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LAGRANGIAN SYSTEMS WITH LIPSCHITZ OBSTACLE ON MANIFOLDS

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Abstract. Lagrangian systems constrained on the closure of an open subset with Lipschitz boundary in a manifold are considered. Under suitable assumptions, the existence of infinitely many periodic solutions is proved.

1. Introduction

The study of Lagrangian functionals of the form

$$f(\gamma) = \int_0^1 L(s, \gamma(s), \gamma'(s))\, ds \tag{1.1}$$

on a manifold M, where $L(s,(q,v))\colon \mathbb{R} \times TM \to \mathbb{R}$, costitutes a well studied topic in Mechanics and Global analysis. In particular, about the existence and multiplicity of periodic solutions γ of the associated Euler equation, we refer the reader to [1], where the case in which M is a compact manifold without boundary is considered. Starting from [1], some extensions have been considered in the literature, when M is embedded in an Euclidean space. In [3] the case where M is a compact submanifold with boundary in $\mathbb{R}^n$ has been considered.

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In such a case, the associated Euler equation has the form

$$\frac{d}{ds}(D_v L(s,\gamma,\gamma')) - D_q L(s,\gamma,\gamma') \in \mathrm{N}_{\gamma(s)} M, \tag{1.2}$$

where $\mathrm{N}_q M$ is the outer normal cone to M at q. The main feature is that the natural domain of the functional (1.1) is

$$X = \{\gamma \in W^{1,2}(0,1;\mathbb{R}^n) : \gamma(0) = \gamma(1), \gamma(s) \in M \text{ for all } s\} \tag{1.3}$$

which is naturally a metric space, but not a smooth manifold (even with boundary). Moreover, solutions γ of (1.2) are not of class C^2, but only $W^{2,\infty}$ and satisfy (1.2) almost everywhere. In the same direction, the case in which M is a compact p-convex subset of $\mathbb{R}^n$ has been considered in [4]. The class of p-convex subsets [8] includes in particular subsets with corners of convex type and concave parts of class C^2. This direction of research was started by [18], where the case of an n-dimensional submanifold with boundary of class C^2 in $\mathbb{R}^n$ had been considered.

Another development has been started more recently in [11], [19], considering the case in which M is the closure of a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Also in this case the set X is naturally only a metric space. Moreover, since in this case we cannot expect the solution γ of (1.2) to be of class C^1, the Euler equation itself requires a reformulation.

The purpose of this paper is to consider the intrinsic case in which M is the closure of a bounded open subset of a differentiable manifold N, instead of $\mathbb{R}^n$, and also to relax the convexity condition on L, which was in [19] of uniform quadratic type, to the mere convexity with coercivity of order $p > 1$.

Our approach follows the lines of [19], but it is completely intrinsic. Of course the lack of strict convexity in L causes new technical difficulties.

The paper is organized as follows: in Section 2 we state our main results, while Section 3 is devoted to some recalls of nonsmooth analysis. Finally, in Section 4 we prove the main results.

2. Statement of the main results

Let N be a differentiable manifold without boundary of class C^2 and $M \subseteq N$. In the sequel, each $\gamma \in W^{1,p}(a,b;N)$ will be identified with its continuous representative $\widetilde{\gamma}: [a,b] \to N$. We set

$$W^{1,p}(a,b;M) := \{\gamma \in W^{1,p}(a,b;N) : \gamma(s) \in M \text{ for each } s \in [a,b]\}.$$

Remark 2.1. Let g and $\widetilde{g}$ be two Riemannian structures on N and let d and $\widetilde{d}$ be the induced distances on N. Then there exists a continuous function $c: N \to]0,\infty[$ such that, for all $q \in N$ and all $v \in T_q N$,

$$g(q)(v,v) \le c(q)\widetilde{g}(q)(v,v), \qquad \widetilde{g}(q)(v,v) \le c(q) g(q)(v,v).$$

In particular, for every compact subset $K \subseteq N$ there exists $C > 0$ such that, for all $q_1, q_2 \in K$,

$$d(q_1, q_2) \leq C\widetilde{d}(q_1, q_2), \qquad \widetilde{d}(q_1, q_2) \leq C d(q_1, q_2).$$

Let $1 < p < \infty$ and $L: \mathbb{R} \times TN \to \mathbb{R}$ be a function of class C^1 such that there exist two continuous functions $c, k: M \to]0, \infty[$ and $d \in \mathbb{R}$ such that for every $s \in \mathbb{R}$ and $q \in M$ one has

$$k(q)|v|^p - d \leq L(s, q, v) \leq c(q)(1 + |v|^p) \quad \text{for all } v \in \mathrm{T}_q N, \tag{2.1}$$

$$|D_{(q,v)} L(s, q, v)| \leq c(q)(1 + |v|^p) \quad \text{for all } v \in \mathrm{T}_q N, \tag{2.2}$$

$$L(s, q, \cdot\,) \text{ is convex} \quad \text{on } \mathrm{T}_q N, \tag{2.3}$$

where $|v| = \sqrt{g(q)(v, v)}$.

In (2.1), (2.2) we mean that N is provisionally endowed with a Riemannian structure. By Remark 2.1 the above conditions do not depend on the Riemannian structure chosen on N.

In charts, (2.1), (2.2) mean that for every $s \in \mathbb{R}$ and $q \in M$ it is

$$\begin{aligned} k(q)|v|^p - d \leq L(s, q, v) &\leq c(q)(1 + |v|^p) \quad \text{for all } v \in \mathrm{T}_q N, \\ |D_q L(s, q, v)| &\leq c(q)(1 + |v|^p) \quad \text{for all } v \in \mathrm{T}_q N, \\ |D_v L(s, q, v)| &\leq c(q)(1 + |v|^p) \quad \text{for all } v \in \mathrm{T}_q N. \end{aligned}$$

Let us remark that (2.1), (2.3) imply that for every $s \in \mathbb{R}$, $q \in M$ and any $v, w \in \mathrm{T}_q N$ we have

$$|D_v L(s, q, v) w| \leq \widehat{c}(q)(1 + |v|^{p-1})|w|$$

namely, in charts,

$$|D_v L(s, q, v)| \leq \widehat{c}(q)(1 + |v|^{p-1}),$$

where $\widehat{c}: M \to]0, \infty[$ is continuous.

Define a continuous functional $f_{a,b}: W^{1,p}(a, b; M) \to \mathbb{R}$ by

$$f_{a,b}(\gamma) = \int_a^b L(s, \gamma(s), \gamma'(s))\, ds.$$

Given a Riemannian structure on N, for every $\gamma, \eta \in W^{1,p}(a, b; M)$ we set

$$\begin{aligned} d_1(\gamma, \eta) &= \int_a^b d(\gamma(s), \eta(s))\, ds, \\ d_\infty(\gamma, \eta) &= \max\{d(\gamma(s), \eta(s)) : a \leq s \leq b\}, \end{aligned}$$

where d is the distance on N associated with the Riemannian structure.

Definition 2.2. We say that $\gamma \in W^{1,p}(a,b;M)$ is *L-stationary*, if it is not possibile to find $r, c, \sigma > 0$ and a map

$$\mathcal{H}\colon \{\eta \in W^{1,p}(a,b;M) : d_\infty(\eta,\gamma) < r,\ f_{a,b}(\eta) < f_{a,b}(\gamma) + r\} \times [0,r] \to W^{1,p}(a,b;M)$$

such that:

(a) $\mathcal{H}$ is continuous from the product of the topology of the uniform convergence and that of $\mathbb{R}$ to that of the uniform convergence;
(b) for every $\eta \in W^{1,p}(a,b;M)$ with $d_\infty(\eta,\gamma) < r$, $f_{a,b}(\eta) < f_{a,b}(\gamma) + r$ and $t \in [0,r]$ we have

$$\begin{aligned} \mathcal{H}(\eta,t)(a) &= \eta(a), & \mathcal{H}(\eta,t)(b) &= \eta(b), \\ d_1(\mathcal{H}(\eta,t),\eta) &\le ct, & f_{a,b}(\mathcal{H}(\eta,t)) &\le f_{a,b}(\eta) - \sigma t. \end{aligned}$$

Again we mean that the assertion holds after introducing a Riemannian structure on N. By Remark 2.1 this definition does not depend on the choice of the Riemannian structure itself.

Proposition 2.3. *Let $\gamma \in W^{1,p}(a,b;M)$ be L-stationary. Then for every $[\alpha,\beta] \subseteq [a,b]$ the restriction $\gamma_{|[\alpha,\beta]}$ is L-stationary.*

Proof. Set $\widehat{\gamma} = \gamma_{|[\alpha,\beta]}$. By contradiction, assume that there exist $r, c, \sigma > 0$ and

$$\mathcal{H}\colon \{\eta \in W^{1,p}(\alpha,\beta;M) : d_\infty(\eta,\widehat{\gamma}) < r,\ f_{\alpha,\beta}(\eta) f_{\alpha,\beta}(\widehat{\gamma}) + r\} \times [0,r] \to W^{1,p}(\alpha,\beta;M)$$

according to Definition 2.2.

We claim that there exists $r' \in\ ]0,r[$ such that if $\eta \in W^{1,p}(a,b;M)$ with $d_\infty(\eta,\gamma) < r'$ and $f_{a,b}(\eta) < f_{a,b}(\gamma) + r'$, then $f_{\alpha,\beta}(\widehat{\eta}) < f_{\alpha,\beta}(\widehat{\gamma}) + r$, where $\widehat{\eta} = \eta_{|[\alpha,\beta]}$.

Again by contradiction, let $(\eta_h) \subseteq W^{1,p}(a,b;M)$ with η_h convergent to γ with respect to the uniform convergence and $\limsup_h f_{a,b}(\eta_h) \le f_{a,b}(\gamma)$ such that $f_{\alpha,\beta}(\widehat{\eta}_h) \ge f_{\alpha,\beta}(\widehat{\gamma}) + r$. By (2.1) and (2.3) we have

$$\begin{aligned} \limsup_h f_{\alpha,\beta}(\widehat{\eta}_h) &\le \limsup_h f_{a,b}(\eta_h) - \liminf_h \int_{]a,b[\setminus]\alpha,\beta[} L(s,\eta_h,\eta_h')\,ds \\ &\le f_{a,b}(\gamma) - \int_{]a,b[\setminus]\alpha,\beta[} L(s,\gamma,\gamma')\,ds = f_{\alpha,\beta}(\widehat{\gamma}), \end{aligned}$$

whence a contradiction. Then, for any $\eta \in W^{1,p}(a,b;M)$ define

$$\mathcal{K}\colon \{\eta \in W^{1,p}(a,b;M) : d_\infty(\eta,\gamma) < r',\ f_{a,b}(\eta) < f_{a,b}(\gamma) + r'\} \times [0,r'] \to W^{1,p}(a,b;M)$$

by

$$\mathcal{K}(\eta,t)(s)=\begin{cases}\mathcal{H}(\widehat{\eta},t)(s) & \text{if } s\in[\alpha,\beta],\\ \eta(s) & \text{if } s\notin[\alpha,\beta].\end{cases}$$

It is readily seen that $\mathcal{K}$ has all the properties required in Definition 2.2. It follows that γ is not L-stationary, which is absurd. □

Definition 2.4. Let I be an interval in $\mathbb{R}$ with $\mathrm{int}(I)\neq\emptyset$. A continuous map $\gamma\colon I\to M$ is said to be *a generalized solution* of the Lagrangian system associated to L *on* M, if every $s\in\mathrm{int}(I)$ admits a neighbourhood $[a,b]$ in I such that $\gamma_{|[a,b]}$ belongs to $W^{1,p}(a,b;M)$ and is L-stationary.

Definition 2.5. Given $T>0$, a *T-periodic generalized solution of* the Lagrangian system associated to L *on* M is a generalized solution $\gamma\colon\mathbb{R}\to M$ which is periodic of period T.

We now state our main existence result.

Theorem 2.6. *Assume that M is the closure of an open subset of N with locally Lipschitz boundary. Suppose also that M is compact, 1-connected and non-contractible in itself and that*

(2.4) $\qquad L(s+1,q,v)=L(s,q,v)\quad$ *for all $s\in\mathbb{R}$ and all $(q,v)\in TN$.*

Then there exists a sequence (γ_h) of 1-periodic generalized solutions of the Lagrangian system associated to L on M with

$$\lim_h\int_0^1 L(s,\gamma_h(s),\gamma_h'(s))\,ds=+\infty.$$

The notion of generalized solution we have introduced follows the approach of [11, Definition 3.3] and [19, Definition 2.6] and has the advantage to be intrinsically connected to M, although quite indirect. However, at least in the particular case $p=2$, it is possibile to deduce further informations on the generalized solutions.

For every $q\in M$, denote by N_qM the normal cone to M at q (see e.g. Definition 3.2 below).

Theorem 2.7. *Let $p=2$ and assume that there exists a continuous function $\omega\colon N\to\,]0,\infty[$ such that for every $s\in\mathbb{R}$, $q\in M$ it is*

$$D_vL(s,q,v)(v-w)-D_vL(s,q,w)(v-w)\geq\omega(q)|v-w|^2\quad \textit{for all } v,w\in \mathrm{T}_qN.$$

Let $\gamma\in W^{1,2}(a,b;M)$ be L-stationary. Then $\gamma\in W^{1,\infty}(a,b;M)$, $D_{(q,v)}L(s,\gamma,\gamma')\in L^\infty(a,b;T^*(TN))$ and there exist a finite Borel measure μ on $]a,b[$ and a bounded Borel function $\nu\colon\,]a,b[\,\to T^*N$ such that $\nu(s)\in\mathrm{N}_{\gamma(s)}M$ for μ-a.e. $s\in\,]a,b[$ and

$$\int_a^b D_{(q,v)}L(s,\gamma,\gamma')(\delta,\delta')\,ds=-\int_a^b\nu(\delta)\,d\mu$$

for any $\delta \in W_0^{1,1}(a,b;TN)$ *with* $\delta(s) \in \mathrm{T}_{\gamma(s)}N$ *for every* $s \in [a,b]$.

Also in this assertion we mean that N is provisionally endowed with a Riemannian structure. Since γ is continuous, by Remark 2.1 the assertion is independent of the choice of the structure.

Proof of Theorem 2.7. By Proposition 2.3, we may assume that $\gamma([a,b])$ is contained in a coordinated neighbourhood. Then the assertion follows from [19, Theorem 2.10]. □

3. Some relevant results of nonsmooth analysis

In the first part of this section let N be a differentiable manifold of class C^2 and M be the closure of an open set in N with locally Lipschitz boundary.

If X is a Banach space, $E \subseteq X$ and $x \in E$, we denote by $\mathrm{T}_x E$ the tangent cone to E at x, according to [6]. We also denote by $\mathrm{B}_r(x)$ the open ball of center x and radius r.

Definition 3.1. Let $x \in E$ and $v \in X$. We say that v is *hypertangent to E at x* if there exists $\delta > 0$ such that $\mathrm{B}_\delta(x) + [0,\delta]\mathrm{B}_\delta(v) \subseteq E$. Let us denote by $\mathrm{Hyp}_x E$ the set of the v's hypertangent to E at x.

Definition 3.2. Let $q \in M$ and $v \in \mathrm{T}_q N$. We say that v is *tangent to M at q* if there exists a chart (U,φ) at q such that $d\varphi(q)v \in \mathrm{T}_{\varphi(q)}\varphi(U \cap M)$. The set of the v's tangent to M at q is denoted by $\mathrm{T}_q M$ and is called the *tangent cone to M at q*.

We say that v is *hypertangent to M at q* if there exists a chart (U,φ) at q such that $d\varphi(q)v$ is hypertangent to $\varphi(U \cap M)$ at $\varphi(q)$. The set of the v's hypertangent to M at q is denoted by $\mathrm{Hyp}_q M$ and is called the *hypertangent cone to M at q*. Finally, we set $\mathrm{N}_q M = \{\varphi \in \mathrm{T}_q^* N : \varphi(v) \le 0 \text{ for all } v \in \mathrm{T}_q M\}$. $\mathrm{N}_q M$ is called the *normal cone to M at q*.

Remark 3.3. For every $q \in M$ it is $\mathrm{Hyp}_q M \neq \emptyset$ (see [6]) and $\mathrm{Hyp}_q M \subseteq \mathrm{T}_q M$.

Theorem 3.4. *There exists a section $\nu: N \to TN$ of class C^1 such that*

$$\nu(q) \in \mathrm{Hyp}_q M \quad \textit{for all } q \in M.$$

Proof. For all $q \in N$, let

$$\Psi(q) = \begin{cases} \mathrm{Hyp}_q M & \text{if } q \in M, \\ \mathrm{T}_q N & \text{if } q \in N \setminus M. \end{cases}$$

Then for every $q \in N$, $\Psi(q)$ is convex in $\mathrm{T}_q N$ and for every $q \in N$ there exists a chart (U,φ) at q such that

$$\bigcap_{\xi \in U} (d\varphi(\xi)(\Psi(\xi))) \neq \emptyset.$$

It follows that there exists $\nu\colon N \to TN$ of class C^1 with $\nu(q) \in \Psi(q)$ for every $q \in N$, hence the assertion. $\square$

LEMMA 3.5. *Let $\widetilde{N}$ be a submanifold of class C^2 of $\mathbb{R}^n$, $\widetilde{M}$ be the closure of an open subset of $\widetilde{N}$ with locally Lipschitz boundary, A be an open subset of $\mathbb{R}^n$ with $\widetilde{N} \subseteq A$ and $\pi\colon A \to \widetilde{N}$ be a retraction of class C^2 such that π is Lipschitz continuous of constant 2. Then there exists a map $\nu\colon \widetilde{N} \to \mathbb{R}^n$ of class C^1 such that the following facts hold:*

(a) *for any $q \in \widetilde{N}$ we have $\nu(q) \in \mathrm{T}_q\widetilde{N}$;*
(b) *for any $q \in \widetilde{M}$ there exists $\delta > 0$ such that*

$$\textit{if}\quad \begin{cases} \xi \in \mathrm{B}_\delta(q), \\ \pi(\xi) \in \widetilde{M}, \\ 0 < t \leq \delta, \\ v \in \mathrm{B}_\delta(\nu(q)), \end{cases} \qquad \textit{then}\quad \pi(\xi + tv) \in \mathrm{int}(\widetilde{M});$$

(c) *for every compact subset $K \subseteq \widetilde{M}$ there exist $\widehat{r}, \widehat{c} > 0$ satisfying*

$$\pi((1-t)q + t\pi(\xi + \rho\nu(\xi))) \in \widetilde{M}$$

whenever $q \in \widetilde{M}$, $\xi \in K$, $\widehat{c}|q - \xi| \leq \rho \leq \widehat{r}$ and $t \in [0,1]$.

PROOF. By Theorem 3.4 there exists a map $\nu\colon \widetilde{N} \to \mathbb{R}^n$ of class C^1 such that for any $q \in \widetilde{N}$ it is $\nu(q) \in \mathrm{T}_q\widetilde{N}$.

To prove (b), assume by contradiction that $q \in \widetilde{M}$, $\xi_h \to q$, $t_h \to 0^+$ and $v_h \to \nu(q)$ with $\pi(\xi_h) \in \widetilde{M}$ and $\pi(\xi_h + t_h v_h) \notin \mathrm{int}(\widetilde{M})$.

Let (U, φ) be the chart at q such that $\varphi\colon U \to \mathrm{T}_q\widetilde{N}$, $\varphi(q) = 0$ and $\pi(q + \varphi(\xi)) = \xi$ for any $\xi \in U$; in particular, $\nu(q) \in \mathrm{Hyp}_0\varphi(U \cap \widetilde{M})$.
Then we have

$$\varphi(\pi(\xi_h + t_h v_h)) \notin \mathrm{int}(\varphi(U \cap \widetilde{M})).$$

Since

$$\varphi(\pi(\xi_h + t_h v_h)) = \varphi(\pi(\xi_h)) + t_h(d[\varphi \circ \pi](\xi_h)v_h + \varepsilon_h)$$

with $\varepsilon_h \to 0$ in $\mathrm{T}_q\widetilde{N}$, it follows that $d[\varphi \circ \pi](\xi_h)v_h + \varepsilon_h \in \mathrm{T}_q\widetilde{N}$ and

$$\varphi(\pi(\xi_h + t_h v_h)) \in \mathrm{int}(\varphi(U \cap \widetilde{M}))$$

for large h, which is absurd.

Now let us prove (c). By contradiction, let (q_h) in $\widetilde{M}$, (ξ_h) in K, (t_h) in $[0,1]$, $\rho_h \to 0$ with $h|q_h - \xi_h| \leq \rho_h \leq 1/h$ and

$$\pi((1-t_h)q_h + t_h\pi(\xi_h + \rho_h\nu(\xi_h))) \notin \widetilde{M}.$$

Up to a subsequence $\xi_h \to \xi$ in K, $q_h \to \xi$ in $\widetilde{M}$ and $t_h \to t$ in $[0,1]$. It is

$$\pi((1-t_h)q_h + t_h\pi(\xi_h + \rho_h\nu(\xi_h))) = \pi\left(q_h + t_h\rho_h\left(\frac{\pi(\xi_h + \rho_h\nu(\xi_h)) - q_h}{\rho_h}\right)\right).$$

On the other hand,

$$\frac{\pi(\xi_h+\rho_h\nu(\xi_h))-q_h}{\rho_h}-\nu(\xi)=\frac{\pi(\xi_h+\rho_h\nu(\xi))-\xi_h-\rho_h\nu(\xi)}{\rho_h} + \frac{\xi_h-q_h}{\rho_h}+\frac{\pi(\xi_h+\rho_h\nu(\xi_h))-\pi(\xi_h+\rho_h\nu(\xi))}{\rho_h}.$$

By [11, Theorem 4.4], it is

$$\lim_h \frac{\pi(\xi_h+\rho_h\nu(\xi))-\xi_h-\rho_h\nu(\xi)}{\rho_h}=0.$$

Moreover, by the lipschitzianity of π it is also

$$\left|\frac{\pi(\xi_h+\rho_h\nu(\xi_h))-\pi(\xi_h+\rho_h\nu(\xi))}{\rho_h}\right|\le 2|\nu(\xi_h)-\nu(\xi)|.$$

It follows that

$$\lim_h \frac{\pi(\xi_h+\rho_h\nu(\xi_h))-q_h}{\rho_h}=\nu(\xi),$$

hence by (a) it is

$$\pi\bigg(q_h+t_h\rho_h\bigg(\frac{\pi(\xi_h+\rho_h\nu(\xi_h))-q_h}{\rho_h}\bigg)\bigg)\in\widetilde{M}$$

for large h, which is a contradiction. □

Definition 3.6. A subset E of N is said to be a LNR in N if there exists an open neighbourhood U of E in N and a locally Lipschitzian retraction $r\colon U\to E$.

Theorem 3.7. *The set M is a* LNR *in N.*

Proof. By [14, §2, Theorems 2.10 and 2.14], we may assume that N is a smooth submanifold of $\mathbb{R}^n$. By [14, §4, Theorem 5.1], there exist an open subset A of $\mathbb{R}^n$ with $N\subseteq A$ and a retraction $\pi\colon A\to N$ of class C^∞ such that π is Lipschitz continuous of constant 2. Let $\nu\colon N\to\mathbb{R}^n$ be as in Lemma 3.5. By (b) of Lemma 3.5, for every $q\in M$ there exists $\delta_q>0$ such that

$$\text{if}\quad\begin{cases}\xi\in \mathrm{B}_{\delta_q}(q),\\ \pi(\xi)\in M,\\ 0<t\le\delta_q,\\ v\in \mathrm{B}_{\delta_q}(\nu(q)),\end{cases}\quad\text{then}\quad \pi(\xi+tv)\in\mathrm{int}(M).$$

Let $\delta'_q\in\,]0,\delta_q]$ be such that

$$\text{if}\quad\begin{cases}\xi\in \mathrm{B}_{\delta'_q}(q),\\ 0\le t\le\delta'_q,\end{cases}\quad\text{then}\quad\begin{cases}\xi+t\nu(\xi)\in \mathrm{B}_{\delta_q}(q),\\ \nu(\xi)\in \mathrm{B}_{\delta_q/2}(\nu(q)),\\ |\xi-q|+\delta_q|\nu(\xi)-\nu(q)|\le\delta_q^2/4.\end{cases}$$

For every $q \in M$, define

$$U_q = \{\xi \in \mathrm{B}_{\delta'_q}(q) : \pi(\xi + \delta'_q \nu(\xi)) \in \mathrm{int}(M)\}, \quad U = \bigcup_{q \in M} U_q.$$

For every $\xi \in U$, let $T(\xi) = \min\{t \geq 0 : \pi(\xi + t\nu(\xi)) \in M\}$. It is easy to see that, if $q \in M$ and $\xi \in U_q$, then

$$T(\xi) < \delta'_q, \quad \xi + T(\xi)\nu(\xi) \in \mathrm{B}_{\delta_q}(q), \quad \pi(\xi + T(\xi)\nu(\xi)) \in M$$

and

$$\text{(3.1)} \qquad \text{if } \begin{cases} 0 \leq t \leq \delta_q, \\ v \in \mathrm{B}_{\delta_q}(\nu(q)), \end{cases} \quad \text{then} \quad \pi(\xi + T(\xi)\nu(\xi) + tv) \in M.$$

Let now $q \in M$ and $\xi_1, \xi_2 \in U_q$ with $\xi_1 \neq \xi_2$. We set

$$s = \frac{2}{\delta_q}(|\xi_1 - \xi_2| + T(\xi_1)|\nu(\xi_1) - \nu(\xi_2)|)$$

and

$$v = \nu(\xi_2) - \frac{1}{s}(\xi_1 - \xi_2 + T(\xi_1)(\nu(\xi_1) - \nu(\xi_2))).$$

We have $s \in \,]0, \delta_q]$ and $v \in \mathrm{B}_{\delta_q}(\nu(q))$. If we consider $t = T(\xi_1) + s$, an easy calculation shows that

$$\xi_2 + t\nu(\xi_2) = \xi_1 + T(\xi_1)\nu(\xi_1) + sv.$$

By (3.1) it follows that $\pi(\xi_2 + t\nu(\xi_2)) \in M$, hence $T(\xi_2) \leq t$. Therefore we get

$$T(\xi_2) \leq T(\xi_1) + s \leq T(\xi_1) + \frac{2}{\delta_q}(|\xi_1 - \xi_2| + \delta_q|\nu(\xi_1) - \nu(\xi_2)|);$$

exchanging the role of ξ_1 and ξ_2 we have

$$|T(\xi_1) - T(\xi_2)| \leq \frac{2}{\delta_q}(|\xi_1 - \xi_2| + \delta_q|\nu(\xi_1) - \nu(\xi_2)|),$$

hence T is locally Lipschitzian. If follows that the map $r: U \to M$ defined by $r(\xi) = \pi(\xi + T(\xi)\nu(\xi))$ is a locally Lipschitzian retraction. Therefore M is an LNR in $\mathbb{R}^n$, in particular in N. □

In the second part of this section, we recall some abstract notions and results of nonsmooth analysis.

Let Y be a metric space endowed with the metric d and let $f: Y \to \overline{\mathbb{R}}$ be a function. We set

$$\mathrm{epi}(f) = \{(u, \lambda) \in Y \times \mathbb{R} : f(u) \leq \lambda\}.$$

In the following, $Y \times \mathbb{R}$ will be endowed with the metric

$$d((u, \lambda), (v, \mu)) = (d(u, v)^2 + (\lambda - \mu)^2)^{1/2}$$

and $\mathrm{epi}(f)$ with the induced metric.

Definition 3.8. For every $u \in Y$ with $f(u) \in \mathbb{R}$, we denote by $|df|(u)$ the supremum of the σ's in $[0,\infty[$ such that there exist $r > 0$ and a continuous map

$$\mathcal{H}\colon (\mathrm{B}_r(u, f(u)) \cap \mathrm{epi}(f)) \times [0, r] \to Y$$

satisfying

$$d(\mathcal{H}((v,\mu),t),v) \le t, \qquad f(\mathcal{H}((v,\mu),t)) \le \mu - \sigma t,$$

whenever $(v,\mu) \in \mathrm{B}_r(u, f(u)) \cap \mathrm{epi}(f)$ and $t \in [0, r]$.

The extended real number $|df|(u)$ is called *the weak slope* of f at u.

The above notion has been introduced in [9], following an equivalent approach. When f is continuous, it has been independently introduced also in [17], while a variant appears in [15], [16]. The version we have recalled here is taken from [2].

Proposition 3.9. *Let $u \in Y$ with $f(u) \in \mathbb{R}$. Assume there exist $r, c, \sigma > 0$ and a continuous map*

$$\mathcal{H}\colon \{v \in \mathrm{B}_r(u) : f(v) < f(u) + r\} \times [0, r] \to Y$$

such that for any $v \in \mathrm{B}_r(u)$ with $f(v) < f(u) + r$ and any $t \in [0, r]$ it is

$$d(\mathcal{H}(v,t),v) \le ct, \qquad f(\mathcal{H}(v,t)) \le f(v) - \sigma t.$$

Then we have $|df|(u) \ge \sigma/c$.

Proof. See [11, Proposition 2.3]. □

Now, according to [8], we define a function $\mathcal{G}_f\colon \mathrm{epi}(f) \to \mathbb{R}$ by $\mathcal{G}_f(u,\lambda) = \lambda$. Of course, $\mathcal{G}_f$ is Lipschitzian of constant 1.

Proposition 3.10. *For every $u \in Y$ with $f(u) \in \mathbb{R}$, we have $f(u) = \mathcal{G}_f(u, f(u))$ and*

$$|df|(u) = \begin{cases} \dfrac{|d\mathcal{G}_f|(u,f(u))}{\sqrt{1 - |d\mathcal{G}_f|(u,f(u))^2}} & \text{if } |d\mathcal{G}_f|(u,f(u)) < 1, \\ \infty & \text{if } |d\mathcal{G}_f|(u,f(u)) = 1. \end{cases}$$

Proof. See [2, Proposition 2.3]. □

The previous proposition allows us to reduce, at some extent, the study of the general function f to that of the continuous function $\mathcal{G}_f$. For this purpose, the next result will be useful.

Proposition 3.11. *Let $(u,\lambda) \in \mathrm{epi}(f)$ with $f(u) < \lambda$. Assume that for every $\varepsilon > 0$ there exist $r > 0$ and a continuous map*

$$\mathcal{H}\colon \{v \in \mathrm{B}_r(u) : f(v) < \lambda + r\} \times [0, r] \to Y$$

such that for any $v \in \mathrm{B}_r(u)$ *with* $f(v) < \lambda + r$ *and any* $t \in [0,r]$ *it is*

$$d(\mathcal{H}(v,t),v) \leq \varepsilon t,$$
$$f(\mathcal{H}(v,t)) \leq (1-t)f(v) + t(f(u)+\varepsilon).$$

Then we have $|d\mathcal{G}_f|(u,\lambda) = 1$.

PROOF. See [10, Corollary 2.11]. □

Definition 3.8 may be simplified, when f is continuous.

PROPOSITION 3.12. *Let* $f: Y \to \mathbb{R}$ *be continuous. Then* $|df|(u)$ *is the supremum of the* σ*'s in* $[0,+\infty[$ *such that there exist* $r > 0$ *and a continuous map*

$$\mathcal{H}: \mathrm{B}_r(u) \times [0,r] \to Y$$

satisfying

$$d(\mathcal{H}(v,t),v) \leq t, \qquad f(\mathcal{H}(v,t)) \leq f(v) - \sigma t, \tag{3.2}$$

whenever $v \in \mathrm{B}_r(u)$ *and* $t \in [0,r]$.

PROOF. See [2, Proposition 2.2]. □

By means of the weak slope, we can now introduce the two main notions of critical point theory.

DEFINITION 3.13. We say that $u \in Y$ is *a (lower) critical point* of f, if $f(u) \in \mathbb{R}$ and $|df|(u) = 0$. We say that $c \in \mathbb{R}$ is *a (lower) critical value* of f, if there exists a (lower) critical point $u \in Y$ of f with $f(u) = c$.

REMARK 3.14. Let $\widetilde{d}$ be another metric on Y and let $u \in Y$. Assume that there exist a neighbourhood U of u and $c > 0$ such that, for all $v, w \in U$,

$$d(v,w) \leq c\widetilde{d}(v,w), \qquad \widetilde{d}(v,w) \leq cd(v,w).$$

Then one has $|df|(u) = 0$ if and only if $|\widetilde{d}f|(u) = 0$, where $|\widetilde{d}f|(u)$ is the weak slope with respect to $\widetilde{d}$.

DEFINITION 3.15. Let $c \in \mathbb{R}$. A sequence (u_h) in Y is said to be *a Palais–Smale sequence at level c* (*(PS)$_c$-sequence*, for short) for f, if $f(u_h) \to c$ and $|df|(u_h) \to 0$.

We say that f satisfies *the Palais–Smale condition at level c* ((PS)$_c$, for short), if every (PS)$_c$-sequence (u_h) for f admits a convergent subsequence (u_{h_k}) in Y.

DEFINITION 3.16. A topological space Z is said to be *weakly locally contractible*, if every $u \in Z$ admits a neighbourhood U which is contractible in Z.

Theorem 3.17. *Let Y be weakly locally contractible with* $\operatorname{cat} Y = \infty$, *let* $f\colon Y \to \mathbb{R}$ *be continuous and bounded from below and assume that* $\{u \in Y : f(u) \le c\}$ *is complete and* $(\mathrm{PS})_c$ *hold for every* $c \in \mathbb{R}$. *Then there exists a sequence* (u_h) *of critical points of* f *with* $f(u_h) \to \infty$.

Proof. See [7, Theorem 3.6] and [5, Theorem 1.4.13]. □

Corollary 3.18. *Let Z be a metrizable tolopogical space and* $f\colon Z \to \mathbb{R}$ *a continuous function. Assume that*

(a) *Z is weakly locally contractible and* $\operatorname{cat} Z = \infty$;
(b) *for every* $c \in \mathbb{R}$, *the set* $\{u \in Z : f(u) \le c\}$ *is compact.*

Then, for every compatible metric on Z, there exists a sequence (u_h) of critical points of f with $f(u_h) \to \infty$.

4. Proof of the main results

In the first part of this section, let N be a differentiable manifold of class C^2 and M be a LNR in N. Let us consider

$$\Lambda(M) = \{\gamma \in C([0,1];M) : \gamma(0) = \gamma(1)\}$$

endowed with the uniform topology ($\Lambda(M)$ is called the *free loop space* of M) and

$$X = \{\gamma \in W^{1,p}(0,1;M) : \gamma(0) = \gamma(1)\}.$$

Let $L\colon \mathbb{R} \times TN \to \mathbb{R}$ be a function of class C^1 satisfying (2.1)–(2.4) and define a lower semicontinuous functional $f\colon \Lambda(M) \to \mathbb{R} \cup \{\infty\}$ by

$$f(\gamma) = \begin{cases} \displaystyle\int_0^1 L(s,\gamma(s),\gamma'(s))\,ds & \text{if } \gamma \in X, \\ \infty & \text{if } \gamma \in \Lambda(M) \setminus X. \end{cases}$$

In the following, we will consider the metrizable topological space $\operatorname{epi}(f)$, endowed with the topology induced by $\Lambda(M) \times \mathbb{R}$, and the continuous function $\mathcal{G}_f\colon \operatorname{epi}(f) \to \mathbb{R}$.

Given a Riemannian structure on N, for every $\gamma, \eta \in W^{1,p}(0,1;M)$, we set as before

$$d_1(\gamma,\eta) = \int_0^1 d(\gamma(s),\eta(s))\,ds,$$
$$d_\infty(\gamma,\eta) = \max\{d(\gamma(s),\eta(s)) : 0 \le s \le 1\},$$

where d is the distance on N associated with the Riemannian structure.

Lemma 4.1. *Consider a Riemannian structure on N. Let (γ_h) be a sequence in $W^{1,p}(0,1;M)$ convergent to $\gamma \in W^{1,p}(0,1;M)$ with respect to the topology induced by d_1 and such that $(f(\gamma_h))$ is bounded. Then (γ_h) is convergent to γ with respect to the uniform convergence.*

Proof. Let U be an open subset of M with $\overline{U}$ compact such that $\gamma([0,1]) \subseteq U$. First of all we claim that $\gamma_h([0,1]) \subseteq U$ for h large enought. By contradiction, let $h_k \to \infty$ and $(s_k) \subseteq [0,1]$ such that $\gamma_{h_k}(s_k) \notin U$. Up to a subsequence we have that $s_k \to s \in [0,1]$ and $\gamma_{h_k} \to \gamma$ a.e. in $[0,1]$. Let $a \in [0,1]$ be such that $\gamma_{h_k}(a) \to \gamma(a)$. Assume that $a < s$. It follows that, for k large enough, there exists $b_k \in \,]a, s_k]$ such that $\gamma_{h_k}([a,b_k]) \subseteq \overline{U}$ and $\gamma_{h_k}(b_k) \notin U$. Since $\overline{U}$ is compact, there exists $C > 0$ such that, by (2.1),

$$\int_a^{b_k} L(s, \gamma_{h_k}, \gamma'_{h_k})\, ds \geq \int_a^{b_k} (k(\gamma_{h_k})|\gamma'_{h_k}|^p - d)\, ds \geq \int_a^{b_k} (C|\gamma'_{h_k}|^p - d)\, ds.$$

Moreover, again by (2.1), we have

$$\int_0^a L(s, \gamma_{h_k}, \gamma'_{h_k})\, ds + \int_{b_k}^1 L(s, \gamma_{h_k}, \gamma'_{h_k})\, ds \geq -d(1 - b_k + a).$$

It follows that

$$f(\gamma_{h_k}) = \int_0^1 L(s, \gamma_{h_k}, \gamma'_{h_k})\, ds \geq C \int_a^{b_k} |\gamma'_{h_k}|^p\, ds - d.$$

Hence for every $\sigma, \tau \in [a, b_k]$ with $\tau \leq \sigma$ we have

$$\begin{aligned} d(\gamma_{h_k}(\sigma), \gamma_{h_k}(\tau)) &\leq \int_\tau^\sigma |\gamma'_{h_k}(t)|\, dt \leq \left(\int_\tau^\sigma |\gamma'_{h_k}(t)|^p\, dt\right)^{1/p} |\sigma - \tau|^{1/p'} \\ &\leq \left(\int_a^{b_k} |\gamma'_{h_k}(t)|^p\, dt\right)^{1/p} |\sigma - \tau|^{1/p'} \leq \left(\frac{f(\gamma_{h_k}) + d}{C}\right)^{1/p} |\sigma - \tau|^{1/p'}. \end{aligned}$$

It follows that (γ_{h_k}) is equi-uniformly continuous on $[a, b_k]$. Up to a further subsequence we have that $\gamma_{h_k}(b_k) \to x \in \partial U$. Since $\inf\{d(\gamma(a), y) : y \in \partial U\} > 0$, if a is sufficiently closed to s a contradiction follows.

Arguing as above, for any $s, t \in [0,1]$ we have that

$$d(\gamma_h(s), \gamma_h(t)) \leq \left(\frac{f(\gamma_h) + d}{C}\right)^{1/p} |s - t|^{1/p'}.$$

Since $(f(\gamma_h))$ is bounded, we deduce that (γ_h) is equi-uniformly continuous on $[0,1]$. Therefore it is easy to see that (γ_h) is convergent to γ with respect to the uniform convergence. □

Theorem 4.2. *Consider any Riemannian structure on N and define on* $\mathrm{epi}(f)$ *the metric*

$$d((\gamma,\lambda),(\eta,\mu)) = \sqrt{d_1(\gamma,\eta)^2 + |\lambda-\mu|^2}. \tag{4.1}$$

Then the following facts hold:

(a) *the metric d is compatible with the topology of* $\mathrm{epi}(f)$*;*
(b) *the set of critical points of $\mathcal{G}_f\colon \mathrm{epi}(f)\to\mathbb{R}$ does not depend on the Riemannian structure;*
(c) *if $(\gamma,\lambda)\in\mathrm{epi}(f)$ is a critical point of $\mathcal{G}_f$ with $f(\gamma)=\lambda$, then γ is the restriction to $[0,1]$ of a 1-periodic generalized solution of the Lagrangian system associated to L on M.*

Proof. (a) is an easy consequence of Lemma 4.1; (b) follows from Remarks 2.1 and 3.14. Let us consider property (c). First, let us prove that γ is L-stationary on [0,1]. By contradiction, assume that there exist $r,c,\sigma>0$ and

$$\mathcal{H}\colon \{\eta\in W^{1,p}(0,1;M) : d_\infty(\eta,\gamma)<r,\ f(\eta)<f(\gamma)+r\}\times[0,r]\to W^{1,p}(0,1;M)$$

continuous from the product of the uniform convergence and that of $\mathbb{R}$ to that of the uniform convergence such that

$$\begin{aligned} \mathcal{H}(\eta,t)(0) &= \eta(0), & \mathcal{H}(\eta,t)(1) &= \eta(1),\\ d_1(\mathcal{H}(\eta,t),\eta) &\le ct, & f(\mathcal{H}(\eta,t)) &\le f(\eta)-\sigma t. \end{aligned}$$

If $r'\in\,]0,r[$ is such that if $\eta\in W^{1,p}(0,1;M)$ with $d_1(\eta,\gamma)<r'$ and $f(\eta)<f(\gamma)+r'$, then $d_\infty(\eta,\gamma)<r$. Then the restriction of $\mathcal{H}$ to

$$\{\eta\in W^{1,p}(0,1;M) : d_1(\eta,\gamma)<r',\ f(\eta)<f(\gamma)+r'\}\times[0,r']$$

satisfies the assumptions of Proposition 3.9. It follows that γ is not a critical point of f, a contradiction.

Finally, if we define

$$\widehat{\gamma}(s) = \begin{cases} \gamma\Big(s+\dfrac12\Big) & \text{if } 0\le s\le \dfrac12,\\[2mm] \gamma\Big(s-\dfrac12\Big) & \text{if } \dfrac12\le s\le 1, \end{cases}$$

it turns out that also $\widehat{\gamma}$ is L-stationary on $[0,1]$, whence the assertion. □

Lemma 4.3. *Define $\mathcal{E}\colon\Lambda([0,1];N)\to\mathbb{R}\cup\{\infty\}$ by*

$$\mathcal{E}(\gamma) = \begin{cases} \displaystyle\int_0^1 |\gamma'(s)|^p\,ds & \text{if } \gamma\in X,\\ \infty & \text{if } \gamma\in\Lambda([0,1];N)\setminus X. \end{cases}$$

Then $\mathrm{epi}(f)$ *is homotopically equivalent to* $\mathrm{epi}(\mathcal{E})$.

Proof. By (2.1), for every $\gamma \in X$ we have

$$\mathcal{E}(\gamma) \le \left\| \frac{1}{k \circ \gamma} \right\|_\infty (f(\gamma) + d), \qquad f(\gamma) \le \|c \circ \gamma\|_\infty (\mathcal{E}(\gamma) + 1).$$

Define $\Phi\colon \mathrm{epi}(f) \to \mathrm{epi}(\mathcal{E})$ and $\Psi\colon \mathrm{epi}(\mathcal{E}) \to \mathrm{epi}(f)$ by

$$\Phi(\gamma, \lambda) = \left(\gamma, \left\| \frac{1}{k \circ \gamma} \right\|_\infty (\lambda + d)\right), \qquad \Psi(\gamma, \lambda) = (\gamma, \|c \circ \gamma\|_\infty (\lambda + 1)).$$

Then Ψ and, by Lemma 4.1, Φ are continuous and it is readily seen that $\Psi \circ \Phi$ is homotopic to the identity of $\mathrm{epi}(f)$ while $\Phi \circ \Psi$ is homotopic to the identity of $\mathrm{epi}(\mathcal{E})$. □

Lemma 4.4. *Let* U *be an open subset of* $\mathbb{R}^n$ *and let*

$$\Lambda^1(U) = \{\gamma \in W^{1,p}(0,1;U) : \gamma(0) = \gamma(1)\}$$

endowed with the $W^{1,p}$*-metric. Then there exists a continuous map*

$$\mathcal{K}\colon \Lambda(U) \times [0,1] \to \Lambda(U)$$

such that

$$\begin{gathered} \mathcal{K}(\gamma, 0) = \gamma, \quad \mathcal{K}(\gamma, 1) \in \Lambda^1(U) \quad \textit{for all } \gamma \in \Lambda(U), \\ \mathcal{K}(\,\cdot\,, 1)\colon \Lambda(U) \to \Lambda^1(U) \textit{ is continuous}, \\ \mathcal{K}(\Lambda^1(U) \times [0,1]) \subseteq \Lambda^1(U), \\ \|[\mathcal{K}(\gamma, t)]'\|_p \le \|\gamma'\|_p \quad \textit{for all } \gamma \in \Lambda^1(U) \textit{ and all } t \in [0,1]. \end{gathered}$$

Proof. Let (ρ_ε) be a sequence of mollifiers of class $\mathcal{C}_c^\infty$ on $\mathbb{R}^n$. Let $R_0\gamma = \gamma$ and for every $\varepsilon > 0$ let

$$R_\varepsilon \gamma(s) = \int_{\mathbb{R}} \rho_\varepsilon(s - t)\, \overline{\gamma}(t)\, dt,$$

where $\overline{\gamma}\colon \mathbb{R} \to U$ is 1-periodic such that $\overline{\gamma}_{|[0,1]} = \gamma$. It turns out that there exists a continuous function $\lambda\colon \Lambda(U) \to]0,1]$ such that for every $\gamma \in \Lambda(U)$ it is

$$R_\varepsilon \gamma(s) \in U \quad \text{for all } \varepsilon \in]0, \lambda(\gamma)], \text{ and all } s \in [0,1].$$

Let $\mathcal{K}\colon \Lambda(U) \times [0,1] \to \Lambda(U)$ defined by $\mathcal{K}(\gamma, t) = R_{t\lambda(\gamma)}\gamma$. It is readily seen that $\mathcal{K}$ satisfies all the properties required and the assertion follows. □

Lemma 4.5. *The map* $\widetilde{\pi}\colon \mathrm{epi}(\mathcal{E}) \to \Lambda(M)$ *defined by* $\widetilde{\pi}(\gamma,\lambda) = \gamma$ *is a homotopy equivalence* ($\mathrm{epi}(\mathcal{E})$ *is endowed with the product of the uniform topology and that of* $\mathbb{R}$).

Proof. Arguing as in the proof of Theorem 3.7, we may assume that N is a smooth submanifold of $\mathbb{R}^n$ and we may consider an open subset A of $\mathbb{R}^n$ with $N \subseteq A$ and a retraction $\pi\colon A \to N$ of class C^∞ such that π is Lipschitz continuous of constant 2. Since M is a LNR in N, there exists an open neighbourhood U of M in N and a locally Lipschitzian retraction $r\colon U \to M$. Since $r\circ\pi\colon \pi^{-1}(U) \to M$ is a locally Lipschitzian retraction, then M is also a LNR in $\mathbb{R}^n$. Now taking into account Lemma 4.4 the proof follows the same argument of [11, Theorem 5.3]. □

Theorem 4.6. *The map* $\widehat{\pi}\colon \mathrm{epi}(f) \to \Lambda(M)$ *defined by* $\widetilde{\pi}(\gamma,\lambda) = \gamma$ *is a homotopy equivalence (*$\mathrm{epi}(f)$ *is endowed with the product of the uniform topology and that of* $\mathbb{R}$*).*

Proof. Combining Lemmas 4.3 and 4.5 the assertion follows. □

From now on, we assume that M is the closure of an open subset in N with locally Lipschitz boundary. By Theorem 3.7, M is a LNR in N.

Theorem 4.7. *Consider a Riemannian structure on* N *and the metric defined in* (4.1). *Let* (γ,λ) *be in* $\mathrm{epi}(f)$ *such that* $f(\gamma) < \lambda$. *Then*

$$|d\mathcal{G}_f|(\gamma,\lambda) = 1.$$

Proof. Arguing as in the proof of Theorem 3.7, we may assume that N is a smooth submanifold of $\mathbb{R}^n$ and we may consider an open subset A of $\mathbb{R}^n$ with $N \subseteq A$ and a retraction $\pi\colon A \to N$ of class C^∞ such that π is Lipschitz continuous of constant 2. Therefore we may also consider the function $\widetilde{L}\colon \mathbb{R}\times A\times\mathbb{R}^n \to \mathbb{R}$ such that $\widetilde{L}$ is a C^1-extension of L to $\mathbb{R}\times A\times\mathbb{R}^n$ and such that there exist two continuous functions $\widetilde{c},\widetilde{k}\colon A \to\,]0,\infty[$ and $d\in\mathbb{R}$ such that for every $(s,q,v)\in \mathbb{R}\times A\times\mathbb{R}^n$ one has

$$|D_q\widetilde{L}(s,q,v)| \le \widetilde{c}(q)(1+|v|^p), \tag{4.2}$$

$$|D_v\widetilde{L}(s,q,v)| \le \widetilde{c}(q)(1+|v|^{p-1}), \tag{4.3}$$

$$\widetilde{L}(s,q,v) \ge \widetilde{k}(q)|v|^p - d, \tag{4.4}$$

$$\widetilde{L}(s,q,\,\cdot\,) \text{ is convex.} \tag{4.5}$$

First of all we claim that there exist $\overline{\varepsilon} > 0$ and $\overline{C} > 0$ such that for every $\eta_1,\eta_2\in X$ with $\|\eta_i-\gamma\|_\infty \le \overline{\varepsilon}$ and for every $t\in[0,1]$ it is

$$\begin{aligned}\bigg|\int_0^1 [\widetilde{L}(s,\pi(\eta_1+t(\eta_2-\eta_1)),\pi'(\eta_1+t(\eta_2-\eta_1))\eta_1') - \widetilde{L}(s,\eta_1,\eta_1')]\,ds\bigg| \\ \le \overline{C}t\bigg(1+\int_0^1 \widetilde{L}(s,\eta_1,\eta_1')\,ds + \int_0^1 \widetilde{L}(s,\eta_2,\eta_2')\,ds\bigg)\|\eta_1-\eta_2\|_\infty.\end{aligned}$$

Let $\varepsilon > 0$ be such that if $\eta \in W^{1,p}(0,1;\mathbb{R}^n)$ with $\|\eta - \gamma\|_\infty \le \varepsilon$ then $\eta \in W^{1,p}(0,1;A)$. Since π is of class C^∞ and Lipschitz continuous of constant 2, there exists $\overline{\varepsilon} \in]0,\varepsilon]$ and $\widetilde{C} \ge 2$ such that for every $\eta_1, \eta_2 \in W^{1,p}(0,1;A)$ with $\|\eta_i - \gamma\|_\infty \le \overline{\varepsilon}$ and for every $\xi \in \mathbb{R}^n$ it is

$$|\pi(\eta_1) - \pi(\eta_2)| \le \widetilde{C}|\eta_1 - \eta_2|, \qquad |[\pi'(\eta_1) - \pi'(\eta_2)]\xi| \le \widetilde{C}|\eta_1 - \eta_2||\xi|.$$

Now let $\eta_1, \eta_2 \in X$ with $\|\eta_i - \gamma\|_\infty \le \overline{\varepsilon}$ and let $t \in [0,1]$. For every $\vartheta \in [0,1]$ we have

$$\begin{aligned}(4.6)\qquad &|\eta_1' + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1' - \eta_1')| \\ &= |\eta_1' + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1' - \pi'(\eta_1)\eta_1')| \\ &\le |\eta_1'| + \widetilde{C}|\eta_2 - \eta_1||\eta_1'| \le \widehat{C}(|\eta_1'| + |\eta_2'|)\end{aligned}$$

for some $\widehat{C} > 0$. Unless reducing $\overline{\varepsilon}$, we may suppose that $\widetilde{c}$, $\widetilde{k}$ are constants on $\{\eta \in W^{1,p}(0,1;A) : d_\infty(\eta,\gamma) < \overline{\varepsilon}\}$. Furthermore, applying Lagrange's Theorem, (4.2), (4.3) and (4.6) it is, for some $\vartheta \in [0,1]$,

$$\begin{aligned}&\widetilde{L}(s,\pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1') - \widetilde{L}(s,\eta_1,\eta_1') \\ &= D_q\widetilde{L}(s, \eta_1 + \vartheta(\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1), \eta_1' \\ &\quad + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1' - \eta_1')) \cdot (\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1) \\ &\quad + D_v\widetilde{L}(s, \eta_1 + \vartheta(\pi(\eta_1 + t(\eta_2 - \eta_1)) - \eta_1), \eta_1' \\ &\quad + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1' - \eta_1')) \cdot (\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1' - \eta_1') \\ &\le C(1 + |\eta_1' + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1' - \eta_1')|^p)|\pi(\eta_1 + t(\eta_2 - \eta_1)) - \pi(\eta_1)| \\ &\quad + C(1 + |\eta_1' + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1' - \eta_1')|^{p-1}) \\ &\quad \cdot |\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1' - \pi'(\eta_1)\eta_1'| \\ &\le C_2 t(1 + |\eta_1' + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1' - \eta_1')|^p)|\eta_1 - \eta_2| \\ &\quad + C_2 t(1 + |\eta_1' + \vartheta(\pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1' - \eta_1')|^{p-1})|\eta_1'||\eta_1 - \eta_2| \\ &\le C_3 t(1 + |\eta_1'|^p + |\eta_2'|^p)|\eta_1 - \eta_2| + C_3 t(1 + |\eta_1'|^{p-1} + |\eta_2'|^{p-1})|\eta_1'||\eta_1 - \eta_2| \\ &= C_3 t(1 + |\eta_1'|^p + |\eta_2'|^p)|\eta_1 - \eta_2| + C_3 t(|\eta_1'| + |\eta_1'|^p + |\eta_1'||\eta_2'|^{p-1})|\eta_1 - \eta_2|\end{aligned}$$

for some $C_3 > 0$. It follows that

$$\begin{aligned}&\left|\int_0^1 [\widetilde{L}(s, \pi(\eta_1 + t(\eta_2 - \eta_1)), \pi'(\eta_1 + t(\eta_2 - \eta_1))\eta_1') - \widetilde{L}(s,\eta_1,\eta_1')]\, ds\right| \\ &\quad \le C_3 t(1 + 2\|\eta_1'\|_p^p + \|\eta_2'\|_p^p + \|\eta_1'\|_1 + \|\eta_1'\|_p\|\eta_2'\|_p^{p-1})\|\eta_1 - \eta_2\|_\infty \\ &\quad \le C_4 t(1 + \|\eta_1'\|_p^p + \|\eta_2'\|_p^p)\|\eta_1 - \eta_2\|_\infty\end{aligned}$$

for some $C_4 > 0$. Finally, applying (4.4) we may find $\overline{C} > 0$ such that

$$\begin{aligned}\bigg|\int_0^1 [\widetilde{L}(s,\pi(\eta_1+t(\eta_2-\eta_1)),\pi'(\eta_1+t(\eta_2-\eta_1))\eta_1') - \widetilde{L}(s,\eta_1,\eta_1')]\,ds\bigg| \\ \leq \overline{C}t\bigg(1+\int_0^1 \widetilde{L}(s,\eta_1,\eta_1')\,ds + \int_0^1 \widetilde{L}(s,\eta_2,\eta_2')\,ds\bigg)\|\eta_1-\eta_2\|_\infty\end{aligned}$$

and the claim follows. Let $\varepsilon > 0$, $K = \gamma([0,1])$ and let $\overline{\varepsilon}, \overline{C} > 0$ be as before. Let $C_2 = \overline{C}(1+2\lambda+\varepsilon)$. Let now $\widehat{r}$ and $\widehat{c}$ be as in (c) of Lemma 3.5, and let

$$\widehat{\gamma}(s) = \gamma(s) + \rho\nu(\gamma(s)),$$

where $\rho \in \,]0,\widehat{r}]$ is such that

$$\|\pi(\widehat{\gamma}) - \gamma\|_\infty \leq \min\bigg\{\frac{\varepsilon}{4}, \frac{\varepsilon}{8C_2}, \overline{\varepsilon}\bigg\}, \quad f(\pi\circ\widehat{\gamma}) \leq f(\gamma) + \frac{\varepsilon}{4}.$$

Let $r \in \,]0,\varepsilon/2[$ be such that if $\|\eta-\gamma\|_1 < r$ with $f(\eta) < \lambda + r$, then $\|\eta - \gamma\|_\infty \leq \min\{\rho/\widehat{c}, \varepsilon/4, \varepsilon/8C_2, \overline{\varepsilon}\}$. Then, again by (c) of Lemma 3.5 it is possible to define a continuous map

$$\mathcal{H}\colon \{\eta \in X : \|\eta-\gamma\|_1 < r,\ f(\eta) < \lambda + r\} \times [0,r] \to X$$

by

$$\mathcal{H}(\eta,t) = \pi((1-t)\eta + t\pi(\widehat{\gamma})).$$

It is

$$\|\mathcal{H}(\eta,t) - \eta\|_\infty \leq 2t\|\pi(\widehat{\gamma}) - \eta\|_\infty \leq 2t(\|\pi(\widehat{\gamma}) - \gamma\|_\infty + \|\gamma-\eta\|_\infty) \leq \varepsilon t$$

and hence also

$$\|\mathcal{H}(\eta,t) - \eta\|_1 \leq \varepsilon t.$$

Since $\widetilde{L}$ is convex with respect to the third variable, we get

$$\begin{aligned}&f(\mathcal{H}(\eta,t)) \\ &\quad= \int_0^1 \widetilde{L}(s,\pi(\eta+t(\pi(\widehat{\gamma})-\eta)),\pi'(\eta+t(\pi(\widehat{\gamma})-\eta))(\eta'+t((\pi\circ\widehat{\gamma})'-\eta')))\,ds \\ &\quad\leq \int_0^1 \widetilde{L}(s,\pi(\eta+t(\pi(\widehat{\gamma})-\eta)),\pi'(\eta+t(\pi(\widehat{\gamma})-\eta))\eta')\,ds \\ &\qquad+ t\bigg[\int_0^1 \widetilde{L}(s,\pi(\eta+t(\pi(\widehat{\gamma})-\eta)),\pi'(\eta+t(\pi(\widehat{\gamma})-\eta))(\pi\circ\widehat{\gamma})')\,ds \\ &\qquad- \int_0^1 \widetilde{L}(s,\pi(\eta+t(\pi(\widehat{\gamma})-\eta)),\pi'(\eta+t(\pi(\widehat{\gamma})-\eta))\eta')\,ds\bigg].\end{aligned}$$

Furthermore, it is

$$\begin{aligned}\bigg|\int_0^1 &[\widetilde{L}(s,\pi(\eta+t(\pi(\widehat{\gamma})-\eta)),\pi'(\eta+t(\pi(\widehat{\gamma})-\eta))\eta')-\widetilde{L}(s,\eta,\eta')]\,ds\bigg|\\ &\le \overline{C}t(1+f(\eta)+f(\pi\circ\widehat{\gamma}))\|\pi(\widehat{\gamma})-\eta\|_\infty\\ &< \overline{C}t(1+2\lambda+\varepsilon)(\|\pi(\widehat{\gamma})-\gamma\|_\infty+\|\gamma-\eta\|_\infty)\le\frac{\varepsilon}{4}t\end{aligned}$$

and

$$\begin{aligned}\bigg|\int_0^1 &[\widetilde{L}(s,\pi(\eta+t(\pi(\widehat{\gamma})-\eta)),\pi'(\eta+t(\pi(\widehat{\gamma})-\eta))(\pi\circ\widehat{\gamma})')\\ &\qquad -\widetilde{L}(s,\pi\circ\widehat{\gamma},(\pi\circ\widehat{\gamma})')]\,ds\bigg|\\ &\le \overline{C}t(1+f(\eta)+f(\pi\circ\widehat{\gamma}))\|\pi(\widehat{\gamma})-\eta\|_\infty\\ &< \overline{C}t(1+2\lambda+\varepsilon)(\|\pi(\widehat{\gamma})-\gamma\|_\infty+\|\gamma-\eta\|_\infty)\le\frac{\varepsilon}{4}t.\end{aligned}$$

Therefore we finally get

$$f(\mathcal{H}(\eta,t))\le f(\eta)+\frac{\varepsilon}{4}t+\bigg(f(\pi\circ\widehat{\gamma})-f(\eta)+\frac{\varepsilon}{2}\bigg)t\le f(\eta)+t(f(\gamma)-f(\eta)+\varepsilon)$$

and the assertion follows from Proposition 3.11. □

Finally, we can prove Theorem 2.6.

PROOF. Now assume also that M is compact, 1-connected and non-contractible in itself. By Theorem 3.7, we have that M is a LNR in N, in particular an absolute neighbourhood retract. From [13, Corollary 1.4] it follows that $\operatorname{cat}\Lambda(M)=\infty$. Moreover, $\Lambda(M)$ also is an absolute neighbourhood retract, hence weakly locally contractible. On the other hand, by Theorem 4.6 $\Lambda(M)$ is homotopically equivalent to $\operatorname{epi}(f)$. Therefore $\operatorname{cat}\operatorname{epi}(f)=\infty$ and $\operatorname{epi}(f)$ is weakly locally contractible. Let now $c\in\mathbb{R}$ and consider the sublevel

$$\mathcal{G}_f^c=\{(\gamma,\lambda)\in\Lambda(M)\times\mathbb{R}:f(\gamma)\le\lambda\le c\}.$$

Since M is compact, from (2.1) and Ascoli's theorem we deduce that $\mathcal{G}_f^c$ is compact. By Corollary 3.18, there exists a sequence (γ_h,λ_h) of critical points of $\mathcal{G}_f^c$ with respect to the metric (4.1) with $\lambda_h\to\infty$. By Theorem 4.7 we have that $\lambda_h=f(\gamma_h)$. From (c) of Theorem 4.2 the assertion follows. □

The next two results correspond to the well-known equation $d/ds\,H=-D_sL$, where H is the Hamiltonian function associated with L.

THEOREM 4.8. *Let $\gamma\in W^{1,p}(a,b;M)$ be L-stationary. Assume that L does not depend on s. Then the map $\{s\mapsto D_vL(\gamma,\gamma')\gamma'-L(\gamma,\gamma')\}$ is constant a.e.*

PROOF. Arguing as in the proof of Theorem 4.7, we may assume that N is a smooth submanifold of $\mathbb{R}^n$, A is an open subset of $\mathbb{R}^n$ with $N\subseteq A$ and

$\widetilde{L}: A \times \mathbb{R}^n \to \mathbb{R}$ is a C^1-extension of L to $A \times \mathbb{R}^n$ satisfying (4.2)–(4.5). Assume, for a contradiction, that there exists $\varphi \in C_c^\infty(a,b)$ such that

$$\sigma := \frac{1}{2}\int_a^b \{[D_v\widetilde{L}(\gamma,\gamma')\cdot\gamma' - \widetilde{L}(\gamma,\gamma')]\varphi'\}\,ds > 0.$$

Let $r > 0$ be such that $r\|\varphi'\|_\infty < 1$ and let $\psi: [a,b]\times[0,r] \to [a,b]$ be the smooth function such that

$$\lambda = \psi(\lambda,t) - t\varphi\,(\psi(\lambda,t)) \quad \text{for all } \lambda \in [a,b] \text{ and all } t \in [0,r].$$

Unless reducing r we may suppose that the functions c, k in (4.2)–(4.4) are constants on $\{\eta \in W^{1,p}(a,b;M) : d_\infty(\eta,\gamma) < r\}$. Define $\mathcal{H}: \{\eta \in W^{1,p}(a,b;M) : d_\infty(\eta,\gamma) < r,\ f_{a,b}(\eta) < f_{a,b}(\gamma) + r\} \times [0,r] \to W^{1,p}(a,b;M)$ by

$$\mathcal{H}(\eta,t)(\mu) = \eta\,(\mu - t\varphi(\mu))\ .$$

It is easy to see that $\mathcal{H}$ is continuous from the product topology of the uniform convergence and of $\mathbb{R}$ to that of the uniform convergence and that

$$\mathcal{H}(\eta,t)(a) = \eta(a), \qquad \mathcal{H}(\eta,t)(b) = \eta(b).$$

Moreover, by (4.4)

$$\begin{aligned}
d_1(\mathcal{H}(\eta,t),\eta) &= \int_a^b |\eta(\mu - t\varphi(\mu)) - \eta(\mu)|\,d\mu \\
&= t\int_a^b |\eta'(\mu - \theta\varphi(\mu))||1 - t\varphi'(\mu)|\,d\mu \\
&\leq t\bigg(\int_a^b |\eta'(\lambda)|^p \frac{1}{|1-\theta\varphi'(\psi(\lambda,\theta))|^p}\,d\lambda\bigg)^{1/p}\bigg(\int_a^b |1 - t\varphi'(\mu)|^{p'}\,d\mu\bigg)^{1/p'} \\
&\leq \frac{t}{(1-\theta\|\varphi'\|_\infty)^p}\bigg(\int_a^b |\eta'(\lambda)|^p\,d\lambda\bigg)^{1/p}\bigg(\int_a^b |1 - t\varphi'(\mu)|^{p'}\,d\mu\bigg)^{1/p'} \\
&\leq \overline{C}t\bigg(\int_a^b (L(\eta(\lambda),\eta'(\lambda)) + d)\,d\lambda\bigg)^{1/p} < \widehat{C}t(f_{a,b}(\gamma) + r + d(b-a))^{1/p},
\end{aligned}$$

for some $\widehat{C} > 0$. Following the same argument of the proof of [19, Theorem 5.10] we also have

$$f_{a,b}(\mathcal{H}(\eta,t)) = f_{a,b}(\eta) + t\Theta(\eta,t)$$

where

$$\begin{aligned}
\Theta(\eta,t) = \int_a^b \bigg[&- D_v\widetilde{L}(\eta(\lambda),(1 - t\varphi'(\psi(\lambda,t)))\eta'(\lambda))\cdot\eta'(\lambda)\varphi'(\psi(\lambda,t)) \\
&+ \widetilde{L}(\eta(\lambda),(1 - t\varphi'(\psi(\lambda,t)))\eta'(\lambda))\frac{\varphi'(\psi(\lambda,t))}{1 - t\varphi'(\psi(\lambda,t))}\bigg]\,d\lambda.
\end{aligned}$$

We claim that, for r sufficiently small, we have $\Theta(\eta,t) \le -\sigma$ for any $\eta \in W^{1,p}(a,b;M)$ with $d_\infty(\eta,\gamma) < r$, $f_{a,b}(\eta) < f_{a,b}(\gamma)+r$ and $0 \le t \le r$. By contradiction, let (η_h) be a sequence in $W^{1,p}(a,b;M)$ uniformly convergent to γ with $f_{a,b}(\eta_h) < f_{a,b}(\gamma)+1/h$ and (t_h) be a non negative sequence convergent to 0 such that $\Theta(\eta_h,t_h) > -\sigma$. Because of (4.4) and $f_{a,b}$ is lower semicontinuous, we have that $f_{a,b}(\eta_h) \to f_{a,b}(\gamma)$. Again by (4.4) (η_h) is bounded in $W^{1,p}(a,b;M)$ and up to a subsequence $\eta_h' \rightharpoonup \gamma'$ in $L^p(a,b;M)$. Therefore $[1-t_h\varphi'(\psi(\,\cdot\,,t_h))]\eta_h' \rightharpoonup \gamma'$ in $L^p(a,b;M)$. We have that

$$\begin{aligned}\int_a^b &[\widetilde{L}(\gamma(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda)) - \widetilde{L}(\gamma(\lambda),\gamma'(\lambda))]\,d\lambda\\ &= \int_a^b D_v\widetilde{L}(\gamma(\lambda),(1-\tau)\gamma'(\lambda)+\tau\eta_h'(\lambda))\cdot(\eta_h'(\lambda)-\gamma'(\lambda))\,d\lambda\\ &\quad + t_h\int_a^b \varphi'(\psi(\lambda,t_h))D_v\widetilde{L}(\gamma(\lambda),(1-\vartheta)\eta_h'(\lambda)\\ &\quad + \vartheta[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda))\cdot\eta_h'(\lambda)\,d\lambda.\end{aligned}$$

By (4.3) we have that $D_v\widetilde{L}(\gamma,(1-\tau)\gamma'+\tau\eta_h') \in L^{p'}(a,b;M)$ and hence

$$\int_a^b D_v\widetilde{L}(\gamma(\lambda),(1-\tau)\gamma'(\lambda)+\tau\eta_h'(\lambda))\cdot(\eta_h'(\lambda)-\gamma'(\lambda))\,d\lambda \to 0.$$

Again by (4.3) we have that

$$\int_a^b \varphi'(\psi(\lambda,t_h))D_v\widetilde{L}(\gamma(\lambda),(1-\vartheta)\eta_h'(\lambda)+\vartheta[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda))\cdot\eta_h'(\lambda)\,d\lambda$$

is bounded. Therefore we have that

$$\int_a^b \widetilde{L}(\gamma,[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h')\,d\lambda \to \int_a^b \widetilde{L}(\gamma(\lambda),\gamma'(\lambda))\,d\lambda.$$

By [12, Lemma 3.1] applied to the function $\mathcal{F}(\lambda,\xi) = \widetilde{L}(\gamma(\lambda),\xi)$ we obtain that

$$\begin{aligned}\widetilde{L}(\gamma,[1-t_h\varphi'(\psi(\,\cdot\,,t_h))]\eta_h') &\rightharpoonup \widetilde{L}(\gamma,\gamma') &&\text{in } L^1(a,b;M),\\ D_v\widetilde{L}(\gamma,[1-t_h\varphi'(\psi(\,\cdot\,,t_h))]\eta_h') &\to D_v\widetilde{L}(\gamma,\gamma') &&\text{in } L^{p'}(a,b;M)\end{aligned}$$

and there exists $\Psi \in L^1(a,b;M)$ such that $|\eta_h'|^p \le \Psi$. For some $t \in\,]0,1[$ we have that

$$\begin{aligned}&\widetilde{L}(\eta_h(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda)) - \widetilde{L}(\gamma(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda))\\ &\quad = D_q\widetilde{L}((1-t)\gamma(\lambda)+t\eta_h(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda))\cdot(\eta_h(\lambda)-\gamma(\lambda)).\end{aligned}$$

By (4.2) we deduce that $D_q\widetilde{L}((1-t)\gamma+t\eta_h,[1-t_h\varphi'(\psi(\,\cdot\,,t_h))]\eta_h') \in L^{p'}(a,b;M)$ and hence

$$[\widetilde{L}(\eta_h,[1-t_h\varphi'(\psi(\,\cdot\,,t_h))]\eta_h') - \widetilde{L}(\gamma,[1-t_h\varphi'(\psi(\,\cdot\,,t_h))]\eta_h')] \rightharpoonup 0 \quad \text{in } L^1(a,b;M).$$

It follows that

$$\widetilde{L}(\eta_h,[1-t_h\varphi'(\psi(\,\cdot\,,t_h))]\eta_h') \rightharpoonup \widetilde{L}(\gamma,\gamma') \quad \text{in } L^1(a,b;M).$$

Fix $\varepsilon > 0$, let $\delta > 0$ such that for any $\mathcal{L}^1$-measurable subset $\Omega \subseteq \,]a,b[$ with $\mathcal{L}^1(\Omega) < \delta$ we have

$$\int_\Omega \Phi(\lambda)\,d\lambda < \frac{\varepsilon}{2} \quad \text{for all } \Phi \in L^1(a,b;M).$$

Let $R > 0$ be such that $\mathcal{L}^1(\{\lambda \in [a,b] : |\eta_h'(\lambda)| > R\}) < \delta$. Let $\Omega_h = \{\lambda \in [a,b] : |\eta_h'(\lambda)| > R\}$ and $\Omega_h' = \{\lambda \in [a,b] : |\eta_h'(\lambda)| \le R\}$. By (4.3) we have

$$\begin{aligned}
\int_a^b &|D_v\widetilde{L}(\eta_h(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda)) \\
&\qquad - D_v\widetilde{L}(\gamma(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda))|^{p'}\,d\lambda \\
&\le \int_{\Omega_h} \overline{C}(1+\Psi(\lambda))\,d\lambda + \int_{\Omega_h'} |D_v\widetilde{L}(\eta_h(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda)) \\
&\qquad - D_v\widetilde{L}(\gamma(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda))|^{p'}\,d\lambda \\
&< \frac{\varepsilon}{2} + \int_{\Omega_h'} |D_v\widetilde{L}(\eta_h(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda)) \\
&\qquad - D_v\widetilde{L}(\gamma(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda))|^{p'}\,d\lambda.
\end{aligned}$$

Since the map

$$\begin{aligned}\{\lambda \to [D_v\widetilde{L}(\eta_h(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda)) - D_v\widetilde{L}(\gamma(\lambda),& \\ [1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda))]\}&\end{aligned}$$

is uniformly continuous on Ω_h', for h sufficiently large we have

$$\begin{aligned}\int_{\Omega_h'} |D_v\widetilde{L}(\eta_h(\lambda),[1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda)) - D_v\widetilde{L}(\gamma(\lambda),& \\ [1-t_h\varphi'(\psi(\lambda,t_h))]\eta_h'(\lambda))|^{p'}\,d\lambda < \frac{\varepsilon}{2}.&\end{aligned}$$

It follows that

$$\|D_v\widetilde{L}(\eta_h,[1-t_h\varphi'(\psi(\,\cdot\,,t_h))]\eta_h') - D_v\widetilde{L}(\gamma,[1-t_h\varphi'(\psi(\,\cdot\,,t_h))]\eta_h')\|_{p'} \to 0.$$

Therefore

$$D_v\widetilde{L}(\eta_h,[1-t_h\varphi'(\psi(\,\cdot\,,t_h))]\eta_h') \to D_v\widetilde{L}(\gamma,\gamma') \quad \text{in } L^{p'}(a,b;M)$$

and we deduce that

$$\Theta(\eta_h,t_h) \to \int_a^b \{[-D_v\widetilde{L}(\gamma,\gamma')\cdot\gamma' + \widetilde{L}(\gamma,\gamma')]\varphi'\}\,d\lambda = -2\sigma,$$

a contradiction. Finally, we have $f_{a,b}(\mathcal{H}(\eta,t)) \le f_{a,b}(\eta) - \sigma t$. It follows that γ is not L-stationary, a contradiction. □

Theorem 4.9. *Let $\gamma \in W^{1,p}(a,b;M)$ be L-stationary. Assume that for every $s \in \mathbb{R}$ and $q \in M$ one has*

$$|D_s L(s,q,v)| \le c(q)(1+|v|^p), \quad \text{for all } v \in \mathrm{T}_q N, \tag{4.7}$$

$$L(s,q,\,\cdot\,) \text{ is strictly convex on } \mathrm{T}_q N. \tag{4.8}$$

Then the map $\{s \mapsto D_v L(s,\gamma,\gamma')\gamma' - L(s,\gamma,\gamma')\}$ belongs to $W^{1,1}(a,b)$ and we have

$$\int_a^b [D_v L(s,\gamma,\gamma')\gamma' - L(s,\gamma,\gamma')]\varphi'\,ds = \int_a^b D_s L(s,\gamma,\gamma')\varphi\,ds$$

for all $\varphi \in C_c^\infty(a,b)$.

Proof. Arguing as in the proof of Theorem 4.7, we may assume that N is a smooth submanifold of $\mathbb{R}^n$, A is an open subset of $\mathbb{R}^n$ with $N \subseteq A$ and $\widetilde{L}\colon \mathbb{R}\times A\times\mathbb{R}^n \to \mathbb{R}$ is a C^1-extension of L to $\mathbb{R}\times A\times\mathbb{R}^n$ satisfying (4.2)–(4.4) and such that for every $(s,q,v) \in \mathbb{R}\times A\times\mathbb{R}^n$ one has

$$|D_s\widetilde{L}(s,q,v)| \le \widetilde{c}(q)(1+|v|^p), \tag{4.9}$$

$$\widetilde{L}(s,q,\,\cdot\,) \text{ is strictly convex.} \tag{4.10}$$

Assume, for a contradiction, that there exists $\varphi \in C_c^\infty(a,b)$ such that

$$\sigma := \frac{1}{2}\int_a^b \{[D_v\widetilde{L}(s,\gamma,\gamma')\cdot\gamma' - \widetilde{L}(s,\gamma,\gamma')]\varphi' - D_s\widetilde{L}(s,\gamma,\gamma')\varphi\}\,ds > 0.$$

Arguing as in the proof of Theorem 4.8 we may introduce the continuous map

$$\mathcal{H}\colon \{\eta \in W^{1,p}(a,b;M) : d_\infty(\eta,\gamma) < r,\ f_{a,b}(\eta) < f_{a,b}(\gamma) + r\} \times [0,r] \to W^{1,p}(a,b;M)$$

defined by

$$\mathcal{H}(\eta,t)(\mu) = \eta\,(\mu - t\varphi(\mu))$$

satisfying the following facts:

$$\mathcal{H}(\eta,t)(a) = \eta(a), \quad \mathcal{H}(\eta,t)(b) = \eta(b),$$
$$d_1(\mathcal{H}(\eta,t),\eta) < \widehat{C}t\,(f_{a,b}(\gamma) + r + d(b-a))^{1/p},$$
$$f_{a,b}(\mathcal{H}(\eta,t)) \le f_{a,b}(\eta) + t\Theta(\eta,t)$$

where $\widehat{C} > 0$,

$$\begin{aligned}\Theta(\eta,t) = \int_a^b \Big[& D_s\widetilde{L}(\lambda + t\vartheta(\lambda,t)\varphi(\psi(\lambda,t)), \eta, (1 - t\vartheta(\lambda,t)\varphi'(\psi(\lambda,t)))\eta')\varphi(\psi(\lambda,t)) \\ & - D_v\widetilde{L}(\lambda + t\vartheta(\lambda,t)\varphi(\psi(\lambda,t)), \eta(\lambda), (1 - t\varphi'(\psi(\lambda,t)))\eta'(\lambda))\cdot\eta'(\lambda)\varphi'(\psi(\lambda,t)) \\ & + \widetilde{L}(\psi(\lambda,t), \eta(\lambda), (1 - t\varphi'(\psi(\lambda,t)))\eta'(\lambda))\frac{\varphi'(\psi(\lambda,t))}{1 - t\varphi'(\psi(\lambda,t))}\Big]\,d\lambda\end{aligned}$$

and $0 < \vartheta(\lambda,t) < 1$.

We claim that, for r sufficiently small, we have $\Theta(\eta,t) \leq -\sigma$ for any $\eta \in W^{1,p}(a,b;M)$ with $d_\infty(\eta,\gamma) < r$, $f_{a,b}(\eta) < f_{a,b}(\gamma) + r$ and $0 \leq t \leq r$. By contradiction, let (η_h) be a sequence in $W^{1,p}(a,b;M)$ uniformly convergent to γ with $f_{a,b}(\eta_h) < f_{a,b}(\gamma) + \frac{1}{h}$ and (t_h) be a non negative sequence convergent to 0 such that $\Theta(\eta_h,t_h) > -\sigma$. Because of (4.4) and $f_{a,b}$ is lower semicontinuous, we have that $f_{a,b}(\eta_h) \to f_{a,b}(\gamma)$. Again by (4.4) (η_h) is bounded in $W^{1,p}(a,b;M)$ and up to a subsequence $\eta_h \rightharpoonup \gamma$ in $W^{1,p}(a,b;M)$. On the other hand, we have

$$\begin{aligned}\int_a^b \widetilde{L}(\lambda,\gamma(\lambda),\eta_h'(\lambda))\,d\lambda - \int_a^b \widetilde{L}(\lambda,\gamma(\lambda),\gamma'(\lambda))\,d\lambda \\ = f_{a,b}(\eta_h) - f_{a,b}(\gamma) - \int_a^b \widetilde{L}(\lambda,\eta_h(\lambda),\eta_h'(\lambda))\,d\lambda + \int_a^b \widetilde{L}(\lambda,\gamma(\lambda),\eta_h'(\lambda))\,d\lambda.\end{aligned}$$

Taking into account (4.2), we get that

$$\int_a^b \widetilde{L}(\lambda,\gamma(\lambda),\eta_h'(\lambda))\,d\lambda \to \int_a^b \widetilde{L}(\lambda,\gamma(\lambda),\gamma'(\lambda))\,d\lambda.$$

By [20, Theorem 3] applied to the function $\Phi(\lambda,\xi) = \widetilde{L}(\lambda,\gamma(\lambda),\xi)$ it follows that η_h' is strongly convergent to γ' in $L^p(a,b;M)$; hence $\eta_h \to \gamma$ in $W^{1,p}(a,b;M)$. Because of (4.2), (4.3) and (4.9), we have that

$$\Theta(\eta_h,t_h) \to \int_a^b \{[-D_v\widetilde{L}(\lambda,\gamma,\gamma')\cdot\gamma' + \widetilde{L}(\lambda,\gamma,\gamma')]\varphi' + D_s\widetilde{L}(\lambda,\gamma,\gamma')\varphi\}\,d\lambda = -2\sigma,$$

a contradiction. Finally, we have $f_{a,b}(\mathcal{H}(\eta,t)) \leq f_{a,b}(\eta) - \sigma t$. It follows that γ is not L-stationary, a contradiction. $\square$

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