MINIMIZERS AND SYMMETRIC MINIMIZERS
FOR PROBLEMS WITH CRITICAL SOBOLEV EXPONENT

Shoyeb Waliullah

Abstract. In this paper we will be concerned with the existence and non-existence of constrained minimizers in Sobolev spaces $D^{k,p}(\mathbb{R}^N)$, where the constraint involves the critical Sobolev exponent. Minimizing sequences are not, in general, relatively compact for the embedding $D^{k,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N, Q)$ when Q is a non-negative, continuous, bounded function. However if Q has certain symmetry properties then all minimizing sequences are relatively compact in the Sobolev space of appropriately symmetric functions. For Q which does not have the required symmetry, we give a condition under which an equivalent norm in $D^{k,p}(\mathbb{R}^N)$ exists so that all minimizing sequences are relatively compact. In fact we give an example of a Q and an equivalent norm in $D^{k,p}(\mathbb{R}^N)$ so that all minimizing sequences are relatively compact.

1. Introduction

In this paper we will be concerned with the existence and non-existence of constrained minimizers in Sobolev spaces $D^{k,p}(\mathbb{R}^N)$, where $p > 1$ and the constraint involves the critical Sobolev exponent. It is well known that such minimizers correspond to non-trivial solutions of nonlinear elliptic partial differential equations. After the minimization problem has been formulated one can easily state conditions under which non-trivial solutions to the minimization problem will not exist. One can then go on to state conditions under which the problem

will have a solution. In general these conditions are not easy to check, but in some cases this can be done.

We would also like to mention that some of the problems we look at here have already been considered by other authors, but our method is technically somewhat simpler.

The paper is organized as follows. We initially consider the problem of finding a minimizer associated with the embedding $D^{k,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N, Q)$, with the usual norm in $D^{k,p}(\mathbb{R}^N)$. To this end, we use some preliminary results to establish the well known concentration-compactness lemma. We then give a proof of the known result, that minimizers in general do not exist if Q is not constant and $Q \geq 0$, and in this case minimizing sequences concentrate at the maximum of Q. However, such concentration does not take place if Q has certain symmetry properties, which will be defined later on, and provided we can show that a certain inequality is strict. Examples show the existence of Q so that the afore mentioned inequality is strict. In Section 7 we apply our results to nonlinear partial differential equations to show the existence of solutions. There, we derive some more conditions on Q so that solutions to the partial differential equations exist, and give results which are similar to results given in [6].

In the section following that one we obtain results concerning the weighted Sobolev embedding $D^{k,p}(\mathbb{R}^N, H) \hookrightarrow L^{p^*}(\mathbb{R}^N, Q)$, where we choose the weight H to be a continuous bounded positive function such that $\inf_{x \in \mathbb{R}^N} H > 0$. This ensures that $D^{k,p}(\mathbb{R}^N, H)$ is just the space $D^{k,p}(\mathbb{R}^N)$ equipped with an equivalent norm. We proceed by first proving the existence of minimizers, provided a certain condition is satisfied. An example is then provided to verify the existence of functions H and Q so that the above mentioned condition is satisfied. Before ending the section with a treatment of the symmetric case, we give conditions under which minimizers do not exist.

The final section is devoted to problems with singular weights. These problems arise from the well-known Caffarelli–Kohn–Nirenberg inequality. Our work here is related to the work in [33] and [11].

2. Notation and conventions

In order to prevent ourselves from repeating let us state here some notation and conventions we will use throughout this paper. Q will denote a continuous, bounded, non-negative function in $\mathbb{R}^N$. $Q_0 := Q(0)$, $Q_\infty := \overline{\lim}_{|x| \to \infty} Q(x)$ and if we write $Q_\infty = \lim_{|x| \to \infty} Q(x)$ we assume that the limit exists. This distinction is made because many of our results do not require the existence of this limit.

We will denote by G any closed subgroup of $O(N)$, the group of orthogonal transformations. Let $G_x = \{gx : g \in G\}$ be the orbit of x; $|G_x|$ denote the number of elements in G_x and $|G_\infty| := 1$. Note that we necessarily have $|G_0| = 1$.

As usual, $D^{k,p}(\mathbb{R}^N)$ is the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$(2.1) \qquad \|\nabla^k u\|_p := \left(\int_{\mathbb{R}^N} |\nabla^k u|^p\, dx\right)^{1/p}$$

and $|\nabla^k u|^p := \sum_{|\alpha|=k} |D^\alpha u|^p$.

The equivalent norm for $1 < p < \infty$ which will be useful is the following one:

$$(2.2) \qquad \begin{array}{ll} \|(-\Delta)^{k/2}u\|_p & \text{if } k \text{ is even,} \\ \|\nabla(-\Delta)^{(k-1)/2}u\|_p & \text{if } k \text{ is odd.} \end{array}$$

This is a consequence of the inequality $\|\nabla^2 u\|_p \le C\|\Delta u\|_p$, which can be found in [25], [27], [17]. Since many of our results are independent of the norm used in $D^{k,p}(\mathbb{R}^N)$, we will denote both of them by $\|u\|_{k,p}$. Where necessary we will specify which norm is being used.

For the sake of convenience we will write $L^p(\mathbb{R}^N, Q) = L^p(Q\,dx)$ where the norm is denoted by $\|u\|_{p,Q} = (\int |u|^p Q\, dx)^{1/p}$. Also we will usually write $\int_\Omega u$ instead of $\int_\Omega u(x)\, dx$, and if no region of integration is mentioned then the integration is to be taken over $\mathbb{R}^N$. Further, following the notation used in distribution theory, we will use the symbol $\mu(\phi)$ to mean $\int_{\mathbb{R}^N} \phi\, d\mu$.

3. Preliminary remarks

We will begin by considering the following minimization problem:

$$(3.1) \qquad \overline{S} = \inf\left\{\|u\|_{k,p}^p : u \in D^{k,p}(\mathbb{R}^N), \int Q|u|^{p^*} = 1\right\},$$

where $p^* := Np/(N-kp)$ is the critical Sobolev exponent and $pk < N$. As we have mentioned, one can use any one of the norms (2.1) or (2.2) in (3.1).

REMARK 3.1. By applying the Lagrange multiplier method, we see that any properly normalized minimizer of (3.1), when $k = 1$, solves

$$-\sum_{|\alpha|=1} D^\alpha(|D^\alpha u|^{p-2}D^\alpha u) = Q(x)|u|^{p^*-2}u,$$

if we use the norm in (2.1) and

$$-\text{div}(|\nabla u|^{p-2}\nabla u) = Q(x)|u|^{p^*-2}u,$$

if we use the norm in (2.2). Of course, the value of the constant $\overline{S}$ depends on the norm as well.

REMARK 3.2. For general Q we will show that minimizers of (3.1) usually do not exist. This is a well-known fact which can be deduced from the work of P.-L. Lions [23], [24]. Our motivation for presenting it here is to show the contrast between the results when Q does and does not have any symmetry.

When Q is invariant under the action of G we have the following minimization problem

$$(3.2) \qquad \overline{S}_G = \inf\left\{ \|u\|_{k,p}^p : u \in D_G^{k,p}(\mathbb{R}^N), \ \int Q|u|^{p^*} = 1 \right\}.$$

Here $D_G^{k,p}(\mathbb{R}^N)$ is the subspace of $D^{k,p}(\mathbb{R}^N)$ consisting of functions which are G-symmetric (or G-invariant). We say that $u : \mathbb{R}^N \to \mathbb{R}$ is G-symmetric if $u(gx) = u(x)$ for all $g \in G$ and almost every $x \in \mathbb{R}^N$. In the sequel the minimizers of (3.2) will be called symmetric minimizers. We note that by the principle of symmetric criticality [34, Theorem 1.28] minimizers of the above problem also give us solutions to partial differential equations.

The partial differential equation associated with (3.2) was studied in [6] when $p = 2$, $k = 1$ and the second norm was used. There the authors used the mountain-pass theorem and the principle of symmetric criticality to show the existence of G-symmetric solutions. Here we will not appeal to the mountain-pass theorem but will use more direct methods.

The case when $Q = 1$ was studied by P.-L. Lions in [23], where it was shown that there exists a $u \neq 0$ which achieves

$$S = \inf_{\substack{u \in D^{k,p}(\mathbb{R}^N) \\ u \neq 0}} \frac{\|u\|_{k,p}^p}{\left(\int |u|^{p^*} \right)^{p/p^*}}.$$

Equivalently we have

$$(3.3) \qquad S = \inf\left\{ \|u\|_{k,p}^p : u \in D^{k,p}(\mathbb{R}^N), \ \int |u|^{p^*} = 1 \right\}.$$

The crucial tool here is the concentration-compactness lemma, originally due to P.-L. Lions, with extensions made by G. Bianchi, J. Chabrowski, A. Szulkin, Ben–Naoum, Troestler, M. Willem [23], [24], [6], [34].

4. The concentration-compactness lemma

Before we go on to state and prove the concentration-compactness lemma, we prove a few preliminary results. We denote by $C_{0,G}^\infty(\mathbb{R}^N)$ the subspace of $C_0^\infty(\mathbb{R}^N)$ consisting of G-symmetric functions. We note that for every $\varepsilon > 0$ there exists a constant $C(\varepsilon, p) > 0$ such that

$$(4.1) \qquad \big||x + y|^p - |x|^p\big| \leq \varepsilon|x|^p + C(\varepsilon, p)|y|^p \quad \text{for all } x, y \in \mathbb{R}.$$

PROPOSITION 4.1. *Suppose* $kp < N$, $|\alpha| = k$, $\xi \in C_{0,G}^\infty(\mathbb{R}^N)$ *and* $u_n \rightharpoonup 0$ *in* $D_G^{k,p}(\mathbb{R}^N)$, *then*

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} |D^\alpha(\xi u_n)|^p \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} |\xi D^\alpha u_n|^p dx,$$

provided the limit exists.

Proof. The Leibniz formula gives

$$D^\alpha(\xi u_n) = \xi D^\alpha u_n + \sum_{0 < \beta \leq \alpha} C_{\alpha,\beta} D^\beta \xi D^{\alpha-\beta} u_n.$$

For $\varepsilon > 0$ put $x = \xi D^\alpha u_n$ and $y = \sum_{0 < \beta \leq \alpha} C_{\alpha,\beta} D^\beta \xi D^{\alpha-\beta} u_n$ in (4.1) to get

$$||D^\alpha(\xi u_n)|^p - |\xi D^\alpha u_n|^p| \leq \varepsilon |\xi D^\alpha u_n|^p + C(\varepsilon, p) | \sum_{0 < \beta \leq \alpha} C_{\alpha,\beta} D^\beta \xi D^{\alpha-\beta} u_n|^p.$$

Now an application of Hölder's inequality (for sums) gives

$$||D^\alpha(\xi u_n)|^p - |\xi D^\alpha u_n|^p| \leq \varepsilon |\xi D^\alpha u_n|^p + C_1(\varepsilon, p) \sum_{0 < \beta \leq \alpha} |C_{\alpha,\beta} D^\beta \xi D^{\alpha-\beta} u_n|^p.$$

Since $D^\beta \xi \in C_0^\infty(\mathbb{R}^N)$, we have $D^\beta \xi D^{\alpha-\beta} u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $0 < \beta \leq \alpha$, by the Rellich–Kondrachov theorem. So

$$\lim_{n \to \infty} \left| \int_{\mathbb{R}^N} |D^\alpha(\xi u_n)|^p \, dx - \int_{\mathbb{R}^N} |\xi D^\alpha u_n|^p \, dx \right|$$

$$\leq \lim_{n \to \infty} \int_{\mathbb{R}^N} ||D^\alpha(\xi u_n)|^p - |\xi D^\alpha u_n|^p| \, dx \leq \varepsilon \lim_{n \to \infty} \int_{\mathbb{R}^N} |\xi D^\alpha u_n|^p dx.$$

Since ε is arbitrary, we reach the desired conclusion. $\qquad\square$

We next give a proposition which is an essential part in the proof of the concentration-compactness lemma. The proof can be found in P.-L. Lions [23], but we give a slightly different argument.

Proposition 4.2. *Let μ, ν be two bounded nonnegative measures on $\mathbb{R}^N$ satisfying for some constant $C \geq 0$*

$$(4.2) \qquad \left(\int_{\mathbb{R}^N} |\phi|^q \, d\nu \right)^{1/q} \leq C \left(\int_{\mathbb{R}^N} |\phi|^p \, d\mu \right)^{1/p}, \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^N)$$

where $1 \leq p < q < \infty$, and let μ_s be the atomic part of μ. Then there exists an at most countable set $(x_j)_{j \in J}$ of distinct points in $\mathbb{R}^N$ and a set of numbers $(\nu_j)_{j \in J}$ in $]0, \infty[$ such that

$$\nu = \sum_{j \in J} \nu_j \delta_{x_j}, \qquad \mu_s \geq C^{-p} \sum_{j \in J} \nu_j^{p/q} \delta_{x_j}.$$

Proof. From inequality (4.2) we obtain

$$(\nu(A))^{p/q} \leq C^p \mu(A) \quad \text{for all Borel sets } A.$$

We decompose ν into the atomic and non-atomic parts, i.e. we write

$$\nu = \widetilde{\nu} + \sum_{j \in J} \nu_j \delta_{x_j}.$$

The set J is at most countable since ν is a bounded measure. Since $\nu(\{x\}) = \lim_{\varepsilon \to 0} \nu(B(x, \varepsilon))$, we have

$$(\nu_j)^{p/q} = \nu(\{x_j\})^{p/q} \leq C^p \mu(\{x_j\}).$$

We further conclude that $\widetilde{\nu}$ is absolutely continuous with respect to μ, and by the Radon–Nikodym theorem $\widetilde{\nu} = f\mu$ where $f \in L^1_+(\mu)$. For μ-a.e. x which is not an atom of μ we have

$$C^{-p} f(x)^{p/q} = \lim_{\rho \to 0} \frac{C^{-p}(\int_{B_\rho(x)} d\widetilde{\nu})^{p/q}}{(\int_{B_\rho(x)} d\mu)^{p/q}} \leq \lim_{\rho \to 0} \left(\int_{B_\rho(x)} d\mu \right)^{(q-p)/q} = 0.$$

Since $\widetilde{\nu}$ is atom free and μ has at most countably many atoms, the result follows. $\square$

We point out here that if the reverse inequality in (4.2) also holds then μ and ν concentrate at a single point (see [23]). Recall the definition (3.2) of $\overline{S}_G$ and let $M(\mathbb{R}^N)$ denote the space of finite measures in $\mathbb{R}^N$. When G is the trivial group we will denote $\overline{S}_G$ by $\overline{S}$.

LEMMA 4.3 (Concentration-compactness lemma). *Let G be any closed subgroup of $O(N)$ and Q a non-negative continuous bounded G-symmetric function on $\mathbb{R}^N$. Further let $\{u_n\}_{n=1}^\infty \subset D_G^{k,p}(\mathbb{R}^N)$ be a sequence such that*

$$\begin{aligned}
u_n &\rightharpoonup u &&\text{in } D_G^{k,p}(\mathbb{R}^N), \\
|\nabla^k(u_n - u)|^p &\overset{*}{\rightharpoonup} \mu &&\text{in } M(\mathbb{R}^N), \\
Q|(u_n - u)|^{p^*} &\overset{*}{\rightharpoonup} \nu &&\text{in } M(\mathbb{R}^N), \\
u_n &\to u &&\text{a.e. on } \mathbb{R}^N
\end{aligned}$$

and define

$$\mu_\infty := \lim_{R \to \infty} \overline{\lim_{n \to \infty}} \int_{|x|>R} |\nabla^k u_n|^p, \qquad \nu_\infty := \lim_{R \to \infty} \overline{\lim_{n \to \infty}} \int_{|x|>R} Q|u_n|^{p^*}.$$

If μ_s is the atomic part of μ, then it follows that

$$\nu = \sum_{j \in J} \nu_j \delta_{x_j}, \tag{4.3}$$

$$\|\nu\|^{p/p^*} \leq \overline{S}_G^{-1} \|\mu_s\|, \tag{4.4}$$

$$\nu_\infty^{p/p^*} \leq \overline{S}_G^{-1} \mu_\infty, \tag{4.5}$$

$$\overline{\lim_{n \to \infty}} \|u_n\|_{k,p}^p \geq \|u\|_{k,p}^p + \|\mu_s\| + \mu_\infty, \tag{4.6}$$

$$\overline{\lim_{n \to \infty}} \|u_n\|_{p^*,Q}^{p^*} = \|u\|_{p^*,Q}^{p^*} + \|\nu\| + \nu_\infty. \tag{4.7}$$

Moreover, if $u = 0$ and $\|\nu\|^{p/p^*} = \overline{S}_G^{-1} \|\mu\|$, then ν and μ are concentrated at a single orbit.

PROOF. Our argument is patterned on the proof of Lemma 1.40 in [34]. We note that $\overline{S}_G \geq \overline{S} \geq S\|Q\|_\infty^{-p/p^*} > 0$.

(i) Assume first $u = 0$. Let $\xi \in C_0^\infty(\mathbb{R}^N)$, then we have

$$\left(\int Q|\xi u_n|^{p^*}\, dx \right)^{p/p^*} \leq \overline{S}^{-1} \int |\nabla^k(\xi u_n)|^p\, dx.$$

Taking limits on both sides and using Proposition 4.1 gives

$$\left(\int |\xi|^{p^*}\, d\nu \right)^{p/p^*} \leq \overline{S}^{-1} \int |\xi|^p\, d\mu.$$

Equality (4.3) now follows from Proposition 4.2. In order to obtain inequality (4.4) we observe that ν is G-symmetric and so if ν concentrates at x then it concentrates at gx for all $g \in G$, i.e. ν concentrates on G_x. Further, the amount of mass at each point is the same. Since ν has finite mass, we conclude that the orbit G_x of concentration can have only finitely many distinct points.

Now let $\xi \in C_{0,G}^\infty(\mathbb{R}^N)$, then we have

$$\left(\int Q|\xi u_n|^{p^*}\, dx \right)^{p/p^*} \leq \overline{S}_G^{-1} \int |\nabla^k(\xi u_n)|^p\, dx.$$

Taking limits on both sides and using Proposition 4.1 gives

(4.8) $$\left(\int |\xi|^{p^*}\, d\nu \right)^{p/p^*} \leq \overline{S}_G^{-1} \int |\xi|^p\, d\mu.$$

In particular

$$(\nu(G_x))^{p/p^*} \leq \overline{S}_G^{-1} \mu(G_x) = \overline{S}_G^{-1} \mu_s(G_x)$$

if G_x is an orbit of concentration.

Inequality (4.4) now follows from the strict concavity of the map $\lambda \to \lambda^{p/p^*}$.

(ii) For $R > 1$, let $\psi_R \in C^\infty(\mathbb{R}^N)$ be a radially symmetric function such that $\psi_R(x) = 1$ for $|x| > R + 1$, $\psi_R(x) = 0$ for $|x| < R$ and $0 \leq \psi_R(x) \leq 1$ on $\mathbb{R}^N$. We then obtain

$$\left(\int Q|\psi_R u_n|^{p^*}\, dx \right)^{p/p^*} \leq \overline{S}_G^{-1} \int |\nabla^k(\psi_R u_n)|^p\, dx.$$

Since $D^{\alpha-\beta} u_n \to 0$ in $L_{\text{loc}}^p(\mathbb{R}^N)$ and $D^\beta \psi_R \in C_0^\infty(\mathbb{R}^N)$ for $0 < \beta \leq \alpha$, we obtain the following inequality by applying Proposition 4.1:

(4.9) $$\varlimsup_{n\to\infty} \left(\int Q|\psi_R u_n|^{p^*}\, dx \right)^{p/p^*} \leq \overline{S}_G^{-1} \varlimsup_{n\to\infty} \int |\nabla^k u_n|^p \psi_R^p\, dx.$$

We also have that

$$\int_{|x|>R+1} |\nabla^k u_n|^p\, dx \leq \int |\nabla^k u_n|^p \psi_R^p\, dx \leq \int_{|x|>R} |\nabla^k u_n|^p\, dx$$

and

$$\int_{|x|>R+1} Q|u_n|^{p^*}\,dx \le \int Q|u_n|^{p^*}\psi_R^{p^*}\,dx \le \int_{|x|>R} Q|u_n|^{p^*}\,dx.$$

Hence

$$\mu_\infty = \lim_{R\to\infty}\,\overline{\lim_{n\to\infty}}\int |\nabla^k u_n|^p\psi_R^p dx, \qquad \nu_\infty = \lim_{R\to\infty}\,\overline{\lim_{n\to\infty}}\int Q|u_n|^{p^*}\psi_R^{p^*} dx.$$

Inequality (4.5) now follows from (4.9).

(iii) Further assume that $\|\nu\|^{p/p^*} = \overline{S}_G^{-1}\|\mu\|$. From Hölder's inequality we have, for $\xi \in C_{0,G}^\infty(\mathbb{R}^N)$

$$\left(\int |\xi|^p\,d\mu\right)^{1/p} \le \|\mu\|^{k/N}\left(\int |\xi|^{p^*}\,d\mu\right)^{1/p^*}.$$

Combining this with (4.8) gives

$$\left(\int |\xi|^{p^*}\,d\nu\right)^{1/p^*} \le \overline{S}_G^{-1/p}\|\mu\|^{k/N}\left(\int |\xi|^{p^*}\,d\mu\right)^{1/p^*}.$$

The above inequality gives $\nu \le \overline{S}_G^{-p^*/p}\|\mu\|^{kp^*/N}\mu$, which combined with the equality $\|\nu\|^{p/p^*} = \overline{S}_G^{-1}\|\mu\|$ implies

$$\nu = \overline{S}_G^{-p^*/p}\|\mu\|^{kp^*/N}\mu \quad\text{and}\quad \mu = \overline{S}_G\|\nu\|^{-pk/N}\nu.$$

So for $\xi \in C_{0,G}^\infty(\mathbb{R}^N)$ we have from (4.8)

$$\left(\int |\xi|^{p^*}\,d\nu\right)^{p/p^*} \le \int |\xi|^p\|\nu\|^{-pk/N}\,d\nu$$

that is,

$$\|\nu\|^{k/N}\left(\int |\xi|^{p^*}\,d\nu\right)^{1/p^*} \le \left(\int |\xi|^p\,d\nu\right)^{1/p}.$$

Hence for each open G-symmetric set $\Omega \subset \mathbb{R}^N$

$$\nu(\Omega)^{1/p^*}\nu(\mathbb{R}^N)^{k/N} \le \nu(\Omega)^{1/p}.$$

It follows that either $\nu(\Omega) = 0$ or $\nu(\mathbb{R}^N) \le \nu(\Omega)$. Therefore ν is concentrated at a single orbit, and so is μ.

(iv) Consider now the general case. Set $v_n = u_n - u$, then $v_n \rightharpoonup 0$ in $D_G^{k,p}(\mathbb{R}^N)$ and inequality (4.4) follows from part (i) of the proof.

(v) For any $\varepsilon > 0$, set $x = D^\alpha u_n$ and $y = -D^\alpha u$ in inequality (4.1) to obtain,

$$||D^\alpha v_n|^p - |D^\alpha u_n|^p| \le \varepsilon|D^\alpha u_n|^p + C(\varepsilon,p)|D^\alpha u|^p.$$

It follows that

$$\left| \int_{|x|>R} \left(|\nabla^k v_n|^p - |\nabla^k u_n|^p \right) dx \right| = \left| \int_{|x|>R} \sum_{|\alpha|=k} \left(|D^\alpha v_n|^p - |D^\alpha u_n|^p \right) dx \right|$$

$$\leq \int_{|x|>R} \sum_{|\alpha|=k} \left(||D^\alpha v_n|^p - |D^\alpha u_n|^p| \right) dx$$

$$\leq \varepsilon \int_{|x|>R} \sum_{|\alpha|=k} |D^\alpha u_n|^p \, dx + C(\varepsilon,p) \int_{|x|>R} \sum_{|\alpha|=k} |D^\alpha u|^p \, dx$$

$$= \varepsilon \int_{|x|>R} |\nabla^k u_n|^p \, dx + C(\varepsilon,p) \int_{|x|>R} |\nabla^k u|^p \, dx.$$

Since ε is arbitrary, by letting $n \to \infty$ and $R \to \infty$, we conclude that

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} |\nabla^k v_n|^p = \mu_\infty.$$

From the Brézis–Lieb lemma (see [34, Lemma 1.32]) we have

$$\overline{\lim_{n\to\infty}} \left(\int_{|x|>R} Q|u_n|^{p^*} \, dx - \int_{|x|>R} Q|v_n|^{p^*} \, dx \right) = \int_{|x|>R} Q|u|^{p^*} \, dx.$$

So

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int_{|x|>R} Q|v_n|^{p^*} = \nu_\infty.$$

Inequality (4.5) now follows from part (ii) of the proof.

(vi) There exists a finite measure $\widetilde{\mu}$ such that $|\nabla^k u_n|^p \overset{*}{\rightharpoonup} \widetilde{\mu}$ in $M(\mathbb{R}^N)$.

Let $\phi_\eta \in C_0^\infty(B(x_j, \eta))$, $0 \leq \phi_\eta \leq 1$ and $\phi_\eta(x_j) = 1$ where x_j is an atom of μ. Set $x = D^\alpha v_n$ and $y = D^\alpha u$ in inequality (4.1) to get

$$|\widetilde{\mu}(\phi_\eta) - \mu(\phi_\eta)| \leq \lim_{n\to\infty} \int_{\mathbb{R}^N} \sum_{|\alpha|=k} \phi_\eta ||D^\alpha u_n|^p - |D^\alpha v_n|^p|$$

$$\leq \lim_{n\to\infty} \int_{\mathbb{R}^N} \sum_{|\alpha|=k} \left(\varepsilon \phi_\eta |D^\alpha v_n|^p + C(\varepsilon,p)\phi_\eta |D^\alpha u|^p \right)$$

$$= \varepsilon \mu(\phi_\eta) + C(\varepsilon,p) \int_{\mathbb{R}^N} |\nabla^k u|^p \phi_\eta.$$

Letting $\eta \to 0$, we have

$$|\widetilde{\mu}(\{x_j\}) - \mu_s(\{x_j\})| \leq \varepsilon \mu_s(\{x_j\}).$$

From the fact that ε is arbitrary, we see that the atomic part of $\widetilde{\mu}$ is equal to μ_s. Since $\xi D^\alpha u_n \rightharpoonup \xi D^\alpha u$ in $L^p(\mathbb{R}^N)$ for all positive $\xi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\lim_{n\to\infty} \int |\xi \nabla^k u_n|^p \geq \int |\xi \nabla^k u|^p.$$

Now, $|\nabla^k u|^p$ seen as a measure is relatively singular to the Dirac measures δ_{x_j}, and it follows that

$$\|\widetilde{\mu}\| \geq \|u\|_{k,p}^p + \|\mu_s\|.$$

For $R > 1$ we have

$$\overline{\lim_{n\to\infty}} \int |\nabla^k u_n|^p = \overline{\lim_{n\to\infty}} \left(\int \psi_R |\nabla^k u_n|^p + \int (1 - \psi_R)|\nabla^k u_n|^p \right).$$

As $R \to \infty$, by Lebesgue's dominated convergence theorem we have

$$(4.10) \qquad \overline{\lim_{n\to\infty}} \|u_n\|_{k,p}^p = \mu_\infty + \|\widetilde{\mu}\| \geq \mu_\infty + \|u\|_{k,p}^p + \|\mu_s\|.$$

An application of the Brézis–Lieb lemma gives, for $R > 1$,

$$\overline{\lim_{n\to\infty}} \int Q|u_n|^{p^*} = \overline{\lim_{n\to\infty}} \left(\int \psi_R Q|u_n|^{p^*} + \int (1 - \psi_R)Q|u_n|^{p^*} \right)$$

$$= \overline{\lim_{n\to\infty}} \int \psi_R Q|u_n|^{p^*} + \int (1 - \psi_R)d\nu + \int (1 - \psi_R)Q|u|^{p^*}.$$

As $R \to \infty$,

$$\overline{\lim_{n\to\infty}} \|u_n\|_{p^*,Q}^{p^*} = \|u\|_{p^*,Q}^{p^*} + \|\nu\| + \nu_\infty$$

follows from Lebesgue's dominated convergence theorem. Hence we have proved (4.6) and (4.7). $\qquad\square$

It is important to make the following remarks.

REMARK 4.4. There are many variants of the above lemma as we will see later on. We mention here one of them. It is clear that we could have used the norm in (2.2). The only difference in this case would be that the conclusion of Proposition 4.1 needs to be replaced by

$$\lim_{n\to\infty} \|(-\Delta)^{k/2}(\psi_n u)\|_p = \lim_{n\to\infty} \|\psi(-\Delta)^{k/2}u\|_p \qquad \text{(even } k),$$

$$\lim_{n\to\infty} \|\nabla(-\Delta)^{(k-1)/2}(\psi_n u)\|_p = \lim_{n\to\infty} \|\psi\nabla(-\Delta)^{(k-1)/2}u\|_p \quad \text{(odd } k).$$

The argument is similar.

REMARK 4.5. Looking back at the proof of the above lemma, we see that part (vi) is rather cumbersome and forces (4.6) to be an inequality rather than an equality. However, in the case when $p = 2$, we can avoid the argument in part (vi) of the proof above by using the following argument which exploits the Hilbert space structure of $D_G^{k;2}(\mathbb{R}^N)$.

$$\lim_{n\to\infty} \int |\nabla^k v_n|^2 \psi_R^2 \, dx = \lim_{n\to\infty} \int |\nabla^k u_n|^2 \psi_R^2 \, dx - \int |\nabla^k u|^2 \psi_R^2 \, dx,$$

since the Brézis–Lieb lemma holds when almost every convergence is replaced by weak convergence (see [34, Remarks 1.33]). Hence,

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int |\nabla^k v_n|^2 \psi_R^2 \, dx = \lim_{R\to\infty} \overline{\lim_{n\to\infty}} \int |\nabla^k u_n|^2 \psi_R^2 \, dx = \mu_\infty.$$

For $R > 1$ we have, once again by the Brézis–Lieb lemma

$$\varlimsup_{n \to \infty} \int |\nabla^k u_n|^2 = \varlimsup_{n \to \infty} \left(\int \psi_R |\nabla^k u_n|^2 + \int (1 - \psi_R) |\nabla^k u_n|^2 \right)$$
$$= \varlimsup_{n \to \infty} \int \psi_R |\nabla^k u_n|^2 + \int (1 - \psi_R) d\mu + \int (1 - \psi_R) |\nabla^k u|^2.$$

As $R \to \infty$, by Lebesgue's dominated convergence theorem we have

$$\varlimsup_{n \to \infty} \|u_n\|_{k,2}^2 = \mu_\infty + \|\mu\| + \int |\nabla^k u|^2.$$

So we arrive at the stronger conclusion

$$\varlimsup_{n \to \infty} \|u_n\|_{k,2}^2 = \|u\|_{k,2}^2 + \|\mu\| + \mu_\infty.$$

Further, one can replace μ_s with μ in inequality (4.4).

REMARK 4.6. If $u = 0$, then by definition, $\widetilde{\mu} = \mu$. Hence it follows from (4.10) that $\varlimsup_{n \to \infty} \|u_n\|_{k,p}^p = \mu_\infty + \|\mu\|$.

If $\{u_n\}_{n=1}^\infty \subset D^{k,p}(\mathbb{R}^N)$ is a bounded sequence such that $Q|(u_n - u)|^{p^*} \overset{*}{\rightharpoonup} \nu$, then we may assume that $|u_n - u|^p \overset{*}{\rightharpoonup} \gamma$. Hence, by defining γ_∞ in the same way as ν_∞, we see that $\nu(\{x\}) = Q(x)\gamma(\{x\})$ and $\nu_\infty \le Q_\infty \gamma_\infty$. So γ and ν concentrate at exactly the same points, if $Q > 0$. Further, $\nu_\infty = Q_\infty \gamma_\infty$ if $Q_\infty = \lim_{|x| \to \infty} Q(x)$.

5. Non-existence result

The proposition given below is the essential part in showing that for general Q a minimizer of (3.1) does not exist.

PROPOSITION 5.1. *If Q is a bounded nonnegative continuous function in $\mathbb{R}^N$, then $S = \overline{S} \|Q\|_\infty^{p/p^*}$.*

PROOF. We have,

$$\overline{S} = \inf_{\substack{u \in D^{k,p}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int |\nabla^k u|^p}{(\int Q|u|^{p^*})^{p/p^*}} \ge \inf_{\substack{u \in D^{k,p}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int |\nabla^k u|^p}{\|Q\|_\infty^{p/p^*} (\int |u|^{p^*})^{p/p^*}} = \frac{S}{\|Q\|_\infty^{p/p^*}}.$$

So, $S \le \overline{S} \|Q\|_\infty^{p/p^*}$ follows. Let u be a function which achieves S in (3.3) and for $x_0 \in \mathbb{R}^N$ set

$$u_\varepsilon(x) = \varepsilon^{-N/p^*} u\left(\frac{x - x_0}{\varepsilon} \right).$$

Through a variable substitution we have

$$\overline{S} \le \frac{\int |\nabla^k u_\varepsilon|^p \, dx}{(\int Q(x)|u_\varepsilon|^{p^*} \, dx)^{p/p^*}} = \frac{\int |\nabla^k u|^p \, dy}{(\int Q(\varepsilon y + x_0)|u|^{p^*} \, dy)^{p/p^*}}.$$

As $\varepsilon \to 0$, by Lebesgue's dominated convergence theorem we obtain

$$\overline{S} \leq \frac{S}{Q(x_0)^{p/p^*}}.$$

The assertion follows, since we have $(Q(x_0))^{p/p^*}\overline{S} \leq S \leq \overline{S}\|Q\|_\infty^{p/p^*}$, for all $x_0 \in \mathbb{R}^N$. $\qquad\square$

To see that minimizers of (3.1) usually do not exist, assume that u is such a minimizer. Then in view of Proposition 5.1 we have

$$\left(\int Q|u|^{p^*}\right)^{p/p^*} \leq \|Q\|_\infty^{p/p^*}\left(\int |u|^{p^*}\right)^{p/p^*}$$

$$\leq \|Q\|_\infty^{p/p^*}S^{-1}\int |\nabla^k u|^p = \left(\int Q|u|^{p^*}\right)^{p/p^*}.$$

So it follows that

$$\left(\int_{\mathbb{R}^N}(\|Q\|_\infty - Q)|u|^{p^*}\right) = 0.$$

We now deduce that if the set $E = \{x \in \mathbb{R}^N : \|Q\|_\infty = Q(x)\}$ has measure zero, then a minimizer of (3.1) does not exist. We can further conclude, since the minimizers for S are positive everywhere when $p > 1$, $k = 1$ or $p = 2$ and $k > 1$ (see Section 7), that the minimizers of (3.1) exist if and only if Q is constant. We state these observations in the following proposition.

PROPOSITION 5.2. *If the set* $E = \{x \in \mathbb{R}^N : \|Q\|_\infty = Q(x)\}$ *has measure zero, then problem* (3.1) *has no minimizer. Further, when* $p > 1$, $k = 1$ *or when* $p = 2$, $k \geq 2$, *minimizers of* (3.1) *exist if and only if* Q *is constant.*

6. Sufficient condition for existence of minimizers

In this section we assume that Q is invariant under the action of the group G. We give a sufficient condition for the existence of symmetric minimizers for problem (3.2). We will then give examples which show that there are functions Q so that this condition holds.

THEOREM 6.1. *If* $\overline{S}_G \sup_{x \in \mathbb{R}^N \cup \{\infty\}} Q(x)^{p/p^*}|G_x|^{p/p^*-1} < S$ *then problem* (3.2) *has a minimizer.*

PROOF. Let $\{u_n\}$ be a minimizing sequence for $\overline{S}_G$ such that $\|u_n\|_{p^*,Q} = 1$. For some subsequence, still denoted $\{u_n\}$, we may assume that the conditions of Lemma 4.3 are fulfilled, and so the conclusions hold. We need to show that $\|\nu\| = \nu_\infty = 0$. We have

$$\overline{S}_G = \lim_{n\to\infty}\|u_n\|_{k,p}^p \geq \|u\|_{k,p}^p + \|\mu_s\| + \mu_\infty,$$

$$1 = \lim_{n\to\infty}\|u_n\|_{p^*,Q}^{p^*} = \|u\|_{p^*,Q}^{p^*} + \|\nu\| + \nu_\infty.$$

Combining these with inequalities (4.4) and (4.5) gives

(6.1) $\overline{S}_G(\|u\|_{p^*,Q}^{p^*} + \|\nu\| + \nu_\infty)^{p/p^*} \geq \|u\|_{k,p}^p + \|\mu_s\| + \mu_\infty$

$$\geq \overline{S}_G((\|u\|_{p^*,Q}^{p^*})^{p/p^*} + \|\nu\|^{p/p^*} + \nu_\infty^{p/p^*}).$$

So, only one of the three quantities, $\|u\|_{p^*,Q}^{p^*}$, $\|\nu\|$ and ν_∞, is equal to 1 and the other two are zero. If $\nu_\infty = 1$, then using the hypothesis, Remark 4.6, (6.1) and (4.5) with $Q = 1$ and G the trivial group (i.e. $\overline{S} = S$) we have

$$S(\gamma_\infty)^{p/p^*} > \overline{S}_G(Q_\infty \gamma_\infty)^{p/p^*} \geq \overline{S}_G(\nu_\infty)^{p/p^*} \geq \mu_\infty \geq S(\gamma_\infty)^{p/p^*},$$

a contradiction. So $\nu_\infty = 0$. If $\|\nu\| = 1$ then $u = 0$ and $\|\nu\|^{p/p^*} = \overline{S}_G^{-1}\|\mu\|$ by (6.1) and Remark 4.6, and so ν is concentrated at a single orbit G_x. We can conclude that $|G_x| < \infty$ from the fact that the set of concentration points of ν are G-invariant and the concentration mass is the same at each point. Once again we get a contradiction, since

$$S(\gamma(\{x\}))^{p/p^*} > \overline{S}_G|G_x|^{p/p^*-1}(Q(x)\gamma(\{x\}))^{p/p^*}$$
$$= \overline{S}_G|G_x|^{p/p^*-1}(\nu(\{x\}))^{p/p^*} = \mu(\{x\}) \geq S(\gamma(\{x\}))^{p/p^*}.$$

Here we have used the fact that $|G_x|^{p/p^*}(\nu(\{x\}))^{p/p^*} = \|\nu\|^{p/p^*} = \overline{S}_G^{-1}\|\mu\| = \overline{S}_G^{-1}|G_x|\mu(\{x\})$. It follows that $\|u\|_{p^*,Q} = 1$, and u is a minimizer of (3.2).   □

REMARK 6.2. The results presented above are independent of the norm chosen on $D^{k,p}(\mathbb{R}^N)$. But we still have to show that there are functions Q for which the above condition holds. To this end, we will assume that the norm used on $D^{k,p}(\mathbb{R}^N)$ is the norm given in (2.2). This will guarantee that there are radially symmetric, nonnegative and decreasing minimizers for problem (3.3), (see [23, Corollary I.2]). Now, if

$$S_G = \inf\left\{\|u\|_{k,p}^p : u \in D_G^{k,p}(\mathbb{R}^N), \int |u|^{p^*} = 1\right\},$$

then $S_G = S$. This is because there exists a radially symmetric and hence G-symmetric function which minimizes S and $S \leq S_G$.

Let $\delta > 0$, and choose a G-symmetric function $u \in C_0^\infty(\mathbb{R}^N)$ such that

$$S = S_G < \frac{\|u\|_{k,p}^p}{\|u\|_{p^*,Q}^{p^*}} < S + \delta.$$

For any orbit G_x with finite cardinality, let x_i, $i = 1, \ldots, n$, be the distinct elements of the orbit and set

$$u_\varepsilon(x) = \sum_{i=1}^n \varepsilon^{-N/p^*} u\left(\frac{x - x_i}{\varepsilon}\right).$$

If ε is small enough, then the functions in the above sum have disjoint supports. Through a variable substitution we have

$$\overline{S}_G \leq \frac{\|u_\varepsilon\|_{k,p}^p}{(\int Q(x)|u_\varepsilon|^{p^*}\,dx)^{p/p^*}} = \frac{\sum_{i=1}^n \|u\|_{k,p}^p}{(\sum_{i=1}^n \int Q(\varepsilon y + x_i)|u|^{p^*}\,dy)^{p/p^*}}.$$

By Lebesgue's dominated convergence theorem we may take the limit under the integral sign. Hence by letting ε go to 0 we obtain

$$\overline{S}_G \leq Q(x)^{-p/p^*}|G_x|^{1-p/p^*}(S + \delta).$$

Since δ is arbitrary, we have $\overline{S}_G \leq Q(x)^{-p/p^*}|G_x|^{1-p/p^*}S$ for all $x \in \mathbb{R}^N$. We may also take the function $u_\varepsilon(x) = \varepsilon^{-N/p^*}u(x/\varepsilon)$ and allow ε to go to ∞ to obtain $\overline{S}_G \leq Q_\infty^{-p/p^*}S$ provided $Q_\infty = \lim_{|x|\to\infty} Q(x)$. Since $S/\|Q\|_\infty^{p/p^*} \leq \overline{S}_G$ (see the argument of Proposition 5.1), we may conclude that

$$\frac{S}{\|Q\|_\infty^{p/p^*}} \leq \overline{S}_G \leq S \inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-p/p^*}}{Q(x)^{p/p^*}}.$$

We further observe that if $\|Q\|_\infty^{-p/p^*} = \inf_{x \in \mathbb{R}^N \cup \{\infty\}} |G_x|^{1-p/p^*}Q(x)^{-p/p^*}$, which can occur only at points $x \in \mathbb{R}^N \cup \{\infty\}$ with $|G_x| = 1$, then the assumptions of Theorem 6.1 cannot be satisfied. In this case we can state a result similar to Proposition 5.2.

EXAMPLE 6.3. The most trivial example of Q satisfying the assumption of Theorem 6.1 is when $Q_0 = Q_\infty = 0$ and $|G_x| = \infty$ for $x \in \mathbb{R}^N - \{0\}$. These conditions immediately guarantee that concentration can neither occur at infinity nor at any point of $\mathbb{R}^N$, and the assumption of Theorem 6.1 is satisfied.

The following example shows that the condition

$$\overline{S}_G \sup_{x \in \mathbb{R}^N \cup \{\infty\}} Q(x)^{p/p^*}|G_x|^{p/p^*-1} < S$$

is not always necessary to conclude that minimizers of $\overline{S}_G$ exist.

EXAMPLE 6.4. Suppose that

$$|G_x|^{p^*/p-1}Q_0 \geq Q(x) \geq Q_0 = Q_\infty = \lim_{|x|\to\infty} Q(x) > 0 \quad \text{for all } x \in \mathbb{R}^N.$$

We know that

$$\overline{S}_G \leq S \inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-p/p^*}}{Q(x)^{p/p^*}}.$$

If strict inequality holds then a minimizer for $\overline{S}_G$ exists by Theorem 6.1. On the other hand if

$$\overline{S}_G = S \inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-p/p^*}}{Q(x)^{p/p^*}} = \frac{S}{Q_0^{p/p^*}}$$

then a minimizer for $\overline{S}_G$ also exists. To see this, let u be a minimizer for S. We then have

$$(6.2) \qquad \overline{S}_G \leq \frac{\|u\|_{k,p}^p}{\|u\|_{p^*,Q}^p} \leq \frac{\|u\|_{k,p}^p}{Q_0^{p/p^*}\|u\|_{p^*}^p} = \frac{S}{Q_0^{p/p^*}} = \overline{S}_G.$$

So u is a minimizer for $\overline{S}_G$ as well.

We have thus proved:

COROLLARY 6.5. *Problem* (3.2) *with the norm in* (2.2) *has a minimizer if*

$$|G_x|^{p^*/p-1}Q_0 \geq Q(x) \geq Q_0 = Q_\infty = \lim_{|x|\to\infty} Q(x) > 0 \quad \text{for all } x \in \mathbb{R}^N.$$

The above corollary together with Proposition 5.2 shows that $\overline{S} < \overline{S}_G$ if $E = \{x \in \mathbb{R}^N : \|Q\|_\infty = Q(x)\}$ has measure zero and $|G_x|^{p^*/p-1}Q_0 \geq Q(x) \geq Q_0 = Q_\infty = \lim_{|x|\to\infty} Q(x) \geq 0$ for all $x \in \mathbb{R}^N$.

REMARK 6.6. From (6.2) we can conclude that Q is constant on sets where u, the minimizer for S, does not vanish. In particular, when $p = 2$ or $k = 1$ we know that u is strictly positive and so Q must be constant.

7. Application to partial differential equations

In this section $\|u\|_{k,p}$ will denote the norm in (2.2).

7.1. The case $p = 2$, $k = 1$. In [6] the authors studied the solutions to the following problem

$$(7.1) \qquad -\Delta u = Q(x)|u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \ u \in D_G^{1,2}(\mathbb{R}^N),$$

where $N > 2$, $2^* = 2N/(N-2)$ and Q is G-symmetric. We know that any minimizer of problem (3.2) with $p = 2$ and $k = 1$ will then give a solution of the above problem. In Proposition 2 in [6] the authors show that a solution to problem (3.2) exists if $\overline{S}_G \max\{Q_0^{2/2^*}, Q_\infty^{2/2^*}, |G|^{-2/N}\|G\|_\infty^{2/2^*}\} < S$ where $|G| = \inf_{x \in \mathbb{R}^N, x\neq 0} |G_x|$ and $|G_x|$ is as before. Comparing this to Theorem 6.1 shows that our result is the one given there.

We can now state some conditions on Q, taken from [6], which will guarantee that the assumption of Theorem 6.1 is satisfied. The proofs are similar to those of Corollaries 1 and 2 in [6]. We include them for further reference.

COROLLARY 7.1. *Suppose that Q is G-symmetric,*

$$\inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-2/2^*}}{Q(x)^{2/2^*}} = \frac{1}{Q_0^{2/2^*}}$$

and either

(a) $Q(x) \geq Q_0 + \varepsilon|x|^N$ *for some $\varepsilon > 0$ and $|x|$ small or*

(b) $|Q(x) - Q_0| \le C|x|^\alpha$ for some constant $C > 0$, $\alpha > N$, $|x|$ small and

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-2N} \, dx > 0.$$

Then there exists a nontrivial solution to problem (7.1).

PROOF. We know that the instanton $v(x) = (1 + |x|^2)^{-N/2^*}$ is the unique minimizer for (3.3) with $k = 1$ and $p = 2$, up to translation and dilation. In view of Theorem 6.1 it suffices to show that, for some $\eta > 0$,

$$\overline{S}_G^{-2^*/2} \ge \int_{\mathbb{R}^N} Q(x)|Av(x/\eta)|^{2^*} > \int_{\mathbb{R}^N} Q_0|Av(x/\eta)|^{2^*} = Q_0 S^{-2^*/2},$$

where $A > 0$ is a constant chosen so that $\|Av(x/\eta)\|_{1,2} = 1$. Of course this is equivalent to showing that, for some $\eta > 0$,

$$\int_{\mathbb{R}^N} Q(x)\left(\frac{1}{\eta^2 + |x|^2}\right)^N - \int_{\mathbb{R}^N} Q_0\left(\frac{1}{\eta^2 + |x|^2}\right)^N > 0.$$

(a) By the hypothesis, for some $\delta > 0$,

$$\int_{|x| \le \delta} (Q(x) - Q_0)\left(\frac{1}{\eta^2 + |x|^2}\right)^N \ge \varepsilon \int_{|x| \le \delta} \left(\frac{|x|}{\eta^2 + |x|^2}\right)^N \to \infty$$

as $\eta \to 0$. On the other hand, for all $\eta > 0$ we have

$$\left| \int_{|x| > \delta} (Q(x) - Q_0)\left(\frac{1}{\eta^2 + |x|^2}\right)^N \right| \le C_1 \int_{|x| > \delta} \frac{1}{|x|^{2N}} = C_2$$

for some constants C_1, C_2 greater than zero and independent of η. We now obtain the required conclusion.

(b) By the hypothesis, $|Q(x) - Q_0||x|^{-2N} \in L^1(\mathbb{R}^N)$, and by Lebesgue's dominated convergence theorem we have

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)\left(\frac{1}{\eta^2 + |x|^2}\right)^N \to \int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-2N}$$

as $\eta \to 0$. Hence, we deduce the required conclusion. $\qquad \square$

COROLLARY 7.2. Suppose that Q is G-symmetric, $Q_\infty = \lim_{|x| \to \infty} Q(x)$,

$$\inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-2/2^*}}{Q(x)^{2/2^*}} = \frac{1}{Q_\infty^{2/2^*}}$$

and either

(a) $Q(x) \ge Q_\infty + \varepsilon|x|^{-N}$ for some $\varepsilon > 0$ and $|x|$ large or
(b) $|Q(x) - Q_\infty| \le C|x|^{-\alpha}$ for some constant $C > 0$, $\alpha > N$, $|x|$ large and

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty) \, dx > 0.$$

Then there exists a nontrivial solution to problem (7.1).

PROOF. As mentioned in the proof of the previous corollary, in view of Theorem 6.1 it suffices to show that for some $\eta > 0$

$$\overline{S}_G^{-2^*/2} \geq \int_{\mathbb{R}^N} Q(x)|Av(x/\eta)|^{2^*} > \int_{\mathbb{R}^N} Q_\infty |Av(x/\eta)|^{2^*} = Q_\infty S^{-2^*/2}.$$

(a) Hence, we need to show that

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty)\left(\frac{1}{1 + |x/\eta|^2}\right)^N > 0$$

for some $\eta > 0$. By the hypothesis, we can find $R > 0$ such that $Q(x) \geq Q_\infty + \varepsilon|x|^{-N}$ for all $|x| \geq R$. It follows that

$$\int_{|x|>R} (Q(x) - Q_\infty)\left(\frac{1}{1 + |x/\eta|^2}\right)^N \to \infty$$

as $\eta \to \infty$. We also have

$$\left| \int_{|x|\leq R} (Q(x) - Q_\infty)\left(\frac{1}{1 + |x/\eta|^2}\right)^N \right| \leq C_1$$

where $C_1 > 0$ is independent of η. By putting these two observations together, we obtain the desired result.

(b) By the hypothesis, $|Q(x) - Q_\infty| \in L^1(\mathbb{R}^N)$ and so

$$\lim_{\eta \to \infty} \int_{\mathbb{R}^N} (Q(x) - Q_\infty)\left(\frac{1}{1 + |x/\eta|^2}\right)^N = \int_{\mathbb{R}^N} (Q(x) - Q_\infty)\, dx > 0,$$

we immediately conclude the desired result. □

REMARK 7.3. We observe that $Q_\infty = \lim_{|x|\to\infty} Q(x)$ is now a part of the assumption. In the case when

$$\inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-p/p^*}}{Q(x)^{p/p^*}} \leq \min\{Q_0^{-p/p^*}, Q_\infty^{-p/p^*}\}$$

one can construct Q such that assumption of Theorem 6.1 is satisfied (see [6]).

7.2. The case $p = 2$ and $k > 1$. We continue with a higher order variant of the above example. We wish to find non-trivial solutions to the following semi-linear partial differential equation

(7.2) $$(-\Delta)^k u = Q(x)|u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \ u \in D_G^{1,2}(\mathbb{R}^N),$$

where $N > 2k$, $2^* = 2N/(N - 2k)$ and Q is G-symmetric. Keeping in mind the norm (2.2), a minimizer for (3.2) with $p = 2$, will then give a solution of the above problem. In the previous example, by knowing explicitly the instanton which minimizes (3.3), we could state explicit conditions on Q under which problem (3.2) has a minimizer. We do the same thing here, since we know that

up to translation and dilation the instanton $v(x) = (1+|x|^2)^{-N/2^*}$ is a minimizer for (3.2) (see [29]). By the same arguments as in Corollaries 7.1 and 7.2 we see that the following results hold.

COROLLARY 7.4. *Suppose that Q is G-symmetric,*

$$\inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-2/2^*}}{Q(x)^{2/2^*}} = \frac{1}{Q_0^{2/2^*}}$$

and either

(a) $Q(x) \geq Q_0 + \varepsilon|x|^N$ *for some $\varepsilon > 0$ and $|x|$ small or*
(b) $|Q(x) - Q_0| \leq C|x|^\alpha$ *for some constant $C > 0$, $\alpha > N$, $|x|$ small and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-2N}\, dx > 0.$$

Then there exists a nontrivial solution to problem (7.2).

COROLALRY 7.5. *Suppose that Q is G-symmetric, $Q_\infty = \lim_{|x| \to \infty} Q(x)$,*

$$\inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-2/2^*}}{Q(x)^{2/2^*}} = \frac{1}{Q_\infty^{2/2^*}}$$

and either

(a) $Q(x) \geq Q_\infty + \varepsilon|x|^{-N}$ *for some $\varepsilon > 0$ and $|x|$ large or*
(b) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ *for some constant $C > 0$, $\alpha > N$, $|x|$ large and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty)\, dx > 0.$$

Then there exists a nontrivial solution to problem (7.2).

REMARK 7.6. Once again, when

$$\inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-p/p^*}}{Q(x)^{p/p^*}} \leq \min\{Q_0^{-p/p^*}, Q_\infty^{-p/p^*}\}$$

one can construct Q such that assumption of Theorem 6.1 is satisfied.

For results in the non-critical case we refer to [5] and references therein.

7.3. The case $p > 1$ and $k = 1$. Here we obtain an equation involving the p-Laplace operator. We have

(7.3) $\qquad -\Delta_p u = Q(x)|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N, \ u \in D_G^{1,2}(\mathbb{R}^N),$

where $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$, $N > p$, $p^* = Np/(N-p)$ and Q is G-symmetric. It is known from the work of T. Aubin [2] and G. Talenti [32] that $v(x) = (1 + |x|^{p/(p-1)})^{-N/p^*}$ is the unique minimizer up to translation and dilation, for problem (3.3) with $k = 1$. In this case also Corollaries 7.1 and 7.2 hold with minor changes. Since the proofs are similar we skip them.

Corollary 7.7. *Suppose that Q is G-symmetric,*

$$\inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-p/p^*}}{Q(x)^{p/p^*}} = \frac{1}{Q_0^{p/p^*}}$$

and either

(a) $Q(x) \geq Q_0 + \varepsilon |x|^{N/(p-1)}$ *for some $\varepsilon > 0$ and $|x|$ small or*
(b) $|Q(x) - Q_0| \leq C|x|^\alpha$ *for some constant $C > 0$, $\alpha > N/(p-1)$, $|x|$ small and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-pN/(p-1)} \, dx > 0.$$

Then there exists a nontrivial solution to problem (7.3).

Corollary 7.8. *Suppose that Q is G-symmetric, $Q_\infty = \lim_{|x| \to \infty} Q(x)$,*

$$\inf_{x \in \mathbb{R}^N \cup \{\infty\}} \frac{|G_x|^{1-p/p^*}}{Q(x)^{p/p^*}} = \frac{1}{Q_\infty^{p/p^*}}$$

and either

(a) $Q(x) \geq Q_\infty + \varepsilon |x|^{-N}$ *for some $\varepsilon > 0$ and $|x|$ large or*
(b) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ *for some constant $C > 0$, $\alpha > N$, $|x|$ large and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty) dx > 0.$$

Then there exists a nontrivial solution to problem (7.3).

The p-Laplace operator in equation (7.3) has been the object of many studies, where both critical and non-critical exponents have been considered. We refer the reader e.g. to [30], [26], [1], [12], [28] and the references therein.

7.4. The p-biharmonic operator. Let

$$F(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\Delta u|^p,$$

then

$$F'(u)\phi = \frac{1}{p} \int_{\mathbb{R}^N} |\Delta u|^{p-2} \Delta u \Delta \phi \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^N),$$

i.e. any minimizer of problem (3.2) with $k = 2$ will satisfy

(7.4) $\qquad \Delta(|\Delta u|^{p-2}\Delta u) = Q|u|^{p^*-2}u \quad \text{in } \mathbb{R}^N, \ u \in D_G^{2,p}(\mathbb{R}^N).$

In this case the explicit form of the minimizers of S_G is not known, therefore we are not able to give explicit conditions on Q so that a solution to (7.4) exists. However, by using Corollary 6.5 we may conclude that if $|G_x|^{p^*/p-1}Q_0 \geq Q(x) \geq Q_0 = Q_\infty = \lim_{|x| \to \infty} Q(x) > 0$ then equation (7.4) has a G-invariant solution.

The operator $\Delta(|\Delta u|^{p-2}\Delta u)$ is called the p-biharmonic operator. In comparison to the p-Laplace operator, very little is known about it. However see [14], [19] and [31].

8. Double weights

In this section, we will apply the methods developed in the previous sections to a more general problem. Let H be a bounded continuous function in $\mathbb{R}^N$. Assume that $\overline{H} = \inf_{x \in \mathbb{R}^N} H(x) > 0$ and $H_\infty := \lim_{|x| \to \infty} H(x)$ exists. We will look at the following problem:

$$I = \inf\{\|u\|_{k,p,H}^p : u \in D^{k,p}(\mathbb{R}^N),\ \|u\|_{p^*,Q} = 1\}.$$

Here $\|u\|_{k,p,H}$ can either be $\|\nabla^k u\|_{p,H}$ or $\|(-\Delta)^{k/2} u\|_{p,H}$ when k is even and $\|\nabla(-\Delta)^{(k-1)/2} u\|_{p,H}$ when k is odd. There is no problem in doing so since our hypothesis on H shows that, for even k,

$$\int_{\mathbb{R}^N} H|\Delta^{k/2} u|^p \sim \int_{\mathbb{R}^N} |\Delta^{k/2} u|^p \sim \int_{\mathbb{R}^N} |\nabla^k u|^p \sim \int_{\mathbb{R}^N} H|\nabla^k u|^p,$$

where $\sim$ indicates the equivalence of norms. The same is true for odd k. Similarly as in Section 4, we first assume that $\|u\|_{k,p,H} = \|\nabla^k u\|_{p,H}$.

We note that the condition $\overline{H} > 0$ guaranties the positivity of I and also that $\| \cdot \|_{k,p,H}$ is an equivalent norm to $\| \cdot \|_{k,p}$ in $D^{k,p}(\mathbb{R}^N)$. To keep things simple we will also assume that $Q_\infty := \lim_{|x| \to \infty} Q(x)$ exists. It is easy to see that the methods applied in the previous sections can be adapted to handle the case of double weights.

This type of problems with double weights have been studied by some authors. We refer the reader to [3], [4], [9], [18] and references therein.

We start by studying the effect of dilation and translation in order to obtain a relationship between the values I and S. Let u be a function which achieves S in (3.3) and for $x_0 \in \mathbb{R}^N$ set

$$u_\varepsilon(x) = \varepsilon^{-N/p^*} u\left(\frac{x - x_0}{\varepsilon}\right).$$

Through a variable substitution we have

$$I \leq \frac{\int H(x)|\nabla^k u_\varepsilon|^p\, dx}{(\int Q(x)|u_\varepsilon|^{p^*}\, dx)^{p/p^*}} = \frac{\int H(\varepsilon y + x_0)|\nabla^k u|^p\, dy}{(\int Q(\varepsilon y + x_0)|u|^{p^*}\, dy)^{p/p^*}}.$$

As $\varepsilon \to 0$, by Lebesgue's dominated convergence theorem we obtain,

$$I \leq \frac{SH(x_0)}{Q(x_0)^{p/p^*}}.$$

Since the above inequality holds for all $x_0 \in \mathbb{R}^N$, we conclude

$$I \leq S \inf_{x \in \mathbb{R}^N} \frac{H(x)}{(Q(x))^{p/p^*}}.$$

On the other hand, we have

$$\left(\int Q|u|^{p^*}\right)^{p/p^*} \leq \|Q\|_\infty^{p/p^*}\left(\int |u|^{p^*}\right)^{p/p^*} \leq S^{-1}\frac{\|Q\|_\infty^{p/p^*}}{\overline{H}}\int H|\nabla^k u|^p$$

for all $u \in D^{k,p}(\mathbb{R}^N)$. Hence we deduce that

$$(8.2) \qquad S\frac{\overline{H}}{\|Q\|_\infty^{p/p^*}} \leq I \leq S \inf_{x \in \mathbb{R}^N} \frac{H(x)}{(Q(x))^{p/p^*}}.$$

Next, we require the concentration-compactness lemma, which gives us information regarding weakly converging sequences and in particular minimizing sequences. Since Proposition 4.1 holds even when we use $H\,dx$ as weights, we can state another version of the concentration-compactness lemma. Since the proof is similar to that of Lemma 4.3 we omit it.

LEMMA 8.1 (Concentration-compactness lemma). *Assume that our hypothesis on H and Q hold, and $\{u_n\}_{n=1}^\infty \subset D^{k,p}(\mathbb{R}^N)$ is a sequence such that*

$$u_n \rightharpoonup u \quad in\ D^{k,p}(\mathbb{R}^N),$$
$$H|\nabla^k(u_n - u)|^p \overset{*}{\rightharpoonup} \mu \quad in\ M(\mathbb{R}^N),$$
$$Q|(u_n - u)|^{p^*} \overset{*}{\rightharpoonup} \nu \quad in\ M(\mathbb{R}^N),$$
$$u_n \to u \quad a.e.\ on\ \mathbb{R}^N,$$

and define

$$\mu_\infty := \lim_{R\to\infty} \varlimsup_{n\to\infty} \int_{|x|>R} H|\nabla^k u_n|^p, \qquad \nu_\infty := \lim_{R\to\infty} \varlimsup_{n\to\infty} \int_{|x|>R} Q|u_n|^{p^*}.$$

If μ_s is the atomic part of μ, then it follows that

$$\nu = \sum_{j\in J} \nu_j \delta_{x_j}, \qquad \|\nu\|^{p/p^*} \leq I^{-1}\|\mu_s\|, \qquad \nu_\infty^{p/p^*} \leq I^{-1}\mu_\infty,$$

$$\varlimsup_{n\to\infty} \|u_n\|_{k,p,H}^p \geq \|u\|_{k,p}^p + \|\mu_s\| + \mu_\infty,$$

$$\varlimsup_{n\to\infty} \|u_n\|_{p^*,Q}^{p^*} = \|u\|_{p^*,Q}^{p^*} + \|\nu\| + \nu_\infty.$$

Moreover, if $u = 0$ and $\|\nu\|^{p/p^*} = I^{-1}\|\mu\|$, then ν and μ are concentrated at a single point.

REMARK 8.2. It is obvious that we could have taken into account the action of a closed subgroup $G \subset O(N)$ provided H and Q are G-invariant and obtain a result similar to Lemma 4.3.

REMARK 8.3. If $\{u_n\}_{n=1}^\infty \subset D^{k,p}(\mathbb{R}^N)$ is a bounded sequence such that

$$H|\nabla^k(u_n - u)|^p \overset{*}{\rightharpoonup} \mu, \qquad Q|(u_n - u)|^{p^*} \overset{*}{\rightharpoonup} \nu,$$

then we may assume that

$$|\nabla^k(u_n - u)|^p \overset{*}{\rightharpoonup} \alpha \quad \text{and} \quad |(u_n - u)|^{p^*} \overset{*}{\rightharpoonup} \beta.$$

Hence, by defining α_∞ and β_∞ in the way μ_∞ is defined, we see that $\mu(\{x\}) = H(x)\alpha(\{x\})$, $\nu(\{x\}) = Q(x)\beta(\{x\})$, $\mu_\infty = H_\infty\alpha_\infty$ and $\nu_\infty = Q_\infty\beta_\infty$.

We can now state a result which basically, is a necessary and sufficient condition for all minimizing sequences to be relatively compact. That it is sufficient follows from the following theorem. To see that this is also necessary we refer the reader to the work of P.-L. Lions [21]–[24]. We would like to mention that the hypothesis of the next theorem is hard to check, but we give an example which will show that there exist H and Q such that the assumption is satisfied.

THEOREM 8.4. *If* $I < S \inf_{x \in \mathbb{R}^N} H(x)/(Q(x))^{p/p^*}$ *then all minimizing sequences are relatively compact. In particular, a minimizer for I exists.*

PROOF. Let $\{u_n\}_{n=1}^\infty \subset D^{k,p}(\mathbb{R}^N)$ be a minimizing sequence for I. Arguing exactly as in Theorem 6.1 we see that only one of the three quantities, $\|u\|_{p^*,Q}^{p^*}$, $\|\nu\|$ and ν_∞, is equal to 1 and the other two are zero.

(a) If $\nu_\infty = 1$, then

$$I = I(\nu_\infty)^{p/p^*} = I(Q_\infty \beta_\infty)^{p/p^*} \geq \mu_\infty = H_\infty \alpha_\infty \geq SH_\infty(\beta_\infty)^{p/p^*} = S\frac{H_\infty}{Q_\infty^{p/p^*}}.$$

Hence, $I \geq SH_\infty/Q_\infty^{p/p^*} \geq S\inf_{x\in\mathbb{R}^N} H(x)/(Q(x))^{p/p^*}$ contradicts our assumption.

(b) If $\|\nu\| = 1$, then $u = 0$, $I\|\nu\|^{p/p^*} \geq \|\mu\|$ and so by the previous lemma ν concentrates at a point $x \in \mathbb{R}^N$. We now have

$$I = I(\nu(\{x\}))^{p/p^*} = I(Q(x)\beta(\{x\}))^{p/p^*} \geq \mu(\{x\})$$
$$= H(x)\alpha(\{x\}) \geq SH(x)(\beta(\{x\}))^{p/p^*}.$$

Once again $I \geq SH(x)/(Q(x))^{p/p^*}$ will contradict our assumption. It follows that $\|u\|_{p^*,Q}^{p^*} = 1$ and so the proof is complete. $\qquad\square$

We now give the example mentioned above.

EXAMPLE 8.5. Let $k = 1$ and $H = Q^{p/p^*}$. We shall construct a Q such that

$$I < S \inf_{x\in\mathbb{R}^N} \frac{H(x)}{(Q(x))^{p/p^*}} = S.$$

Set $u(x) = (1 + |x|^{p/(p-1)})^{-N/p^*}$, so that $|\nabla u|^p = C|x|^{p/(p-1)}|u|^{p^*}$ and

$$S = \frac{\int |\nabla u|^p}{(\int |u|^{p^*})^{p/p^*}}.$$

For some $\eta > 0$, let $1 \leq Q(x) \leq 1 + \eta$ and set $Q(x) = 1$ if $|x| > 2\delta$, $Q(x) = 1 + \eta$ if $|x| < \delta$. We shall show that $\delta > 0$ can be chosen such that

(8.3) $$I \leq \frac{\int Q^{p/p^*}|\nabla u|^p}{(\int Q|u|^{p^*})^{p/p^*}} < S.$$

We have

$$\int Q^{p/p^*}|\nabla u|^p = \int_{|x|<\delta}(1+\eta)^{p/p^*}|\nabla u|^p + \int_{\delta<|x|<2\delta}Q^{p/p^*}|\nabla u|^p + \int_{2\delta<|x|}|\nabla u|^p$$

$$= \int_{|x|<\delta}(1+\eta)^{p/p^*}|\nabla u|^p + \int_{\delta<|x|<2\delta}Q^{p/p^*}|\nabla u|^p + \int|\nabla u|^p - \int_{|x|<2\delta}|\nabla u|^p$$

$$= \int_{|x|<\delta}((1+\eta)^{p/p^*}-1)C|x|^{p/(p-1)}|u|^{p^*}$$

$$\quad + \int_{\delta<|x|<2\delta}(Q^{p/p^*}-1)C|x|^{p/(p-1)}|u|^{p^*} + S\Big(\int|u|^{p^*}\Big)^{p/p^*}$$

$$\leq \frac{p}{p^*}\eta C\delta^{p/(p-1)}\int_{|x|<\delta}|u|^{p^*} + C\frac{p}{p^*}\eta(2\delta)^{p/(p-1)}\int_{\delta<|x|<2\delta}|u|^{p^*} + S\Big(\int|u|^{p^*}\Big)^{p/p^*}$$

$$\leq \frac{p}{p^*}\eta C\delta^{p/(p-1)}\int_{|x|<\delta}|u|^{p^*} + C_1\frac{p}{p^*}\eta(2\delta)^{p/(p-1)}\int_{|x|<\delta}|u|^{p^*} + S\Big(\int|u|^{p^*}\Big)^{p/p^*}.$$

We have used the inequalities

$$(1+\eta)^{p/p^*} \leq 1 + \frac{p}{p^*}\eta \quad \text{and} \quad \int_{\delta<|x|<2\delta}|u|^{p^*} \leq C_2\int_{|x|<\delta}|u|^{p^*}.$$

The second one follows easily from the fact that u is decreasing in $|x|$. Also,

$$\Big(\int Q|u|^{p^*}\Big)^{p/p^*} = \Big(\eta\int_{|x|<\delta}|u|^{p^*} + \int_{\delta<|x|<2\delta}(Q-1)|u|^{p^*} + \int|u|^{p^*}\Big)^{p/p^*}$$

$$\geq \Big(\eta\int_{|x|<\delta}|u|^{p^*} + \int|u|^{p^*}\Big)^{p/p^*}.$$

Taylor expansion of $f(x) = x^{p/p^*}$ about $\int|u|^{p^*}$ gives

$$\Big(\int Q|u|^{p^*}\Big)^{p/p^*} \geq \Big(\int|u|^{p^*}\Big)^{p/p^*} + \frac{p}{p^*}\Big(\int|u|^{p^*}\Big)^{p/p^*-1}\eta\int_{|x|<\delta}|u|^{p^*}$$

$$+ o\Big(\eta\int_{|x|<\delta}|u|^{p^*}\Big).$$

So, we see that (8.3) holds if we can show that

$$\frac{p}{p^*}\eta C\delta^{p/(p-1)}\int_{|x|<\delta}|u|^{p^*} + C_1\frac{p}{p^*}\eta(2\delta)^{p/(p-1)}\int_{|x|<\delta}|u|^{p^*}$$

$$< S\frac{p}{p^*}\Big(\int|u|^{p^*}\Big)^{p/p^*-1}\eta\int_{|x|<\delta}|u|^{p^*} + o\Big(\eta\int_{|x|<\delta}|u|^{p^*}\Big).$$

Since $\int_{|x|<\delta}|u|^{p^*} = o(1)$, the above inequality can be re-written in the form $A_1\delta^{p/(p-1)} < A_2 + o(1)$. Hence it suffices to choose $\delta > 0$ small enough.

Remark 8.6. The above theorem together with the example shows a rather surprising fact regarding the embedding $D^{k,p}(\mathbb{R}^N) \hookrightarrow L^{p^*}(\mathbb{R}^N, Q)$. In Section 5

we saw that in general not all minimizing sequences are relatively compact if the norms (2.1) or (2.2) are used in $D^{k,p}(\mathbb{R}^N)$. But, for some Q there exists an equivalent norm in $D^{k,p}(\mathbb{R}^N)$ so that all minimizing sequences are relatively compact.

Returning to inequality (8.2) we see that if

$$\inf_{x\in\mathbb{R}^N} \frac{H(x)}{(Q(x))^{p/p^*}} = \frac{\overline{H}}{\|Q\|_\infty^{p/p^*}}$$

then the hypothesis of Theorem 8.4 cannot be satisfied. In this case minimizing sequences are not relatively compact and minimizers do not exist. More precisely, we have the following proposition which of course is a straight forward generalization of the observations made in Section 5.

PROPOSITION 8.7. *Suppose that* $I = S\overline{H}/\|Q\|_\infty^{p/p^*}$. *If* $E_Q = \{x \in \mathbb{R}^N : \|Q\|_\infty = Q(x)\}$ *or* $E_H = \{x \in \mathbb{R}^N : \overline{H} = H(x)\}$ *has measure zero, then there are no minimizers to problem* (8.1).

PROOF. We argue as we did in Proposition 5.2. Suppose that $I = S\overline{H}/\|Q\|_\infty^{p/p^*}$ and $u \in D^{k,p}(\mathbb{R}^N)$ is a minimizer for I. Then

$$\left(\int Q|u|^{p^*}\right)^{p/p^*} \leq \|Q\|_\infty^{p/p^*}\left(\int |u|^{p^*}\right)^{p/p^*} \leq \frac{S^{-1}\|Q\|_\infty^{p/p^*}}{\overline{H}}\int \overline{H}|\nabla^k u|^p$$

$$= I^{-1}\int \overline{H}|\nabla^k u|^p \leq I^{-1}\int H|\nabla^k u|^p = \left(\int Q|u|^{p^*}\right)^{p/p^*}$$

So it follows that $\int(\|Q\|_\infty - Q)|u|^{p^*}\,dx = 0$ and $\int(H - \overline{H})|\nabla^k u|^p = 0$. Hence there are no minimizers. $\square$

REMARK 8.8. Combining the above proposition and Theorem 8.4 gives another interesting result. Suppose that $E_Q = \{x \in \mathbb{R}^N : \|Q\|_\infty = Q(x)\}$ or $E_H = \{x \in \mathbb{R}^N : \overline{H} = H(x)\}$ has measure zero, then $I < S\inf_{x\in\mathbb{R}^N} H(x)/(Q(x))^{p/p^*}$ implies $I > S\overline{H}/\|Q\|_\infty^{p/p^*}$.

Now we turn to the problem of finding symmetric minimizers. Assuming that H and Q are G-invariant, we consider the following problem

(8.4) $I_G = \inf\{\|u\|_{k,p,H}^p : u \in D_G^{k,p}(\mathbb{R}^N),\ \|u\|_{p^*,Q} = 1\},$

where $\|u\|_{k,p,H} = \|(-\Delta)^{k/2}u\|_{p,H}$ when k is even and $\|\nabla(-\Delta)^{(k-1)/2}u\|_{p,H}$ when k is odd. We can now state the conditions under which a minimizer to the above problem exists. We use the same notation for H as we do for Q.

THEOREM 8.9. *If* $I_G < S\inf_{x\in\mathbb{R}^N\cup\{\infty\}} |G_x|^{1-p/p^*}H(x)Q(x)^{-p/p^*}$ *then the infimum in* (8.4) *is attained.*

The above theorem is a straight forward generalization of Theorem 6.1 and the proof is an obvious adaptation of that of Theorem 6.1. From the above

theorem we can immediately conclude that if $Q_0 = Q_\infty = 0$ and $|G_x| = \infty$ for $x \in \mathbb{R}^N - \{0\}$, then a minimizer to problem (8.4) exists. By using explicitly the properties of the minimizers of problem (3.3) we can state explicit conditions on H and Q so that the minimizer of problem (8.4) exists.

The following corollaries are generalizations of Corollaries 7.7 and 7.8.

COROLLARY 8.10. *Assume that H and Q are G-symmetric functions,*

$$\frac{H_0}{Q_0^{p/p^*}} = \inf_{x \in \mathbb{R}^N \cup \{\infty\}} |G_x|^{1-p/p^*} H(x) Q(x)^{-p/p^*}$$

and $H_0 = \sup H$. If either

(a) $Q(x) \geq Q_0 + \varepsilon |x|^{N/(p-1)}$ *for some $\varepsilon > 0$ and $|x|$ small or*

(b) $|Q(x) - Q_0| \leq C|x|^\alpha$ *for some constant $C > 0$, $\alpha > N/(p-1)$, $|x|$ small and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-pN/(p-1)} \, dx > 0$$

then there exists a minimizer for problem (8.4) with $p > 1$ and $k = 1$.

PROOF. We know that the instanton $v(x) = (1 + |x|^{p/(p-1)})^{-N/p^*}$ is the unique minimizer for problem (3.3) with $k = 1$ and $p > 1$, up to translation and dilation. In view of Theorem 8.9, we need to show that for some $\eta > 0$,

$$I_G \leq \frac{\int H|\nabla A(v(x/\eta))|^p}{(\int Q|Av(x/\eta)|^{p^*})^{p/p^*}} < \frac{H_0 \int |\nabla(Av(x/\eta))|^p}{(Q_0 \int |Av(x/\eta)|^{p^*})^{p/p^*}} = S \frac{H_0}{Q_0^{p/p^*}},$$

where $A > 0$ is a constant chosen so that $\|Av(x/\eta)\|_{1,p} = 1$. Since

$$\int H|\nabla(Av(x/\eta))|^p \leq H_0 \int |\nabla(Av(x/\eta))|^p,$$

it suffices to show that, for some $\eta > 0$,

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)\left(\frac{1}{\eta^{p/(p-1)} + |x|^{p/(p-1)}}\right)^N > 0.$$

The proof is as in Corollary 7.2 (cf. Corollary 7.8). $\qquad\square$

COROLLARY 8.11. *Assume that H and Q are G-symmetric functions,*

$$\frac{H_\infty}{Q_\infty^{p/p^*}} = \inf_{x \in \mathbb{R}^N \cup \{\infty\}} |G_x|^{1-p/p^*} H(x) Q(x)^{-p/p^*}$$

and $H_\infty = \sup H$. If either

(a) $Q(x) \geq Q_\infty + \varepsilon |x|^{-N}$ *for some $\varepsilon > 0$ and $|x|$ large or*

(b) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ *for some constant $C > 0$, $\alpha > N$, $|x|$ large and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty) \, dx > 0,$$

then there exists a minimizer for problem (8.4) with $p > 1$ and $k = 1$.

PROOF. The instanton $v(x) = (1 + |x|^{p/(p-1)})^{-N/p^*}$ is the unique minimizer for problem (3.3) with $k = 1$ and $p > 1$, up to translation and dilation. In view of Theorem 8.9, we have to show that for some $\eta > 0$

$$I_G \leq \frac{\int H |\nabla A(v(x/\eta))|^p}{(\int Q |Av(x/\eta)|^{p^*})^{p/p^*}} < \frac{H_\infty \int |\nabla(Av(x/\eta))|^p}{(Q_\infty \int |Av(x/\eta)|^{p^*})^{p/p^*}} = S \frac{H_\infty}{Q_\infty^{p/p^*}},$$

where the $A > 0$ is a constant chosen such that $\|Av(x/\eta)\|_{1,2} = 1$. Since $\int H |\nabla A(v(x/\eta))|^p \leq H_\infty \int |\nabla(Av(x/\eta))|^p$, it suffices to show that for some $\eta > 0$

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty) \left(\frac{1}{1 + |x/\eta|^{p/(p-1)}} \right)^N > 0.$$

The proof is as in Corollary 7.2 (cf. Corollary 7.8). $\qquad\qquad\square$

We see that similar proofs to the ones given for the two preceding corollaries above is valid even when $p = 2$ and $k \geq 1$, and so we have

CORLLARY 8.12. *Assume that H and Q are G-symmetric functions,*

$$\frac{H_0}{Q_0^{2/2^*}} = \inf_{x \in \mathbb{R}^N \cup \{\infty\}} |G_x|^{1 - 2/2^*} H(x) Q(x)^{-2/2^*}$$

and $H_0 = \sup H$, $2^* = 2N/(N - 2k)$. If either

(a) $Q(x) \geq Q_0 + \varepsilon |x|^N$ *for some $\varepsilon > 0$ and $|x|$ small or*

(b) $|Q(x) - Q_0| \leq C |x|^\alpha$ *for some constant $C > 0$, $\alpha > N$, $|x|$ small and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_0) |x|^{-2N} \, dx > 0.$$

Then there exists a minimizer for problem (8.4) with $p = 2$ and $k \geq 1$.

CORLLARY 8.13. *Assume that H and Q are G-symmetric functions,*

$$\frac{H_\infty}{Q_\infty^{2/2^*}} = \inf_{x \in \mathbb{R}^N \cup \{\infty\}} |G_x|^{1 - 2/2^*} H(x) Q(x)^{-2/2^*}$$

and $H_\infty = \sup H$, $2^* = 2N/(N - 2k)$. If either

(a) $Q(x) \geq Q_\infty + \varepsilon |x|^{-N}$ *for some $\varepsilon > 0$ and $|x|$ large or*

(b) $|Q(x) - Q_\infty| \leq C |x|^{-\alpha}$ *for some constant $C > 0$, $\alpha > N$, $|x|$ large and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty) \, dx > 0.$$

Then there exists a minimizer for problem (8.4) with $p = 2$ and $k \geq 1$.

9. Singular weights

Let $D_a^{1,2}(\mathbb{R}^N)$ be the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\left(\int_{\mathbb{R}^N} ||x|^{-a} \nabla u|^2 \, dx \right)^{1/2}.$$

We define

$$(9.1) \qquad S(a,b) := \inf_{\substack{u \in D_a^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} ||x|^{-a} \nabla u|^2 \, dx}{(\int_{\mathbb{R}^N} ||x|^{-b} u|^p \, dx)^{2/p}},$$

$$(9.2) \qquad S(a,b,\lambda) := \inf_{\substack{u \in D_a^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} ||x|^{-a} \nabla u|^2 + \lambda ||x|^{-(a+1)} u|^2 \, dx}{(\int_{\mathbb{R}^N} ||x|^{-b} u|^p \, dx)^{2/p}},$$

where $N \geq 3$, $0 \leq a < (N-2)/2$, $a \leq b < a+1$,

$$p = p(a,b) := \frac{2N}{N-2+2(b-a)}$$

and λ is a negative parameter. Due to an inequality by L. Caffarelli, R. Kohn and L. Nirenberg [7] $S(a,b)$ and $S(a,b,\lambda)$ are positive for $a \leq b \leq a+1$ and suitable λ (see [33]).

The first problem was studied in [20] when $a = 0$, and for positive a it was studied in [10]. There one can also find an explicit form of the minimizer. Both problems were then studied in [33] by using a different method. There the authors proved the existence of minimizers provided $-S(a,a+1) < \lambda < 0$. More results can also be found in [8]. Due to these results, the method we have developed in the previous sections allows us now to study

$$(9.3) \qquad I(a,b) = \inf_{\substack{u \in D_a^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} ||x|^{-a} \nabla u|^2 \, dx}{(\int_{\mathbb{R}^N} Q ||x|^{-b} u|^p \, dx)^{2/p}},$$

$$(9.4) \qquad I(a,b,\lambda) = \inf_{\substack{u \in D_a^{1,2}(\mathbb{R}^N) \\ u \neq 0}} \frac{\int_{\mathbb{R}^N} ||x|^{-a} \nabla u|^2 + \lambda ||x|^{-(a+1)} u|^2 \, dx}{(\int_{\mathbb{R}^N} Q ||x|^{-b} u|^p \, dx)^{2/p}}.$$

In a recent paper by Y. Deng and L. Jin [11] the authors studied the second problem when $a = 0$ and Q is G-symmetric. Our method will allow us to improve the results given in [11] for the case $b > 0$. We mention here that the above problems are delicate when $a = b$ since then we are dealing with the critical Sobolev exponent.

In our present work, we are mainly interested in the case when Q is G-symmetric, but as an illustration of the advantage of our method, we give the following simple result. Since problems (9.1) and (9.2) are dilation invariant, we

have by the same argument as in the beginning of the previous section that

$$\frac{S(a,b)}{\|Q\|_\infty^{2/p}} \le I(a,b) \le \min\{(Q_0)^{-2/p}, (Q_\infty)^{-2/p}\}S(a,b),$$

$$\frac{S(a,b,\lambda)}{\|Q\|_\infty^{2/p}} \le I(a,b,\lambda) \le \min\{(Q_0)^{-2/p}, (Q_\infty)^{-2/p}\}S(a,b,\lambda)$$

provided $Q_\infty = \lim_{|x|\to\infty} Q(x)$ exists (see the argument following Remark 6.2; it will become clear later why we have neglected $Q(x)$ for other x). This shows that the assumption of the following proposition is satisfied by some Q. With this in mind, we state conditions under which minimizers to problems (9.3) and (9.4) will not exist.

PROPOSITION 9.1. *If*

$$I(a,b) = \frac{S(a,b)}{\|Q\|_\infty^{2/p}} \quad or \quad I(a,b,\lambda) = \frac{S(a,b,\lambda)}{\|Q\|_\infty^{2/p}}$$

and if $E = \{x \in \mathbb{R}^N : Q(x) = \|Q\|_\infty\}$ *has measure zero then there are no minimizers, respectively for* $I(a,b)$ *and* $I(a,b,\lambda)$.

PROOF. The argument is the same as in Proposition 8.7 but somewhat simpler. $\square$

Assume now that Q is a G-symmetric function. Denote by $D_{a,G}^{1,2}(\mathbb{R}^N)$ the subspace of $D_a^{1,2}(\mathbb{R}^N)$ consisting of G-symmetric functions. $S_G(a,b)$, $S_G(a,b,\lambda)$, $I_G(a,b)$ and $I_G(a,b,\lambda)$ will denote the infima as in (9.1)–(9.4), but with $D_a^{1,2}(\mathbb{R}^N)$ replaced by $D_{a,G}^{1,2}(\mathbb{R}^N)$. Of course we have a similar result to Proposition 9.1 with identical proof, in this symmetric case.

PROPOSITION 9.2. *If*

$$I_G(a,b) = \frac{S_G(a,b)}{\|Q\|_\infty^{2/p}}, \qquad I_G(a,b,\lambda) = \frac{S_G(a,b,\lambda)}{\|Q\|_\infty^{2/p}}$$

and if $E = \{x \in \mathbb{R}^N : Q(x) = \|Q\|_\infty\}$ *has measure zero then there are no minimizers for* $I_G(a,b)$ *and* $I_G(a,b,\lambda)$.

We start by stating one more version of the concentration-compactness lemma. When G is the group consisting of only the identity element, the lemma and its proof can be found in [33].

LEMMA 9.3 (Concentration-compactness lemma). *Assume that* Q *is a G-symmetric continuous, bounded function and let* $N \ge 3$, $0 \le a < (N-2)/2$, $a \le b < a+1$, $p = p(a,b)$ *and* $-I(a,a+1) < \lambda$. *Let* $\{u_n\}_{n=1}^\infty \subset D_{a,G}^{1,2}(\mathbb{R}^N)$ *be*

a sequence such that

$$u_n \rightharpoonup u \quad in \ D^{1,2}_{a,G}(\mathbb{R}^N),$$

$$|||x|^{-a}\nabla(u_n - u)|^2 \overset{*}{\rightharpoonup} \mu \quad in \ M(\mathbb{R}^N),$$

$$Q|||x|^{-b}(u_n - u)|^p \overset{*}{\rightharpoonup} \nu \quad in \ M(\mathbb{R}^N),$$

$$|||x|^{-a}\nabla(u_n - u)|^2 + \lambda|||x|^{-(a+1)}u - u_n|^2 \overset{*}{\rightharpoonup} \gamma \quad in \ M(\mathbb{R}^N),$$

$$u_n \to u \quad a.e. \ on \ \mathbb{R}^N,$$

and define

$$\mu_\infty := \lim_{R \to \infty} \overline{\lim_{n \to \infty}} \int_{|x|>R} |||x|^{-a}\nabla u|^2,$$

$$\nu_\infty := \lim_{R \to \infty} \overline{\lim_{n \to \infty}} \int_{|x|>R} Q|||x|^{-b}u|^p,$$

$$\gamma_\infty := \lim_{R \to \infty} \overline{\lim_{n \to \infty}} \int_{|x|>R} |||x|^{-a}\nabla u|^2 + \lambda|||x|^{-(a+1)}u|^2.$$

Then it follows that

(9.5)
$$\|\nu\|^{2/p} \le I_G(a,b)^{-1}\|\mu\|,$$

$$\|\nu\|^{2/p} \le I_G(a,b,\lambda)^{-1}\|\gamma\|,$$

(9.6)
$$\nu_\infty^{2/p} \le I_G(a,b)^{-1}\mu_\infty,$$

$$\nu_\infty^{2/p} \le I_G(a,b,\lambda)^{-1}\gamma_\infty,$$

$$\overline{\lim_{n \to \infty}} \, \||x|^{-a}\nabla u_n\|_2^2 = \||x|^{-a}\nabla u\|_2^2 + \|\mu\| + \mu_\infty,$$

$$\overline{\lim_{n \to \infty}} \, \||x|^{-a}\nabla u_n\|_2^2 + \lambda\||x|^{-(a+1)}u_n\|_2^2$$

$$= \||x|^{-a}\nabla u\|_2^2 + \lambda\||x|^{-(a+1)}u\|_2^2 + \|\gamma\| + \gamma_\infty,$$

$$\overline{\lim_{n \to \infty}} \, \||x|^{-b}u_n\|_{p,Q}^p = \||x|^{-b}u\|_{p,Q}^p + \|\nu\| + \nu_\infty.$$

Further, suppose $u = 0$, *then* $\|\nu\|^{2/p} = I_G(a,b)^{-1}\|\mu\|$ *implies that* ν, μ *are concentrated at a single orbit and* $\|\nu\|^{2/p} = I_G(a,b,\lambda)^{-1}\|\gamma\|$ *implies that* ν, γ *are concentrated at a single orbit.*

The proof is similar to that of Lemma 4.3, keeping in mind that $D^{1,2}_{a,G}(\mathbb{R}^N)$ is a Hilbert space, and so Remark 4.5 is applicable. The only technical point is the verification of a result similar to Proposition 4.1. This can be easily deduced by using the following lemma, which is actually similar to Lemma 2 in [33] and its proof is easily adapted.

LEMMA 9.4. *Let* $N \geq 3$ *and* $0 \leq a < (N-2)/2$. *If* $u_n \rightharpoonup u$ *in* $D_{a,G}^{1,2}(\mathbb{R}^N)$ *then* $|x|^{-a} u_n \to |x|^{-a} u$ *in* $L_{loc}^2(\mathbb{R}^N)$.

REMARK 9.5. If $\{u_n\}_{n=1}^\infty \subset D_{a,G}^{1,2}(\mathbb{R}^N)$ is a bounded sequence such that $Q||x|^{-b}(u_n - u)|^p \overset{*}{\rightharpoonup} \nu$ then we may assume that $||x|^{-b}(u_n - u)|^p \overset{*}{\rightharpoonup} \alpha$, for some α. Hence, by defining α_∞ in the way ν_∞ is defined, we see that $\nu(\{x\}) = Q(x)\alpha(\{x\})$ and $\nu_\infty \leq Q_\infty \alpha_\infty$ where $Q_\infty = \overline{\lim}_{|x| \to \infty} Q(x)$. Further, $\nu_\infty = Q_\infty \alpha_\infty$ if $Q_\infty = \overline{\lim}_{|x| \to \infty} Q(x) = \lim_{|x| \to \infty} Q(x)$.

REMARK 9.6. Our present problem is somewhat different from the previous ones. This is due to the fact that the quotient associated with $S(a,b)$ is not translation invariant. To understand the effect of this, we choose $u \in C_0^\infty(\mathbb{R}^N)$ and see that

$$\frac{\int_{\mathbb{R}^N} ||x|^{-a} \nabla u_\varepsilon^y|^2 \, dx}{(\int_{\mathbb{R}^N} ||x|^{-b} u_\varepsilon^y|^p \, dx)^{2/p}} = \frac{\varepsilon^{-2(b-a)} \int_{\mathbb{R}^N} ||\varepsilon x + y|^{-a} \nabla u|^2 \, dx}{(\int_{\mathbb{R}^N} ||\varepsilon x + y|^{-b} u|^p \, dx)^{2/p}},$$

where $u_\varepsilon^y(x) = u((x-y)/\varepsilon)$. Hence if $b > a$ and $y \neq 0$ the quotient turns to infinity as ε turns to zero. This also shows the different nature of the problem when $b = a$.

With reference to Lemma 9.3 we have the following lemma.

LEMMA 9.7. *Assume that* $\{u_n\}$ *is a minimizing sequence for* $S_G(a,b)$ *such that* $\|\nu\|^{2/p} = S_G(a,b)^{-1}\|\mu\|$. *Then concentration can only occur at zero and infinity.*

PROOF. Assume first that G is the group consisting of only the identity element and concentration occurs at some $y \neq 0$.

We may assume that $\int ||x|^{-b} u_n|^p = 1$. From Lemma 9.3 we have that $||x|^{-b} u_n|^p \overset{*}{\rightharpoonup} \delta_y$ and $||x|^{-a} \nabla u_n|^2 \overset{*}{\rightharpoonup} S(a,b)\delta_y$ in the sense of measures.

Now, for any $\varepsilon > 0$,

$$\frac{\int_{\mathbb{R}^N} ||x|^{-a} \nabla u_n|^2 \, dx}{(\int_{\mathbb{R}^N} ||x|^{-b} u_n|^p \, dx)^{2/p}} < S(a,b) + \varepsilon$$

provided n is large. Also, for any $\eta > 0$ there is a $\lambda(\eta) > 0$ such that $|y|^{-2a} - \eta < |x|^{-2a}$ and $|x|^{-bp} < |y|^{-bp} + \eta$ for $x \in B(y, \lambda(\eta))$. Choose $\lambda < \lambda(\eta)$ and let $\phi \in C_0^\infty(B(y, \lambda))$ be a radially symmetric function about y with $0 \leq \phi \leq 1$ and $\phi = 1$ in $B(y, \lambda/2)$. From our assumptions we have

$$\lim_{n \to \infty} \int |\phi|^p ||x|^{-b} u_n|^p = 1$$

and since $u_n \to 0$ in $L^2(B(y,\lambda))$ the first two integrals on the right hand side of the following equality turn to zero

$$\int ||x|^{-a}\nabla(\phi u_n)|^2 = \int |x|^{-2a}u_n^2|\nabla\phi|^2$$
$$+ 2\int |x|^{-2a}u_n\phi\nabla\phi\cdot\nabla u_n + \int |x|^{-2a}\phi^2|\nabla u_n|^2.$$

Hence

$$\lim_{n\to\infty}\int ||x|^{-a}\nabla v_n|^2 = S(a,b),$$

where $v_n = \phi u_n$. By using Sobolev's inequality and then Hölder's inequality we arrive at

$$(S(a,b)+\varepsilon)\left((|y|^{-bp}+\eta)\int |v_n|^p\,dx\right)^{2/p} > (|y|^{-2a}-\eta)\int |\nabla v_n|^2$$
$$\geq (|y|^{-2a}-\eta)S\|v_n\|_{2^*}^2 \geq (|y|^{-2a}-\eta)S\|v_n\|_p^2|B(y,\lambda)|^{2/2^*-2/p},$$

where n is large, $S = S(0,0)$, $2^* = p(0,0)$ and $|B(y,\lambda)|$ is the N dimensional Lebesgue measure of $B(y,\lambda)$. By letting n turn to ∞, we obtain

$$(S(a,b)+\varepsilon)(|y|^{-bp}+\eta)^{2/p} \geq (|y|^{-2a}-\eta)S|B(y,\lambda)|^{2/2^*-2/p}.$$

Since λ can be chosen arbitrarily small and $2^* > p$ for $b > a$ the above inequality leads to a contradiction. When $b = a$, $p = 2^*$ and from the above inequality we obtain $S(a,a) \geq S$. This however contradicts the fact, proven in [33], that $S(a,a) < S$. In the general case we know that the orbit of concentration G_y has finite cardinality. If y_i, $i = 1,\ldots,m$, are the distinct points of G_y, we choose small disjoint balls $B(y_i,\lambda)$. We then choose functions, radially symmetric about y_i, $\phi_i \in C_0^\infty(B(y_i,\lambda))$ with $0 \leq \phi_i \leq 1$ and $\phi_i = 1$ in $B(y_i,\lambda/2)$. One can now argue as above but instead of ϕ one uses the function $\phi_G = \sum_{i=1}^m \phi_i$, which can be assumed to be G-symmetric. $\qquad\square$

With a concentration-compactness lemma at our disposal, we may proceed to compare $I_G(a,b)$ and $S_G(a,b)$ as required by our method. We know from [10] that function

$$(9.7) \qquad u(x) = (1+|x|^{2a-bp+2})^{(N-2a-2)/(2a-bp+2)}$$

is, up to dilation and multiplication by a constant, a minimizer for $S(a,b)$. Since $S(a,b) \leq S_G(a,b)$ and the above minimizer is radially symmetric, we have $S(a,b) = S_G(a,b)$.

The following theorem is the main result of this section.

THEOREM 9.8. *If $I_G(a,b) < \min\{Q_0^{-2/p}, Q_\infty^{-2/p}\} S_G(a,b)$ then all minimizing sequences are relatively compact. In particular, there is a minimizer for $I_G(a,b)$.*

PROOF. The argument is similar to the ones given in the previous sections. Therefore we omit some details. Let $\{u_n\}_{n=1}^\infty \subset D_{a,G}^{1,2}(\mathbb{R}^N)$ be a minimizing sequence for $I_G(a,b)$. Going if necessary to a subsequence, still denoted by u_n, we may assume that the conditions of Lemma 9.3 are fulfilled. Hence

$$I_G(a,b) = \overline{\lim_{n\to\infty}} \||x|^{-a}\nabla u_n\|_2^2 = \||x|^{-a}\nabla u\|_2^2 + \|\mu\| + \mu_\infty$$

and

$$1 = \overline{\lim_{n\to\infty}} \|u_n\|_{p,Q}^p = \|u\|_{p,Q}^p + \|\nu\| + \nu_\infty.$$

So we have using inequalities (9.5) and (9.6)

$$
\begin{aligned}
I_G(a,b)(\|u\|_{p,Q}^p + \|\nu\| + \nu_\infty)^{2/p} &= \||x|^{-a}\nabla u\|_2^2 + \|\mu\| + \mu_\infty \\
&\geq I_G(a,b)((\|u\|_{p,Q}^p)^{2/p} + \|\nu\|^{2/p} + \nu_\infty^{2/p}).
\end{aligned}
$$

Since $p > 2$, we deduce that only one of the quantities $\|u\|_{p,Q}^p$, $\|\nu\|$ and ν_∞ is 1 and the other two are zero. If $\nu_\infty = 1$, we obtain a contradiction, since from Remark 9.5 and Lemma 9.3 we have

$$
\begin{aligned}
S_G(a,b)(\alpha_\infty)^{2/p} &> I_G(a,b)(Q_\infty \alpha_\infty)^{2/p} \\
&\geq I_G(a,b)(\nu_\infty)^{2/p} = \mu_\infty \geq S_G(a,b)(\alpha_\infty)^{2/p}.
\end{aligned}
$$

If $\|\nu\| = 1$ then $u = 0$ and $\|\nu\|^{2/p} = I_G(a,b)^{-1}\|\mu\|$ and so ν is concentrated at the origin by Lemma 9.7. Once again we obtain a contradiction since

$$
\begin{aligned}
S_G(a,b)(\alpha(\{0\}))^{2/p} &> I_G(a,b)(Q_0\alpha(\{0\}))^{2/p} \\
&= I_G(a,b)(\nu(\{0\}))^{2/p} = \|\mu\| \geq S_G(a,b)(\alpha(\{0\}))^{2/p}.
\end{aligned}
$$

So it follows that $\|u\|_{p,Q}^p = 1$ and we reach the desired conclusion. $\square$

Set $u_\eta(x) = u(x/\eta) = (1 + |x/\eta|^{2a-bp+2})^{(N-2a-2)/(2a-bp+2)}$, then

$$S_G(a,b) = \frac{\int_{\mathbb{R}^N} \||x|^{-a}\nabla u_\eta|^2\, dx}{\int_{\mathbb{R}^N} \||x|^{-b}u_\eta|^p\, dx} \quad \text{for all } \eta > 0.$$

If we assume that $Q_\infty = \lim_{|x|\to\infty} Q(x)$, then by placing the above function in the quotient associated with $I_G(a,b)$ and letting η tend to 0 and ∞, we obtain as before, $I_G(a,b) \leq \min\{Q_0^{-2/p}, Q_\infty^{-2/p}\} S_G(a,b)$. At this point we can easily deduce that if $\min\{Q_0^{-2/p}, Q_\infty^{-2/p}\} = \|Q\|_\infty^{-2/p}$ then by Proposition 9.2, minimizers in general will not exist. However, we have the following corollary to Theorem 9.8, which is similar to Corollary 6.5.

COROLLARY 9.9. *If Q is G-symmetric and*

$$Q(x) \geq Q_0 = Q_\infty = \lim_{|x| \to \infty} Q(x) \geq 0$$

then there is a minimizer for $I_G(a, b)$.

PROOF. When $Q_0 = Q_\infty = 0$ it is easy to see that concentration cannot occur at zero or infinity. So a minimizer exists. Assume $Q_0 = Q_\infty > 0$. If $I_G(a, b) < Q_0^{-2/p} S_G(a, b)$ then we are done by Theorem 9.8. If $I_G(a, b) = Q_0^{-2/p} S_G(a, b)$, let u be the function in (9.7). u is then a minimizer of $S_G(a, b)$, and

$$\begin{aligned}
I_G(a, b) &\leq \frac{\int_{\mathbb{R}^N} ||x|^{-a} \nabla u|^2 \, dx}{(\int_{\mathbb{R}^N} Q ||x|^{-b} u|^p \, dx)^{2/p}} \\
&\leq \frac{\int_{\mathbb{R}^N} ||x|^{-a} \nabla u|^2 \, dx}{Q_0^{2/p} (\int_{\mathbb{R}^N} ||x|^{-b} u|^p \, dx)^{2/p}} = \frac{S_G(a, b)}{Q_0^{2/p}} = I_G(a, b).
\end{aligned}$$

It follows that u is a minimizer of $I_G(a, b)$. $\qquad\square$

We note that a remark similar to Remark 6.6 is also applicable here.

Of course knowing the explicit form of the minimizer for $S_G(a, b)$ allows us to give conditions on Q, similar to those given in the previous sections, so that minimizers exist.

COROLLLARY 9.10. *Suppose that Q is G-symmetric, $Q_0 \geq Q_\infty \geq 0$ and either*

(a) $Q(x) \geq Q_0 + \varepsilon |x|^{N-bp}$ *for some $\varepsilon > 0$ and $|x|$ small or*
(b) $|Q(x) - Q_0| \leq C|x|^\alpha$ *for some constant $C > 0$, $\alpha > N - bp$, $|x|$ small and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_0) |x|^{-2N+bp} \, dx > 0.$$

Then there exists a minimizer for $I_G(a, b)$.

COROLLARY 9.11. *Suppose that Q is G-symmetric, $Q_\infty \geq Q_0 \geq 0$ and either*

(a) $Q(x) \geq Q_\infty + \varepsilon |x|^{-N+bp}$ *for some $\varepsilon > 0$ and $|x|$ large or*
(b) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ *for some constant $C > 0$, $\alpha > N - bp$, $|x|$ large and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty) |x|^{-bp} \, dx > 0.$$

Then there exists a minimizer for $I_G(a, b)$.

The proofs are similar to those of Corollaries 7.1 and 7.2.

Having established the resluts above we can prove the following result by using similar arguments.

Theorem 9.12. *If $I_G(a, b, \lambda) < \min\{Q_0^{-2/p}, Q_\infty^{-2/p}\} S_G(a, b, \lambda)$ then all minimizing sequences are relatively compact. In particular, there exists a minimizer for $I_G(0, b, \lambda)$.*

Corollary 9.13. *If Q is G-symmetric, $Q(x) \geq Q_0 = Q_\infty = \lim_{|x| \to \infty} Q(x) \geq 0$ then $I_G(a, b, \lambda)$ has a minimizer.*

In Deng's and Jin's article (see [11, Theorem 2.1]) the authors presented a result which in effect says that there exists a minimizer for $I_G(0, b, \lambda)$ provided that

$$I_G(0, b, \lambda) < \inf_{x \in \mathbb{R}^N \cup \{\infty\}} Q(x)^{-2/p} |G_x|^{1-2/p} S_G(0, b, \lambda).$$

We see that Theorem 9.12 improves this result for $b > 0$, since our condition does not require any knowledge of $|G_x|$.

In order to obtain explicit conditions on Q so that the assumption of Theorem 9.12 is satisfied, we require the explicit knowledge of the minimizer for $S_G(a, b, \lambda)$. This explicit form is not known to the author. However, in the case when $a = 0$ and $0 > \lambda > \bar{\lambda} = -((n-2)/2)^2$ we know from [11] that, up to multiplication by a constant and dilation, $S(0, b, \lambda)$ is achieved by

$$u(x) = \frac{1}{|x|^{\sqrt{-\bar{\lambda}} - \beta}(1 + |x|^{(2-bp)\beta/\sqrt{-\bar{\lambda}}})^{(N-2)/(2-bp)}},$$

where $\beta = (\lambda - \bar{\lambda})^{1/2}$. Since the above function is radially symmetric, we deduce that $S_G(0, b, \lambda)$ has a minimizer. We may now proceed to formulate explicit conditions on Q so that a minimizer for $I_G(0, b, \lambda)$ exists.

Corollary 9.14. *Suppose that Q is G-symmetric, $Q_0 \geq Q_\infty \geq 0$ and either*

(a) $Q(x) \geq Q_0 + \varepsilon |x|^{2\beta(N-bp)/(N-2)}$ *for some $\varepsilon > 0$ and $|x|$ small or*
(b) $|Q(x) - Q_0| \leq C|x|^\alpha$ *for some constant $C > 0$, $\alpha > 2\beta(N - bp)/(N - 2)$, $|x|$ small and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_0)|x|^{-N-2\beta(N-bp)/(N-2)}\, dx > 0.$$

Then there exists a minimizer for $I_G(0, b, \lambda)$.

Corllary 9.15. *Suppose that Q is G-symmetric $Q_\infty \geq Q_0 \geq 0$ and either*

(a) $Q(x) \geq Q_\infty + \varepsilon |x|^{-2\beta(N-bp)/(N-2)}$ *for some $\varepsilon > 0$ and $|x|$ large or*
(b) $|Q(x) - Q_\infty| \leq C|x|^{-\alpha}$ *for some constant $C > 0$, $\alpha > 2\beta(N - bp)/(N - 2)$, $|x|$ large and*

$$\int_{\mathbb{R}^N} (Q(x) - Q_\infty)|x|^{n-2\beta(N-bp)/(N-2)}\, dx > 0.$$

Then there exists a minimizer for $I_G(0, b, \lambda)$.

The proofs are similar to those of Corollaries 7.1 and 7.2.

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Shoyeb Waliullah
Department of Mathematics
Stockholm University
106 91 Stockholm, SWEDEN

E-mail address: shoyeb@math.su.se