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TOPOLOGIES ON THE GROUP OF HOMEOMORPHISMS OF A CANTOR SET

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Abstract. Let Homeo(Ω) be the group of all homeomorphisms of a Cantor set Ω. We study topological properties of Homeo(Ω) and its subsets with respect to the uniform (τ) and weak (τ_w) topologies. The classes of odometers and periodic, aperiodic, minimal, rank 1 homeomorphisms are considered and the closures of those classes in τ and τ_w are found.

1. Introduction

The present paper is a continuation of our article [1] about topologies on the group $\mathrm{Aut}(X,\mathcal{B})$ of all Borel automorphisms of a standard Borel space. In the introduction to that article, we discussed our approach to the study of topologies on groups of transformations of an underlying space. As we mentioned there, we were motivated, first of all, by remarkable results in ergodic theory concerning topological properties of the group of all automorphisms of a standard measure space. We refer to the classical articles of Halmos [10] and Rokhlin [16] where the uniform and weak topologies appeared as "key players" in ergodic theory.

The central object of the present paper is the group Homeo(Ω) of all homeomorphisms of a Cantor set Ω. Although we consider several topologies on

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Homeo(Ω), this group is mostly studied under two topologies, τ and τ_w. These are analogues of the uniform and weak topologies in measurable dynamics. We should remark that τ_w is, in fact, the usual sup-topology of uniform convergence which has occurred in many papers on topological dynamics (see, e.g. [7], [9]). Many interesting questions can be asked about the topological properties of Homeo(Ω) and its subsets. For instance, E. Glasner and B. Weiss studied in [9] the Rokhlin property on $(\mathrm{Homeo}(\Omega), \tau_w)$ showing that the action of Homeo(Ω) on itself by conjugation has dense orbits. In this article, we will concentrate on the following directions, which we believe are natural initial questions in this theory: (i) global properties of some basic topologies on Homeo(Ω), (ii) finding closures of subsets of Homeo(Ω) consisting of periodic, aperiodic, minimal, topologically transitive, rank 1 homeomorphisms, and odometers in τ and τ_w.

It might be asked why we consider only Cantor sets as the underlying space. First of all, we remark that Cantor sets and their homeomorphisms arise naturally in various areas of dynamical systems, for example in fractals, low-dimensional dynamics etc. Although topological and measurable dynamics are, strictly speaking, completely different theories, we believe that Cantor dynamics has several features in common with measurable dynamics. To support this point of view we refer to the results on orbit equivalence of minimal homeomorphisms and full groups proved in [6]–[8], [11], [3], [4]. We believe that the following properties of Cantor sets underlie this similarity: (a) all Cantor sets are homeomorphic; (b) for every Cantor set, there exists a countable family of clopen sets generating the topology; (c) any Cantor set can be partitioned into a finite collection of clopen subsets. Nevertheless, we are optimistic that some ideas of this paper may be used in the context of general topological dynamics.

The paper is organized as follows. In Section 2, we introduce several topologies on Homeo(Ω) and study global topological properties of Homeo(Ω) mostly with respect to τ and τ_w. All possible relations between these topologies are found in Theorem 2.3. We mention the curious fact that $(\mathrm{Homeo}(\Omega), \tau_w)$ is a zero-dimensional Polish space. It turns out that τ_w is equivalent to the topology p whose base of neighbourhoods is defined by $W(T; F_1, \ldots, F_k) = \{S \in \mathrm{Homeo}(\Omega) \mid SF_i = TF_i,\ i = 1, \ldots, k\}$ where F_i is clopen. This fact is a justification of the name "weak" topology which we use for τ_w. Section 3 deals principally with the problem of approximation by periodic homeomorphisms. We prove a topological version of the Rokhlin lemma for minimal homeomorphisms for both τ and τ_w. On the other hand, we show that pointwise periodic homeomorphisms are not dense in $(\mathrm{Homeo}(\Omega), \tau_w)$. Amongst other results, we obtain a description of periodic and aperiodic homeomorphisms from the topological full group of a minimal homeomorphism. In Section 4, we consider homeomorphisms of rank 1 and show that they are necessarily odometers. In the last section, we study closures of various subsets in Homeo(Ω) with respect to τ and τ_w. In particular, we prove that the τ_w-closure of the set of minimal homeomorphisms is the same as the closure of the set of odometers. Moreover, we give a dynamical description of homeomorphisms which belong to the closure: $T \in$ Homeo(Ω) belongs to the τ_w-closure of the set of minimal homeomorphisms if and only if T has the following property: for every non-trivial clopen F, the sets $TF \setminus F$ and $F \setminus TF$ are non-empty.

Throughout the paper, we use the following standard notation:

- Ω is a Cantor set;
- CO(Ω) is the family of all clopen subsets in Ω;
- Homeo(Ω) is the group of all homeomorphisms of Ω with identity map $\mathbb{I} \in$ Homeo(Ω);
- Aut$(X,\mathcal{B})(X,B)$ is the group of all one-to-one Borel automorphisms of a standard Borel space $(X,\mathcal{B})$;
- $\mathcal{A}p$ is the set of all aperiodic homeomorphisms;
- $\mathcal{P}er$ is the set of all pointwise periodic homeomorphisms and $\mathcal{P}er_0$ is the subset of $\mathcal{P}er$ consisting of homeomorphisms with finite period;
- $\mathcal{M}in$ is the set of all minimal homeomorphisms;
- $\mathcal{M}ix$ is the set of all mixing homeomorphisms;
- $\mathcal{M}_1(\Omega)$ is the set of all Borel probability measures on Ω;
- δ_x is the Dirac measure at $x \in \Omega$;
- $E(S,T) = \{x \in \Omega \mid Tx \neq Sx\} \cup \{x \in X \mid T^{-1}x \neq S^{-1}x\}$ where $S,T \in$ Homeo(Ω);
- $\mu(f) = \int_X f\, d\mu$ where f is in $C(\Omega)_1$ (= the set of continuous real-valued functions with $\|f\| := \sup\{|f(x)| : x \in \Omega\} \leq 1$), and $\mu \in \mathcal{M}_1(\Omega)$;
- $\mu \circ S(A) := \mu(SA)$ and $\mu \circ S(f) := \int_\Omega f\, d(\mu \circ S) = \int_\Omega f(S^{-1}x)\, d\mu(x)$ where $S \in$ Homeo(Ω);
- $A^c = \Omega \setminus A$.

2. Topologies on Homeo(Ω)

In this section, we define several topologies on Homeo(Ω). These topologies are similar to those studied in [1] for Aut$(X,\mathcal{B})$. We make the following (rather obvious) changes to the settings of [1]: a standard Borel space $(X,\mathcal{B})$ is replaced by a Cantor set Ω, and Borel sets and functions are replaced by clopen sets and continuous functions.

Definition 2.1 (cf. [1, Definition 2.1]).

(a) The uniform topology τ on Homeo(Ω) is defined as the relative topology on Homeo(Ω) induced from (Aut$(\Omega,\mathcal{B}),\tau$). The base of neighbourhoods is formed by

$$U(T;\mu_1,\dots,\mu_n;\varepsilon)=\{S\in\mathrm{Homeo}(\Omega))\mid \mu_i(E(S,T))<\varepsilon,\ i=1,\dots,n\}. \tag{2.1}$$

(b) The topology τ' is defined on Homeo(Ω) by the base of neighbourhoods

$$\begin{aligned}&U'(T;\mu_1,\dots,\mu_n;\varepsilon)\\&\quad=\Big\{S\in\mathrm{Homeo}(\Omega)\ \Big|\ \sup_{F\in\mathrm{CO}(\Omega)}\mu_i(TF\ \Delta\ SF)<\varepsilon,\ i=1,\dots,n\Big\}.\end{aligned}\tag{2.2}$$

(c) The topology τ'' is defined on Homeo(Ω) by the base of neighbourhoods

$$\begin{aligned}&U''(T;\mu_1,\dots,\mu_n;\varepsilon)\\&\quad=\Big\{S\in\mathrm{Homeo}(\Omega)\ \Big|\ \sup_{f\in C(\Omega)_1}|\mu_i\circ S(f)-\mu_i\circ T(f)|<\varepsilon,\ i=1,\dots,n\Big\}.\end{aligned}\tag{2.3}$$

(d) The topology p is defined on Homeo(Ω) by the base of neighbourhoods

$$W(T;F_1,\dots,F_k)=\{S\in\mathrm{Homeo}(\Omega)\mid SF_i=TF_i,\ i=1,\dots,k\}. \tag{2.4}$$

(e) The topology $\overline{p}$ is defined on Homeo(Ω) by the base of neighbourhoods

$$\begin{aligned}&\overline{W}(T;F_1,\dots,F_k;\mu_1,\dots,\mu_n;\varepsilon)=\{S\in\mathrm{Homeo}(\Omega)\mid\\&\quad\mu_j(SF_i\ \Delta\ TF_i)+\mu_j(S^{-1}F_i\ \Delta\ T^{-1}F_i)<\varepsilon,\ i=1,\dots,n;j=1,\dots,k\}.\end{aligned}\tag{2.5}$$

In all the above definitions, we have taken $T\in\mathrm{Homeo}(\Omega)$, $\mu_i\in\mathcal{M}_1(\Omega)$, and $F_i\in\mathrm{CO}(\Omega)$, $i=1,\dots,n$.

It is a simple exercise to verify that the collections of sets so defined do indeed form bases of topologies.

As in [1], we will study the topologies which are defined by their bases of neighbourhoods. With some abuse of definition, we will say that two topologies are *equivalent* if they are defined by equivalent bases of neighbourhoods (actually such topologies coincide).

We remark that in the Borel case (see [1]) we have also defined the topologies $\tau_B,\tau'_B,\tau''_B,p_B$, and $\overline{p}_B$ (the subindex B stands for Borel; in [1] these topologies were denoted without B). In fact, only one of them, τ, is the relative topology induced on Homeo(Ω) from $(\mathrm{Aut}(\Omega,\mathcal{B}),\tau_B)$. The others are not relative topologies on Homeo(Ω) because in their definition we use clopen subsets and continuous functions instead of Borel ones (see (2.2)–(2.5)). Obviously, τ'_B,τ''_B,p_B, and $\overline{p}_B$ being induced onto Homeo(Ω) from $\mathrm{Aut}(\Omega,\mathcal{B})$ are at least not weaker than the corresponding topologies τ',τ'',p, and $\overline{p}$. Thus, we have to deal here with the topological counterparts of topologies studied in [1]. Nevertheless, we will see that the greater part of our results about relations between the topologies proved in [1] is still true in the context of homeomorphisms of Cantor sets. In most cases, the proofs for homeomorphisms are either word for word repetition of those in the Borel case or can be easily adapted.

Definition 2.2. For $S, T \in \mathrm{Homeo}(\Omega)$, define

$$d_w(S,T) = \sup_{x\in\Omega} d(Sx,Tx) + \sup_{x\in\Omega} d(S^{-1}x,T^{-1}x). \tag{2.6}$$

Denote by τ_w the topology on $\mathrm{Homeo}(\Omega)$ generated by the metric d_w.

The topology τ_w is well known in topological dynamics and probably is generally considered as the most natural topology on $\mathrm{Homeo}(\Omega)$. In particular, it can easily be seen that τ_w is equivalent to the topology defined by the base of neighbourhoods $\widetilde{W}(T; f_1,\dots,f_n;\varepsilon) = \{S \in \mathrm{Homeo}(\Omega) \mid \|f_i\circ T - f_i \circ S\| < \varepsilon$, $i = 1,\dots,n\}$ where $f_1,\dots,f_n$ are $\mathbb{Z}$-valued continuous functions on Ω. The proof of this fact is similar to that of [1, Theorem 4.7] for Borel dynamics.

We call it the *weak topology* following our point of view explained in [3], [4] (see also Theorem 2.3(d) below).

It is well known that $(\mathrm{Homeo}(\Omega), \tau_w)$ is a Polish space (for every compact metric space Ω). By $B_\delta(T)$, we denote the set $\{S \in \mathrm{Homeo}(\Omega) \mid d_w(S,T) < \delta\}$, $T \in \mathrm{Homeo}(\Omega)$.

Our first main result is the following

Theorem 2.3.

(a) *The topologies τ and τ' are equivalent.*
(b) *The topology τ $(\sim \tau')$ is strictly stronger than τ''.*
(c) *The topology τ $(\sim \tau')$ is strictly stronger than $\overline{p}$.*
(d) *The topology τ_w is equivalent to p.*
(e) *The topology p $(\sim \tau_w)$ is strictly stronger than $\overline{p}$.*
(f) *The topology τ is not comparable with τ_w $(\sim p)$ and the topology τ'' is not comparable with $\overline{p}$.*

Proof. A direct analogue of this theorem was proved in the context of Borel dynamics in [1]. The principal difference is here that one needs to work with clopen sets instead of Borel sets. We will indicate only what modifications need to be made for the use of $\mathrm{Homeo}(\Omega)$.

(a) We follow the idea of the proof of [1, Theorem 4.2]. The fact that $\tau \succ \tau'$ may be proved as in [1]. To prove that $\tau' \succ \tau$, we show that each neighbourhood $U = U(\mathbb{I}; \mu_1,\dots,\mu_n;\varepsilon)$ contains $U' = U'(\mathbb{I};\mu_1,\dots,\mu_n;\varepsilon/2)$. By definition, $T \in U'(\mathbb{I};\mu_1,\dots,\mu_n;\varepsilon/2)$ if $\mu_i(TF \,\Delta\, F) < \varepsilon/2$ for all clopen F and all $i = 1,\dots,n$. We first note that $E(T,\mathbb{I})$ is open, T-invariant, and contains some clopen E_0 such that $E_0 \cap TE_0 = \emptyset$. Thus one of the following alternatives must hold:

(i) for every clopen $F \subset E(T,\mathbb{I})$ with $F\cap TF = \emptyset$ there exists a clopen set $F' \supset F$ such that $F' \cap TF' = \emptyset$; or
(ii) there exists a clopen $F_0 \subset E(T,\mathbb{I})$, $F_0 \cap TF_0 = \emptyset$ which cannot be extended to a large set F' preserving disjointness of F' and TF' (in other words, each clopen set $F' \supset F_0$ has a nonempty intersection with TF').

Clearly, condition (ii) is equivalent to the following property: $F_0 \cup TF_0 \cup T^2F_0 = E(T,\mathbb{I})$. Therefore, in this case,

$$\mu_i(E(T,\mathbb{I})) \le \mu_i(TF_0 \,\Delta\, F_0) + \mu_i(TF_0 \,\Delta\, T^2F_0) < \varepsilon.$$

If (i) holds, then one can find $F_1 \in \mathrm{CO}(\Omega)$ such that $F_1 \cap TF_1 = \emptyset$ and $\mu_i(E(T,\mathbb{I}) - (F_1 \cup TF_1)) < \varepsilon/2$. This implies that $\mu_i(E(T,\mathbb{I})) < \varepsilon$, $i = 1,\dots,n$, and by (2.1), we are done.

(b) The method of proof that τ is strictly stronger than τ'' is the same as in [1, Theorem 4.6]. To show that a statement analogous to [1, Proposition 4.5] holds, we need to use clopen sets in the definitions of two auxiliary topologies $\widetilde{\tau}$ and $\overline{\tau}$ as well as continuous functions in the proof of that proposition (see [1, Remark 4.9]). In particular, one sees that the topology τ'' is equivalent on $\mathrm{Homeo}(\Omega)$ to the topology $\overline{\tau}$ defined by the base

$$\begin{aligned}(2.7)\quad &\overline{V}(T;\mu_1,\dots,\mu_n;\varepsilon)\\ &= \Big\{S \in \mathrm{Aut}(X,\mathcal{B}) \;\Big|\; \sup_{F\in \mathrm{CO}(\Omega)} |\mu_j(TF) - \mu_j(SF)| < \varepsilon,\ j = 1,\dots,n\Big\},\end{aligned}$$

where $T \in \mathrm{Homeo}(\Omega)$ and $\mu_i \in \mathcal{M}_1(\Omega)$.

(c) The proof is a word for word repetition of [1, Proposition 4.3].

(d) Fix some $\delta > 0$ and let $\mathcal{Q} = (F_i)_{i=1}^n$ be a partition of Ω into clopen sets such that $\mathrm{diam}(F_i) < \delta$, $i = 1,\dots,n$. If $S \in W(\mathbb{I};F_1,\dots,F_n)$, then $SF_i = F_i$, and therefore $\sup_{x\in\Omega} d(Sx,x) + \sup_{x\in\Omega} d(S^{-1}x,x) \le 2\delta$. This proves that $B_{2\delta}(\mathbb{I}) \supset W(\mathbb{I};F_1,\dots,F_n)$.

Conversely, let $W(\mathbb{I};\ F_1,\dots,F_n)$ be given. Take the partition $\mathcal{Q} = (E_i)_{i\in I}$ which is generated by all F_i and $F_i^c = \Omega - F_i$, $i = 1,\dots,n$. Take $\varepsilon > 0$ such that

$$\varepsilon < \min\{\min_{i\ne j} \mathrm{dist}(E_i,E_j),\ \min_i(\mathrm{diam}(E_i))\}.$$

Then, every $S \in B_\varepsilon(\mathbb{I})$ has the property $SE_i = E_i$, i.e. $\mathcal{Q}$ is fixed. Therefore $SF_k = F_k$, $k = 1,\dots,n$, because every F_k is a union of some E_i's. Thus, $S \in W(\mathbb{I};F_1,\dots,F_k)$.

(e) As an immediate corollary of the equivalence proved in (d), we obtain $\tau_w \succ \overline{p}$. To see that $\overline{p}$ is strictly weaker than τ_w, we note that $\overline{p}$ is weaker than the topology $\tau' \sim \tau$. If we assumed that $\overline{p}$ was equivalent to τ_w, we would have that τ is always stronger than τ_w. But the latter is false (see (f) or [3]).

(f) See [1, Proposition 4.8] where the pairs τ'' and $\overline{p}$ have been considered. The fact that τ and τ_w are not comparable is a direct consequence of [3, Theorem 4.8]. □

Now we formulate several statements concerning topological properties of $\mathrm{Homeo}(\Omega)$.

Proposition 2.4.

(a) $(\mathrm{Homeo}(\Omega), p)$ *is a* 0*-dimensional complete metric space with no isolated points.*

(b) $\mathrm{Homeo}(\Omega)$ *is a Hausdorff topological group with respect to the topologies* τ, τ', τ'', p, τ_w.

(c) $\mathrm{Homeo}(\Omega)$ *is not closed in* $(\mathrm{Aut}(\Omega, \mathcal{B}), \tau)$.

(d) $\mathrm{Homeo}(\Omega)$ *is dense in* $(\mathrm{Aut}(\Omega, \mathcal{B}), \tau)$.

Proof. The proof of the first statement follows easily from [1, Proposition 2.11], replacing Borel sets by clopen sets. In fact it can be shown that *the sets* $W(T; F_1, \dots, F_n)$ *(see* (2.4)*) are closed with respect to the topologies* τ, τ'', p *and* $\overline{p}$. The second statement of the proposition is based on a routine verification (see [1]). The third assertion is taken from [3].

(d) We need to show that for any Borel automorphism T of $(\Omega, \mathcal{B})$, for any $\varepsilon > 0$, and for any $\mu_1, \dots, \mu_n \in \mathcal{M}_1(\Omega)$ there exists a homeomorphism S of Ω such that $\mu_i(E(S,T)) < \varepsilon$, $1 = 1, \dots, n$. By Lusin's theorem, we can find a closed subset F_i of Ω such that the restriction of T to F_i is a one-to-one continuous map from F_i onto $T(F_i)$ and

$$\mu_i(\Omega \setminus F_i) < \frac{\varepsilon}{2}, \quad \mu_i \circ T(\Omega \setminus F_i) < \frac{\varepsilon}{2}, \quad i = 1, \dots, n.$$

Let $F = \bigcup_{i=1}^n F_i$. Then F is closed, T is continuous on F, and $\mu_i(\Omega \setminus F) + \mu_i(\Omega \setminus T(F)) < \varepsilon$ for all i.

Since F and TF are closed, we can represent $\Omega \setminus F$ and $\Omega \setminus TF$ as unions of infinitely many clopen sets: $\Omega \setminus F = \bigcup_{j=1}^\infty A_j$ and $\Omega \setminus TF = \bigcup_{j=1}^\infty A'_j$. Then by Theorem 1 of [13], the continuous map $T \colon F \to TF$ can be extended to a homeomorphism S of Ω such that $Tx = Sx$, $x \in F$ and $T^{-1}x = S^{-1}x$, $x \in TF$. Clearly, $\mu_i(E(S,T)) < \varepsilon$, $i = 1, \dots, n$. $\square$

Convention. As mentioned above, $\mathrm{Homeo}(\Omega)$ is not closed in $\mathrm{Aut}(\Omega, \mathcal{B})$ in the uniform topology τ, therefore the τ-closure of a subset $Y \subset \mathrm{Homeo}(\Omega)$ does not belong to $\mathrm{Homeo}(\Omega)$, in general. For convenience, we will use the following convention $\overline{Y}^\tau := \overline{Y}^\tau \cap \mathrm{Homeo}(\Omega)$ without further explanation.

Proposition 2.5. *Let* (T_n) *be a sequence of homeomorphisms of* Ω. *Then:*

(a) $T_n \xrightarrow{\tau} S$ *if and only if for all* $x \in \Omega$ *there exists* $n(x) \in \mathbb{N}$ *such that* $T_n x = Sx$ *for all* $n > n(x)$.

(b) $T_n \xrightarrow{p} \mathbb{I}$ *if and only if for all* $F \in \mathrm{CO}(\Omega)$ *there exists* $n(F)$ *such that* $T_n F = F$ *for all* $n > n(F)$.

(c) $T_n \xrightarrow{\overline{p}} \mathbb{I}$ *if and only if for all* $\mu \in \mathcal{M}_1(\Omega)$ *and for all* $F \in \mathrm{CO}(\Omega)$

$$\mu(T_n F \,\Delta\, F) + \mu(T_n^{-1} F \,\Delta\, F) \to 0 \tag{2.8}$$

or if and only if for all $F \in \mathrm{CO}(\Omega)$,

$$F = \limsup_{n\to\infty} T_n F = \limsup_{n\to\infty} T_n^{-1} F, \tag{2.9}$$

where

$$\limsup_{n\to\infty} T_n F = \bigcup_m \bigcap_{n>m} T_n F.$$

Proof. Notice that (a) is proved in [3] and (b) is obvious. Relation (2.8) is a direct consequence of the definitions. To prove the other equivalence in (c), we note that for any $x \in \Omega$ and $F \in \mathrm{CO}(\Omega)$, the convergence $T_n \xrightarrow{\overline{p}} \mathbb{I}$ implies that

$$\delta_x(T_n F \,\Delta\, F) + \delta_x(T^{-1}F \,\Delta\, F) \to 0$$

as $n \to \infty$. This means that if $x \in F$, then there exists $n_0 = n_0(x,F)$ such that $x \in T_n F$ and $x \in T_n^{-1}F$ for all $n > n_0$. Thus, we have proved that $F \subset \bigcup_m \bigcap_{n>m} T_n F$ and $F \subset \bigcup_m \bigcap_{n>m} T_n^{-1} F$. In fact, these inclusions are equalities. Indeed, if we assume that there exists $x_0 \in F^c = \Omega - F$ with $x_0 \in \bigcap_{n>m} T_n F$ for some m, then we have a contradiction to the fact that x_0 also belongs to $\bigcup_k \bigcap_{n>k} T_n F^c$. Thus, (2.9) holds.

Conversely, let $E_m = \bigcap_{n>m} T_n F$ and $\bigcup_m E_m = F$. Since $E_m \subset E_{m+1}$, we see that for any measure $\mu \in \mathcal{M}_1(\Omega)$, $\mu E_m \to \mu F$ $(m \to \infty)$. Remark that $E_m \subset T_n F$ for all $n > m$. Therefore $E_m = E_m \cap T_n F \subset F \cap T_n F \subset F$. Thus, we have $\mu(F \cap T_n F) \to \mu F$ as $n \to \infty$. Similarly $\mu(F \cap T_n^{-1} F) \to \mu F$. By (2.8), the proof is complete. □

3. Periodic approximation

Let Ω be a Cantor set equipped with a metric d compatible with the clopen topology. It is natural to distinguish two principal classes of homeomorphisms of Ω, the periodic and the aperiodic. We will say that $P \in \mathrm{Homeo}(\Omega)$ is *pointwise periodic* if every P-orbit is finite. If $T \in \mathrm{Homeo}(\Omega)$ has no periodic points, then T is called *aperiodic*. Denote these classes by $\mathcal{P}er$ and $\mathcal{A}p$, respectively.

In the paper [9] a new interesting notion of simple homeomorphisms was defined. Recall that, by definition, $S \in \mathrm{Homeo}(\Omega)$ is *simple* if it satisfies the following conditions:

(i) There exist clopen subsets F_j and integers $r_j \geq 1$, $j = 1, \ldots, k$, such that the collection $\{S^i F_j \mid i = 0, 1, \ldots, r_j,\ j = 1, \ldots, k\}$ is pairwise disjoint and S has period r_j on F_j.

(ii) There exist clopen subsets C_s, $s = 1, \ldots, l$, and, for each s, two disjoint periodic orbits $(y_s^+, S y_s^+, \ldots, S^{q_s^+ - 1} y_s^+)$, $(y_s^-, S y_s^-, \ldots, S^{q_s^- - 1} y_s^-)$ such that the sets $(S^n C_s \mid n \in \mathbb{Z},\ s = 1, \dots, l)$ are pairwise disjoint and spiral towards the periodic orbits of y_s^+ and y_s^-, that is

$$\lim_{n\to\pm\infty} \operatorname{dist}(S^n C_s, S^n y_s^{\pm}) = 0.$$

(iii) The space Ω may be represented as

$$\Omega = \bigcup_{j=1}^{k} \bigcup_{i=0}^{r_j-1} S^i F_j \cup \bigcup_{s=1}^{l} \bigcup_{n\in\mathbb{Z}} S^n C_s \cup \bigcup_{s=1}^{l} [(y_s^+, \dots, S^{q_s^+-1} y_s^+) \cup (y_s^-, \dots, S^{q_s^- -1} y_s^-)]. \tag{3.1}$$

It was shown in [9, Theorem 2.2] that the set $\mathcal{S}$ of simple homeomorphisms is dense in $(\mathrm{Homeo}(\Omega), \tau_w)$.

Theorem 3.1.

(a) *$\mathcal{A}p$ is closed in* $\mathrm{Homeo}(\Omega)$ *with respect to the topologies τ and τ''.*

(b) *$\mathcal{A}p$ is dense in* $\mathrm{Homeo}(\Omega)$ *with respect to τ_w and $\overline{p}$.*

Proof. (a) The fact that $\overline{\mathcal{A}p}^{\tau} = \mathcal{A}p$ may be proved in the same way as in [1]. (Recall that by the convention from Section 2 we take the part of the τ-closure of $\mathcal{A}p$ that lies in $\mathrm{Homeo}(\Omega)$). Furthermore, since $\tau \succ \tau''$, we have $\overline{\mathcal{A}p}^{\tau''} \supseteq \overline{\mathcal{A}p}^{\tau}$. To prove (a), we need to show that the above inclusion is in fact equality. We will use the equivalence of τ'' and $\overline{\tau}$ (see (2.7)). Let $S \in \overline{\mathcal{A}p}^{\overline{\tau}}$ and assume S has a point x_0 of period n. Then $(x_0, x_1, \dots, x_{n-1})$ is a finite S-periodic orbit, where $x_i = S^i x_0$ and $S^n x_0 = x_0$. Take $\mu_i = \delta_{x_i}$, $i = 0, \dots, n-1$, and consider an arbitrary homeomorphism T from $\overline{V} = \overline{V}(S; \mu_0, \dots, \mu_{n-1}; \varepsilon)$. It follows that T has the same periodic orbit $(x_0, \dots, x_{n-1})$. To see this, assume that $Tx_0 \neq Sx_0$. Then there exists a clopen set F containing Tx_0 which does not contain Sx_0. By (2.7) this contradicts the fact that $T \in \overline{V}$. Similarly, one can show that $Tx_i = Sx_i$ for $i = 1, \dots, n-1$. Thus, every such homeomorphism T has a periodic orbit. This contradicts our assumption that there exists some $S \in \overline{\mathcal{A}p}^{\tau''} \setminus \mathcal{A}p$.

We remark that $\mathcal{A}p$ is not closed in τ_w. Indeed, one can easily find a sequence of aperiodic homeomorphisms that converges to the identity map in τ_w.

(b) To prove that $\mathcal{A}p$ is dense in $(\mathrm{Homeo}(\Omega), \tau_w)$ (hence in $(\mathrm{Homeo}(\Omega), \overline{p})$), it suffices to show that each simple homeomorphism can be approximated by an aperiodic homeomorphism. We use the above notation from the definition of simple homeomorphisms.

Let S be a simple homeomorphism and let $\varepsilon > 0$. Denote by

$$\widetilde{F}_j = \bigcup_{i=0}^{r_j-1} S^i F_j, \quad j = 1, \dots, k,$$

$$\widetilde{C}_s = \bigcup_{n\in\mathbb{Z}} S^n C_s \cup \{y_s^+, \dots, S^{q_s^+-1} y_s^+\} \cup \{y_s^-, \dots, S^{q_s^- -1} y_s^-\}, \quad s = 1, \dots, l.$$

The sets $\widetilde{F}_1, \dots, \widetilde{F}_k$ and $\widetilde{C}_1, \dots, \widetilde{C}_l$ are clopen, disjoint, S-invariant and by (3.1)

$$\Omega = \bigcup_{j=1}^{k} \widetilde{F}_j \cup \bigcup_{s=1}^{l} \widetilde{C}_s.$$

Given ε, we will find an aperiodic homeomorphism T such that

$$d_w(S,T) = \sup_{x\in\Omega} d(Tx, Sx) + \sup_{x\in\Omega} d(T^{-1}x, S^{-1}x) < \varepsilon. \tag{3.2}$$

To do this, it suffices to find aperiodic homeomorphisms $P_j: \widetilde{F}_j \to \widetilde{F}_j$ and $R_s: \widetilde{C}_s \to \widetilde{C}_s$ satisfying (3.2) on the sets $\widetilde{F}_j$ and $\widetilde{C}_s$ for all j, s. To construct P_j, $j = 1, \dots, k$, we divide the S-tower $(F_j, \dots, S^{r_j-1}F_j)$ into finitely many clopen subtowers $(F_{jm}, \dots, S^{r_j-1}F_{jm})$, $m = 1, \dots, m_j$, such that $\operatorname{diam}(S^i F_{jm}) < \varepsilon$ for all i and m. Let $P_j(m)$ be an aperiodic homeomorphism of F_{jm}. Define $P_j x = Sx$ for $x \in \bigcup_{i=1}^{r_j-1} S^i F_{jm}$ and $P_j x = SP_j(m)x$ for $x \in F_{jm}$, $m = 1, \dots, m_j$. By construction, P_j maps $\widetilde{F}_j$ onto itself and $d_w(P_j, S) < \varepsilon$ on each $\widetilde{F}_j$, $j = 1, \dots, k$.

Fix some $s \in \{1, \dots, l\}$. To construct an aperiodic homeomorphism R_s of $\widetilde{C}_s$ such that $d_w(S, R_s) < \varepsilon$, we will use the following property:

- $(*)$ given a proper clopen subset A of a Cantor set Z, one can find a sequence of disjoint clopen sets $A_1 = A, A_2, \dots$ in Z and a homeomorphism $R: Z \to Z \setminus A$ such that
  - (i) the set $Z \setminus \bigcup_{j=1}^{\infty} A_j$ is uncountable,
  - (ii) $RA_j = A_{j+1}$, $R(Z \setminus \bigcup_{j=1}^{\infty} A_j) = X \setminus \bigcup_{j=1}^{\infty} A_j$ and R is aperiodic on $Z \setminus \bigcup_{j=1}^{\infty} A_j$.

Let a be the minimum of distances between the points $\{S^i y_s^+, S^j y_s^- \mid i = 0, \dots, q_s^+ - 1;\ j = 0, \dots, q_s^- - 1\}$. Given $0 < \varepsilon < a/2$, we can find n_0 such that $\operatorname{dist}(S^n C_s, S^n y_s^+) < \varepsilon/4$ and $\operatorname{diam}(S^n C_s) < \varepsilon/4$ for $n \geq n_0$. Without loss of generality we can assume that $n_0 \equiv 0 \mod (q_s^+)$.

Denote by

$$B_p = \bigcup_{i=0}^{\infty} S^{n_0+p+iq_s^+}(C_s) \cup \{S^p y_s^+\}, \quad p = 0, \dots, q_s^+ - 1.$$

The set B_p is clopen and $\operatorname{diam}(B_p) < \varepsilon/2$ for each p. Observe that $SB_p = B_{p+1}, p = 0, \dots, q_s^+ - 2$, and $SB_{q_s^+-1} = B_0 \setminus A$ where $A = S^{n_0} C_s$. Now we can apply property $(*)$ for $Z = B_0$. Choose an infinite sequence of clopen sets

$A_1 = A, A_2, A_3, \dots$ such that every A_i is a subset of B_0 and the set $B_0 \setminus \bigcup_{i=1}^{\infty} A_i$ is uncountable. Take a homeomorphism R_0 (defined on B_0 only) which maps B_0 onto $B_0 \setminus A$ and satisfies the condition:

$$R_0\left(B_0 \setminus \bigcup_{i=1}^{\infty} A_i\right) = B_0 \setminus \bigcup_{i=1}^{\infty} A_i, \quad R_0 A_i = A_{i+1}, \quad i = 1, 2, \dots .$$

Let now R_i be a homeomorphism defined on B_{i-1} such that $R_i B_{i-1} = B_i$, $i = 1, \dots, q_s^+ - 1$. Define the homeomorphism $R_s^+ \colon D_s \to D_s$ where $D_s = \bigcup_{p=0}^{q_s^+ - 1} B_p$ as follows: $R_s^+ x = R_p x$ for $x \in B_p$, $p = 0, \dots, q_s^+ - 2$ and $R_s^+ x = R_0(R_1^{-1} \dots R_{q_s^+ - 1}^{-1})x$ for $x \in B_{q_s^+ - 1}$. We see that $R_s^+ B_{q_s^+ - 1} = B_0 \setminus A$ and therefore $SB_p = R_s^+ B_p$ for all p. It follows that $d(R_s^+ x, Sx) + d(R_s^{+-1} x, S^{-1} x) < \varepsilon$ for $x \in D_s$.

Replacing S by S^{-1} in the above construction, we can similarly define a homeomorphism R_s^- which acts only on the clopen set $D'_s = \bigcup_{r=0}^{q_s^- - 1} B'_r$ where

$$B'_r = \bigcup_{i=0}^{\infty} S^{-m_0 - r - iq_s^-}(C_s) \cup \{S^{-r} z_s^-\}, \quad r = 0, \dots, q_s^- - 1.$$

Here $m_0 \equiv 0 \mod (q_s^-)$ is defined analogously to n_0 and $z_s^- = S^{q_s^- - 1} y_s^-$. It can be easily checked that $d_w(S, R_s^-) < \varepsilon$ on the set D'_s.

Finally, we define $R_s \colon \widetilde{C}_s \to \widetilde{C}_s$, $s = 1, \dots, l$ as follows

$$R_s^- x = \begin{cases} Sx & \text{for } x \in \displaystyle\bigcup_{i=-m_0+1}^{n_0 - 1} S^i C_s, \\ R_s^+ x & \text{for } x \in B_s, \\ R_s^- x & \text{for } x \in B'_s. \end{cases}$$

Thus, the aperiodic homeomorphism T defined by P_j and R_s $(j = 1, \dots, k; s = 1, \dots, l)$ satisfies (3.2). $\square$

We note that every simple homeomorphism has a nontrivial periodic part $Z = \bigcup_{j=1}^{k} \bigcup_{i=0}^{r_j - 1} S^i F_j$. Therefore the two dense subsets, $\mathcal{A}p$ and $\mathcal{S}$, are disjoint in Homeo(Ω).

Let $\mathcal{P}er_0$ be the subset of $\mathcal{P}er$ consisting of all homeomorphisms with finite period, that is $P \in \mathcal{P}er_0$ if and only if there exists $m \in \mathbb{N}$ such that $P^m x = x$ for all $x \in \Omega$. This means that Ω can be decomposed into a finite union of clopen sets Ω_p such that the period of P at each point from Ω_p is exactly p. By $\mathcal{P}er_p$, we denote the subset of $\mathcal{P}er_0$ consisting of homeomorphisms with $\Omega_p = \Omega$. Such homeomorphisms are called *p-periodic*. Clearly, the set of simple homeomorphisms, $\mathcal{S}$ contains $\mathcal{P}er_0$.

Let $P \in \mathcal{P}er_p$, then any P-orbit consists of p different points. A subset $E \subset \Omega$ is called *fundamental* for P if $(E, P(E), \dots, P^{p-1}(E))$ is a partition of Ω.

Lemma 3.2. *Let Ω be a Cantor set and let P be a p-periodic homeomorphism. Then there exists a clopen P-fundamental subset $E \subset \Omega$.*

Proof (suggested by B. Weiss). Let d be a metric on Ω compatible with the clopen topology. We note that there exists some $c > 0$ such that $d(x, P^i(x)) > c$ for all $x \in \Omega$ and all $i = 1, \ldots, p-1$. Indeed, let us fix some $i < p$ and assume that for any $n \in \mathbb{N}$ there exists $x_n \in X$ such that $d(x_n, P^i(x_n)) < 1/n$. Take a convergent subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $x_{n_k} \to x_0$ as $k \to \infty$. Then $P^i(x_{n_k}) \to P(x_0)$, and therefore $d(x_0, P^i(x_0)) = 0$. This contradicts the assumption that $P \in \mathcal{P}er_p$.

Now let $(A_1, \ldots, A_n)$ be a partition of Ω into clopen sets such that

$$\operatorname{diam}(A_i) \leq \frac{c}{2} \quad \text{for all } i = 1, \ldots, n. \tag{3.3}$$

Define $E_1 = A_1$, and for $i = 2, \ldots, n$, set inductively

$$E_i = E_{i-1} \cup (A_i \setminus O_P(E_{i-1})) \tag{3.4}$$

where $O_P(F) = \bigcup_{i=0}^{p-1} P^i(F)$. We first prove that

$$E_k \cap P^i(E_k) = \emptyset, \quad k = 1, \ldots, n,\ i = 1, \ldots, p-1. \tag{3.5}$$

Clearly, (3.5) is true for $k = 1$. Assume that this relation is valid for E_{k-1}. Then it follows from (3.3) and (3.4) that

$$\begin{aligned} E_k \cap P^i(E_k) = {} & [E_{k-1} \cap P^i(E_{k-1})] \cup [E_{k-1} \cap (P^i(A_k) \setminus O_P(E_{k-1})] \\ & \cup [P^i(E_{k-1}) \cap (A_k \setminus O_P(E_{k-1}))] \\ & \cup [(A_k \setminus O_P(E_{k-1})) \cap (P^i(A_k) \setminus O_P(E_{k-1}))] = \emptyset. \end{aligned}$$

Next, we show that

$$\bigcup_{i-0}^{p-1} P^i(E_k) \supset \bigcup_{j=1}^{k} A_j, \quad k = 1, \ldots, n. \tag{3.6}$$

Again assume that (3.6) is proved for E_{k-1}. Then

$$\begin{aligned} \bigcup_{i=0}^{p-1} P^i(E_k) &= \bigcup_{i-0}^{p-1} P^i(E_{k-1}) \cup \bigcup_{i=0}^{p-1} (P^i(A_k) \setminus O_P(E_{k-1})) \\ &= \bigcup_{i=0}^{p-1} P^i(E_{k-1}) \cup (A_k \setminus O_P(E_{k-1})) \cup \bigcup_{i=0}^{p-1} (P^i(A_k) \setminus O_P(E_{k-1})). \end{aligned}$$

The first term contains $A_1 \cup \cdots \cup A_{k-1}$ by assumption. The first and second terms together contain A_k.

Thus, it follows from (3.5) and (3.6) that for the clopen set $E = E_n$ the orbit $O_P(E)$ consists of pairwise disjoint sets and $O_P(E) = \Omega$. $\square$

It follows immediately from Lemma 3.2 that for every $P \in \mathcal{P}er_0$ there exists a P-invariant partition $(\Omega_1, \dots, \Omega_m)$ of Ω into clopen subsets such that $\Omega_i = \bigcup_{j=0}^{k_i-1} P^j E_i$ where E_i is a fundamental clopen subset for P on Ω_i and k_i is the period of P on Ω_i.

We will now consider the closure of $\mathcal{P}er$ in Homeo(Ω) with respect to both τ and τ_w. Firstly, we show that $\overline{\mathcal{P}er}^{\tau_w}$ is a proper subset in Homeo(Ω). This means that there are homeomorphisms which cannot be approximated by periodic homeomorphisms in τ_w.

We call a homeomorphism $T \in$ Homeo(Ω) *dissipative* if there exits a clopen set $F \subset \Omega$ such that either $TF \subsetneqq F$ or $F \subsetneqq TF$. Clearly, dissipative homeomorphisms exist in Homeo(Ω) since any two clopen sets are homeomorphic.

Proposition 3.3.

(a) *The set $\overline{\mathcal{P}er}^{\tau_w}$ is a proper subset in (*Homeo$(\Omega), \tau_w)$*: In fact, if T is a dissipative homeomorphism of Ω, then $T \notin \overline{\mathcal{P}er}^{\tau_w}$.*

(b) $(\mathcal{S} \setminus \mathcal{P}er_0) \cap \overline{\mathcal{P}er}^{\tau_w} = \emptyset$*; in other words, if a simple homeomorphism S has an aperiodic part, then it cannot be approximated in τ_w by pointwise periodic homeomorphisms.*

Proof. (a) Take a dissipative homeomorphism T and let F be a clopen subset such that $TF \subsetneqq F$. We will show that the neighbourhood $W(T; F)$ does not contain any homeomorphism from $\mathcal{P}er$. Assume that this is false and let $P \in \mathcal{P}er$ be such that $PF = TF$. Then $P^n F \subsetneqq \cdots \subsetneqq PF\ (= TF) \subsetneqq F$ for any n. It follows that there are points from F with infinite orbits, and this contradicts the pointwise periodicity of P.

(b) It suffices to show that each S from $(\mathcal{S} \setminus \mathcal{P}er_0)$ is dissipative. We use notation from the definition of simple homeomorphisms. Note that it follows from decomposition (3.1) that every closed set $E_s = (\bigcup_{n=0}^{\infty} S^n C_s) \cup [(y_s^+, \dots, S^{q_s^+-1} y_s^+)]$ is, in fact, clopen because $\Omega \setminus Z$ is a finite disjoint union of closed sets. Clearly, $SE_s \subsetneqq E_s$ and the result follows from (a). □

In Section 5 we will strengthen the above result and give a complete description of the set $\overline{\mathcal{P}er_0}^{\tau_w}$.

Let T be a minimal homeomorphism of a Cantor set Ω. We consider the full group $[T] = \{\gamma \in \text{Homeo}(\Omega) \mid \gamma x = T^{m_\gamma(x)} x, \text{ for all } x \in \Omega\}$ and the *topological full group* $[[T]]$ of homeomorphisms generated by T. Recall that a homeomorphism $\gamma \in$ Homeo(Ω) belongs to $[[T]]$ if and only if $\gamma x = T^{m_\gamma(x)} x$ where $x \mapsto m_\gamma(x)$ is a continuous function $\Omega \to \mathbb{Z}$ (see [3], [7], [8] for details).

It was shown in [3] how one can use Kakutani–Rokhlin partitions to describe the structure of homeomorphisms from $[[T]]$. Here we recall some facts that will be used later on.

Let (A_n) be a sequence of clopen subsets of Ω such that $A_n \supset A_{n+1}$, $n \in \mathbb{N}$, and $\bigcap_n A_n$ is a singleton in Ω. Given T and A_n, we can produce a Kakutani–Rokhlin partition ξ_n of Ω which is determined by the function of first return to A_n under the action of T ([15], [11]). The partition ξ_n consists of a finite collection of T-towers $\xi_n(v)$, $v \in V_n$:

$$\xi_n(v) = \{P_n^i(v) := T^i P_n(v) \mid i = 0, \dots, h_n(v) - 1\}$$

where $P_n^0(v) = P_n(v)$. We note that (A_n) can be also chosen such that ξ_{n+1} refines ξ_n and $\bigcap_n \xi_n$ generates the clopen topology on Ω. Moreover one can assume that $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$.

Suppose $\gamma \in [[T]]$. Then there exists $N \in \mathbb{N}$ such that $E_i = \{x \in \Omega \mid \gamma x = T^i x\}$, $-N \le i \le N$, is a clopen finite partition of Ω (some of E_i's may be empty). Note that for sufficiently large n, each set E_i becomes a ξ_n-set, that is $\gamma P^i(v_n) = T^l P^i(v_n)$ for some $l = l(i, v_n)$. Moreover, we may suppose that $N < 2^{-1} h_n$, where $h_n = \min(h_n(v) : v \in V_n)$.

We will now commence the study of the periodic and aperiodic parts of the topological full group with respect to the uniform and weak topologies τ and τ_w.

Let us denote by $\mathcal{P}er_0(T)$ the set $\mathcal{P}er_0 \cap [[T]]$.

Theorem 3.4. *$\mathcal{M}in \subset \overline{\mathcal{P}er_0}^{\tau}$ and $\mathcal{M}in \subset \overline{\mathcal{P}er_0}^{\tau_w}$. More precisely, let T be a minimal homeomorphism of Ω, then:*

(a) *given a neighbourhood $U(T; \mu_1, \dots, \mu_m; \varepsilon)$, there exists a periodic homeomorphism $P \in \mathcal{P}er_0(T)$ such that $P \in U(T; \mu_1, \dots, \mu_m; \varepsilon)$;*
(b) *given $\varepsilon > 0$ there exists $Q \in \mathcal{P}er_0(T)$ such that $d_w(T, Q) < \varepsilon$.*

Proof. We will prove (a); assertion (b) can be proved similarly.

Every measure $\mu \in \mathcal{M}_1(\Omega)$ has an at most countable set of points of positive measure; denote it by $\{x_\mu(k)\}$. Given $\mu_1, \dots, \mu_m$ and $\varepsilon > 0$, find a finite set $Y = \{x_{\mu_i}(k) \mid i = 1, \dots, m,\ k \in I(\mu_i) \subset \mathbb{N}\}$ where the finite subset $I(\mu_i)$ is determined by the condition

$$\sum_{k \notin I(\mu_i)} \mu_i(\{x_{\mu_i}(k)\}) \le \frac{\varepsilon}{3}, \quad i = 1, \dots, m. \tag{3.7}$$

Let $Y = (y_1, \dots, y_N)$. Choose a point $\overline{x} \in \Omega \setminus Y$ such that $T\overline{x} = \overline{y}$ does not belong to Y. By [11], we can find a sequence (ξ_n), $\xi_n = \{T^i D_j(n) \mid 0 \le i \le k(j,n) - 1,\ j \in K(n)\}$, $|K(n)| < \infty$, of Kakutani–Rokhlin partitions satisfying the following conditions:

(i) ξ_{n+1} refines ξ_n and for the base $B(n) = \bigcup_j B_j(n)$, one has $B(n+1) \subset B(n)$;
(ii) (ξ_n) spans the clopen topology on Ω;
(iii) $\bigcap_n B(n) = \{\overline{y}\}$ and $\bigcap_n C(n) = \{\overline{x}\}$ where $C(n) = \bigcup_j T^{k(j,n)-1} D_j(n)$.

Let n_0 be sufficiently large so that $B(n_0) \cap Y = C(n_0) \cap Y = \emptyset$ and

$$\mu_i(B(n_0)) < \frac{\varepsilon}{2}, \quad \mu_i(C(n_0)) < \frac{\varepsilon}{2}, \quad i = 1, \dots, m. \tag{3.8}$$

The sets $B(n_0)$ and $C(n_0)$ may contain points of positive measure μ_i but, by (3.7), the total contribution of these points to the measures of either of the sets is less than $\varepsilon/3$.

For every T-subtower $\xi^j_{n_0} = \{T^i D_j(n_0) \mid 0 \le i \le k(j, n_0) - 1\}$, $j \in K(n_0)$, we define a periodic homeomorphism $P_j(n_0)$:

$$P_j(n_0)x = \begin{cases} Tx & \text{if } x \notin T^{k(j,n_0)-1}D_j(n_0), \\ T^{-k(j,n_0)+1}x & \text{otherwise.} \end{cases} \tag{3.9}$$

We define the periodic homeomorphism P as follows: $Px = P_j(n_0)x$ if $x \in \xi_{j,n}$. We get from (3.9) that $P \in [[T]]$ and $E(P, T) = B(n_0) \cup C(n_0)$. Thus, by (3.8), we obtain that $P \in U(T; \mu_1, \dots, \mu_m; \varepsilon)$.

To prove (b), we observe that $\operatorname{diam}(B(n))$ and $\operatorname{diam}(C(n))$ tend to 0 as $n \to \infty$. Therefore, the above method allows us to find a periodic homeomorphism $Q \in \mathcal{P}er_0(T)$ which is ε-close to T with respect to τ_w. $\square$

Although we have shown that $\mathcal{P}er_0$ is not dense in $(\mathrm{Homeo}(\Omega), \tau_w)$ it is interesting to decide whether $\mathcal{P}er_0(T)$ is dense in $[T]$ with respect to τ and τ_w.

Theorem 3.5. *Let (Ω, T) be a Cantor minimal system, then:*

(a) $\overline{\mathcal{P}er_0(T)}^{\tau} = [T]$ *and*
(b) $\overline{\mathcal{P}er_0(T)}^{\tau_w} \supset [[T]]$.

Proof. Case (a) will be considered in detail, case (b) is similar.

We use here notation from the preceding proof. Let $\gamma \in [[T]]$. Then there exists $K \in \mathbb{N}$ such that for all $i \in [-K, K]$, the clopen sets $E_i = \{x \in \Omega \mid \gamma x = T^i x\}$ constitute a partition $\eta = \eta(\gamma)$ of Ω. We first prove that for any neighbourhood $U_\gamma = U(\gamma; \mu_1, \dots, \mu_m; \varepsilon)$ there exists a periodic homeomorphism $P \in \mathcal{P}$ such that $P \in U_\gamma$. We apply the method used in the proof of Theorem 3.4. Let Y and (ξ_n) be as above. In addition to (i)–(iii), we may assume that (ξ_n) satisfies the following conditions (see [3] for details):

(iv) the height $k(j, n)$ of every T-subtower ξ^j_n approaches to infinity as $n \to \infty$;
(v) ξ_n refines η, i.e. every E_i is a union of atoms from ξ_n; in particular, for every element $D \in \xi_n$, $\gamma x = T^i x$, $x \in D$, where $i = i(D)$.

Take $M \in \mathbb{N}$ and choose n_1 so large that $\min\{k(j, n) \mid j \in K(n)\} \ge K(M+2)$ for all $n \ge n_1$. Let

$$Z = \left(\bigcup_{i=0}^{K-1} T^i B(n_1) \right) \cup \left(\bigcup_{i=0}^{K-1} T^{-i} C(n_1) \right),$$

where $B(n_1)$ and $C(n_1)$ are the base and top of ξ_{n_1}, respectively. It follows from (i)–(v) that the γ-orbit of any atom D of the partition ξ_n meets Z at least once. Furthermore, by the same reasoning as in the proof of Theorem 3.4, we can assume that n_1 is chosen sufficiently large that $\mu_i(Z) < \varepsilon$, $i = 1, \dots, m$.

Now fix a T-subtower $\xi^j_{n_1}$, consisting of sets $T^i D_j(n_1) = D(i,j)$. Define a periodic homeomorphism $P(j,n_1)\colon \xi^j_{n_1} \to \xi^j_{n_1}$ from $[[T]]$ as follows. Let $D(K,j)$ be the first atom (with respect to the natural order on $\xi^j_{n_1}$) that does not belong to Z. Consider the sets $\gamma^p D(K,j)$, $p = 0, \dots, L$, where $\gamma^L(D(K,j)) \subset Z$ and $\gamma^p D(K,j) \cap Z = \emptyset$. Define $P(j,n_1)x = \gamma x$ on $\bigcup_{0\le p<L} \gamma^p(D(K,j))$ and $P(j,n_1)x = \gamma^{-L}x$ on $\gamma^L(D(K,j))$. Let $D(i_1,j)$ be the first atom in $\xi^j_{n_1}$ where $P(j,n_1)$ has not been defined. We extend the definition of $P(j,n_1)$ on a finite piece of the γ-orbit outgoing from $D(i_1,j)$ that does not meet Z. Repeating this construction we eventually define $P(j,n_1)$ for all $D(i,j)$ with $K \le i < k(j,n_1) - K + 1$ and for some $D(i,j)$ from Z. We set $P(i,n_1)$ to be the identity map for the remaining part of atoms of Z. Let now $Px = P(j,n_1)x$ if x is in $\xi^j_{n_1}$. Then P is a periodic homeomorphism from $[[T]]$ whose period is at most M at every point. By construction, $E(P,\gamma) \subset Z$, that is $P \in U_\gamma$. To complete the proof of (a), use the argument of [3, Theorem 4.5] where the density of $[[T]]$ in $[T]$ was established.

For case (b), first observe that $\operatorname{diam}(Z)$ can be made arbitrarily small by choosing n sufficiently large. Then use the same method to prove that each $\gamma \in [[T]]$ can be approximated in τ_w by homeomorphisms from $\mathcal{P}er_0(T)$. As was shown in [3], the τ_w-closure of $[[T]]$ does not, in general, contain $[T]$. $\square$

Our next goal is to describe periodic and aperiodic homeomorphisms γ from $[[T]]$ where T is a minimal homeomorphism of Ω. We will refine the results proved in [3], describing all possible types of γ-orbits.

Fix some $\xi = \bigcup_{v\in V} \xi(v)$ from the sequence (ξ_n) of Kakutani–Rokhlin partitions built by T and a refining sequence of clopen subsets (A_n) (see above). Given ξ and T, define two partitions α and α' of V: we say that J is an atom of α if J is the smallest subset of V such that $T(\bigcup_{v\in J} T^{h(v)-1}D(v))$ is a ξ-set. Similarly, $J' \in \alpha'$ if $T^{-1}(\bigcup_{v\in J'} D(v))$ is a ξ-set and J' is the smallest subset with this property. Obviously, for every $J \in \alpha$ there exists $J' \in \alpha'$ such that

$$T\Big(\bigcup_{v\in J} T^{h(v)-1} D(v)\Big) = \bigcup_{v\in J'} D(v). \tag{3.10}$$

Notice that (3.10) defines a one-to-one correspondence $m\colon J \to J'$ between atoms of α and those of α'.

For $J \in \alpha$ and $J' \in \alpha'$ define $L_{tj}(J)$ and $L_{bk}(J')$, $0 \le j,\ k \le h/2$, $h = \min_{v\in V} h(v)$ as follows:

$$L_{tj}(J) = \bigcup_{v\in J} T^{h(v)-j-1}D(v), \qquad L_{bk}(J') = \bigcup_{v'\in J'} T^k D(v') \tag{3.11}$$

(here t stands for "top" and b stands for "base"). Remark that the indexes j and k in $L_{tj}(J)$ and $L_{bk}(J')$ indicate the distance of $D^{h(v)-j-1}(v)$, $v \in J$, and $D^k(v)$, $v' \in J'$, from the top and from the base of the corresponding towers.

Since (ξ_n) generates the clopen topology, we note that given $\gamma \in [[T]]$ there exists $\xi \in (\xi_n)$ such that for every $i \in \mathbb{Z}$ the set $\{x \in \Omega \mid \gamma x = T^i x\}$, is a ξ-set. Then for every ξ-atom $D^i(v)$, there exists an integer $l = l(D^i(v))$ such that

$$\gamma x = T^l x, \quad x \in D^i(v). \tag{3.12}$$

On the other hand, it was proved in [3] that for γ and ξ as above, the following property holds: if $\gamma(D^i(v)) = T^l(D^i(v))$ and $l+i \ge h(v)$ (i.e. $D^i(v)$ goes through the top of $\xi(v)$ under action of γ), then the entire set $L_{tj}(J)$, $j = h(v) - i - 1$, containing $D^i(v)$, also goes through the top of $\bigcup_{v\in J} \xi(v)$. Furthermore, $L_{tj}(J)$ is mapped by γ onto $L_{bk}(J')$ where $J' = m(J)$ and k is uniquely determined by j, l. A similar property holds when $\gamma(D^k(v')) = T^s(D^k(v'))$, $v' \in J'$, and $k + s < 0$. In this case, the set $L_{bk}(J')$ goes through the base and is mapped by γ onto some $L_{tj}(J)$.

It turns out that the above result allows us to solve the inverse problem, that is, to find a finite collection of objects that uniquely determine a homeomorphism $\gamma \in [[T]]$. For this, we take the following data:

(a) a positive integer $N < h/2$;

(b) subsets $\mathrm{I}_L(J)$, $\mathrm{II}_L(J')$, $\mathrm{I}_A(J)$, $\mathrm{II}_{\mathcal{A}}(J')$ of $\{0, \dots, N\}$ such that $|\mathrm{I}_L(J)| = |\mathrm{II}_A(J')|$, $|\mathrm{II}_L(J')| = |\mathrm{I}_A(J)|$, $J \in \alpha$, $J' \in \alpha'$;

(c) one-to-one maps $\rho(J)\colon \mathrm{I}_L(J) \to \mathrm{II}_A(J')$, $\sigma(J')\colon \mathrm{II}_L(J') \to \mathrm{I}_A(J)$ where $J \in \alpha,\ J' \in \alpha'$, and $m(J) = J'$;

(d) a one-to-one map $\pi(v)\colon \mathrm{I}_{NL}(v) \to \mathrm{I}_{NA}(v)$ where $v \in J \cap J'$, $J \in \alpha$, $J' \in \alpha'$, and the set $\mathrm{I}_{NL}(v)$ (resp. $\mathrm{I}_{NA}(v)$) consists of those $k \in \{0, \dots, h(v)-1\}$ such that $k \notin \mathrm{II}_L(J')$ (resp. $k \notin \mathrm{II}_A(J')$) and $h(v) - k - 1 \notin \mathrm{I}_L(J)$ (resp. $h(v) - k - 1 \notin \mathrm{I}_A(J)$).

In the above notation, the indexes A, L, NA, NL mean the first letters in words "arriving", "leaving", "not arriving", "not leaving".

Using these data, we can define a homeomorphism γ from $[[T]]$ by the following rule:

For every ξ-atom $D^j(v) = T^j(D(v))$, $v \in J \cap J'$, we have that: either

(1) j is in $\mathrm{I}_{NL}(v)$, or

(2) $h(v) - j - 1 \in \mathrm{I}_L(J)$, or

(3) $j \in \mathrm{II}_L(J')$.

(Note that in view of (a)–(d) only one of the above possibilities can occur). If (3) holds, then $D^j(v)$ belongs to $L_{bj}(J')$; if (2) holds, then $D^j(v)$ belongs to $L_{tk}(J)$ where $k = h(v) - j - 1$.

According to cases (1)–(3), we define

$$\gamma(T^j D(v)) = T^{\pi(v)(j)} D(v) \quad \text{if } j \in \mathrm{I}_{NL}(v), \tag{3.13}$$

$$\gamma(L_{tk}(J)) = L_{b,\rho(J)(k)}(J') \quad \text{if } j \in \mathrm{I}_L(J), \tag{3.14}$$

$$\gamma(L_{bj}(J')) = L_{t,\sigma(J')(j)}(J) \quad \text{if } j \in \mathrm{II}_L(J'). \tag{3.15}$$

We observe that the image of $D^j(v)$ under the γ-action is a ξ-set if γ is defined by (3.13) and is no longer a ξ-set if γ is defined by (3.14) and (3.15).

Denote by $\Gamma(\xi)$ the set of all homeomorphisms that can be constructed from the data (a)–(d) by (3.13)–(3.15). It was proved in [3] that $\Gamma(\xi_n)$ is an increasing sequence of subsets in $[[T]]$ and, for every $\gamma \in [[T]]$, there exists ξ from (ξ_n) such that $\gamma \in \Gamma(\xi)$.

Fix a homeomorphism $\gamma \in [[T]]$. Let $\xi \in (\xi_n)$ satisfy (3.12). Then γ determines the subsets $\mathrm{I}_L(J)$, $\mathrm{II}_L(J')$, $\mathrm{I}_A(J)$, $\mathrm{II}_A(J')$ and maps $\rho(J), \sigma(J')$ such that (3.13)–(3.15) hold. Let us consider γ-orbits in terms of these subsets and maps.

First suppose that $\gamma \in [[T]]$ is periodic. There are two possible types of periodic behaviour for γ.

Case 1. We start with some $D^{j_0}(v)$ where $j_0 \in \mathrm{I}_{NL}(v) \cap \mathrm{I}_{NA}(v)$. Suppose that the γ-orbit of $D^{j_0}(v)$ does not leave $\xi(v)$. By (3.13), this means that the entire sequence $(j_k)_{k=0}^s$, $j_k = \pi(v)(j_{k-1})$, belongs to $\mathrm{I}_{NL}(v) \cap \mathrm{I}_{NA}(v)$ and $j_s = j_0$, $j_k \neq j_0,\ k < s$. Let

$$\eta_1 = \{T^{j_k} D(v) \mid k = 0, \dots, s-1\} = \{\gamma^i(T^{j_0} D(v)) \mid i = 0, \dots, s-1\} \tag{3.16}$$

be the γ-orbit where $\gamma(T^{j_k} D(v)) = T^{j_{k+1}}(D(v))$. Then, it follows that the γ-orbit of $D^{j_0}(v)$ returns to this set and because the orbit is a part of T-tower, we get that γ is periodic.

Case 2. The other type of cyclic γ-orbit has the following structure. Fix some $J \in \alpha$ and let $J' = m(J)$. Suppose that that $j_0 \in \mathrm{I}_L(J) \cap \mathrm{I}_A(J)$, $j_1 = \rho(J)(j_0) \in \mathrm{II}_A(J') \cap \mathrm{II}_L(J')$, $j_2 = \sigma(J')(j_1) \in \mathrm{I}_A(J) \cap \mathrm{I}_L(J), \dots, j_{2s-1} = \rho(J)(j_{2s-2}) \in \mathrm{II}_A(J') \cap \mathrm{II}_L(J')$ and $\sigma(J')(j_{2s-1}) = j_0$. By (3.14) and (3.15), this case corresponds to the following periodic γ-orbit

$$\eta_2 = \{\gamma^i(L_{tj_0}(J)) \mid i = 0, \dots, 2s-1\} \tag{3.17}$$

where $\gamma(L_{tj_0}(J)) = L_{bj_1}(J')$, $\gamma(L_{bj_1}(J')) = L_{tj_2}(J))$ etc. until it returns to $L_{tj_0}(J)$.

It follows from the above proof that γ belongs to $\mathcal{P}er_0$. We summarize the above observations in the following statement.

Lemma 3.6. *Let γ be a periodic homeomorphism from $[[T]]$. Then there exists ξ from (ξ_n) such that every periodic γ-orbit either has the form* (3.16) *or the form* (3.17).

We can study γ-orbits for an aperiodic $\gamma \in [[T]]$ in a similar manner. Given $\varepsilon > 0$ and $\gamma \in [[T]]_{ap}$ where $[[T]]_{ap} = [[T]] \cap \mathcal{A}p$, we take ξ from the sequence (ξ_n) such that $\gamma \in \Gamma(\xi)$. Since $\mathrm{diam}(A_n) \to 0$ as $n \to \infty$, we can assume that

$$\mathrm{diam}\Big(\bigcup_{J\in\alpha}\bigcup_{j=0}^{N} L_{tj}(J)\Big) + \mathrm{diam}\Big(\bigcup_{J'\in\alpha'}\bigcup_{j=0}^{N} L_{bj}(J')\Big) < \varepsilon. \tag{3.18}$$

Considering γ-orbits of atoms of ξ, one can find a finite partition ζ of Ω into clopen subsets, γ-towers, such that those towers have their bases and tops into the sets $\bigcup_{j=0}^{N} L_{bj}(J')$ and $\bigcup_{j=0}^{N} L_{tj}(J)$. Then we can construct an odometer which is, by (3.18), ε-close to γ in τ_w (see e.g. the proof of Theorem 5.3). The case of the topology τ is considered similarly. We should note that given $\varepsilon > 0$ and $\mu_1, \dots, \mu_n \in \mathcal{M}_1(\Omega)$ we can chose ξ such that for all i,

$$\mu_i\Big(\bigcup_{J\in\alpha}\bigcup_{j=0}^{N} L_{tj}(J)\Big) + \mu_i\Big(\bigcup_{J'\in\alpha'}\bigcup_{j=0}^{N} L_{bj}(J')\Big) < \varepsilon.$$

This relation guarantees that the odometer which we have found is also ε-close to γ in τ.

Using these ideas, it is now easy to prove the following theorem. We leave details to the reader. Another proof of the first relation of the theorem can also be obtained from Theorem 5.9 (see Remark 5.13 below).

Theorem 3.7. *If T is a minimal homeomorphism of Ω, then $[[T]]_{ap} \subset \overline{\mathcal{M}in}^{\tau}$ and $[[T]]_{ap} \subset \overline{\mathcal{M}in}^{\tau_w}$ where $[[T]]_{ap} = [[T]] \cap \mathcal{A}p$.*

Let T be a homeomorphism of a Cantor set Ω. Then T can be also considered as a Borel automorphism of $(\Omega, \mathcal{B})$. Therefore, one can define two full groups $[T]_C$ and $[T]_B$ where

$$\begin{aligned}
[T]_C &= \{S \in \mathrm{Homeo}(\Omega) \mid Sx \in \{T^n x \mid n \in \mathbb{Z}\} \text{ for all } x \in \Omega\},\\
[T]_B &= \{S \in \mathrm{Aut}(\Omega, \mathcal{B}) \mid Sx \in \{T^n x \mid n \in \mathbb{Z}\} \text{ for all } x \in \Omega\}.
\end{aligned}$$

Here the subindeces C and B correspond to the cases of Cantor and Borel dynamics. Obviously, $[T]_C \subset [T]_B$ and $[T]_B$ is closed in $\mathrm{Aut}(\Omega, \mathcal{B})$ with respect to τ [1].

If $S \in [T]_C$ (or $S \in [T]_B$), then S generates two partitions $\pi(S) = (X_n \mid n \in \mathbb{Z})$ and $\pi'(S) = (X'_n \mid n \in \mathbb{Z})$ of Ω into closed (Borel) subsets $X_n = \{x \in \Omega \mid Sx = T^n x\}$ and $X'_n = S(X_n) = T^n(X_n)$. Those homeomorphisms from $[T]_C$, for which the sets X_n are clopen, form the so called topological full group $[[T]]_C$.

It was shown in Section 2 that $\mathrm{Homeo}(\Omega)$ is non-closed and dense in $\mathrm{Aut}(\Omega,\mathcal{B})$. On the other hand, we proved in [3] that for a minimal homeomorphism T,

$$\overline{[[T]]_C}^{\tau} \cap \mathrm{Homeo}(\Omega) = [T]_C. \tag{3.19}$$

But the problem of finding the entire closure of $[[T]]_C$ in $\mathrm{Aut}(\Omega,\mathcal{B})$ with respect to τ remained open. We answer this question in the following theorem.

Theorem 3.8. *Let T be a minimal homeomorphism of a Cantor set Ω. Then* $\overline{[[T]]_C}^{\tau} = [T]_B$.

Proof. It is clear that $\overline{[[T]]_C}^{\tau} \subset [T]_B$. Take a Borel automorphism $S \in [T]_B$. Let $U(S) = U(S;\mu_1,\dots,\mu_m;\varepsilon)$ be a τ-neighbourhood of S. To prove the theorem we need to show that $U(S)$ contains a homeomorphism R from $[[T]]_C$. By (3.19), it suffices to prove that there exists some $R_1 \in [T]_C \cap U(S)$.

Consider the partitions $\pi(S)$ and $\pi'(S)$ defined above. Choose n_0 such that

$$\sum_{|n|>n_0} \mu_i(X_n) < \frac{\varepsilon}{4}, \quad \sum_{|n|>n_0} \mu_i(X'_n) < \frac{\varepsilon}{4}, \quad i = 1,\dots,m. \tag{3.20}$$

For each X_n, $|n| \le n_0$, find a closed $F_n \subset X_n$ such that for all $i = 1,\dots,m$,

$$\mu_i(X_n \setminus F_n) < \frac{\varepsilon}{4(2n_0+1)}, \qquad \mu_i(X_n \setminus S(F_n)) < \frac{\varepsilon}{4(2n_0+1)}. \tag{3.21}$$

Let $F'_n = S(F_n)$. Clearly, F'_n is also a closed subset in X'_n. The collections (F_n) and (F'_n) consist of pairwise disjoint sets. Then there exist clopen sets $G_n \supset F_n$ and $G'_n \supset F'_n$ such that $(G_n : |n| \le n_0)$ and $(G'_n : |n| \le n_0)$ are pairwise disjoint collections of sets.

Let $\xi_k = \bigcup_{v\in V_k} \xi_k(v)$ be a sequence of Kakutani-Rokhlin partitions built by T and a decreasing sequence of clopen sets (see above). For every k, find partitions $\alpha = \alpha_k$ and $\alpha' = \alpha'_k$ with atoms J and J' satisfying (3.10). Define a new partition η_k of Ω which consists of the sets $L_{tj}(J), L_{bj}(J')$, $J \in \alpha$, $J' \in \alpha'$, $j = 1,\dots,n_0$ (see (3.11)) and the remaining atoms of ξ_k. Clearly, every atom of η_k is a ξ_k-set and the sequence (η_k) generates the clopen topology on Ω.

Choose k sufficiently large such that

$$\min_{v\in V_k} h(k,v) > 3n_0,$$

where $h(k,v)$ is the height of $\xi_k(v)$ and every set G_n, G'_n, $|n| \le n_0$, is an η_k-set. Without loss of generality, we can assume that if E is an atom of $\eta = \eta_k$ such that $E \subset G_n$ (or $E \subset G'_n$), then $E \cap F_n \ne \emptyset$ (or $E \cap F'_n \ne \emptyset$), $|n| \le n_0$.

Fix some G_n. By construction, G_n is a union of atoms $D_1,\dots,D_p$ of ξ_k and sets $L_{tj}(J), L_{bj}(J')$ and each of these sets intersects F_n. Define $\overline{S}x = T^n x$ if $x \in \bigcup_{s=1}^{p} D_s$. We also define $\overline{S}x = T^n x$ if $x \in L_{tj}(J)$ (or $x \in L_{bj}(J')$) and the set $L_{tj}(J)$ ($L_{bj}(J')$) goes through the top (base) of ξ_k under the action of T^n. If $L_{tj}(J)$ (or $L_{bj}(J')$) is a subset of G_n which does not go through the top (base) under T^n, we discard from $L_{tj}(J)$ ($L_{bj}(J')$) those atoms of ξ_k which do not meet F_n and set $\overline{S}x = T^n x$ for x from the remaining atoms. In such a way, we have found a clopen subset $\overline{G}_n \subset G_n$ and a map $\overline{S}$ defined on $\overline{G}_n$. Clearly, $\overline{G}_n \supset F_n$. Similarly, we define a clopen set $\overline{G'}_n$ such that $F'_n \subset \overline{G'}_n \subset G'_n$. It can be easily seen that $\overline{SG}_n = \overline{G'}_n$. Indeed, if $E \subset \overline{G}_n$ and $E \in \eta$, then there exists $y \in E \cap F_n$. Hence $T^n y \in F'_n$. At the same time, $T^n(E)$ is an η-set. Thus, $T^n(E) \subset \overline{G'}_n$. In such a way, we have found a partially defined homeomorphism $\overline{S}$ such that $\overline{S}(A) = B$ where

$$A = \bigcup_{|n|\le n_0} \overline{G}_n \quad \text{and} \quad B = \bigcup_{|n|\le n_0} \overline{G'}_n.$$

Since $\overline{S} \in [[T]]_C$ on the clopen set $\bigcup_{|n|\le n_0} \overline{G}_n$, we get that for any T-invariant measure ν

$$\nu\Big(X \setminus \bigcup_{|n|\le n_0} \overline{G}_n\Big) = \nu\Big(X \setminus \bigcup_{|n|\le n_0} \overline{G'}_n\Big).$$

By [8, Proposition 2.6], there exists a homeomorphism $R' \in [T]_C$ which maps $X\setminus\bigcup_{|n|\le n_0} \overline{G}_n$ onto $X\setminus\bigcup_{|n|\le n_0} \overline{G'}_n$. Now define $Rx = \overline{S}x$ if $x \in A$ and $Rx = R'x$ if $x \in X \setminus A$. Then $R \in [T]_C$ and it remains to show that $R \in U(S)$. Note that if $x \in F_n$, then $Sx = \overline{S}x = Rx$. Therefore

$$E(S,R) \subset \Big(X \setminus \bigcup_{|n|\le n_0} F_n\Big) \cup \Big(X \setminus \bigcup_{|n|\le n_0} F'_n\Big).$$

Then, for given measures μ_i, $i = 1, \dots, m$, we have by (3.20) and (3.21)

$$\begin{aligned} \mu_i(E(S,R)) &< \mu_i\Big(X \setminus \bigcup_{|n|\le n_0} F_n\Big) + \mu_i\Big(X \setminus \bigcup_{|n|\le n_0} F'_n\Big) \\ &= \mu_i\Big(X \setminus \bigcup_{|n|\le n_0} X_n\Big) + \mu_i\Big(X \setminus \bigcup_{|n|\le n_0} X'_n\Big) \\ &\quad + \mu_i\Big(\bigcup_{|n|\le n_0} (X_n \setminus F_n)\Big) + \mu_i\Big(\bigcup_{|n|\le n_0} (X'_n \setminus F'_n)\Big) \\ &< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + 2(2n_0+1)\frac{\varepsilon}{4(2n_0+1)} = \varepsilon. \end{aligned}$$

□

REMARK 3.9. We note that the τ_w-closures of the full group $[T]$ and the topological full group $[[T]]$, $T \in \mathcal{A}p$, can be easily found. Indeed, it was noted $\overline{[T]}^{\tau_w} = \{S \in \mathrm{Homeo}(\Omega) \mid \mu \circ S = \mu \text{ for all } \mu \in M_1(T)\}$ where $M_1(T)$ is the set of T-invariant Borel probability measures [7]. It is not hard to show that $R \in \overline{[[T]]}^{\tau_w}$ if and only if for every clopen set E there exists $\gamma \in [[T]]$ such that $RE = \gamma E$. In other words, $R \in \overline{[[T]]}^{\tau_w}$ if and only if the clopen sets E and RE are $[[T]]$-equivalent for every E. Recall that in [3] we defined the notion of saturated homeomorphisms: a minimal homeomorphism T is called saturated if any two clopen sets A and B such that $\mu(A) = \mu(B)$ for all $\mu \in M_1(T)$ are $[[T]]$-equivalent. Obviously, every odometer is a saturated homeomorphism. It was proved there that T is saturated if and only if $\overline{[[T]]}^{\tau_w} = \overline{[T]}^{\tau_w}$. Now this result follows easily from the above description of the τ_w-closures.

4. Rank of a homeomorphism

The concept of the rank of an automorphism of a standard measure space is an important invariant in ergodic theory. This notion has been studied in many papers. M. Nadkarni [14] has recently defined rank for Borel automorphisms. Here we consider *rank for homeomorphisms* of a Cantor set.

We first recall the definition of *odometer* (or *adding machine*). Let $\{\lambda_t\}_{t=0}^{\infty}$ be a sequence of integers such that $\lambda_t \geq 2$. Denote by $p_{-1} = 1,\ p_t = \lambda_0\lambda_1\cdots\lambda_t,\ t = 0, 1, \ldots$ Let Δ be the group of all p_t-adic numbers; then any element of Δ can be represented as an infinite formal series:

$$\Delta = \left\{ x = \sum_{i=0}^{\infty} x_i p_{i-1} \;\middle|\; x_i \in (0, 1, \ldots , \lambda_i - 1) \right\}.$$

It is well known that Δ is a compact metric abelian group. An odometer, S, is the transformation acting on Δ as follows: $Sx = x + 1,\ x \in \Delta$, where $1 = 1p_{-1} + 0p_0 + 0p_1 + \ldots \in \Delta$. From the topological point of view, (Δ, S) is a strictly ergodic Cantor system.

Let

$$D_0^t = \left\{ x = \sum_{i=0}^{\infty} x_i p_{i-1} \;\middle|\; x_0 = x_1 = \ldots = x_t = 0 \right\}.$$

We see that the sets $(D_0^t, \ldots , D_{p_t-1}^t)$, $D_i^t = S^i(D_0^t)$ form a partition ξ_t of Δ into clopen sets. Clearly, (ξ_t), $t \geq 0$, is a refining sequence of of S-towers. Moreover, $S(\xi)_t = \xi_t$ for every t.

We will denote by $\mathcal{O}d = \mathcal{O}d(\Omega)$ the set of homeomorphisms of Ω homeomorphic to an odometer (Δ, S). Elements from $\mathcal{O}d$ will be also called odometers.

Lemma 4.1. *Let $T \in \mathrm{Homeo}(\Omega)$ and let $(F_1, \ldots , F_n)$ be a partition of Ω into clopen sets such that $TF_i = F_{i+1}$, $1 \leq i \leq n - 1$, $TF_n = F_1$. Then there exists an odometer S such that $SF_i = TF_i$ for all i, i.e. $S \in W(T; F_1, \ldots , F_n)$.*

The proof of the lemma is based on the definition of odometer and left to the reader.

Definition 4.2. Let $T \in \mathrm{Homeo}(\Omega)$ and let $\mathcal{F}_n$, $n \in \mathbb{N}$, be a partition of Ω which is a union of r disjoint T-towers consisting of clopen sets, that is

$$\mathcal{F}_n = \bigcup_{j=1}^{r} \bigcup_{i=0}^{h_n(i)-1} T^i F_n(j)$$

where $F_n(j) \in \mathrm{CO}(\Omega)$ is the base of jth tower and $h_n(j)$ is its height. We say that T has rank at most r if $\mathcal{F}_{n+1}$ refines $\mathcal{F}_n$ and all $\mathcal{F}_n$'s generate the clopen topology on Ω. We say that T has rank r if T has rank at most r but does not have rank at most $r-1$. T has infinite rank if it does not have rank r for any finite r.

Obviously, every odometer is a homeomorphism of rank one.

We denote the set of all homeomorphisms having rank at most r by $\mathcal{R}(\leq r)$ and the set of homeomorphisms of rank r by $\mathcal{R}(r)$.

PROPOSITION 4.3. *Let $T \in \mathrm{Homeo}(\Omega)$ and suppose rank$(T) = r < \infty$. If $(\mathcal{F}_n)$ is a sequence of clopen subsets of Ω as in Definition 4.2, then $\mathcal{F}_n = \Omega$ for all sufficiently large n.*

PROOF. We note that $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n \in \mathbb{N}$ by Definition 4.2. If we assume that $\mathcal{F}_n \neq \Omega$ for all n, then we get that $\bigcap_n (\Omega \setminus \mathcal{F}_n) \neq \emptyset$. This contradicts the assumption that the $\mathcal{F}_n$'s generate the topology on Ω. □

Suppose that the clopen set $\mathcal{F}$ is a disjoint union of r disjoint families of clopen sets: $\mathcal{F} = \bigcup_{i,j} D_{ij}$ where $i = 1, \ldots, r$ and $j = 0, 1, \ldots, h(i)-1$. Let σ be a total order on the set $(0, 1, \ldots, h(i)-1)$ for $i = 1, \ldots, r$. Denote by $\sigma(j)$ the successor of $j \in (0, \ldots, h(i)-1)$. Define

$$\mathcal{Z}_{r,\sigma}(\mathcal{F}) = \{R \in \mathrm{Homeo}(\Omega) \mid RD_{ij} = D_{i\sigma(j)},\quad i = 1, \ldots, r,\ j = 0, \ldots, h(i)-2\}. \tag{4.1}$$

Let $S \in \mathcal{Z}_{r,\sigma}(\mathcal{F})$. Then S transforms the sets $(D_{ij} : j = 0, \ldots, h(i)-1)$ into an S-tower, provided we introduce a new enumeration such that $SD_{ij} = D_{ij+1}$. Note that

$$\begin{aligned} \mathcal{Z}_{r,\sigma}(\mathcal{F}) &= \bigcap_{i=1}^{r} \bigcap_{j=0}^{h(i)-2} W(S; D_{ij}) \\ &= W(S; D_{10}, \ldots, D_{1h(1)-2}, D_{20}, \ldots, D_{rh(r)-2}). \end{aligned} \tag{4.2}$$

THEOREM 4.4. *For every finite r, the set $\mathcal{R}(\leq r)$ is a G_δ-set in τ_w-topology and $\mathcal{R}(\leq r)$ is an $F_{\sigma\delta}$-set in the topologies τ, τ'', $\overline{p}$. In particular, $\mathcal{R}(1)$, the set of homeomorphisms of rank 1, is a G_δ-set in τ_w and $\mathcal{R}(1)$ is a $F_{\sigma\delta}$ in τ, τ'', $\overline{p}$.*

PROOF. Let (Q_n) be a refining sequence of finite partitions into clopen sets generating the clopen topology. Then for a finite partition $\mathcal{F}$ into clopen subsets there exists Q_n such that Q_n refines F, $Q_n \succ F$. One can easily check that

$$\mathcal{R}(\leq r) = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} \bigcup_{Q_n \prec \mathcal{F} \prec Q_k} \bigcup_{\sigma} \mathcal{Z}_{r,\sigma}(\mathcal{F}). \tag{4.3}$$

By (4.2) and Proposition 2.4, every set $\mathcal{Z}_{r,\sigma}(\mathcal{F})$ is clopen in τ_w and closed in τ, τ'', $\overline{p}$. The theorem follows. □

Theorem 4.5. *If T has rank 1, then T is minimal.*

Proof. Let $(\mathcal{F})_n$ be a refining sequence of T-towers generating the topology on Ω. Without loss of generality, we may assume that every $\mathcal{F}_n$ is a clopen partition of Ω. Take some $x \in \Omega$ and let E be a nonempty clopen subset. Find n such that E is an $\mathcal{F}_n$-set. Then there exists an integer m such that $T^m x \in E$. This means that T-orbit of x is dense in Ω. □

Theorem 4.6. *$T \in \mathcal{R}(1)$ if and only if T is topologically conjugate to an odometer. In other words, $\mathcal{R}(1) = \mathcal{O}d$.*

Proof. Let $(\mathcal{F})_n$ be a sequence of T-towers corresponding to a rank 1 homeomorphism T. Then

$$\mathcal{F}_n = \bigcup_{j=0}^{h_n-1} D_j^n, \quad TD_j^n = D_{j+1}^n,\ 0 \le j \le h_n - 2.$$

Since $\mathcal{F}_{n+1} \succ \mathcal{F}_n$ and every $\mathcal{F}_n$ is a clopen partition of Ω, we see that h_n divides h_{n+1}, say $h_{n+1} = \lambda_n h_n$. We have that

$$D_j^n = \bigcup_{k=0}^{\lambda_n-1} D_{kh_n+j}^{n+1}, \quad j = 0, \dots, h_n - 1.$$

This proves that T is completely defined, up to conjugacy, by a sequence of positive integers (λ_n) and therefore T is conjugate to an odometer by Lemma 4.1.□

Let $C(T)$ denote the centralizer of a homeomorphism T, that is $C(T) = \{R \in \mathrm{Homeo}(\Omega) \mid RT = TR\}$. Denote by $\mathrm{Wcl}(T)$ the τ_w-closure of the cyclic group $\{T^n \mid n \in \mathbb{Z}\}$. We prove now the weak closure theorem for homeomorphisms of rank 1.

Theorem 4.7. *Let $T \in \mathrm{Homeo}(\Omega)$ be a homeomorphism of rank 1. Then $C(T) = \mathrm{Wcl}(T)$.*

Proof. To show that $C(T) \supset \mathrm{Wcl}(T)$, it suffices to check that $C(T)$ is closed in $\tau_w \sim p$. To do this, we take a sequence $(S_n) \subset C(T)$ converging to S. Since $\mathrm{Homeo}(\Omega)$ is a topological group in τ_w, then $(S_n, T) \mapsto S_n T$ and $(T, S_n) \mapsto TS_n$ are continuous and therefore, taking the limit, we get that $ST = TS$.

To see that $\mathrm{Wcl}(T) \supset C(T)$, we use the fact that T can be taken as an odometer by Theorem 4.6. Let $R \in C(T)$ and let $\xi = (F, TF, \dots, T^{n-1}F)$ be a T-tower covering Ω (then $T^n F = F$). Given $\delta > 0$, we can find sufficiently large n such that $\mathrm{diam}(F) < \delta$. Denote by $\varepsilon_0 = \min(\mathrm{dist}(F, T^j F) | 0 < j < n)$. Then given $0 < \varepsilon < \varepsilon_0$ find some δ such that $d(Rx, Ry) < \varepsilon$ whenever $d(x, y) < \delta$

where d is a metric on Ω. It follows that RF is a subset of some $T^i F, 0 \le i_0 < n$. If we assume that RF is a proper subset of $T^{i_0}F$, then we come to a contradiction since, in this case, $R(T^j F)$ must be also a proper subset of $T^{i_0+j}F$, $1 \le j < n$, where $i_0 + j$ is understood by mod 0. In other words, we have shown that $R \in W(T^{i_0}; F, TF, \dots, T^{n-1}F)$. $\square$

We note that J. King proved in [12] the weak closure theorem in the context of measurable dynamics.

Remark 4.8. Let T be an odometer and $\xi = (F, TF, \dots, T^{n-1}F)$ be a T-tower covering Ω. Then for any $S \in W(T; F, \dots, T^{n-1}F)$ we have that $S^i \in W(T^i; F, \dots, T^{n-1}F)$, $i \in \mathbb{Z}$. More general,

$$W(T; F, \dots, T^{n-1}F)^i \subset W(T^i; F, \dots, T^{n-1}F), \quad i \in \mathbb{Z}.$$

5. Minimal and mixing homeomorphisms

Let $\mathcal{M}in$ denote the set of minimal homeomorphisms and let $\mathcal{M}ix$ be the set of all mixing homeomorphisms of Ω (recall that T is mixing if for any non-empty clopen sets E and F there exists $n \in \mathbb{N}$ such that $T^i E \cap F \neq \emptyset,\ \forall i \ge n$). The following statement shows that minimality and mixing are not typical properties.

Proposition 5.1. *The following properties hold:*

(a) $\overline{\mathcal{M}in}^{\tau} \subset \mathcal{A}p$, $\overline{\mathcal{M}ix \cap \mathcal{A}p}^{\tau} \subset \mathcal{A}p$.

(b) *For any neighbourhood* $W = W(\mathbb{I}; F_1, \dots, F_n)$, $\overline{\mathcal{M}in}^{\tau_w} \cap W = \emptyset$ *and* $\overline{\mathcal{M}ix}^{\tau_w} \cap W = \emptyset$.

(c) *If* $R \in \mathcal{S}$ *is a simple homeomorphism* (*in particular, R can belong to* $\mathcal{P}er_0$), *then there exists a neighbourhood* $W = W(R; F_1, \dots, F_n)$ *such that* $\overline{\mathcal{M}in}^{\tau_w} \cap W = \emptyset$, $\overline{\mathcal{M}ix}^{\tau_w} \cap W = \emptyset$.

Proof. (a) The result follows from Theorem 3.1(a).

(b) Obviously, $\mathcal{M}in$ and $\mathcal{M}ix$ do not meet any neighbourhood $W = W(\mathbb{I}; F_1, \dots, F_n)$ of the identity. Since W is a clopen subset in $(\mathrm{Homeo}(\Omega), \tau_w)$, we get that $\overline{\mathcal{M}in}^{\tau_w} \subset W^c$ and $\overline{\mathcal{M}ix}^{\tau_w} \subset W^c$. In particular, we see that $\overline{\mathcal{M}in}^{\tau_w}$ and $\overline{\mathcal{M}ix}^{\tau_w}$ are proper subsets of $\mathrm{Homeo}(\Omega)$.

(c) Since $R \in \mathcal{S}$, we can find a clopen subset E of Ω such that the sets $R^i E,\ i = 0, 1, \dots, n-1$, are disjoint and $R^n E = E$. Denote $\Omega_n = \bigcup_{i=0}^{n-1} R^i E$. Then if $S \in W(R; E, RE, \dots, R^{n-1}E)$, then $S(R^i E) = R^{i+1}E$ and therefore S cannot be mixing. Now let F be a non-empty clopen subset of E. The subsets $F, RF, \dots, R^{n-1}F$ are still disjoint and if $S \in W(R; F, RF, \dots, R^{n-1}F)$, then we have that $S(\bigcup_{i=0}^{n-1} R^i F) = \bigcup_{i=0}^{n-1} R^i F$, that is S is not minimal. $\square$

Since the set $\mathcal{S}$ of simple homeomorphisms is dense in $(\mathrm{Homeo}(\Omega), \tau_w)$, we easily obtain the following result.

Corollary 5.2. *$\overline{\mathcal{M}in}^{\tau_w}$ and $\overline{\mathcal{M}ix}^{\tau_w}$ are nowhere dense in* $(\mathrm{Homeo}(\Omega), \tau_w)$.

Theorem 5.3.

(a) $\overline{\mathcal{O}d}^{\tau} = \overline{\mathcal{R}(1)}^{\tau} = \overline{\mathcal{M}in}^{\tau}$,

(b) $\overline{\mathcal{O}d}^{\tau_w} = \overline{\mathcal{R}(1)}^{\tau_w} = \overline{\mathcal{M}in}^{\tau_w}$.

Proof. It follows from Theorem 4.6 that we need to prove the relations $\mathcal{M}in \subset \overline{\mathcal{R}(1)}^{\tau}$ and $\mathcal{M}in \subset \overline{\mathcal{R}(1)}^{\tau_w}$ only. First consider the closure $\overline{\mathcal{R}(1)}^{\tau_w}$. Let T be a minimal homeomorphism. Take a sequence of clopen subsets (A_n) such that $A_n \supset A_{n+1}$ and $\bigcap_{n\geq 1} A_n$ is a singleton. Let ξ_n be the Kakutani–Rokhlin partition defined as in Section 3 by T with fixed base A_n. It is known that (A_n) may be chosen in such a way that the refining sequence (ξ_n) generates the topology on Ω. Every ξ_n is a finite union of T-towers $\xi_n(i)$, $i = 1, \ldots, k_n$, where $\xi_n(i) = \{T^j A_n(i) \mid j = 0, \ldots, h_i - 1\}$. Given $\varepsilon > 0$ choose n sufficiently large so that $\mathrm{diam}(A_n) < \varepsilon/2$ and $\mathrm{diam}(T^{-1}A_n) < \varepsilon/2$. For n, consider two collections of clopen subsets: $(T^{h_1-1}A_n(1), \ldots, T^{h_{k_n}-1}A_n(k_n))$ and $(A_n(1), \ldots, A_n(k_n))$. Let S_i be a one-to-one continuous map from $T^{h_i-1}A_n(i)$ onto $A_n(i+1)$, $i = 1, \ldots, k_n - 1$. Let $Y_1 = \Omega \setminus T^{h_{k_n}-1}A_n(k_n)$ and $Y_2 = \Omega \setminus A_n(1)$. Define a one-to-one continuous map S from Y_1 onto Y_2 by

$$Sx = \begin{cases} Tx & \text{if } x \notin \bigcup_{i=1}^{k_n} T^{h_i-1}A_n(i), \\ S_i x & \text{if } x \in T^{h_i-1}A_n(i),\ i = 1, \ldots, k_n - 1. \end{cases}$$

We note that $d(Tx, Sx) \leq \mathrm{diam}(T^{-1}A_n)$, $x \in Y_1$ and $d(T^{-1}x, S^{-1}x) \leq \mathrm{diam}(A_n)$, $x \in Y_2$.

This construction defines an S-tower with base $A_n(1)$ and top $T^{h_{k_n}-1}A_n(k_n)$. Next, by using cutting and stacking, we may extend S to a homeomorphism of Ω denoted also by S. Clearly, S has rank 1. By our construction,

$$d_w(S, T) = \sup_{x\in\Omega} d(Sx, Tx) + \sup_{x\in\Omega} d(S^{-1}x, T^{-1}x) < \varepsilon.$$

This proves that T can be approximated in τ_w by a homeomorphism S of rank 1. Thus we have shown that $\mathcal{M}in \subset \overline{\mathcal{R}(1)}^{\tau_w}$.

Now we will prove that $\mathcal{M}in \subset \overline{\mathcal{R}(1)}^{\tau}$. Let $\mu_1, \ldots, \mu_m$ be a given collection of Borel measures and let $\varepsilon > 0$. Take $T \in \mathcal{M}in$ and construct (A_n) and (ξ_n) as in the first part of the proof. It was proved in Theorem 3.5 that n can be chosen so large that $\mu_i(A_n) < \varepsilon/2$ and $\mu_i(T^{-1}A_n) < \varepsilon/2$, $i = 1, \ldots, m$. We apply the above method to define a homeomorphism S of Ω of rank 1. Obviously, $E(S, T) \subset A_n \cup T^{-1}A_n$. Then $\mu_i(E(S, T)) < \varepsilon$ for $i = 1, \ldots, m$. Therefore $S \in U(T; \mu_1, \ldots, \mu_m; \varepsilon) \cap \mathcal{R}(1)$. □

The next statement can be proved by using the techniques of proof of Theorem 5.3.

Corollary 5.4. *Let T be a homeomorphism of Ω which has a finite decomposition into minimal components. Then $T \in \overline{\mathcal{R}(1)}^{\tau}$ and $T \in \overline{\mathcal{R}(1)}^{\tau_w}$.*

It follows from Theorems 4.5 and 5.3 that the following result holds.

Corollary 5.5.

(a) *$\mathcal{O}d$ is a dense G_δ-set in $\overline{\mathcal{M}in}^{\tau_w}$, i.e. $\mathcal{O}d$ is a residual set in $\overline{\mathcal{M}in}^{\tau_w}$.*
(b) *$\mathcal{O}d$ is a dense $F_{\sigma\delta}$-set in $\overline{\mathcal{M}in}^{\tau}$.*
(c) *$\mathcal{M}in$ is closed neither in τ nor in τ_w.*

It follows from Corollary 5.5 that a typical minimal homeomorphism is saturated (see Remark 3.9).

Theorem 5.6. *$\mathcal{M}ix \cap \overline{\mathcal{M}in}^{\tau_w}$ is nowhere dense in $(\overline{\mathcal{M}in}^{\tau_w}, \tau_w)$.*

Proof. Let $T \in \overline{\mathcal{M}in}^{\tau_w}$. Take a τ_w-neighbourhood $W(T; F_1, \dots , F_m)$. We showed in Theorem 5.3 that there exists an odometer $S \in W(T; F_1, \dots , F_m)$. Let $\mathcal{F} = (E, SE, \dots , S^{n-1}E)$ be an S-tower such that $\bigcup_{i=0}^{n-1} S^iE = \Omega$ and $S^nE = E$. Then $W(S; E, SE, \dots , S^{n-1}E)$ consists of the homeomorphisms $R \in \mathrm{Homeo}(\Omega)$ such that $R(S^iE) = S^{i+1}E$, $i = 0, \dots , n-1$. Note that, choosing n sufficiently large, we have that every F_i is an $\mathcal{F}$-set. Therefore $W(S; E, SE, \dots , S^{n-1}E) \subset W(T; F_1, \dots , F_m)$ since $SF_i = TF_i$ and each F_i is a union of some atoms from $\mathcal{F}$. Then for any $RW(S; E, SE, \dots , S^{n-1}E)$, we have $\{m \in \mathbb{Z} \mid R^mE \cap E \neq \emptyset\} = n\mathbb{Z}$ and therefore R cannot be mixing. This proves that $\mathcal{M}ix$ is nowhere dense in $\overline{\mathcal{M}in}^{\tau_w}$. □

Remark 5.7. Since $\mathcal{M}ix \cap \mathcal{O}d = \emptyset$, Theorem 5.6 is consistent with the conclusion of Corollary 5.5 that $\mathcal{O}d$ is a set of second category in $\overline{\mathcal{M}in}^{\tau_w}$.

We now introduce a concept which will allow us to characterize the closure of $\mathcal{M}in$ in the weak topology τ_w.

Definition 5.8. We call a homeomorphism T moving if for any non-trivial clopen set F each of the sets $TF \setminus F$ and $F \setminus TF$ is not empty. A homeomorphism T is called weakly moving if $TF \neq F$ for every non-trivial $F \in \mathrm{CO}(\Omega)$.

We denote by $\mathcal{M}ov$ and w-$\mathcal{M}ov$ the sets of moving and weakly moving homeomorphisms, respectively. Obviously, $\mathcal{M}ov \subset$ w-$\mathcal{M}ov$ and if $T \in \mathcal{M}ov$ (or $T \in$ w-$\mathcal{M}ov$) then also $T^{-1} \in \mathcal{M}ov$ (or $T^{-1} \in$ w-$\mathcal{M}ov$).

It follows immediately that $\mathcal{M}in$ and $\mathcal{M}ix$ are subsets of w-$\mathcal{M}ov$. We make this statement more precise in Theorem 5.9 below.

Note that the set w-$\mathcal{M}ov$ is closed in τ_w. Indeed, it is easily seen that

$$\text{w-}\mathcal{M}ov = \bigcap_{F \in \mathrm{CO}(\Omega)} W(\mathbb{I}; F)^c. \tag{5.1}$$

Another simple observation from (5.1) says that if $P \in \mathcal{P}er_0$, then $P \notin$ w-$\mathcal{M}ov$.

Theorem 5.9. *$T \in \mathcal{M}ov$ if and only if $T \in \overline{\mathcal{O}d}^{\tau_w} = \overline{\mathcal{M}in}^{\tau_w}$.*

Proof. Let $T \in \overline{\mathcal{O}d}^{\tau_w}$. Consider a neighbourhood $W(T;F)$ where $F \in \mathrm{CO}(\Omega)$. Then $W(T;F)$ contains an odometer S such that $SF = TF$. Let ξ be an S-tower such that F is a ξ-set. Since every atom of ξ is shifted by S to the next level of the tower, we get that $TF \setminus F = SF \setminus F \neq \emptyset$ and $F \setminus TF = F \setminus SF \neq \emptyset$.

Conversely, suppose that T is moving. Let $W(T; F_1, \ldots, F_n)$ be a neighbourhood where $\zeta = (F_1, \ldots, F_n)$ is a partition of Ω into clopen sets. We know that for every i the sets $TF_i \setminus F_i$ and $F_i \setminus TF_i$ are non-empty. Take the intersection $\zeta \wedge T(\zeta)$ of partitions ζ and $T\zeta$ consisting of atoms $TF_i \cap F_j = F_{ij}$ where $i, j = 1, \ldots, n$ and some of F_{ij}'s may be empty.

We first consider a particular case when for every $i = 1, \ldots, n$,

$$|\{1 \leq j \leq n \mid TF_i \cap F_j \neq \emptyset\}| = |\{1 \leq j \leq n \mid TF_j \cap F_i \neq \emptyset\}|. \tag{5.2}$$

Let $A = (a_{ij})$ be an $n \times n$ matrix where $a_{ij} = 1$ if $TF_i \cap F_j \neq \emptyset$ and $a_{ij} = 0$ otherwise. In other words, we consider the directed graph Γ with the set of vertices $(1, \ldots, n)$ and the set of arrows defined by A: an arrow from i to j exists if and only if $a_{ij} = 1$. Note that Γ may have loops, i.e. arrows that begin and end at the same vertex. In other words, relation (5.2) says that the number of arrows coming to a vertex i equals the number of arrows outgoing from i. We claim that (Γ, A) is a connected graph. Indeed, let $Z(i)$ be the set of vertices that can be connected with i by a path. We show that $Z(i) = (1, \ldots, n)$ for every i. If $j \in Z(i)$ then there exist $j_0 = i, j_1, \ldots, j_s = j$ such that $TF_{j_k} \cap F_{j_{k+1}} \neq \emptyset$, $k = 0, \ldots, s-1$. Assume that $\Lambda = (1, \ldots, n) \setminus Z(i) \neq \emptyset$ and denote by $E = \bigcup_{j \in \Lambda} F_j$. Since $T \in \mathcal{M}ov$, we get that $TE^c \setminus E^c \neq \emptyset$ or $TE^c \cap E \neq \emptyset$ which contradicts the assumption.

Since we have a connected graph (Γ, A) satisfying (5.2), we can choose an Euler path L, consisting of arrows, which goes through all vertices and takes every arrow only once (see [5] for details).

We now construct a homeomorphism S of Ω using the path L. To every arrow from i to j we associate the set F_{ij}. We start with some vertex i_0 and let S be a homeomorphism from $F_{i_0 j}$ onto F_{jk} if the arrow from j to k follows that from i_0 to j in L. Then we extend the definition of S going along L. Thus, S is defined on Ω and atoms of $T(\zeta) \wedge \zeta$ form an S-tower, and since the path L is annular, the top of the tower is mapped by S onto the base. Moreover, one can verify that by definition of S, $TF_i = SF_i$ for all i. Indeed, fix some i and consider the sets F_{ji} and F_{ik} when j, k run over $(1, \ldots, n)$. We see that $\bigcup_j F_{ji} = F_i$ and $\bigcup_k F_{ik} = TF_i$. Hence S maps F_i onto TF_i. Therefore we have found an odometer S which belongs to the neighbourhood $W(T; F_1, \ldots, F_n)$.

In general, (5.2) does not hold. But we can slightly modify the above construction to obtain the result. We note that there are positive integers m_{ij} such that

$$\sum_{\{j|a_{ij}=1\}} m_{ij} = \sum_{\{j|a_{ji}=1\}} m_{ji}. \tag{5.3}$$

This means that we can consider a new graph (Γ, A') over the same set of vertices $(1, \dots, n)$ and with an extended set of arrows: if i and j are such that $a_{ij} = 1$, then we take exactly m_{ij} directed arrows from i to j. Given two connected vertices i and j, we can assign a number from 1 to m_{ij} to each arrow from i to j. In this way, we see by (5.3) that the connected graph (Γ, A') has the following property: the number of arrows arriving at each vertex is the same as the number of arrows leaving that vertex. Therefore, there exists a closed path L' that goes through all vertices (visiting each vertex several times) and includes each arrow only once.

To construct an odometer S, we divide every non-empty set F_{ij} into m_{ij} non-empty clopen subsets $E_{ij,k}$, $k = 1, \dots, m_{ij}$. To define S, we start with a vertex i_0 and some outgoing arrow $l(i_0, i_1; k_1)$ from i_0 to i_1 with the assigned number $1 \le k_1 \le m_{i_0 i_1}$. Then S is defined by the following rule: if the arrow $l(i_1, i_2, k_2)$ from i_1 to i_2 with the number k_2 is next to $l(i_0, i_1; k_1)$ in L' the we set $S\colon E_{i_0 i_1, k_1} \to E_{i_1 i_2, k_2}$. One can continue this procedure going along L' until the last arrow in L' has been used. This arrow has the largest number $m_{i_{n-1} i_0}$ amongst those that return to i_0. As before, it is easy to check that the sets $(E_{ij,k},\ i, j = 1, \dots, n; k = 1, \dots, m_{ij})$ define an S-tower and $SF_i = TF_i$ for all i. □

Let $\mathcal{T}t$ denote the set of all topologically transitive homeomorphisms of Ω.

Corollary 5.10.

(a) $\mathcal{T}t \subset \mathcal{M}ov$ *and* $\overline{\mathcal{T}t}^{\tau_w} = \overline{\mathcal{M}in}^{\tau_w}$;
(b) $\overline{\mathcal{M}in}^{\tau} \neq \overline{\mathcal{M}in}^{\tau_w}$.

Proof. It is easily seen that every $T \in \mathcal{T}t$ satisfies Definition 5.8. Thus, we have

$$\mathcal{M}ov \supset \mathcal{T}t \supset \mathcal{M}in,$$

and the result follows from Theorem 5.9.

On the other hand, if some T from $\mathcal{T}t$ has a periodic orbit then such a T cannot be in $\overline{\mathcal{M}in}^{\tau}$ in view of Theorem 3.1. □

Corollary 5.11. $\mathcal{M}in \subset \overline{\mathcal{O}d}^{\tau_w} = \overline{\mathcal{M}in}^{\tau_w} \subset$ w-$\mathcal{M}ov$ *and* $\mathcal{M}ix \subset \overline{\mathcal{O}d}^{\tau_w} = \overline{\mathcal{M}in}^{\tau_w} \subset$ w-$\mathcal{M}ov$.

Proof. We need only show that if T is either minimal or mixing then T belongs to $\overline{\mathcal{O}d}^{\tau_w}$. For this, assume that it is not true, i.e. $T \notin \overline{\mathcal{O}d}^{\tau_w} = \mathcal{M}ov$. Then there exists a proper clopen subset F such that either

(i) $TF = F$ or

(ii) $TF \subset F$ or
(iii) $F \subset TF$.

If (i) holds then T is neither minimal nor mixing. If (ii) holds, then $T^{n+1}F \subset T^n F$, $n \in \mathbb{N}$, and the set $\bigcap_{n\geq 0} T^n F$ is a closed T-invariant set. Thus, T cannot be minimal. Denote by $E = F\setminus TF$. Then $E, TE, \dots, T^n E, \dots$ are pairwise disjoint and therefore T cannot be mixing. Similar arguments are used in case (iii). □

Analyzing the proof of Theorem 5.9 we can immediately deduce the following consequence.

Theorem 5.12. *A homeomorphism T of Ω belongs to $\overline{\mathcal{P}er_0}^{\tau_w}$ if and only if for each clopen F either $TF = F$ or $TF \setminus F \neq \emptyset$ or $F \setminus TF \neq \emptyset$. Therefore every homeomorphism $T \in \overline{\mathcal{M}in}^{\tau_w}$ can be approximated by a periodic homeomorphism $P \in \mathcal{P}er_0$ in the topology τ_w, that is $\overline{\mathcal{M}in}^{\tau_w} \subset \overline{\mathcal{P}er_0}^{\tau_w}$.*

Proof. From the proof of Theorem 5.9, we see that if the hypotheses of the Theorem hold, one can construct a periodic homeomorphism S of Ω using the Euler path L in the same way as for odometers. □

A simple consequence of Theorem 5.12 is the following fact: $\overline{\mathcal{P}er}^{\tau_w} = \overline{\mathcal{P}er_0}^{\tau_w}$.

Remark 5.13. We may also use Theorem 5.9 to show that for a minimal homeomorphism $T \in \mathrm{Homeo}(\Omega)$, $[[T]]_{ap} \subset \overline{\mathcal{M}in}^{\tau}$ as stated in Theorem 3.7. For this, we need to prove that if $\gamma \in [[T]]_{ap}$, then $\gamma \in \mathcal{M}ov$, that is for any clopen E the sets $\gamma E\setminus E$, $E\setminus \gamma E$ are non-empty. Assume that there exists $F \in \mathrm{CO}(\Omega)$ such that $\gamma F \subseteq F$ and deduce from this that γ must have a periodic part. Choose a Kakutani–Rokhlin partition ξ such that $\gamma \in \Gamma(\xi)$ and the clopen sets F and γF are ξ-sets (we use here notation of Section 3). Then F and γF are unions of some atoms $D^j(v)$ of ξ. We have two possibilities: either every $D^j(v) \subset F$ does not leave the subtower $\xi(v)$ under the action of γ, or there exists $D^{j_0}(v_0)$ in F such that $D^{j_0}(v_0)$ belongs to some $L_{ti}(J)$ (or $L_{bk}(J')$). From the first case we get that γ must have a periodic part inside of $\xi(v)$. The second case implies that if $D^{j_0}(v_0) \subset F \cap L_{ti}(J)$, then $L_{ti}(J) \subset F$ (as well as $\gamma(L_{ti}(J))$) is also a subset in F since the set γF is a ξ-set. Then γ again has a periodic part.

We can strengthen the first statement of Theorem 5.3 by giving a complete description of closures of various classes of homeomorphisms in the topology τ.

Theorem 5.14. $\overline{\mathcal{O}d}^{\tau} = \overline{\mathcal{R}(1)}^{\tau} = \overline{\mathcal{M}in}^{\tau} = \overline{\mathcal{T}t}^{\tau} \cap \mathcal{A}p = \mathcal{A}p$.

Proof. Let T be an aperiodic homeomorphism of Ω and let $U(T) = U(T; \mu_1, \dots, \mu_k; \varepsilon)$ be a τ-neighbourhood of T.

We will apply the following result established in [2] to prove the Rokhlin lemma.

LEMMA 5.15. *Let T be an aperiodic homeomorphism of a Cantor set Ω, $\mu_1, \dots, \mu_k \in \mathcal{M}_1(\Omega)$, $\varepsilon > 0$. Given a positive integer $n \geq 2$, there exists a partition of Ω into a finite number of clopen T-towers $(\eta_1, \dots, \eta_q)$ such that the height $h(\eta_i)$ of every tower is at least n. Moreover, these towers can be chosen such that*

$$\mu_i\left(\bigcup_{j=0}^{n-1} T^{-j} B\right) > 1 - \varepsilon, \tag{5.4}$$

where $B = \bigcup_{i=1}^{q} B_i$ and B_i is the base of η_i.

SKETCH OF PROOF. We begin with a clopen finite disjoint cover $(U_1, \dots, U_k)$ of Ω such that $T^j U_i \cap U_i = \emptyset$, $j = 1, \dots, n-1$, $i = 1, \dots, k$. Consider $\xi_1 = (U_1, TU_1, \dots, T^{n-1}U_1)$ and define $C_1 = \bigcup_{j=0}^{n-1} T^j U_1$. Let $U_i^1 = U_1 - C_1$, $i = 2, \dots, k$. Define

$$U_2^1(i) = \{x \in U_2^1 \mid T^i x \in U_1,\ T^j x \notin U_1,\ 0 \leq j \leq i-1\}, \quad i = 1, \dots, n-1,$$

and

$$U_2^1(0) = \{x \in U_2^1 \mid T^j x \notin U_1, \text{ for all } 1 \leq j \leq n-1\}.$$

Each set $U_2^1(i)$ is the base of the T-tower

$$\xi_2^1(i) = \{U_2^1(i), TU_2^1(i), \dots, T^{n-1+i}U_2^1(i)\}, \quad \text{for all } i = 0, \dots, n-1.$$

Take the set

$$U_1^1 = U_1 \setminus \bigcup_{i=1}^{n-1} T^i U_2^1(i)$$

as the base of a subtower ξ_1^1 of ξ_1. We get the collection of disjoint T-towers $\Xi(1) = \{\xi_1^1, \xi_2^1(0), \xi_2^1(1), \dots, \xi_2^1(n-1)\}$ each of which is of height at least n. Denote by $C_1^1, C_2^1(0), C_2^1(1), \dots, C_2^1(n-1)$ the supports of the corresponding towers from $\Xi(1)$. Note that $C_1^1 \cup C_2^1(0) \cup C_2^1(1) \cup \ldots \cup C_2^1(n-1) = C_1 \cup C_2^1$ where $C_2^1 = \bigcup_{s=0}^{n-1} T^i U_2^1$. Define the sets $U_i^2 = U_i^1 \setminus C_2^1$, for all $i = 3, \dots, k$. For each tower ξ from $\Xi(1)$, denote by $U_3^2(\xi)$ the subset of U_3^2 which consists of those points whose T-orbits meet ξ for at most $n-1$ iterations. We can now apply the construction used above to the set $U_3^2(\xi)$ and the tower $\xi \in \Xi(1)$. Repeating this procedure at most $k-1$ times we will finally obtain a collection $\Xi = (\xi_1, \dots, \xi_s)$ of disjoint T-towers which covers Ω such that the height of each tower $\xi \in \Xi$ is at least n.

Choose $m \in \mathbb{N}$ such that $1/m < \varepsilon$ and let V_j denote the top of ξ_j. Using the obvious fact that amongst m pairwise disjoint subsets of Ω at least one must have μ_i-measure less than ε, we may choose a clopen set $B = T^{-K}V$, where $0 \leq K < n$ and $V = \bigcup_{j=1}^{s} V_j$, such that (5.4) holds. To get the T-towers $\eta_1, \dots, \eta_q$, one refines the existing towers $\xi_1, \dots, \xi_s$ such that the base of each

η_i is a subset of B and the top is a subset of $T^{-1}B$. Full details of the proof are in [2]. □

We now return to the proof of the Theorem. Define a homeomorphism S as follows. Let S_i be a homeomorphism sending the top $T^{h(\xi_i)-1}B_i$ of ξ_i onto the base B_{i+1} of ξ_{i+1} $(i = 1, \ldots, m-1)$ and S_n maps $T^{h(\xi_m)-1}B_m$ onto B_1. Take $n = 2$ in Lemma 5.15 and define

$$Sx = \begin{cases} Tx & \text{if } x \in \bigcup_{i=1}^{m} \bigcup_{j=0}^{h(\xi_i)-2} T^j B_j, \\ S_i x & \text{if } x \in T^{h(\xi_i)-1}B_i \text{ for some } i = 1, \ldots, m. \end{cases}$$

It follows from (5.4) that the homeomorphism S belongs to $U(T)$. On the other hand, there exists a homeomorphism $S_0 \in \mathcal{O}d$ such that $S_0 x \neq Sx$ only if $x \in T^{h(\xi_m)-1}B_m$, hence $S_0 \in U(T)$. This proves that $\overline{\mathcal{O}d}^{\tau} = \mathcal{A}p$. □

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