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EXISTENCE OF POSITIVE SOLUTIONS TO SYSTEMS OF NONLINEAR INTEGRAL OR DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we are concerned with existence of positive solutions for systems of nonlinear Hammerstein integral equations, in which one nonlinear term is superlinear and the other is sublinear. The discussion is based on the product formula of fixed point index on product cone and fixed point index theory in cones. As applications, we consider existence of positive solutions for systems of second-order ordinary differential equations with different boundary conditions.

1. Introduction

In this paper, we consider existence of positive solutions for the following system of nonlinear Hammerstein integral equations

$$\text{(S)} \qquad \begin{cases} u(x) = \displaystyle\int_{\Omega} k_1(x,y) f_1(y,u(y),v(y))\,dy & \text{for } x \in \overline{\Omega}, \\ v(x) = \displaystyle\int_{\Omega} k_2(x,y) f_2(y,u(y),v(y))\,dy & \text{for } x \in \overline{\Omega}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $k_i \in C(\overline{\Omega}\times\overline{\Omega}, \mathbb{R}^+)$, $f_i \in C(\overline{\Omega}\times\mathbb{R}^+\times\mathbb{R}^+, \mathbb{R}^+)$ $(i = 1, 2)$ and $\mathbb{R}^+ = [0, \infty)$.

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Definition 1.1. Let $C^+(\overline{\Omega}) = \{u \in C(\overline{\Omega}) \mid u(x) \geq 0$, for all $x \in \overline{\Omega}\}$, we say that (u,v) is one positive solution, if $(u,v) \in [C^+(\overline{\Omega}) \setminus \{\theta\}] \times [C^+(\overline{\Omega}) \setminus \{\theta\}]$ satisfies system (S).

The study of nonlinear Hammerstein integral equations was initiated by Hammerstein, see [4]. Subsequently a number of papers have dealt with existence of nontrivial solutions of nonlinear Hammerstein integral equations, see the monographs [3], [5] and the references therein. To the best of our knowledge, the papers dealing with existence of nontrivial solutions, especially positive solutions for system (S) are few, see [7]–[10]. They mainly studied existence of nontrivial or nonnegative solutions for systems of nonlinear Hammerstein integral equations by use of topological methods or fixed point index theory in cones.

Recently, in [1] Cheng and Zhong considered existence of positive solutions for a super-sublinear system of second-order ordinary differential equations by applying the product formula of fixed point index on product cone and fixed point index theory in cones. Motivated by these works, we shall deal with existence of positive solutions for system (S), in which one nonlinear term is superlinear and the other is sublinear. As applications, we consider existence of positive solutions to systems of second-order ordinary differential equations with superlinear and sublinear nonlinearities under different boundary conditions.

Throughout this paper, we suppose that kernel functions $k_i(x,y)$ $(i=1,2)$ satisfy the following conditions:

(i) $k_i(x,y) = k_i(y,x)$, for all $x, y \in \overline{\Omega}$;

(ii) there exist $p_i \in C(\overline{\Omega})$, $0 \leq p_i(x) \leq 1$ such that

$$k_i(x,y) \geq p_i(x) k_i(z,y), \quad \text{for all } x, y, z \in \overline{\Omega};$$

(iii) $\max_{x \in \overline{\Omega}} \int_{\overline{\Omega}} k_i(x,y) p_i(y)\, dy$ is positive.

Lemma 1.2. *Let $B_i \colon C(\overline{\Omega}) \to C(\overline{\Omega})$ be defined by*

$$B_i u(x) = \int_{\overline{\Omega}} k_i(x,y) u(y)\, dy, \quad i = 1,2.$$

Then the spectral radius of B_i, $r(B_i)$ is positive.

Proof. From the definition of B_i and the conditions about k_i, we have

$$B_i p_i(x) = \int_{\overline{\Omega}} k_i(x,y) p_i(y)\, dy \geq p_i(x) \int_{\overline{\Omega}} k_i(z,y) p_i(y)\, dy, \quad x \in \overline{\Omega},$$

and

$$B_i p_i(x) \geq p_i(x) \|B_i p_i\|, \quad x \in \overline{\Omega},$$

here

$$\|B_i p_i\| = \max_{x \in \overline{\Omega}} \int_{\overline{\Omega}} k_i(x,y) p_i(y)\, dy > 0.$$

By induction, we get

$$B_i^n p_i(x) \geq p_i(x)\|B_i p_i\|^n, \quad x \in \overline{\Omega},$$

thus $\|B_i^n\| \geq \|B_i p_i\|^n$. From the formula of spectral radius, we know that

$$r(B_i) = \lim_{n\to\infty} \|B_i^n\|^{1/n} \geq \|B_i p_i\| > 0. \qquad \square$$

DEFINITION 1.3. If f_1, f_2 in system (S) satisfy the following assumptions:

$$(\mathrm{H}_1) \qquad \limsup_{u\to 0^+} \max_{x\in\overline{\Omega}} \frac{f_1(x,u,v)}{u} < \frac{1}{r(B_1)} \quad \text{uniformly w.r.t. } v \in \mathbb{R}^+;$$

$$(\mathrm{H}_2) \qquad \liminf_{v\to 0^+} \min_{x\in\overline{\Omega}} \frac{f_2(x,u,v)}{v} > \frac{1}{r(B_2)} \quad \text{uniformly w.r.t. } u \in \mathbb{R}^+,$$

then we say that f_1 is superlinear with respect to u at the origin and that f_2 is sublinear with respect to v at the origin.

DEFINITION 1.4. If f_1, f_2 in system (S) satisfy the following assumptions:

$$(\mathrm{H}_3) \qquad \liminf_{u\to\infty} \min_{x\in\overline{\Omega}} \frac{f_1(x,u,v)}{u} > \frac{1}{r(B_1)} \quad \text{uniformly w.r.t. } v \in \mathbb{R}^+;$$

$$(\mathrm{H}_4) \qquad \limsup_{v\to\infty} \max_{x\in\overline{\Omega}} \frac{f_2(x,u,v)}{v} < \frac{1}{r(B_2)} \quad \text{uniformly w.r.t. } u \in \mathbb{R}^+,$$

then we say that f_1 is superlinear with respect to u at infinity and that f_2 is sublinear with respect to v at infinity.

Our main result is the following.

THEOREM 1.5. *Assume that f_1 is superlinear w.r.t. u at the origin and infinity, f_2 is sublinear w.r.t. v at the origin and infinity and satisfies the following condition:*

(G) $\limsup_{u\to\infty} \max_{x\in\overline{\Omega}} f_2(x,u,v) = g(v)$ *uniformly w.r.t.* $v \in [0, M]$ (*for all* $M > 0$), *here g is a locally bounded function.*

Then system (S) *has at least one positive solution.*

REMARK 1.6. If f_1 is superlinear w.r.t. u at the origin and infinity and f_2 is also superlinear w.r.t. v at the origin and infinity, then system (S) has at least one positive solution, which can be seen from Steps 1 and 2 in the proof of our theorem.

REMARK 1.7. If f_1 is sublinear w.r.t. u at the origin and infinity and f_2 is also sublinear w.r.t. v at the origin and infinity, furthermore, there exist locally bounded functions g_1 and g_2 such that

$$\limsup_{v\to\infty} \max_{x\in\overline{\Omega}} f_1(x,u,v) = g_1(u)$$

uniformly w.r.t. $u \in [0, M]$ and

$$\limsup_{u\to+\infty} \max_{x\in\overline{\Omega}} f_2(x,u,v) = g_2(v)$$

uniformly w.r.t. $v \in [0, M]$ (for all $M > 0$), then system (S) has at least one positive solution, which can be seen from Steps 3 and 4 in the proof of our theorem.

The paper is organized as follows: in Section 2, we make some preliminaries; in Section 3, we prove our main result; in Section 4, as applications, we prove existence of positive solutions to systems of second order differential equations with different boundary conditions.

2. Preliminaries

In this section, we shall construct a cone which is the Cartesian product of two cones and change the problem (S) into a fixed point problem in the constructed cone. At the same time, we will give some useful preliminary results for the proof of our theorem.

It is well known that $C(\overline{\Omega})$ is a Banach space with the maximum norm $\|u\| = \max_{x\in\overline{\Omega}} |u(x)|$, and $C^+(\overline{\Omega})$ is a total cone of $C(\overline{\Omega})$. Choose bounded domains $\Omega_i \subset \Omega$ $(i = 1, 2)$ such that $\delta_i \overset{\text{def}}{=} \min_{x\in\overline{\Omega}_i} p_i(x) > 0$, which is feasible by the hypotheses of p_i. Now construct sub-cones and subsets as following:

$$K_i = \{u \in C^+(\overline{\Omega}) \mid u(x) \geq \delta_i\|u\|, \text{ for all } x \in \overline{\Omega}_i\},$$
$$K_{r_i} = \{u \in K_i \mid \|u\| < r_i\}, \quad \partial K_{r_i} = \{u \in K_i \mid \|u\| = r_i\}, \quad \text{for all } r_i > 0.$$

Noticing that B_i $(i = 1, 2)$ is completely continuous and positive, it follows from Lemma 1.2 and Krein–Rutman theorem (see [6]) that $r(B_i)$ is one of eigenvalues for B_i and there exist positive eigenfunctions corresponding to $r(B_i)$.

Lemma 2.1. *Let $\psi_i(x)$ be the positive eigenfunctions of B_i corresponding to $r(B_i)$ with $\int_{\overline{\Omega}} \psi_i(x)\,dx = 1$, then the following conclusions are valid:*

(a) *$\int_{\overline{\Omega}} \psi_i(x)u(x)\,dx \leq \|u\|$, for all $u \in K_i$;*
(b) *$\psi_i(x) \geq p_i(x)\|\psi_i\|$, for all $x \in \overline{\Omega}$;*
(c) *there exist constants $c_i > 0$ such that $\int_{\overline{\Omega}} \psi_i(x)u(x)\,dx \geq c_i\|u\|$, for all $u \in K_i$.*

Proof. (a) Obviously,

$$\int_{\overline{\Omega}} \psi_i(x)u(x)\,dx \leq \int_{\overline{\Omega}} \psi_i(x)\,dx \cdot \|u\| = \|u\|.$$

(b) Noticing that $k_i(x,y) \geq p_i(x)k_i(z,y)$, for all $x, y, z \in \overline{\Omega}$, we have

$$\int_{\overline{\Omega}} k_i(x,y)\psi_i(y)\,dy \geq \int_{\overline{\Omega}} p_i(x)k_i(z,y)\psi_i(y)\,dy, \quad \text{for all } x, z \in \overline{\Omega}$$

and

$$r(B_i)\psi_i(x) \geq r(B_i)p_i(x)\psi_i(z), \quad \text{for all } x, z \in \overline{\Omega},$$

which implies that $\psi_i(x) \geq p_i(x)\|\psi_i\|$, for all $x \in \overline{\Omega}$.

(c) It follows from (b) and the definition of K_i. □

For $\lambda \in [0,1]$, $u, v \in C^+(\overline{\Omega})$, we define the mappings:

$$\begin{gathered} T_{\lambda,1}(\,\cdot\,,\,\cdot\,), T_{\lambda,2}(\,\cdot\,,\,\cdot\,)\colon C^+(\overline{\Omega}) \times C^+(\overline{\Omega}) \to C^+(\overline{\Omega}), \\ T_\lambda(\,\cdot\,,\,\cdot\,)\colon C^+(\overline{\Omega}) \times C^+(\overline{\Omega}) \to C^+(\overline{\Omega}) \times C^+(\overline{\Omega}) \end{gathered}$$

by

$$\begin{aligned} T_{\lambda,1}(u,v)(x) &= \int_{\overline{\Omega}} k_1(x,y)[\lambda f_1(y,u(y),v(y)) + (1-\lambda) f_1(y,u(y),0)]\,dy, \\ T_{\lambda,2}(u,v)(x) &= \int_{\overline{\Omega}} k_2(x,y)[\lambda f_2(y,u(y),v(y)) + (1-\lambda) f_2(y,0,v(y))]\,dy \end{aligned}$$

and

$$T_\lambda(u,v)(x) = (T_{\lambda,1}(u,v)(x), T_{\lambda,2}(u,v)(x)).$$

It is obvious that *the existence of positive solutions of system* (S) *is equivalent to the existence of nontrivial fixed points of* T_1 *in* $K_1 \times K_2$.

To compute the fixed point index of T_1, we need the following results.

LEMMA 2.2. *$T_\lambda\colon K_1 \times K_2 \to K_1 \times K_2$ is completely continuous.*

PROOF. For $(u,v) \in K_1 \times K_2$, we show that $T_\lambda(u,v) \in K_1 \times K_2$, i.e. $T_{\lambda,1}(u,v) \in K_1$ and $T_{\lambda,2}(u,v) \in K_2$. By the above definitions and the conditions about $k_i(x,y)$, we have

$$\begin{aligned} T_{\lambda,1}(u,v)(x) &= \int_{\overline{\Omega}} k_1(x,y)[\lambda f_1(y,u(y),v(y)) + (1-\lambda) f_1(y,u(y),0)]\,dy \\ &\geq p_1(x) \int_{\overline{\Omega}} k_1(z,y)[\lambda f_1(y,u(y),v(y)) + (1-\lambda) f_1(y,u(y),0)]\,dy \\ &= p_1(x) T_{\lambda,1}(u,v)(z), \end{aligned}$$

for all $x, z \in \overline{\Omega}$, which implies that

$$T_{\lambda,1}(u,v)(x) \geq \delta_1 \|T_{\lambda,1}(u,v)\|, \quad x \in \overline{\Omega}_1.$$

Similarly,

$$T_{\lambda,2}(u,v)(x) \geq \delta_2 \|T_{\lambda,2}(u,v)\|, \quad x \in \overline{\Omega}_2.$$

Hence, $T_\lambda(K_1 \times K_2) \subset K_1 \times K_2$. By the Arzelà–Ascoli theorem, we know that $T_\lambda\colon K_1 \times K_2 \to K_1 \times K_2$ is completely continuous. □

REMARK 2.3. Denoting $T(\lambda,u,v)(x) = T_\lambda(u,v)(x)$, $\overline{T([0,1] \times K_{r_1} \times K_{r_2})}$ is a compact set by the Arzelà–Ascoli theorem.

Next, we recall some concepts about the fixed point index (see [2], [11]), which will be used in the proof of our theorem. Let X be a Banach space and let $P \subset X$ be a closed convex cone in X. Assume that W is a bounded open subset of X with boundary ∂W, and let $A: P \cap \overline{W} \to P$ be a completely continuous operator. If $Au \neq u$ for $u \in P \cap \partial W$, then the fixed point index $i(A, P \cap W, P)$ is defined. One important fact is that if $i(A, P \cap W, P) \neq 0$, then A has a fixed point in $P \cap W$. The following results are useful in our proof.

Lemma 2.4 ([2], [11]). *Let E be a Banach space and let $P \subset E$ be a closed convex cone in E. For $r > 0$, denote*

$$P_r = \{u \in P \mid \|u\| < r\}, \quad \partial P_r = \{u \in P \mid \|u\| = r\}.$$

Let $A: P \to P$ be completely continuous. Then the following conclusions are valid:

(a) *if $\mu A u \neq u$ for every $u \in \partial P_r$ and $\mu \in (0,1]$, then $i(A, P_r, P) = 1$;*
(b) *if mapping A satisfies the following two conditions:*
(b1) $\inf_{u \in \partial P_r} \|Au\| > 0$;
(b2) *$\mu A u \neq u$ for every $u \in \partial P_r$ and $\mu \geq 1$,*
then $i(A, P_r, P) = 0$.

Lemma 2.5 ([1]). *Let E be a Banach space and let $P_i \subset E$ $(i = 1, 2)$ be a closed convex cone in E. For $r_i > 0$ $(i = 1, 2)$, denote*

$$P_{r_i} = \{u \in P_i \mid \|u\| < r_i\}, \quad \partial P_{r_i} = \{u \in P_i \mid \|u\| = r_i\}.$$

Suppose $A_i: P_i \to P_i$ is completely continuous. If $u_i \neq A_i u_i$, for all $u_i \in \partial P_{r_i}$, then

$$i(A, P_{r_1} \times P_{r_2}, P_1 \times P_2) = i(A_1, P_{r_1}, P_1) \cdot i(A_2, P_{r_2}, P_2),$$

where $A(u, v) \stackrel{\text{def}}{=} (A_1 u, A_2 v)$, for all $(u, v) \in P_1 \times P_2$.

3. Proof of Theorem 1.5

We will choose a bounded open set $D = (K_{R_1} \setminus \overline{K_{r_1}}) \times (K_{R_2} \setminus \overline{K_{r_2}})$ in product cone $K_1 \times K_2$ and verify that a family of operators $\{T_\lambda\}_{\lambda \in I}$ satisfy the sufficient conditions for the homotopy invariance of fixed point index on ∂D. Next, we separate the proof into four steps.

Step 1. From the superlinear assumption of f_1 at the origin, there are $\varepsilon \in (0, 1/r(B_1))$ and $r_1 > 0$ such that

$$\lambda f_1(x, u, v) + (1 - \lambda) f_1(x, u, 0) \leq (1/r(B_1) - \varepsilon) u, \tag{3.1}$$

for all $x \in \overline{\Omega}$, $u \in [0, r_1]$ and $v \in \mathbb{R}^+$. We claim that

$$\mu T_{\lambda,1}(u, v) \neq u, \quad \text{for all } \mu \in (0, 1] \text{ and } (u, v) \in \partial K_{r_1} \times K_2. \tag{3.2}$$

In fact, if it is not true, then there exist $\mu_0 \in (0,1]$ and $(u_0, v_0) \in \partial K_{r_1} \times K_2$, such that $\mu_0 T_{\lambda,1}(u_0, v_0) = u_0$. In combination with (3.1), it follows that

$$u_0(x) \le T_{\lambda,1}(u_0, v_0)(x) \le \int_{\overline{\Omega}} k_1(x,y)(1/r(B_1) - \varepsilon) u_0(y)\, dy.$$

Multiplying the both sides of this inequality by $\psi_1(x)$ and integrating on $\overline{\Omega}$, we get that

$$\int_{\overline{\Omega}} u_0(x)\psi_1(x)\, dx \le \int_{\overline{\Omega}} \int_{\overline{\Omega}} k_1(x,y)(1/r(B_1) - \varepsilon) u_0(y)\psi_1(x)\, dy\, dx,$$

and

$$\int_{\overline{\Omega}} u_0(x)\psi_1(x)\, dx \le (1/r(B_1) - \varepsilon) \int_{\overline{\Omega}} \int_{\overline{\Omega}} k_1(x,y)\psi_1(x)\, dx\, u_0(y)\, dy,$$

that is

$$\int_{\overline{\Omega}} u_0(x)\psi_1(x)\, dx \le (1 - r(B_1)\varepsilon) \int_{\overline{\Omega}} u_0(y)\psi_1(y)\, dy.$$

Noticing that $\int_{\overline{\Omega}} u_0(x)\psi_1(x)\, dx > 0$, hence $1 \le 1 - r(B_1)\varepsilon$, which is a contradiction!

Step 2. By use of the superlinear hypothesis of f_1 at infinity, there exist $\varepsilon > 0$ and $m > 0$ such that

$$\lambda f_1(x,u,v) + (1-\lambda) f_1(x,u,0) \ge (1/r(B_1) + \varepsilon)u, \tag{3.3}$$

for all $x \in \overline{\Omega}$, $u \ge m$ and $v \in \mathbb{R}^+$, thus

$$\lambda f_1(x,u,v) + (1-\lambda) f_1(x,u,0) \ge (1/r(B_1) + \varepsilon)u - C_0, \tag{3.4}$$

for all $x \in \overline{\Omega}$ and $u, v \in \mathbb{R}^+$, here $C_0 = (1/r(B_1) + \varepsilon)m$.

We can prove that there exists a $R_1 > r_1$ such that

$$\mu T_{\lambda,1}(u,v) \ne u \quad \text{and} \quad \inf_{u \in \partial K_{R_1}} \|T_{\lambda,1}(u,v)\| > 0, \tag{3.5}$$

for all $\mu \ge 1$, $(u,v) \in \partial K_{R_1} \times K_2$.

First, if there are $(u_0, v_0) \in K_1 \times K_2$ and $\mu_0 \ge 1$ such that $u_0 = \mu_0 T_{\lambda,1}(u_0, v_0)$, together with (3.4), we get that

$$u_0(x) \ge T_{\lambda,1}(u_0, v_0)(x) \ge \int_{\overline{\Omega}} k_1(x,y)(1/r(B_1) + \varepsilon) u_0(y)\, dy - C.$$

It follows that

$$\int_{\overline{\Omega}} u_0(x)\psi_1(x)\, dx \ge \int_{\overline{\Omega}} \int_{\overline{\Omega}} k_1(x,y)(1/r(B_1) + \varepsilon) u_0(y)\, dy \psi_1(x) dx - C,$$

and

$$\int_{\overline{\Omega}} u_0(x)\psi_1(x)\, dx \ge (1 + r(B_1)\varepsilon) \int_{\overline{\Omega}} u_0(y)\psi_1(y)\, dy - C,$$

which yields

$$\int_{\overline{\Omega}} u_0(x)\psi_1(x)\,dx \le \frac{C}{r(B_1)\varepsilon}.$$

Furthermore, in view of Lemma 2.1(c), we know that

$$\|u_0\| \le \frac{C}{c_1 r(B_1)\varepsilon} \stackrel{\text{def}}{=} R^*.$$

Therefore, as $R > R^*$, $u \ne \mu T_{\lambda,1}(u,v)$ for all $(u,v) \in \partial K_R \times K_2$ and $\mu \ge 1$. In addition, if $R > m/\delta_1$, then by use of (3.3) we know that for all $(u,v) \in \partial K_R \times K_2$,

$$\begin{aligned} \|T_{\lambda,1}(u,v)\| &\ge \int_{\overline{\Omega}} T_{\lambda,1}(u,v)(x)\psi_1(x)\,dx \\ &\ge \int_{\overline{\Omega}}\int_{\overline{\Omega}_1} k_1(y,x)(1/r(B_1)+\varepsilon)u(y)\,dy\,\psi_1(x)\,dx \\ &\ge (1+r(B_1)\varepsilon)\int_{\overline{\Omega}_1} u(y)\psi_1(y)\,dy \ge (1+r(B_1)\varepsilon)\text{mes}(\overline{\Omega}_1)\delta_1^2\|\psi_1\|R, \end{aligned}$$

which implies that $\inf_{u\in\partial K_R}\|T_{\lambda,1}(u,v)\| > 0$. Hence, we choose

$$R_1 > \max\{r_1, R^*, m/\delta_1\}.$$

Step 3. In view of the sublinear assumption of f_2 at the origin, there are $\varepsilon > 0$ and $r_2 > 0$ such that

$$\lambda f_2(x,u,v) + (1-\lambda)f_2(x,u,0) \ge (1/r(B_1)+\varepsilon)v, \tag{3.6}$$

for all $x \in \overline{\Omega}$, $v \in [0, r_2]$ and $u \in \mathbb{R}^+$.

By (3.6) and the proof similar to Steps 1 and 2, we can deduce that

$$\mu T_{\lambda,2}(u,v) \ne v \quad \text{and} \quad \inf_{v\in\partial K_{r_2}} |\, T_{\lambda,2}(u,v)\| > 0, \tag{3.7}$$

for all $\mu \ge 1$, $(u,v) \in K_1 \times \partial K_{r_2}$.

Step 4. By virtue of the sublinear hypothesis and condition (G) of f_2 at infinity, there exist $\varepsilon \in (0, 1/r(B_2))$, $n > 0$ and $C > 0$ such that

$$\lambda f_2(x,u,v) + (1-\lambda)f_2(x,0,v) \le (1/r(B_2)-\varepsilon)v, \tag{3.8}$$

for all $x \in \overline{\Omega}$, $v \ge n$ and $u \in \mathbb{R}^+$, and

$$\lambda f_2(x,u,v) + (1-\lambda)f_2(x,0,v) \le (1/r(B_2)-\varepsilon)v + C, \tag{3.9}$$

for all $x \in \overline{\Omega}$ and $u, v \in \mathbb{R}^+$.

From (3.8), (3.9) and the similar argument used in Step 2, it can be proved that if $v_0 = \mu_0 T_{\lambda,2}(u_0,v_0)$ for $(u_0,v_0) \in K_1 \times K_2$ and $\mu_0 \in (0,1]$, then

$$\|v_0\| \le R' \stackrel{\text{def}}{=} \frac{C}{c_2 r(B_2)\varepsilon}.$$

Hence, we choose $R_2 > \max\{r_2, R'\}$, then

$$\mu T_{\lambda,2}(u,v) \neq v, \quad \text{for all } \mu \in (0,1] \text{ and } (u,v) \in K_1 \times \partial K_{R_2}. \tag{3.10}$$

Now we choose an open set $D = (K_{R_1} \setminus \overline{K_{r_1}}\,) \times (K_{R_2} \setminus \overline{K_{r_2}}\,)$. Based on the expressions (3.2), (3.5), (3.7) and (3.10), it is easy to verify that $\{T_\lambda\}_{\lambda\in I}$ satisfy the sufficient conditions for the homotopy invariance of fixed point index on ∂D; on the other hand, in combination with the classical fixed point index results (see Lemma 2.4), we have

$$\begin{aligned} i(T_{0,1}, K_{r_1}, K_1) &= i(T_{0,2}, K_{R_2}, K_2) = 1, \\ i(T_{0,1}, K_{R_1}, K_1) &= i(T_{0,2}, K_{r_2}, K_2) = 0. \end{aligned}$$

Applying the homotopy invariance of fixed point index and the product formula for the fixed point index (see Lemma 2.5), we obtain that

$$\begin{aligned} i(T_1, D, K_1 \times K_2) &= i(T_0, D, K_1 \times K_2) = \prod_{j=1}^{2} i(T_{0,j}, K_{R_j} \setminus \overline{K_{r_j}}, K_j) \\ &= \prod_{j=1}^{2} [i(T_{0,j}, K_{R_j}, K_j) - i(T_{0,j}, K_{r_j}, K_j)] = -1. \end{aligned}$$

Therefore, system (S) has at least one positive solution. □

4. Applications

As applications, we consider existence of positive solutions for the following system of second-order ordinary differential equations

$$\begin{cases} -u''(x) = f_1(x, u(x), v(x)) & \text{for } x \in \Omega \equiv (0,1), \\ -v''(x) = f_2(x, u(x), v(x)) & \text{for } x \in \Omega \equiv (0,1), \\ u(0) = u(1) = 0, \\ v(0) = v'(1) = 0, \end{cases} \tag{4.1}$$

here $f_1, f_2 \in C([0,1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$.

Theorem 4.1. *Assume that f_2 satisfies condition* (G) *and f_1, f_2 satisfy:*

(H_1^*) $$\limsup_{u\to 0^+} \max_{x\in[0,1]} \frac{f_1(x,u,v)}{u} < \pi^2 < \liminf_{u\to\infty} \min_{x\in[0,1]} \frac{f_1(x,u,v)}{u}$$ *uniformly w.r.t. $v \in \mathbb{R}^+$;*

(H_2^*) $$\liminf_{v\to 0^+} \min_{x\in[0,1]} \frac{f_2(x,u,v)}{v} > \frac{\pi^2}{4} > \limsup_{v\to\infty} \max_{x\in[0,1]} \frac{f_2(x,u,v)}{v}$$ *uniformly w.r.t. $u \in \mathbb{R}^+$.*

Then system (4.1) *has at least one positive solution.*

Proof. We know that system (4.1) is equivalent to the following system of nonlinear Hammerstein integral equations

$$\begin{cases} u(x) = \displaystyle\int_0^1 k_1(x,y) f_1(y,u(y),v(y))\,dy & \text{for } x \in [0,1], \\ v(x) = \displaystyle\int_0^1 k_2(x,y) f_2(y,u(y),v(y))\,dy & \text{for } x \in [0,1], \end{cases}$$

where

$$k_1(x,y) = \begin{cases} x(1-y) & \text{if } x \le y, \\ y(1-x) & \text{if } y \le x, \end{cases} \quad \text{and} \quad k_2(x,y) = \begin{cases} x & \text{if } x \le y, \\ y & \text{if } y \le x. \end{cases}$$

It is easy to verify that kernel functions k_i satisfy conditions (i)–(iii).

According to Theorem 1.5, we need only show that $r(B_1) = \pi^{-2}$ and $r(B_2) = 4\pi^{-2}$. On that purpose, we need only to verify that the minimal eigenvalue of B_1^{-1} and B_2^{-1} is π^2 and $\pi^2/4$, respectively. It follows from the following linear eigenvalue problems:

$$\begin{cases} -u''(x) = \lambda_n u(x), \\ u(0) = u(1) = 0, \end{cases} \quad \text{and} \quad \begin{cases} -v''(x) = \mu_n v(x), \\ v(0) = v'(1) = 0. \end{cases}$$

In fact, $\lambda_n = n^2\pi^2$ and $\mu_n = (n-1/2)^2\pi^2$, $n \in \mathbb{N}$. □

Remark 4.2. For instance, $f_1(x,u,v) = \max\{|\sin u|, u^2\}(1 + \tan^{-1} v)$ and $f_2(x,u,v) = \pi^2(1 + \cot^{-1} u)|\sin v|$, then f_1 and f_2 satisfy conditions (H_1^*), (H_2^*) and (G).

References

[1] X. Cheng and C. Zhong, *Existence of positive solutions for a second-order ordinary differential system*, J. Math. Anal. Appl. **312** (2005), 14–23.

[2] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, 1988.

[3] D. Guo and J. Sun, *Nonlinear Integral Equations*, Shandong Press of Science and Technology, Jinan, 1987. (in Chinese)

[4] A. Hammerstein, *Nichtlineare intergralgleichungen nebst anwendungen*, Acta Math. **54** (1929), 117–176.

[5] M. A. Krasnosel'skiĭ, *Topological methods in the Theory of Nonlinear Integral Equations*, Pergamon, Oxford, 1964.

[6] M. G. Krein and M. A. Rutman, *Linear operators leaving invariant a cone in a Banach space*, Transl. Amer. Math. Soc. **10** (1962), 199–325.

[7] J. Sun and X. Liu, *Computation for topological degree and its applications*, J. Math. Anal. Appl. **202** (1996), 785–796.

[8] J. Wang, *Existence of nontrivial solutions to nonlinear systems of Hammerstein integral equations and applications*, Indian J. Pure Appl. Math. **31** (2001), 1303–1311.

[9] Z. Yang and D. O'Regan, *Positive solvability of systems of nonlinear Hammerstein integral equations*, J. Math. Anal. Appl. **311** (2005), 600–614.

[10] Z. Zhang, *Existence of nontrivial solutions for superlinear systems of integral equations and applications*, Acta Math. Sinica **15** (1999), 153–162.

[11] C. Zhong, X. Fan and W. Chen, *An Introduction to Nonlinear Functional Analysis*, Lanzhou University Press, Lanzhou, 1998. (in Chinese)

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