorem. So, taking the limit as $R\to\infty$, we obtain (2.2).

We have

$$\overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p\,dx \geq \underline{\lim_{n\to\infty}} \int_{\Omega\setminus\Omega_R} |\nabla u_n|^p\,dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} |\nabla u_n|^p\,dx.$$

Hence, as the mapping $v \mapsto \int_{\Omega\setminus\Omega_R} |\nabla v|^p\,dx$ is convex, we get

$$\overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p\,dx \geq \int_{\Omega\setminus\Omega_R} |\nabla u|^p\,dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} |\nabla u_n|^p\,dx.$$

Taking the limit as $R\to\infty$, we obtain (2.3) by Levi's theorem.

Now we introduce the truncation $\psi_R \in \mathcal{C}^\infty(\mathbb{R}^N)$ with $\psi_R(x) = 0$ if $x \in \Omega\setminus\Omega_R$, $\psi_R(x) = 1$ if $x \in \Omega_{R+1}$ and $0 \leq \psi_R \leq 1$. For each $v \in \mathcal{D}^{1,p}_0(\Omega)$, Lemma 2.2 implies that

$$\lambda_p \int_\Omega \frac{|\psi_R v|^p}{|x|^p}\,dx \leq \int_\Omega |\nabla(\psi_R v)|^p\,dx. \tag{2.5}$$

On the other hand, using the fact that for each $\varepsilon > 0$, there exists a constant $c(\varepsilon, p) > 0$ such that

$$||a+b|^p - |a|^p| \leq \varepsilon |a|^p + c(\varepsilon,p)\,|b|^p, \tag{2.6}$$

it is easy to verify that

$$\begin{aligned}\mu_0 &= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} |\nabla(u_n - u)|^p\,dx,\\ \nu_0 &= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n-u|^p}{|x|^p}\,dx.\end{aligned}$$

Using the truncation ψ_R, this implies that

$$\begin{aligned} \mu_0 &= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_\Omega |\nabla(u_n - u)|^p \psi_R^p \, dx, \\ \nu_0 &= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_\Omega \frac{|u_n - u|^p \psi_R^p}{|x|^p} \, dx. \end{aligned} \tag{2.7}$$

Using once again (2.6) and the compactness of the embedding $\mathcal{D}_0^{1,p}(\Omega) \subset L^p(\Omega)$ for bounded domains, we get

$$\mu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_\Omega |\nabla((u_n - u)\psi_R)|^p \, dx. \tag{2.8}$$

Now, putting together (2.5), (2.7) and (2.8) we obtain (2.4), and the lemma is proved. □

Proposition 2.4. *Under assumption* (2.1) *and if* $\lambda < \lambda_p$, *the functionnal* J *is weakly lower semi-continuous.*

Proof. Let (u_n) be such that $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. Going if necessary to a subsequence, we can assume that $J(u_n)$ is convergent. So, we have by the preceding lemma

$$\begin{aligned} \lim_{n\to\infty} J(u_n) &= \overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p \, dx - \lambda \overline{\lim_{n\to\infty}} \int_\Omega \frac{|u_n|^p}{|x|^p} \, dx - \langle f, u\rangle, \\ &\geq J(u) + \mu_0 - \lambda\nu_0, \\ &\geq J(u) + (\lambda_p - \lambda)\nu_0, \\ &\geq J(u), \end{aligned}$$

concluding the proof. □

Theorem 2.1 is an obvious consequence of Proposition 2.4. Moreover, the proof of Proposition 2.4 shows that $\mu_0 = \nu_0 = 0$, so the minimizing sequence (u_n) is strongly convergent up to a subsequence.

Remark 2.5. In the linear case $p = 2$, the strict convexity of the functional ensure the uniqueness of solution. If $p \neq 2$, the uniqueness is in general not true. A counter-example is given by García and Peral [3] for $p > 2$ and by Zhong [13] for $1 < p < 2$.

3. Cylindrical weight-case

In this section, we study

$$\begin{cases} -\mathrm{div}\Big(\dfrac{|\nabla u|^{p-2}\nabla u}{|y|^{ap}}\Big) = \lambda \dfrac{|u|^{p-2}u}{|y|^{(a+1)p}} + \dfrac{f(x)}{p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{3.1}$$

where $x=(y,z)\in\mathbb{R}^k\times\mathbb{R}^{N-k}$. We introduce the corresponding functional

$$J(u):=\int_\Omega \frac{|\nabla u|^p}{|y|^{ap}}\,dx-\lambda\int_\Omega \frac{|u|^p}{|y|^{(a+1)p}}\,dx-\langle f,u\rangle,$$

defined on $\mathcal{D}^{1,p}_{a,k}(\Omega)$, which is the closure of $\mathcal{D}(\Omega)$ with respect to the norm

$$\|u\|=\bigg(\int_\Omega \frac{|\nabla u|^p}{|y|^{ap}}\,dx\bigg)^{1/p}$$

and

$$1<p<\infty, \tag{3.2}$$

$$a\geq 0, \tag{3.3}$$

$$(a+1)p<k, \tag{3.4}$$

$$f\in(\mathcal{D}^{1,p}_{a,k}(\Omega))'.$$

For simplicity we assume that Ω is either $B(0,R)$ or the complement of the corresponding closed ball. The Hardy-type inequality associated to this problem is due to Secchi, Smets and Willem [9].

Lemma 3.1. *Assume that* (3.2)–(3.4) *are satisfied. If* $u\in\mathcal{D}^{1,p}_{a,k}(\Omega)$, *then*

$$\lambda_{kap}\int_\Omega \frac{|u|^p}{|y|^{(a+1)p}}\,dx\leq\int_\Omega \frac{|\nabla u|^p}{|y|^{ap}}\,dx,\quad \textit{where } \lambda_{kap}:=\bigg(\frac{k-(a+1)p}{p}\bigg)^p.$$

The aim of this section is to prove the following result.

Theorem 3.2. *Assume that* (3.2)–(3.4) *are satisfied. If* $\lambda<\lambda_{kap}$, *then* inf J *on* $\mathcal{D}^{1,p}_{a,k}(\Omega)$ *is achieved and problem* (3.1) *has a solution.*

Lemma 3.1 implies that the functional J is continuous, G-differentiable and coercive if $\lambda<\lambda_{kap}$. Let $(u_n)\subset\mathcal{D}^{1,p}_{a,k}(\Omega)$ a minimizing sequence for J, i.e.

$$J(u_n)\to\inf J:=\inf_{u\in\mathcal{D}^{1,p}_{a,k}(\Omega)}J(u)\quad\text{as } n\to\infty.$$

The goal is to prove that this sequence (u_n) contains a subsequence converging to a minimizer of J. As J is coercive, (u_n) is bounded so we can assume that $u_n\rightharpoonup u$ in $\mathcal{D}^{1,p}_{a,k}(\Omega)$. Our first result is the following compactness lemma.

Lemma 3.3. *Assume that* (3.2)–(3.4) *are satisfied. If* $u_n\rightharpoonup u$ *in* $\mathcal{D}^{1,p}_{a,k}(\Omega)$, *then* $u_n/|y|^a\to u/|y|^a$ *in* $L^p(A_c)$ *where* $A_c:=\{x=(y,z)\in\Omega: 1/c<|y|$ *and* $|x|<c\}$ *for every* $c>1$.

Proof. The result is clear as A_c is bounded and as the weight $|y|^{-a}$ is bounded on the set considered. $\square$

We can now obtain the following decomposition lemma.

LEMMA 3.4. *Let* $(u_n) \subset \mathcal{D}^{1,p}_{a,k}(\Omega)$ *be such that* $u_n \rightharpoonup u$ *in* $\mathcal{D}^{1,p}_{a,k}(\Omega)$. *We define*

$$\Omega_R := \Omega \cap ((B^k(0,R^{-1}) \times \mathbb{R}^{N-k}) \cup (\mathbb{R}^N \setminus B^N(0,R))),$$
$$\mu_0 := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|\nabla u_n|^p}{|y|^{ap}}\,dx,$$
$$\nu_0 := \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{|y|^{(a+1)p}}\,dx,$$

then we have

$$\overline{\lim_{n\to\infty}} \int_{\Omega} \frac{|u_n|^p}{|y|^{(a+1)p}}\,dx = \int_{\Omega} \frac{|u|^p}{|y|^{(a+1)p}}\,dx + \nu_0, \tag{3.5}$$
$$\overline{\lim_{n\to\infty}} \int_{\Omega} \frac{|\nabla u_n|^p}{|y|^{ap}}\,dx \ge \int_{\Omega} \frac{|\nabla u|^p}{|y|^{ap}}\,dx + \mu_0, \tag{3.6}$$
$$\lambda_{kap}\,\nu_0 \le \mu_0. \tag{3.7}$$

PROOF. As $u_n \to u$ in $L^p_{loc}(\Omega)$, we have for every $R > 0$,

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_{\Omega} \frac{|u_n|^p}{|y|^{(a+1)p}}\,dx &= \lim_{n\to\infty} \int_{\Omega\setminus\Omega_R} \frac{|u_n|^p}{|y|^{(a+1)p}}\,dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{|y|^{(a+1)p}}\,dx,\\ &= \int_{\Omega\setminus\Omega_R} \frac{|u|^p}{|y|^{(a+1)p}}\,dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n|^p}{|y|^{(a+1)p}}\,dx.\end{aligned}$$

So, taking the limit as $R \to \infty$, we obtain (3.5).

We have

$$\begin{aligned}\overline{\lim_{n\to\infty}} \int_{\Omega} \frac{|\nabla u_n|^p}{|y|^{ap}}\,dx &\ge \underline{\lim_{n\to\infty}} \int_{\Omega\setminus\Omega_R} \frac{|\nabla u_n|^p}{|y|^{ap}}\,dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|\nabla u_n|^p}{|y|^{ap}}\,dx,\\ &\ge \int_{\Omega\setminus\Omega_R} \frac{|\nabla u|^p}{|y|^{ap}}\,dx + \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|\nabla u_n|^p}{|y|^{ap}}\,dx,\end{aligned}$$

as the mapping $v \mapsto \int_{\Omega\setminus\Omega_R} (|\nabla v|^p/|y|^{ap})\,dx$ is convex. Taking now the limit as $R \to \infty$, we obtain (3.6).

To prove inequality (3.7) we begin finding new formulations for μ_0 and ν_0. Using inequality (2.6), we obtain :

$$\mu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|\nabla (u_n - u)|^p}{|y|^{ap}}\,dx,$$
$$\nu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \frac{|u_n - u|^p}{|y|^{(a+1)p}}\,dx.$$

We introduce the truncation $\psi_R \in \mathcal{C}^\infty(\mathbb{R}^N)$ with $\psi_R(x) = 0$ if $x \in \Omega \setminus \Omega_R$, $\psi_R(x) = 1$ if $x \in \Omega_{R+1}$ and $0 \le \psi_R \le 1$. With this truncation, we get

$$\mu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega} \frac{|\nabla (u_n - u)|^p \psi_R^p}{|y|^{ap}}\,dx,$$
$$\nu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega} \frac{|u_n - u|^p \psi_R^p}{|y|^{(a+1)p}}\,dx.$$

Using Lemma 3.3 and (2.6) once again we obtain

$$\mu_0 = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_\Omega \frac{|\nabla((u_n - u)\psi_R)|^p}{|y|^{ap}}\, dx.$$

Applying Lemma 3.1 to $(u_n - u)\psi_R$ and taking the limits we get inequality (3.7).□

This decomposition lemma implies the weak lower semi-continuity of J, leading to Theorem 3.2 as in Section 2.

Remark 3.5. When $k = N$, the singularity in (3.1) is radial, i.e. the problem is

$$\begin{cases} -\mathrm{div}\Big(\dfrac{|\nabla u|^{p-2}\nabla u}{|x|^{ap}}\Big) = \lambda \dfrac{|u|^{p-2}u}{|x|^{(a+1)p}} + \dfrac{f(x)}{p} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

By Theorem 3.2, there exist a solution in the closure of $\mathcal{D}^{1,p}_{a,N}(\Omega)$ when $a \geq 0$, $1 < p < N/(a+1)$, $\lambda < \lambda_{Nap}$ and $f \in (\mathcal{D}^{1,p}_{a,N}(\Omega))'$. Moreover, the proof works for every regular domain Ω such that $0 \notin \partial\Omega$.

4. One dimensional case

Here we study the one dimensional problem

$$\begin{cases} -\Big(\dfrac{|u'|^{p-2}u'}{x^{ap}}\Big)' = \lambda \dfrac{|u|^{p-2}u}{x^{(a+1)p}} + \dfrac{f(x)}{p} & \text{in } \mathbb{R}^+, \\ u(0) = 0, \\ \lim\limits_{x\to+\infty} u(x) = 0. \end{cases} \tag{4.1}$$

To obtain solutions of problem (4.1), we consider the associated energy functional

$$J(u) := \int_0^\infty \frac{|u'|^p}{x^{ap}}\, dx - \lambda \int_0^\infty \frac{|u|^p}{x^{(a+1)p}}\, dx - \langle f, u\rangle$$

defined on $\mathcal{D}^{1,p}_a(\mathbb{R}^+)$, which is the closure of $\mathcal{D}(\mathbb{R}^+)$ with respect to the norm $||u|| = (\int_0^\infty (|u'|^p/x^{ap})\, dx)^{1/p}$, where

$$1 < p < \infty, \tag{4.2}$$

$$a \geq 0, \tag{4.3}$$

$$1 < (a+1)p, \tag{4.4}$$

$$f \in (\mathcal{D}^{1,p}_a(\mathbb{R}^+))'.$$

We will consider

$$\inf J := \inf_{u\in\mathcal{D}^{1,p}_a(\mathbb{R}^+)} J(u).$$

The aim of this section is to prove the following result.

Theorem 4.1. *Assume that* (4.2)–(4.4) *are satisfied. If* $\lambda < \lambda_{ap} := (((a+1)p-1)/p)^p$, *then* $\inf J$ *is achieved and problem* (4.1) *admits a solution.*

In order to prove this theorem, we recall the classical Hardy inequality (see e.g. [4], [12]).

Lemma 4.2. *Assume that* (4.2)–(4.4) *are satisfied. If* $u \in \mathcal{D}_a^{1,p}(\mathbb{R}^+)$, *then*

$$\lambda_{ap} \int_0^\infty \frac{|u|^p}{x^{(a+1)p}}\,dx \le \int_0^\infty \frac{|u'|^p}{x^{ap}}\,dx.$$

This inequality implies that the functional J is continuous, G-differentiable and coercive if $\lambda < \lambda_{ap}$. Let $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^+)$ a minimizing sequence for J, i.e.

$$J(u_n) \to \inf J \quad \text{as } n \to \infty.$$

The following lemma is used in the proof of Lemma 4.4.

Lemma 4.3. *Assume that* (4.2)–(4.4) *are satisfied. If* $u_n \rightharpoonup u$ *in* $\mathcal{D}_a^{1,p}(\mathbb{R}^+)$, *then* $u_n/x^a \to u/x^a$ *in* $L^p([1/c,c])$ *for every* $c > 1$.

Proof. The result is clear as x^{-a} is bounded on the interval considered. □

The following decomposition lemma is proved as before.

Lemma 4.3. *Let* $(u_n) \subset \mathcal{D}_a^{1,p}(\mathbb{R}^+)$ *be such that* $u_n \rightharpoonup u$ *in* $\mathcal{D}_a^{1,p}(\mathbb{R}^+)$. *We define*

$$\begin{aligned}
\Omega_R &:= (0, R^{-1}) \cup (R, \infty),\\
\mu_0 &:= \lim_{R\to\infty} \varlimsup_{n\to\infty} \int_{\Omega_R} \frac{|u_n'|^p}{x^{ap}}\,dx,\\
\nu_0 &:= \lim_{R\to\infty} \varlimsup_{n\to\infty} \int_{\Omega_R} \frac{|u_n|^p}{x^{(a+1)p}}\,dx,
\end{aligned}$$

then we have

$$\begin{aligned}
\varlimsup_{n\to\infty} \int_0^\infty \frac{|u_n|^p}{x^{(a+1)p}}\,dx &= \int_0^\infty \frac{|u|^p}{x^{(a+1)p}}\,dx + \nu_0,\\
\varlimsup_{n\to\infty} \int_0^\infty \frac{|u_n'|^p}{x^{ap}}\,dx &\ge \int_0^\infty \frac{|u'|^p}{x^{ap}}\,dx + \mu_0,\\
\lambda_{ap}\,\nu_0 &\le \mu_0.
\end{aligned}$$

Theorem 4.1 follows as Theorem 3.2.

5. Exterior domain-case

In this section, we study the problem

$$\begin{cases} -\Delta_p u = \lambda \dfrac{|u|^{p-2}u}{\delta(x)^p} + \dfrac{f(x)}{p} & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega,\\ \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases} \tag{5.1}$$

where Ω is an exterior domain, i.e. an open subset of $\mathbb{R}^N$ with a compact boundary of class $\mathcal{C}^2$ and without bounded component and $\delta(x)$ denote the function

$$\delta(x) = \operatorname{dist}(x, \partial\Omega).$$

We also suppose that $N \geq 2$, $1 < p \neq N$ and $f \in (\mathcal{D}_0^{1,p}(\Omega))'$. To obtain a solution of problem (5.1) we introduce the associated functional

$$J(u) := \int_\Omega |\nabla u|^p \, dx - \lambda \int_\Omega \left|\frac{u}{\delta}\right|^p \, dx - \langle f, u\rangle,$$

defined on $\mathcal{D}_0^{1,p}(\Omega)$. Here we use the Hardy's inequality for exterior domains (see [1]).

LEMMA 5.1. *Let $N \geq 2$, $p > 1$ and $p \neq N$, then there exists $\lambda_{Np} > 0$ such that for every $u \in \mathcal{D}_0^{1,p}(\Omega)$,*

$$\lambda_{Np} \int_\Omega \left|\frac{u}{\delta}\right|^p \, dx \leq \int_\Omega |\nabla u|^p \, dx.$$

The main result of this section is the following theorem.

THEOREM 5.2. *Let $N \geq 2$, and $1 < p \neq N$. If $\lambda < \lambda_{Np}$, then* $\inf J$ *on $\mathcal{D}_0^{1,p}(\Omega)$ is achieved and problem* (5.1) *has a solution.*

As the proof of this theorem is very similar to the proofs of Theorems 2.1, 3.2 and 4.1, we only sketch the main ideas.

As in Sections 2–4 the Hardy inequality implies that the functional J is continuous, G-differentiable and coercive if $\lambda < \lambda_{Np}$. So, let $(u_n) \subset \mathcal{D}_0^{1,p}(\Omega)$ a minimizing sequence for J. We can assume that $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. Working as before we obtain the following decomposition lemma.

LEMMA 5.3. *Let $N \geq 2$, $1 < p \neq N$ and $(u_n) \subset \mathcal{D}_0^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ in $\mathcal{D}_0^{1,p}(\Omega)$. We define*

$$\begin{aligned}
\Omega_R &:= \left\{x \in \Omega : \delta(x) < R^{-1} \text{ or } |x| > R\right\}, \\
\mu_0 &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} |\nabla u_n|^p \, dx, \\
\nu_0 &:= \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{\Omega_R} \left|\frac{u_n}{\delta}\right|^p \, dx,
\end{aligned}$$

then we have

$$\begin{gathered}
\overline{\lim_{n\to\infty}} \int_\Omega \left|\frac{u_n}{\delta}\right|^p \, dx = \int_\Omega \left|\frac{u}{\delta}\right|^p \, dx + \nu_0, \\
\overline{\lim_{n\to\infty}} \int_\Omega |\nabla u_n|^p \, dx \geq \int_\Omega |\nabla u|^p \, dx + \mu_0, \\
\lambda_{Np}\, \nu_0 \leq \mu_0.
\end{gathered}$$

To prove Theorem 5.1 it suffices to adapt the proof of Theorem 2.1.

REMARK 5.4. The same result is valid for bounded $\mathcal{C}^2$ domains for any $p > 1$.

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