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SPECTRAL SEQUENCES AND DETAILED CONNECTION MATRICES

Piotr Bartłomiejczyk

Abstract. We introduce detailed connection matrices. We prove that the spectral sequence can be reconstructed from a detailed connection matrix in the category of filtered differential vector spaces.

1. Introduction

Both spectral sequences and connection matrices are a generalization of exact sequences. They express the relationship between certain homology or cohomology groups. The idea of the connection matrix was due to Charles Conley. In [7]–[9], [12] the connection matrix theory was developed for both continuous and discrete dynamical systems. From some point of view, connection matrices can be seen as algebraic representations of the dynamical system. In this paper we introduce detailed connection matrices for filtered differential vector spaces. A filtered differential vector space is a finite increasing filtration of a given vector space together with an endomorphism d such that $d^2 = 0$ and d preserves the filtration. Roughly speaking, a detailed connection matrix is a bigraded subspace of the filtered differential vector space which provides information on some homology groups associated with the filtered differential vector space. It is well known that similar information is contained in spectral sequences. Therefore,

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the main goal of our paper is to establish the clear and purely algebraic relation between detailed connection matrices and spectral sequences. More precisely, we prove that for a given filtered differential vector space there exist a detailed connection matrix that fully reconstructs its spectral sequence.

The organization of the paper is as follows. Section 2 provides a brief exposition of the theory of spectral sequences (see [14] for more details). In Section 3 we introduce detailed connection matrices for filtered differential vector spaces. Section 3 contains also our main result concerning the relation between spectral sequences and detailed connection matrices. This result is proved in Section 4. Section 5 presents a graphic and intuitive approach to detailed connection matrices, especially to the proof of our main theorem. Finally, Section 6 contains a simple example illustrating possible applications of the theory to dynamical systems. For more references to the material presented here, see [1]–[5] and [13].

2. Spectral sequences

It is natural to give the definition of spectral sequences in the category of modules or abelian groups. But since our aim is to compare spectral sequences to detailed connection matrices defined in the category of vector spaces, we restrict ourselves to the case of vector spaces.

Recall that a *filtered vector space* is a vector space A equipped with a finite increasing *filtration*, that is, a sequence $\{A^p\}_0^n$ of subspaces of A such that

$$0 = A^0 \subset A^1 \subset \ldots \subset A^n = A.$$

We will use the following convenient notation

$$\begin{aligned} A^p &= A \quad \text{for } p \geq n, \\ A^p &= 0 \quad \text{for } p \leq 0. \end{aligned}$$

Similarly, a (*bi*)*graded vector space* is, by definition, a vector space A that has a direct sum decomposition

$$A = \bigoplus_{p=1}^{n} A_p \qquad \Big(A = \bigoplus_{p,q=1}^{n} A_{p,q}\Big).$$

The sequences $\{A_p\}_1^n$ and $\{A_{p,q}\}_1^n$ are called a *grading* and *bigrading*, respectively. Moreover, a grading $\{A_p\}_1^n$ of A is called a *splitting* for the filtration $\{A^p\}_0^n$ of A if

$$A^p = \bigoplus_{k=1}^{p} A_k$$

for any $1 \leq p \leq n$. Of course, the splitting is not unique.

Definition 2.1. A *filtered differential vector space* (*f-d space* for short) is a filtered vector space A together with an endomorphism d such that $d^2 = 0$ and d preserves the filtration, i.e. $dA^p \subset A^p$.

Observe that we have two natural finite filtrations associated with the filtered differential vector space. Namely,

$$0 = A^0 \subset A^1 \subset \ldots \subset A^n = A,$$
$$0 = dA^0 \subset dA^1 \subset \ldots \subset dA \subset d^{-1}0 \subset d^{-1}A^1 \subset \ldots \subset d^{-1}A^n = A.$$

If $\{A_p\}_1^n$ is a splitting of the f-d space, then the condition that d is filtration preserving is equivalent to the fact that the components of the differential d in the direct sum decomposition form a triangular matrix.

A *homomorphism of f-d spaces* is any homomorphism of vector spaces $h\colon A \to \widehat{A}$ such that $\widehat{d}h = hd$ and h preserves the filtration, i.e. $hA^p \subset \widehat{A}^p$. It is easy to see that f-d spaces and their homomorphisms form a category, which will be denoted by $\mathcal{FDV}$. Of course, if $h\colon A \to \widehat{A}$ is a homomorphism of f-d spaces, then h induces homology homomorphisms

$$h\colon H\,(A^p/A^q) \to H(\widehat{A}^p/\widehat{A}^q)$$

for any $p \leq q$. Furthermore, it follows from the five lemma that if $h\colon H(A) \to H(\widehat{A})$ is an isomorphism, then so are all $h\colon H(A^p/A^q) \to H(\widehat{A}^p/\widehat{A}^q)$.

Now we are ready to define spectral sequences of f-d spaces. We introduce the following notation. Let

$$Z_p^r := A^p \cap d^{-1}A^{p-r}, \qquad B_p^r := A^p \cap dA^{p+r}$$

for any $r \in \mathbb{Z}^+$ and $p \in \mathbb{Z}$. Since, as is easy to see, $Z_{p-1}^{r-1} \subset Z_p^r$ and $B_p^{r-1} \subset Z_p^r$, the quotient vector space

$$E_p^r := \frac{Z_p^r}{Z_{p-1}^{r-1} + B_p^{r-1}}$$

is well defined. Moreover, since the differential d induces homomorphisms

$$Z_p^r \to Z_{p-r}^r, \qquad Z_{p-1}^{r-1} + B_p^{r-1} \to Z_{p-r-1}^{r-1} + B_{p-r}^{r-1},$$

it also induces the homomorphism of quotient vector spaces, which we will denote by d_p^r:

$$d_p^r\colon E_p^r \to E_{p-r}^r.$$

Observe that $d_p^r([z]) = [dz]$, where $[\,\cdot\,]$ denotes the respective equivalence class. From this we obtain $d_{p-r}^r d_p^r[z] = [ddz] = 0$ and so $d_{p-r}^r d_p^r = 0$. For a fixed r homomorphisms d_p^r induce the homomorphism

$$d^r\colon \bigoplus_p E_p^r \to \bigoplus_p E_p^r.$$

Hence d^r is a differential of a vector space $E^r = \bigoplus_p E^r_p$. A sequence of vector spaces and differentials $\{E^r, d^r\}$, $r = 1, 2, \ldots$, is called the *spectral sequence of the f-d space A*. It is easily seen that if the filtration of the f-d space A has length n, then the spectral sequence stabilizes at the nth term, i.e. $E^r \simeq E^{r+1}$ for $r \geq n$. If $h: A \to \widehat{A}$ is a homomorphism of f-d spaces and $\widehat{Z}^r_p$, $\widehat{B}^r_p$, $\widehat{E}^r_p$, denote the respective vector spaces determined by $\widehat{A}$, then h induces homomorphisms $Z^r_p \to \widehat{Z}^r_p$, $B^r_p \to \widehat{B}^r_p$. Consequently, there exists an induced homomorphism

$$h^r_p: E^r_p \to \widehat{E}^r_p$$

given by $h^r_p[z] = [hz]$. Finally, homomorphisms h^r_p define a homomorphism of vector spaces

$$h^r: E^r \to \widehat{E}^r$$

such that $\widehat{d}^r h^r = h^r d^r$. It is easily seen that the above construction actually defines a functor from the category $\mathcal{FDV}$ to the category of vector spaces $\mathcal{V}$ which maps an f-d space A to the vector space E^r and sends a homomorphism of f-d spaces $h: A \to \widehat{A}$ to the homomorphism of vector spaces $h^r: E^r \to \widehat{E}^r$.

3. Detailed connection matrices

It is worth pointing out that we may define the connection matrix for the filtered differential vector space as an object in various categories. The simplest definition describes the connection matrix as an f-d space (or its splitting) on which the differential lowers the filtration. However, to reconstruct fully the spectral sequence we need a little more complicated version of the connection matrix, which we call the detailed connection matrix. Below we give all the mentioned definitions.

Let us recall that a subspace C of the filtered differential vector space A is called a *simple connection matrix* for A if

(1) $dC^p \subset C^{p-1}$,
(2) the map $i: H(C) \to H(A)$ induced by the inclusion $C \subset A$ is an isomorphism.

Moreover, any graded vector space constructed as a splitting $\{C_p\}_1^n$ of the simple connection matrix C will be called a *classical connection matrix*. It is immediate that if C is a classical connection matrix then the components of the differential d restricted to C form a strictly triangular matrix. Furthermore, from (2) and the five-lemma, we see at once that if C is a simple connection matrix for A, then $i: H(C^p/C^q) \to H(A^p/A^q)$ are isomorphisms for any $q \leq p$. In particular, if $\{C_p\}_1^n$ is a classical connection matrix for A, then

$$H(A^p/A^{p-1}) = H(C^p/C^{p-1}) = C^p/C^{p-1} = C_p.$$

It is high time to introduce the crucial definition of this paper.

DEFINITION 3.1. A bigraded vector space $\{C_{p,q}\}_1^n$ is called a *detailed connection matrix* for the f-d space A if:

(a) $C_{p,q} \subset A^p$ for $1 \leq p, q \leq n$,
(b) $dC_{p+1,q} = 0$ for $p+q \leq n$ and $dC_{p+1,q} \subset C_{p+q-n,n-q+1}$ for $p+q > n$,
(c) the map on homology $i\colon H(C) \to H(A)$ induced by the inclusion $C = \bigoplus_{p,q=1}^n C_{p,q} \subset A$ is an isomorphism.

Of course, a detailed connection matrix may be viewed as a more detailed (i.e. containing more information) version of a classical connection matrix. Observe that, using our standard notation $C_p = \bigoplus_{q=1}^n C_{p,q}$ and $C^p = \bigoplus_{k=1}^p C_k$, we obtain immediately that if $\{C_{p,q}\}_1^n$ is a detailed connection matrix for A, then

(1) the inclusion $C \subset A$ preserves the filtration, i.e. $C^p \subset A^p$,
(2) $\{C^p\}_0^n$ ($\{C_p\}_1^n$) is a simple (classical) connection matrix for A; in particular,

$$H(A^p/A^{p-1}) = C^p/C^{p-1} = C_p = \bigoplus_{q=1}^n C_{p,q}.$$

NOTATION. Let us introduce the notation for some "blocks" in the detailed connection matrix built from the "bricks" $C_{p,q}$. Set

$$\begin{aligned} \min &= \min\{r, n+1-p\}, \\ \max &= \max\{n+1-p, n+1-r\} = n+1-\min\{p,r\} \end{aligned}$$

for every $p, r = 1, \ldots, n$. Here and subsequently, we will use the symbol C_p^r to denote the direct sum

$$\bigoplus_{q=\min}^{\max} C_{p,q}. \tag{3.1}$$

Let A be a fixed filtered differential vector space.

DEFINITION 3.2. We say that a detailed connection matrix $\{C_{p,q}\}_1^n$ for the f-d space A *fully reconstructs* its spectral sequence E if there exist a collection of isomorphisms

$$\{\Phi_p^r\colon C_p^r \to E_p^r\}_{p,r=1}^n$$

such that the following diagram commutes

$$\begin{tikzcd} C_p^r \arrow[r, "d"] \arrow[d, "\Phi_p^r"'] & C_{p-r}^r \arrow[d, "\Phi_{p-r}^r"] \\ E_p^r \arrow[r, "d_p^r"'] & E_{p-r}^r \end{tikzcd} \tag{3.2}$$

for every $p, r = 1, \ldots, n$.

Let us formulate the main result of this paper.

Theorem 3.3 (Main Theorem). *There exists a detailed connection matrix for the filtered differential vector space that fully reconstructs the spectral sequence of this f-d space.*

The significance of the above result comes from the fact that it guarantees that all information contained in the spectral sequence may be recovered and studied using some detailed connection matrix.

4. Proof of the main result

The proof will consist of two parts. In the first part we describe the construction of a bigraded vector space $\{C_{p,q}\}_1^n$ and additional graded vector spaces $\{B^p\}_1^n$, $\{A^p\}_1^n$ satisfying conditions:

(1) $A^p = A^{p-1} \oplus A_p$,
(2) $A_p = B_p \oplus C_p \oplus dB_p$, where $C_p = \bigoplus_{q=1}^n C_{p,q}$,
(3) $B_p \cap d^{-1}0 = 0$,
(4) $dC_{p,q} \subset C_{p+q-n-1,n-q+1}$ assuming that $C_{p,q} = 0$ for $p \leq 0$.

It follows easily that the conditions (1)–(4) guarantee that a bigraded vector space $\{C_{p,q}\}_1^n$ is a detailed connection matrix for the f-d space A. Namely, the conditions (a) and (b) in Definition 3.1 follow immediately from the above conditions (1), (2) and (4). Moreover, since $A = B \oplus C \oplus dB$, where $B = \bigoplus_1^n B_p$ and $C = \bigoplus_1^n C_p$, and

$$H(A) = \frac{\ker d}{\operatorname{Im} d} = \frac{(\ker d|_C) \oplus dB}{(\operatorname{Im} d|_C) \oplus dB} = \frac{\ker d|_C}{\operatorname{Im} d|_C} = H(C),$$

we see that the inclusion $C \subset A$ induces the isomorphism $i\colon H(C) \to H(A)$ on homology, which is precisely the condition (c) of Definition 3.1.

In the second part we prove that applying the formula (3.1) to the bigraded vector space constructed in the first part guarantees commutativity of the diagram 3.2. It may be worth pointing out that in the second part we will refer not only to the above conditions (1)–(4), but also to the details of the construction from the first part.

Part I. For clarity, the construction of subspaces $C_{p,q}$, B_p and A_p will be divided into three steps.

Step 1. First for $p = 1, \dots, n$ we choose B_p to be any complement to $A^p \cap d^{-1}A^{p-1}$ in A^p, i.e.

$$A^p = B_p \oplus (A^p \cap d^{-1}A^{p-1}).$$

By the definition of B_p, we obtain immediately that

- $B_p \cap d^{-1}0 = 0$,
- $dB_p \subset dA^p \cap d^{-1}0 \subset A^p \cap d^{-1}A^{p-1}$,

- $A^{p-1} \cap dB_p = 0$.

Step 2. To simplify the description of the procedure for choosing a subspace $C_{p,q}$ we introduce unified notation for elements of the filtration

$$dA^0 \subset dA^1 \subset \ldots \subset dA^n \subset d^{-1}A^0 \subset \ldots \subset d^{-1}A^n.$$

Namely, let

$$D^m = \begin{cases} dA^m & \text{for } m \leq n, \\ d^{-1}A^{m-n-1} & \text{for } m > n. \end{cases}$$

Then we choose $C_{p,q}$ to be a complement to

$$(A^{p-1} \cap D^{p+q}) + (A^p \cap D^{p+q-1})$$

in $A^p \cap D^{p+q}$ satisfying

$$dC_{p,q} \subset C_{p+q-n-1,n-q+1}. \tag{4.1}$$

The difficulty in the construction of such $C_{p,q}$ depends on whether $C_{p,q}$ is contained in the kernel of d or not. It is clear that the construction of $C_{p,q}$ in the case when $p+q \leq n+1$ poses no problem, because in this case $D^{p+q} \subset D^{n+1} = d^{-1}0$ and, in consequence, the condition (4.1) is trivially satisfied by any complement.

Thus it remains to check the possibility of the choice of $C_{p,q}$ satisfying (4.1) in the case when $p+q > n+1$. We start with the observation that in this case D^{p+q} and D^{p+q-1} are preimages. More precisely,

$$D^{p+q} = d^{-1}A^{p+q-n-1},$$
$$D^{p+q-1} = d^{-1}A^{p+q-n-2}.$$

Furthermore, we observe that since the sum of indices of $C_{p+q-n-1,n-q+1}$ is equal to p and $p \leq n+1$, the subspace $C_{p+q-n-1,n-q+1}$ has been chosen yet as any complement of

$$(A^{p+q-n-2} \cap dA^p) + (A^{p+q-n-1} \cap dA^{p-1})$$

in $A^{p+q-n-1} \cap dA^p$. The existence of a subspace $C_{p,q}$ satisfying the condition (4.1) follows immediately from the equality

$$A^p \cap d^{-1}A^{p+q-n-1} = (A^{p-1} \cap d^{-1}A^{p+q-n-1}) + (A^p \cap d^{-1}A^{p+q-n-2}) + (A^p \cap d^{-1}C_{p+q-n-1,n-q+1})$$

holding for $p+q > n+1$.

It is obvious that right-hand side is a subset of the left-hand side. The reverse inclusion may be concluded as follows. Let

$$a \in A^p \cap d^{-1}A^{p+q-n-1}.$$

Then $da \in A^{p+q-n-1} \cap dA^p$. By the previous stage of the construction (in the kernel)

$$A^{p+q-n-1} \cap dA^p = \\ \big[\big(A^{p+q-n-2} \cap dA^p\big) + \big(A^{p+q-n-1} \cap dA^{p-1}\big)\big] \oplus C_{p+q-n-1,n-q+1},$$

and so there are $x \in A^p \cap d^{-1}A^{p+q-n-2}$, $y \in A^{p-1} \cap d^{-1}A^{p+q-n-1}$ and $c \in C_{p+q-n-1,n-q+1}$ such that

$$da = dx + dy + c$$

or equivalently

$$d(a - x - y) = c.$$

From the last equality

$$a - x - y \in A^p \cap d^{-1}C_{p+q-n-1,n-q+1}.$$

We thus get

$$a = x + y + (a - x - y)$$

with x, y and $a - x - y$ in desired subspaces, which completes the proof of the last inclusion.

Step 3. Finally, we define

$$A_p = B_p \oplus C_p \oplus dB_p.$$

We see at once that A_p, B_p and $C_{p,q}$ satisfy the conditions (1)–(4). Consequently, $\{C_{p,q}\}_{p,q=1}^n$ is a detailed connection matrix for the f-d space A.

Part II. By the definition of the spectral sequence

$$E_p^r := \frac{Z_p^r}{Z_{p-1}^{r-1} + B_p^{r-1}}.$$

However, from the construction of a bigraded vector space $\{C_{p,q}\}_1^n$ in Part I follows that the space C_p^r given by (3.1) is a complement of

$$(A^{p-1} \cap d^{-1}A^{p+\max-1-n-1}) + (A^p \cap d^{-1}A^{p+\min-1})$$

in $A^p \cap d^{-1}A^{p+\max-n-1}$. But by the definition of max

$$d^{-1}A^{p+\max-1-n-1} = d^{-1}A^{\max\{0,p-r\}} = d^{-1}A^{p-r},$$

since $A^k = 0$ for $k \le 0$. Similarly, by the definition of min

$$dA^{p+\min-1} = dA^{\min\{p-r-1,n\}} = dA^{p+r-1},$$

since $A^k = A$ for $k \ge n$.

Therefore C_p^r is a complement of $Z_{p-1}^{r-1} + B_p^{r-1}$ in Z_p^r. Finally, if we recall how d_p^r is defined and how d acts on direct summands $C_{p,q}$, we obtain that the isomorphism $\Phi_p^r \colon C_p^r \to E_p^r$ given by

$$\Phi_p^r(v) := v + (Z_{p-1}^{r-1} + B_p^{r-1})$$

makes the diagram 3.2 commute. □

5. The Zeeman diagram Δ

The detailed connection matrices and spectral sequences may be represented geometrically in the plane using the so-called *Zeeman diagram* Δ (see Figure 1). This diagram was defined by E. C. Zeeman to study the information contained in filtered differential groups (see [14] for more details).

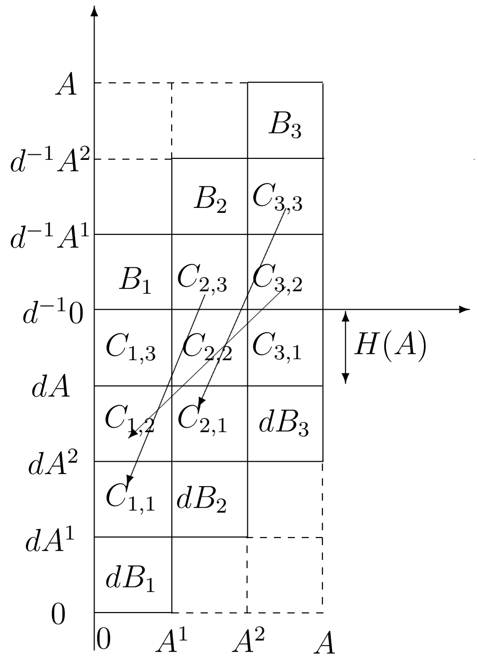

Figure 1. The detailed connection matrix for A

The Zeeman diagram Δ is the union of a collection of unit squares in the plane. The number of these squares depends only on the length of the filtration. The union of any subcollection of squares in Δ is called a *region* of Δ. For example, the region to the left of the vertical line labeled A^i represents the vector subspace A^i in the filtration. Similarly, the regions below the horizontal lines represent the vector spaces dA^i or $d^{-1}A^i$. Since the differential d is filtration preserving, some squares in the diagram represent trivial spaces. In the original

Zeeman diagram the regions represent the quotients of some groups associated with the filtered differential group, but in our diagram the regions represent the components in the direct sum decompositions of some vector spaces.

Since the diagram offers a graphic and intuitive approach to detailed connection matrices and spectral sequences, we will use it to present the general idea behind the proof of Main Theorem.

Let A be an f-d space with the filtration of the length 3, i.e.

$$0 = A^0 \subset A^1 \subset A^2 \subset A^3 = A.$$

We have restricted ourselves to the case $n = 3$ to make all our diagrams easy to analyze. It is clear that the Zeeman diagram Δ of the detailed connection matrix for A constructed in the proof of Main Theorem is as in Figure 1. It is worth pointing out that our picture shows in fact not only the detailed connection matrix $C = \bigoplus_{p,q=1}^{3} C_{p,q}$. but much more, namely the full decomposition of the filtered differential vector space in the notation from the proof of Main Theorem. Observe that the arrows in the picture represent the only possibly nonzero components of the differential d, i.e. maps

$$d_{2,3}\colon C_{2,3} \to C_{1,1}, \qquad d_{3,3}\colon C_{3,3} \to C_{2,1}, \qquad d_{3,2}\colon C_{3,2} \to C_{1,2}.$$

The important point to note here is the form of the relation between the relative homology represented as the columns $C_p = \bigoplus_{q=1}^{3} C_{p,q}$ and the total homology $H(A)$ represented as the row just below the x-axis.

According to the proof of Main Theorem, the above detailed connection matrix allows us to reconstruct the entire spectral sequence of the f-d space A. It is easily seen that in the examined case the spectral sequence stabilizes at the third term, i.e. it has the form $\{E^r, d^r\}_{r=1}^{3}$ (see Figure 2). Denoting the isomorphisms Φ_p^r from Main Theorem by $\simeq$, we obtain the following correspondence between the first term of the spectral sequence and the components of the detailed connection matrix

$$E^1 = \bigoplus_{p=1}^{3} E_p^1, \qquad E_p^1 \simeq \bigoplus_{q=1}^{3} C_{p,q}.$$

Moreover, the first differential d^1 of the spectral sequence is fully described by the action of the components $d_{2,3}$ and $d_{3,3}$ (compare Figures 1 and 2). In the same manner we can see that

$$E^2 = \bigoplus_{p=1}^{3} E_p^2, \qquad E_1^2 \simeq \bigoplus_{q=2}^{3} C_{1,q}, \qquad E_2^2 \simeq C_{2,2}, \qquad E_3^2 \simeq \bigoplus_{q=1}^{2} C_{3,q}.$$

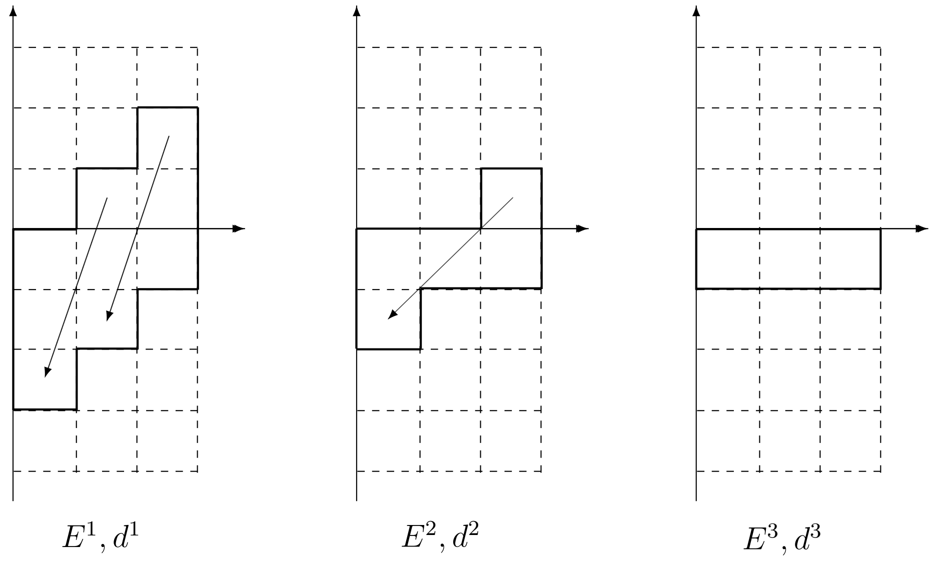

FIGURE 2. The spectral sequence $E = \{E^r, d^r\}_1^3$ for A

Similarly, the second differential d^2 may be identified with the component $d_{3,2}$. Finally,

$$E^3 = \bigoplus_{p=1}^{3} E_p^3, \qquad E_p^3 \simeq C_{p,4-p}.$$

Since the length of the filtration is equal to 3, $d^3 = 0$.

6. Morse decompositions

Both spectral sequences and detailed connection matrices may be used as algebraic tools for studying the dynamics of Morse decompositions. To avoid technicalities we restrict ourselves to flows on compact metric spaces. Let X be such a space.

Recall that a collection $\{M_i\}_1^n$ of mutually disjoint compact invariant subsets of X is called a *Morse decomposition* if for every $x \in X \setminus \bigcup_{i=1}^n M_i$ there are indices $i < j$ such that $\omega^+(x) \subset M_i$ and $\omega^-(x) \subset M_j$. The sets M_i are called *Morse sets*. Moreover, generalized Morse sets for $i \leq j$ are defined as

$$M_{ji} := \left\{ x \in X \;\middle|\; \omega^+(x) \cup \omega^-(x) \subset \bigcup_{k=i}^{j} M_k \right\}.$$

Spectral sequences and detailed connection matrices may be naturally related to a filtration of index pairs associated with the Morse decomposition. Recall that a filtration of compact sets $\{N^i\}_0^n$ is called an *index filtration* if

(1) $\emptyset = N^0 \subset N^1 \subset \ldots \subset N^n = X$,

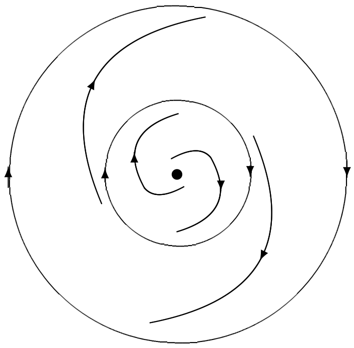

Figure 3. The dynamics of a flow on X

(2) (N^j, N^{i-1}) is an index pair for M_{ji}.

Such filtrations, even regular, exist (see [11] for more details). Instead of giving the precise definition of regularity, we just mention its basic consequence. Namely, if the index filtration is regular, then the Conley index of each Morse set M_{ji} is isomorphic to the singular homology of the pair (N^j, N^{i-1}).

All we need to construct spectral sequences or detailed connection matrices is to start with some f-d space. We show how to find such an f-d space in the context of the Conley index theory. Let $\{N^i\}_0^n$ be an index filtration for the Morse decomposition $\{M_i\}_1^n$. Let $C(N^k)$ be the vector space of singular chains in N^k and $i_k\colon C(N^k) \to C(N)$ be a homomorphism induced by the inclusion $N^k \subset N$. It is evident that a filtration

$$\{i_k(C(N^k))\}_0^n,$$

equipped with the boundary map on singular chains, is an f-d space. Thus, a spectral sequence (detailed connection matrix) for the Morse decomposition $\{M_i\}_1^n$ is exactly a spectral sequence (detailed connection matrix) for the above f-d space.

The main idea behind the concept of spectral sequences or detailed connection matrices for Morse decompositions is that their nonzero differentials provide information on time arrow preserving sequences of connecting orbits between different Morse sets as in the following simple example. Consider a flow on the closed unit ball D^2 in the plane with the dynamics as in Figure 3. Observe that that the fixed point and two periodic orbits form a Morse decomposition of $X = D^2$ with the local Conley indices

$$CH_k(M_1) = \delta_{0k}\mathbb{Q} + \delta_{1k}\mathbb{Q}, \qquad CH_k(M_2) = 0, \qquad CH_k(M_3) = \delta_{2k}\mathbb{Q},$$

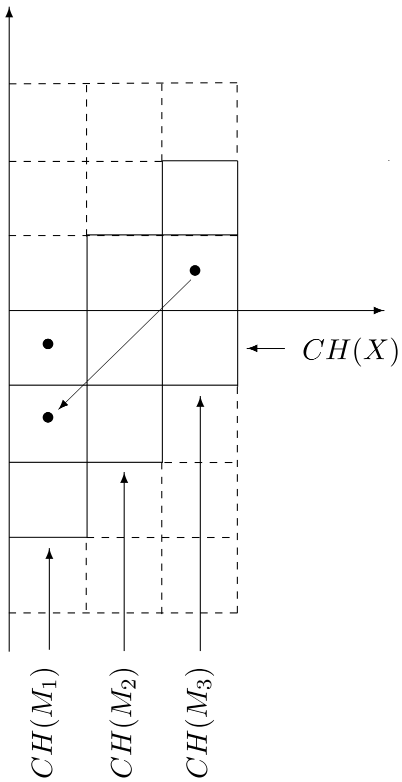

FIGURE 4. The spectral sequence and detailed connection matrix for the Morse decomposition of X

and the global Conley index of the whole space

$$CH_k(X) = \delta_{0k}\mathbb{Q}.$$

An easy comparison of the above Conley indices shows that the spectral sequence and the detailed connection matrix for the Morse decomposition of X is as in Figure 4. The dots in Figure 4 represent elements of some bases of the Conley indices that agree with the direct sum decomposition given by the detailed connection matrix. Observe that the nonzero differential corresponds to the time arrow preserving sequence of connecting orbits from M_3 to M_1.

REFERENCES

[1] P. BARTŁOMIEJCZYK, *The Conley index and spectral sequences*, Topol. Methods Nonlinear Anal. **25** (2005), 195–203.

[2] ______, *Connection graphs*, Fund. Math. **192** (2006), 93–110.

[3] ______, *Simple connection matrices*, Annal. Polon. Math. **90** (2007), 77–87.

[4] C. Conley, *Isolated Invariant Sets and the Morse Index*, CMBS Reg. Conf. Ser. in Math., vol. 38, AMS, Providence, RI, 1978.

[5] C. Conley and E. Zehnder, *Morse type index theory for flows and periodic solutions to Hamiltonian system*, Comm. Pure Appl. Math. **37** (1984), 207–253.

[6] B. Fiedler and K. Mischaikow, *Dynamics of bifurcation for variational problems with $O(3)$ equivariance: a Conley index approach*, Arch. Rational Mech. Anal. **119** (1992), 145–196.

[7] R. Franzosa, *Index filtrations and the homology braid for partially ordered Morse decompositions*, Trans. Amer. Math. Soc. **298** (1986), 193–213.

[8] ———, *The connection matrix theory for Morse decompositions*, Trans. Amer. Math. Soc. **311** (1989), 561–592.

[9] ———, *The continuation theory for Morse decompositions and connection matrices*, Trans. Amer. Math. Soc. **310** (1988), 781–803.

[10] K. Mischaikow and M. Mrozek, *The Conley Index Theory*, Handbook of Dynamical Systems III: Towards Applications, Singapore, 2002, pp. 393–460.

[11] M. R. Razvan, *On Conley's fundamental theorem of dynamical systems*, Internat. J. Math. Math. Sci. **26** (2004), 1397–1401.

[12] J. W. Robbin and D. Salamon, *Lyapunow maps, symplicial complexes and the Stone functor*, Ergodic Theory Dynam. Systems **12** (1992), 153–183.

[13] D. Salamon, *Connected simple systems and the Conley index of isolated invariant sets*, Trans. Amer. Math. Soc. **291** (1985), 1–41.

[14] E. C. Zeeman, *On the filtered differential group*, Ann. Math. **66** (1957), 557–585.

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Piotr Bartłomiejczyk
Institute of Mathematics
University of Gdańsk
Wita Stwosza 57
80-952 Gdańsk, POLAND

E-mail address: pb@math.univ.gda.pl