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NATURAL TOPOLOGIES ON COLOMBEAU ALGEBRAS

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Abstract. We define intrinsic, natural and metrizable topologies $\mathcal{T}_\Omega$, $\mathcal{T}$, $\mathcal{T}_{s,\Omega}$ and $\mathcal{T}_s$ in $\mathcal{G}(\Omega)$, $\overline{\mathbb{K}}$, $\mathcal{G}_s(\Omega)$ and $\overline{\mathbb{K}}_s$, respectively. The topology $\mathcal{T}_\Omega$ induces $\mathcal{T}$, $\mathcal{T}_{s,\Omega}$ and $\mathcal{T}_s$. The topologies $\mathcal{T}_{s,\Omega}$ and $\mathcal{T}_s$ coincide with the Scarpalezos sharp topologies.

1. Introduction

It is well known that the Colombeau's theory was developped aiming to solve non-linear problems of PDEs and ODEs. In a more specific way, Colombeau's full algebra (denoted here as $\mathcal{G}(\Omega)$, where Ω is an open subset of $\mathbb{R}^m$) was introduced to be the universe that contains a large amount of solutions of PDEs and ODEs defined on Ω, which are not solvable classicaly (of course, $\mathcal{G}(\Omega)$ has many other inhabitants, but they do not matter here). The mere fact that $\mathcal{G}(\Omega)$ is an algebra that contains canonically a copy of $\mathcal{D}'(\Omega)$ has represented a great advance because it made possible the multiplication of distributions without any kind of restriction. But this progress, despite its importance, was merely algebraic, letting aside all the resources of Functional Analysis. Since the machinery of classical Functional Analysis has shown to be prolific, it was expected that, soon or late, it would be possible to define a suitable topology (i.e. compatible with the algebraic structure), in order to have, in this context, a complete set of algebraic and topological tools.

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The first and most important step in this direction was done in 1995 by D. Scarpalezos in [12] and [13], who defined the "sharp topology", on the simplified version of Colombeau's algebra. In this work we introduce the topology $\mathcal{T}_\Omega$ on the full algebra $\mathcal{G}(\Omega)$ and prove that the topologies induced by $\mathcal{T}_\Omega$ on $\mathcal{G}_s(\Omega)$ and $\overline{\mathbb{K}}_s$ (ring of Colombeau's generalized scalars (simplified version)) coincide with Scarpalezos's sharp topologies. The topology $\mathcal{T}$ induced by $\mathcal{T}_\Omega$ on $\overline{\mathbb{K}}$ (ring of Colombeau's generalized scalars (full version)) made it a topological ring so that $\mathcal{G}(\Omega)$ is a topological algebra on $\overline{\mathbb{K}}$.

Our starting point differs from the one of Scarpalezos. In fact, using a partial order relation $\leq$ on $\overline{\mathbb{R}}$ (defined in [5] and [9], see Definition 2.2) we have been able to define a notion of "generalized semi-norms" on $\mathcal{G}(\Omega)$ from which the definition of $\mathcal{T}_\Omega$ follows naturally. This way of working suggests an interesting parallel between Colombeau's algebras and the theory of locally convex spaces.

It is pertinent to say that an important part of the motivation for this work was the resolution of a boundary value problem for a nonlinear PDE which we now describe. Let Ω be a non-void bounded open subset of $\mathbb{R}^m$, $T \in \mathbb{R}^*_+$; then $Q := \Omega \times]0,T[$ is a bounded open subset of $\mathbb{R}^{m+1}$.

Theorem 1.1 ([3, Theorem 3.5]). *Consider the* IBVP

$$\begin{cases} u_t - \Delta u + u^3 = 0 & \text{in } \mathcal{G}(Q), \\ u|_{\overline{\Omega}\times\{0\}} = u_0 & \text{in } \mathcal{G}(\overline{\Omega}), \\ u|_{\partial\Omega\times[0,T]} = 0 & \text{in } \mathcal{G}(\partial\Omega \times [0,T]), \end{cases} \tag{1.1}$$

where $u_0 \in \mathcal{G}(\Omega)$. Then, there is a unique $u \in \mathcal{G}(\overline{Q})$ such that u is solution of (1.1) (*with initial data u_0*).

The resolution of this problem involves the completeness of $\mathcal{T}_\Omega$, the definition of a topology $\mathcal{T}_{\overline{\Omega},b}$ on $\mathcal{G}(\overline{\Omega})$ (where Ω is a bounded open subset of $\mathbb{R}^m$), the completeness of $\mathcal{T}_{\overline{\Omega},b}$, besides many other questions.

Let us explain a bit more how to solve (1.1). The first thing to be done is to create the right environment to find a solution. Note that (1.1) was first considered by Brézis–Friedman [7] who showed that, in case $u_0 = \delta$, there is no solution in $\mathcal{D}'$. Colombeau–Langlais [8] showed that if $u_0 \in \mathcal{G}_c(\Omega)$, the ideal of $\mathcal{G}(\Omega)$ consisting of the generalized functions of compact support, then (1.1) has a solution in $\mathcal{G}(\overline{\Omega})$.

To solve the problem, in case $u_0 \in \mathcal{G}(\Omega)$ is any generalized function, we use topological arguments. In fact, we first use the results of this paper to endow $\mathcal{G}(\overline{Q})$ and $\mathcal{G}(\partial\Omega \times [0,T])$ with suitable but yet compatible and complete topologies. Using the density of the ideal $\mathcal{G}_c(\Omega)$, we get a sequence $(u_n)_{n\geq 1}$ of elements of $\mathcal{G}_c(\Omega)$ such that $u_n \to u_0$ in $(\mathcal{G}(\Omega), \mathcal{T}_\Omega)$ when $n \to \infty$ and, from results of Colombeau–Langlais, we construct a sequence $(U_n)_{n\geq 1}$, such that $U_n \in \mathcal{G}(\overline{Q})$

for all $n \geq 1$, is a solution of (1.1) with corresponding initial data u_n. Finally, using algebraic and topological machinery, we show that $(U_n)_{n\geq 1}$ is a Cauchy sequence which converges to an element $U \in \mathcal{G}(\overline{Q})$, the latter being the unique solution of (1.1).

It is important to say that we heavily relied on estimates proved in [8], generalized to an algebraic context, on the topological notion of quasi-regular sets developed in [1] and on the continuity of a nonlinear differential operator which is proved using the here developed topology.

We now present the general ideas on which the present work is based. Let Ω be a non-void open subset of $\mathbb{R}^m$ and $\mathcal{G}(\Omega)$ as in [2, Notation 2.1.1]. If $(\Omega_l)_{l\in\mathbb{N}}$ is an exhaustive sequence of open subsets of Ω then it is well known that the natural locally convex topology t_Ω on $\mathcal{C}^\infty(\Omega)$ can be defined by the fundamental system of semi-norms $|||\cdot|||_{\beta,l}$ given by

$$|||u|||_{\beta,l} := \sup_{x\in\Omega_l} |\partial^\beta u(x)|$$

for $u \in \mathcal{C}^\infty(\Omega)$, $\beta \in \mathbb{N}^m$ and $l \in \mathbb{N}$.

To define our topology $\mathcal{T}_\Omega$ we proceed as follows. First, for every $\beta \in \mathbb{N}^m$ and $l \in \mathbb{N}$, we define the "generalized semi-norm" $||\cdot||_{\beta,l}$ on $\mathcal{G}(\Omega)$ by

$$||f||_{\beta,l} := \mathrm{cl}(\varphi \in A_0 \mapsto |||\widehat{f}(\varphi,\,\cdot)|||_{\beta,l} \in \mathbb{R}),$$

for all $f \in \mathcal{G}(\Omega)$, where $\widehat{f}$ is any representative of f. Next we define a fundamental system of 0-neighbourhoods of $\mathcal{G}(\Omega)$, determining $\mathcal{T}_\Omega$. To this end we need to introduce a partial order relation $\leq$ on $\overline{\mathbb{R}}$ which generalizes the partial order $\leq$ on $\overline{\mathbb{R}}_s$ introduced in [5] and [9].

To get a compatible topology we allow as radius of our balls elements of the set $\mathcal{Q}^\bullet := \{\alpha_r^\bullet \mid r \in \mathbb{R}\}$, which is a linearly ordered subset of $\overline{\mathbb{R}}$ (see Example 2.3) formed by q-positive units of $\overline{\mathbb{R}}$ (i.e. $\mathcal{Q}^\bullet \subset \mathrm{Inv}(\overline{\mathbb{R}}) \cap \overline{\mathbb{R}}_+$). The consideration of the generalized scalars $\alpha_r^\bullet$ is the natural generalization of an idea of [4].

The same ideas, together with the obvious definition of the absolute value for elements of $\overline{\mathbb{K}}$, lead to $\mathcal{T}_\Omega$ and allow us to introduce a natural topology $\mathcal{T}$ on $\overline{\mathbb{K}}$. It is then proved that $\mathcal{T}_\Omega$ and $\mathcal{T}$ induce on the simplified algebras $\mathcal{G}_s(\Omega)$ and $\overline{\mathbb{K}}_s$ the well-known sharp topologies (see [12], [13] and [4]).

The best way to endow an algebra (or ring) E with a topology compatible with its algebraic structure is to give a filter basis $\mathcal{B}$ on E verifying a set of axioms which essentially guarantees the continuity of the algebraic operations of E. This set of axioms is slightly different than the one usually appearing in text books on TVS. This is because a $\overline{\mathbb{K}}$-algebra is not a $\mathbb{K}$-TVS. So, to make the paper self contained, we present, in the Section 2, some results about these axioms.

2. Some basic facts on topological algebras

In this section we present three well know results which guarantee that the topology determined by a given filter basis $\mathcal{B}$ on a ring (or module or algebra) is compatible with the ring (or module or algebra) structure. The proofs of these three results follow easily from [6, Chapter 3, §1, n° 2 (p. 12); §6, n° 3 (p. 75–76)] and are omitted.

In what follows the word "ring" means "commutative ring with unit" and the word "algebra" means "commutative algebra with unit".

Proposition 2.1. *Let A be a ring and $\mathcal{B}$ a filter basis on A verifying the following conditions:*

- $(\mathrm{GA}'_{\mathrm{I}})$ *for every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $V + V \subset U$;*
- $(\mathrm{GA}'_{\mathrm{II}})$ *for every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $-V \subset U$;*
- $(\mathrm{AV}'_{\mathrm{I}})$ *for every $a \in A$ and every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $a.V \subset U$;*
- $(\mathrm{AV}'_{\mathrm{II}})$ *for every $U \in \mathcal{B}$ there exists $V \in \mathcal{B}$ such that $V.V \subset U$.*

Then, there exists a unique topology τ on A, compatible with the ring structure of A, such that $\mathcal{B}$ is a fundamental system of τ-neighbourhoods of 0.

Proposition 2.2. *Let A be a topological ring.*

(a) *Let X be an A-module and $\mathcal{B}$ a filter basis on X verifying the conditions* $(\mathrm{GA}'_{\mathrm{I}})$ *and* $(\mathrm{GA}'_{\mathrm{II}})$ *of Proposition* 2.1 *and, in addition, the three conditions:*

- $(\mathrm{MV}'_{\mathrm{I}})$ *for every $x_0 \in X$ and every $V \in \mathcal{B}$ there exists a* 0*-neighbourhood S in A such that $S.x_0 \subset V$;*
- $(\mathrm{MV}'_{\mathrm{II}})$ *for every $\lambda_0 \in A$ and every $V \in \mathcal{B}$ there exists $W \in \mathcal{B}$ such that $\lambda_0.W \subset V$;*
- $(\mathrm{MV}'_{\mathrm{III}})$ *for every $V \in \mathcal{B}$ there exist $W \in \mathcal{B}$ and a* 0*-neighbourhood S in A such that $S.W \subset V$.*

Then, there exists a unique topology τ on X, compatible with the A-module structure of X, such that $\mathcal{B}$ is a fundamental system of τ-neighbourhoods of 0.

(b) *Let X an A-algebra and $\mathcal{B}$ a filter basis on X verifying the conditions* $(\mathrm{GA}'_{\mathrm{I}})$, $(\mathrm{GA}'_{\mathrm{II}})$, $(\mathrm{MV}'_{\mathrm{I}})$, $(\mathrm{MV}'_{\mathrm{II}})$ *and* $(\mathrm{MV}'_{\mathrm{III}})$ *of* (a) *and, in addition, the two conditions:*

- $(\mathrm{AV}'_{\mathrm{I}})$ *for every $x_0 \in X$ and every $V \in \mathcal{B}$ there exists $W \in \mathcal{B}$ such that $x_0.W \subset V$;*
- $(\mathrm{AV}'_{\mathrm{II}})$ *for every $V \in \mathcal{B}$ there exist $W \in \mathcal{B}$ such that $W.W \subset V$.*

Then, there exists a unique topology τ on X, compatible with the A-algebra structure of X, such that $\mathcal{B}$ is a fundamental system of τ-neighbourhoods of 0.

Let X be a topological ring and A a subring of X (we assume here that the unit element of X belongs to A, see [11]). Then it is clear that the topology

induced on A by the topology on X is compatible with the ring structure of A. Moreover, if we denote by A_0 the topological ring obtained by endowing A with the topology induced by X, it is clear the X becomes an A_0-topological algebra. Now, if $\mathcal{B}$ is a filter basis on X verifying the conditions of Proposition 2.1, it is easily seen that

$$(\mathrm{AV}'_{\mathrm{I}}) \Rightarrow (\mathrm{MV}'_{\mathrm{I}}) \quad \text{and} \quad (\mathrm{MV}'_{\mathrm{II}}); \quad \text{and} \quad (\mathrm{AV}'_{\mathrm{II}}) \Rightarrow (\mathrm{MV}'_{\mathrm{III}})$$

which allow the following simplification of Proposition 2.2(b), which we shall need in the sequel:

Corollary 2.3. *Let X be a ring, A a subring of X (see [11]) and $\mathcal{B}$ a filter basis on X verifying the four conditions* $(\mathrm{GA}'_{\mathrm{I}})$, $(\mathrm{GA}'_{\mathrm{II}})$, $(\mathrm{AV}'_{\mathrm{I}})$ *and* $(\mathrm{AV}'_{\mathrm{II}})$ *of Proposition 2.1 (with X instead of A). Then, there exists a unique topology τ on X with the following properties:*

(a) *τ is compatible with the ring structure of X. The topology τ_A induced by τ on A is compatible with the ring structure of A.*
(b) *X is an A_0-topological algebra (here A_0 denotes the topological ring obtained by endowing A with τ_A).*
(c) *$\mathcal{B}$ is a fundamental system of τ-neighbourhoods of 0.*

3. The scalar full sharp topology

In the remainder of the paper we shall adopt the following conventions. As usual, $\mathbb{K}$ denotes indistinctly $\mathbb{R}$ or $\mathbb{C}$, $\subset :=]0,1]$ and $\subset_\eta :=]0,\eta[$ for each $\eta \in \subset$. We shall assume fixed an arbitrary $m \in \mathbb{N}^*$ and a non void open set Ω in $\mathbb{R}^m$. We shall use freely the symbols $A_q = A_q(m;\mathbb{K})$ $(q \in \mathbb{N})$ [2, Definition 1.5 and Notation 1.8], $\mathcal{G}(\Omega)$ [2, Notation 2.1.1], $\overline{\mathbb{K}} = \overline{\mathbb{K}}(\mathbb{R}^m)$, $\mathcal{E}_M(\mathbb{K}) = \mathcal{E}_M(\mathbb{K};\mathbb{R}^m)$ and $\mathcal{N}(\mathbb{K}) = \mathcal{N}(\mathbb{K};\mathbb{R}^m)$ [2, 3.1, Definition 3.1.2 and Remark 3.1.3]. The definitions of the algebra of the simplified generalized functions and the ring of the simplified generalized numbers,

$$\mathcal{G}_s(\Omega) := \frac{\mathcal{E}_{M,s}[\Omega]}{\mathcal{N}_s[\Omega]} \quad \text{and} \quad \overline{\mathbb{K}}_s := \frac{\mathcal{E}_{M,s}(\mathbb{K})}{\mathcal{N}_s(\mathbb{K})},$$

that we adopt here are the ones given in [4, Section 1], where the notation used are $\mathcal{G}(\Omega)$ and $\overline{\mathbb{K}}$. The only difference between the above definitions in the simplified cases and the ones given in [2, 8.1 and 8.2] is that here (resp. in [2]) the domains of $u \in \mathcal{E}_{M,s}[\Omega]$ and $v \in \mathcal{E}_{M,s}(\mathbb{K})$ are $\subset \times\Omega$ and $\subset$ (resp. $\mathbb{R}^*_+ \times \Omega$ and $\mathbb{R}^*_+$).

As usual, $K \subset\subset \Omega$ denotes the sentence "K is a compact subset of Ω".

We use an exhaustive sequence of open sets $(\Omega_l)_{l\in\mathbb{N}}$ for Ω, which means that $\Omega = \bigcup_{l\in\mathbb{N}} \Omega_l$, Ω_l being a non-void open set and $\overline{\Omega}_l \subset\subset \Omega_{l+1}$, for all $l \in \mathbb{N}$. If $F\colon \Omega \to \mathbb{K}$ is a function and $\emptyset \neq S \subset \Omega$, we define $||F||_S := \sup_{x\in S} |F(x)|$ and,

in the particular case $S = \Omega_l$ (for any $l \in \mathbb{N}$) we write $||F||_l$ instead of $||F||_{\Omega_l}$, i.e. $||F||_l := \sup_{x\in\Omega_l} |F(x)|$.

If $\varphi \in A_0 = A_0(m, \mathbb{K})$ we define $i(\varphi) := \operatorname{diam}\operatorname{supp}(\varphi)$ and since $0 \notin A_0$ we have $i(\varphi) > 0$ for all $\varphi \in A_0$. It is easy to see that $i(\varphi_\varepsilon) = \varepsilon i(\varphi)$ for all $\varepsilon > 0$.

Our next result paves the way for the definition of a partial order relation on $\overline{\mathbb{R}}$ by using the same kind of ideas which leads to the definition of a partial order relation on $\overline{\mathbb{R}}_s$ in [5, Section 3], see also [9].

Lemma 3.1. *For a given $x \in \overline{\mathbb{R}}$ the following are equivalent:*

(a) *every representative $\widehat{x}$ of x satisfies the condition:*

$$\text{(3.1)} \qquad \begin{cases} \text{there exists } N \in \mathbb{N} \text{ such that for all } b > 0 \text{ and for all } \varphi \in A_N \\ \text{there is } \eta = \eta(b, \varphi) \in \mathbf{I} \text{ verifying } \widehat{x}(\varphi_\varepsilon) \geq -\varepsilon^b \text{ for all } \varepsilon \in\subset_\eta; \end{cases}$$

(b) *there exists a representative $\widehat{x}$ of x satisfying* (3.1)*;*

(c) *there exists a representative x_* of x such that $x_*(\varphi) \geq 0$ for all $\varphi \in A_0$;*

(d) *there exist $N \in \mathbb{N}$ and a representative x_* of x such that $x_*(\varphi) \geq 0$ for all $\varphi \in A_N$.*

Proof. (b) $\Rightarrow$ (c) From the representative $\widehat{x}$ of x in (b) we define $h\colon A_0 \to \mathbb{R}$ by $h(\varphi) := 0$ (resp. $-\widehat{x}(\varphi)$) if $\widehat{x}(\varphi) \geq 0$ (resp. $\widehat{x}(\varphi) < 0$), hence $h(\varphi) \geq 0$ for each $\varphi \in A_0$ and it is easily seen that $h \in \mathcal{N}(\mathbb{R})$. Now, it is enough to define $x_*(\varphi) := \widehat{x}(\varphi) + h(\varphi)$ for all $\varphi \in A_0$.

(c) $\Rightarrow$ (a) Let $\widehat{x}$ be an arbitrary representative of x. Since $x_* - \widehat{x} \in \mathcal{N}(\mathbb{R})$ it follows that there is $N \in \mathbb{N}$ such that for all $q \geq N$, for all $\varphi \in A_q$ and for all $b > 0$ we have $|(x_* - \widehat{x}\,)(\varphi_\varepsilon)|\varepsilon^{-b} \to 0$ if $\varepsilon \downarrow 0$. Therefore, there is $\eta = \eta(b, \varphi) \in\subset$ satisfying $|(x_* - \widehat{x}\,)(\varphi_\varepsilon)|\varepsilon^{-b} \leq 1$ for all $\varepsilon \in\subset_\eta$, hence $(x_* - \widehat{x})(\varphi_\varepsilon) \leq \varepsilon^b$ for all $\varepsilon \in\subset_\eta$, which implies $\widehat{x}(\varphi_\varepsilon) \geq x_*(\varphi_\varepsilon) - \varepsilon^b \geq -\varepsilon^b$ for all $\varepsilon \in\subset_\eta$, hence $\widehat{x}$ satisfies condition (3.1).

The implications (a) $\Rightarrow$ (b) and (c) $\Leftrightarrow$ (d) are obvious. □

Definition 3.2. An element $x \in \overline{\mathbb{R}}$ is said to be *quasi-positive* or *q-positive*, if it has a representative satisfying the equivalent conditions of Lemma 3.1. We shall denote this by $x \geq 0$. We shall say also that x is *quasi-negative* or *q-negative* if $-x$ is q-positive and we denote this by $x \leq 0$. If $y \in \overline{\mathbb{R}}$ is another element then we write $x \leq y$ (resp. $x \geq y$) if $y - x$ (resp. $x - y$) is q-positive.

Notation.

$$\overline{\mathbb{K}}^* := \overline{\mathbb{K}} \setminus \{0\}, \quad \overline{\mathbb{R}}_+ := \{x \in \overline{\mathbb{R}} \mid x \geq 0\} \quad \text{and} \quad \overline{\mathbb{R}}_+^* := \{x \in \overline{\mathbb{R}}^* \mid x \geq 0\}.$$

EXAMPLE 3.3. For every $r \in \mathbb{R}$ we shall define an element $\alpha_r^\bullet \in \overline{\mathbb{R}}_+^*$ which will play the same role that the elements $\alpha_r \in \overline{\mathbb{R}}_{s,+}$ (see [4, Section 1]). A representative of $\alpha_r^\bullet$ is the function given by

$$\widehat{\alpha_r^\bullet}\colon \varphi \in A_0 \to i(\varphi)^r \in \mathbb{R}_+^*.$$

Clearly $\widehat{\alpha_r^\bullet}(\varphi) > 0$ for all $\varphi \in A_0$ and $\widehat{\alpha_r^\bullet}(\varphi_\varepsilon) = \varepsilon^r i(\varphi)^r = \widehat{\alpha}_r(\varepsilon) i(\varphi)^r$. Obviously $\widehat{\alpha_r^\bullet} \in \mathcal{E}_M(\mathbb{R})$ and $\alpha_r^\bullet := \mathrm{cl}(\widehat{\alpha_r^\bullet}) \in \overline{\mathbb{R}}_+^*$. From the condition (3.1) it follows at once that

$$r, s \in \mathbb{R} \quad \text{and} \quad s < r \Rightarrow \alpha_r^\bullet \leq \alpha_s^\bullet. \tag{3.2}$$

PROPOSITION 3.4. *The relation $x \leq y$ of Definition* 3.2 *is a partial order relation on $\overline{\mathbb{R}}$.*

PROOF. Follows at once by remarking that from Lemma 3.1(a) we have that, for given $x, y \in \overline{\mathbb{R}}$, the relation $x \leq y$ means that

$$\begin{cases} \text{for any representatives } \widehat{x} \text{ and } \widehat{y} \text{ of } x \text{ and } y, \text{ there exists } N \in \mathbb{N} \\ \text{such that for all } b > 0 \text{ and for all } \varphi \in A_N \text{ there exists } \eta = \eta(b, \varphi) \in \mathbf{I} \\ \text{verifying } \widehat{x}(\varphi_\varepsilon) \leq \widehat{y}(\varphi_\varepsilon) + \varepsilon^b \text{ for all } \varepsilon \in\subset_\eta . \end{cases} \tag{3.3}$$

□

EXAMPLE 3.5. The partial order relation on $\overline{\mathbb{R}}$ it is not linear. Indeed, by defining

$$\widehat{x}(\varphi) := i(\varphi) \sin\left(\frac{1}{i(\varphi)}\right) \quad \text{for all } \varphi \in A_0$$

it is easily seen that $x := \mathrm{cl}(\widehat{x}) \geq 0$ and $x \leq 0$ are false.

PROPOSITION 3.6. *For every $x \in \overline{\mathbb{R}}_+$ and $p \in \mathbb{N}^*$ there is a unique $y \in \overline{\mathbb{R}}_+$ such that $y^p = x$ (this y is denoted by $x^{1/p}$ or $\sqrt[p]{x}$ and it is called q-positive pth-root of x).*

PROOF. Let x_* be the representative of x as in Lemma 3.1(c). Then the function $y_*\colon \varphi \in A_0 \mapsto \sqrt[p]{x_*(\varphi)} \in \mathbb{R}_+$ is well defined and moderate. Clearly $y := \mathrm{cl}(y_*) \in \overline{\mathbb{R}}_+$ and $y^p = x$. □

For every $x \in \overline{\mathbb{K}}$, if $\widehat{x}$ is any representative of x, the function $|\widehat{x}|\colon \varphi \in A_0 \mapsto |\widehat{x}(\varphi)| \in \mathbb{R}_+$ is obviously moderate and since $|\widehat{x}|(\varphi) = |\widehat{x}(\varphi)| \geq 0$ for all $\varphi \in A_0$, it follows that $\mathrm{cl}(|\widehat{x}|) \in \overline{\mathbb{R}}_+$. By the triangle inequality it follows that $\mathrm{cl}(|\widehat{x}|)$ is independent of the representative $\widehat{x}$ of x and only depends of x, hence it is natural to denote this class by $|x|$, i.e.

$$|x| := \mathrm{cl}(|\widehat{x}|)$$

is called absolute value (or module) of x. So we have a natural map

$$x \in \overline{\mathbb{K}} \mapsto |x| \in \overline{\mathbb{R}}_+.$$

The definition below depends on the above concept of absolute value and on Example 3.3.

Definition 3.7. If $x_0 \in \overline{\mathbb{K}}$ and $r \in \mathbb{R}$, then we define

$$V_r[x_0] := \{x \in \overline{\mathbb{K}} \mid |x - x_0| \leq \alpha_r^\bullet\} \quad \text{and} \quad \mathcal{B} := \{V_r[0] \mid r \in \mathbb{R}\}.$$

Remark 3.8. In view of Lemma 3.1(a), the statement $x \in V_r[0]$ is equivalent to the following sentence:

$$\text{(3.4)} \quad \begin{cases} \text{if } \widehat{x} \text{ is any representative of } x \text{ then there exists } N \in \mathbb{N} \text{ such that} \\ \text{for all } b > 0 \text{ and for all } \varphi \in A_N \text{ there exists } \eta = \eta(b, \varphi) \in \subset \\ \text{verifying } |\widehat{x}(\varphi_\varepsilon)| \leq i(\varphi)^r \varepsilon^r + \varepsilon^b, \text{ for all } \varepsilon \in \subset_\eta \end{cases}$$

Lemma 3.9. *The set $\mathcal{B}$ (see Definition 3.7) is a filter basis on $\overline{\mathbb{K}}$ which satisfies the four conditions in Proposition 2.1. Therefore, $\mathcal{B}$ determines a Hausdorff topology $\mathcal{T}$ compatible with the ring structure of $\overline{\mathbb{K}}$.*

Proof. Clearly $\mathcal{B} \neq \emptyset$ and $\emptyset \notin \mathcal{B}$. The implication

$$\text{if } r, s \in \mathbb{R} \text{ then there exists } t \in \mathbb{R} \text{ such that } V_t[0] \subset V_r[0] \cap V_s[0]$$

follows trivially from the statement

$$\text{if } r, s \in \mathbb{R} \text{ and } r < s \text{ then } V_s[0] \subset V_r[0],$$

which follows at once from (3.4).

Statement (3.4) implies also that $\mathcal{B}$ satisfies the four conditions of Proposition 2.1. Indeed, we get easily that $V_{r+1}[0] + V_{r+1}[0] \subset V_r[0]$ for all $r \in \mathbb{R}$, which shows that $(\mathrm{GA}'_{\mathrm{I}})$ holds. The equality $V_r[0] = -V_r[0]$ for all $r \in \mathbb{R}$ shows that $(\mathrm{GA}'_{\mathrm{II}})$ holds. Now we must show that:

$(\mathrm{AV}'_{\mathrm{I}})$ For given $a \in \overline{\mathbb{K}}$ and $V_r[0] \in \mathcal{B}$ there exists $s \in \mathbb{R}$ such that

$$\text{(3.5)} \qquad a.V_s[0] \subset V_r[0].$$

In fact, since $a \in \overline{\mathbb{K}}$, if $\widehat{a}$ is any representative of a there exists $N \in \mathbb{N}$ such that for every $\varphi \in A_N$ there exist $c > 0$ and $\sigma \in \subset$ satisfying

$$|\widehat{a}(\varphi_\varepsilon)| \leq c\varepsilon^{-N} \quad \text{for all } \varepsilon \in \subset_\sigma .$$

Then, it is easily seen that (3.5) holds by defining $s := r + N + 1$.

Next, we shall show that

$(\mathrm{AV}'_{\mathrm{II}})$ Given $V_r[0] \in \mathcal{B}$ there is $s \in \mathbb{R}$ such that $V_s[0]V_s[0] \subset V_r[0]$,

which is trivial by defining $s := (r+1)/2$.

Finally, $\mathcal{T}$ is Hausdorff since it is easily seen that $\bigcap_{r>0} V_r[0] = \{0\}$. □

Definition 3.10. The topology $\mathcal{T}$ on $\overline{\mathbb{K}}$ of Lemma 3.9 is called *scalar full sharp topology*.

The definition of the partial order relation and the absolute value in $\overline{\mathbb{K}}$ just introduced in this Section 3 are rather simple generalizations of the analogous definitions in $\overline{\mathbb{K}}_s$ developed in [5, Lemma 3.1, Definitions 3.2 and 3.4] that we shall use freely in the sequel.

Clearly, we can adapt Definition 3.7 by introducing, for $x_0 \in \overline{\mathbb{K}}_s$ and $r \in \mathbb{R}$ the set

$$V_r(x_0) := \{x \in \overline{\mathbb{K}}_s \mid |x - x_0| \leq \alpha_r\} \quad \text{and} \quad \mathcal{B}_s := \{V_r(0) \mid r \in \mathbb{R}\}.$$

Remark 3.11. From [5, Lemma 3.1] it is easily seen that the statement $x \in V_r(0)$ is equivalent to the following sentence:

$$\text{(3.6)} \quad \begin{cases} \text{if } \widehat{x} \text{ is any representative of } x \text{ then for all } b > 0 \text{ there exists } \eta_b \in \mathbf{I} \\ \text{such that } |\widehat{x}(\varepsilon)| \leq \varepsilon^r + \varepsilon^b \text{ for all } \varepsilon \in \subset_{\eta_b} . \end{cases}$$

With the above notation, introduced just before Remark 3.11, we have the following "simplified version" of Lemma 3.9.

Lemma 3.12. *The set $\mathcal{B}_s$ is a filter basis on $\overline{\mathbb{K}}_s$ which satisfies the four conditions of Proposition* 2.1. *Therefore, $\mathcal{B}_s$ determine a topology $\mathcal{T}_s$ compatible with the ring structure of $\overline{\mathbb{K}}_s$.*

Proof. The argument is a minor modification of the proof of Lemma 3.9 by using (3.6) instead (3.4). □

The topology $\mathcal{T}_s$ of Lemma 3.12 is indeed a familiar one:

Theorem 3.13. *Let τ_s be the "sharp topology" on $\overline{\mathbb{K}}_s$ (see* [12], [13] *and* [4]*), then*

(a) *$\mathcal{T}_s = \tau_s$;*

(b) *$\mathcal{T}_s$ coincides with the topology $j_m^{-1}(\mathcal{T})$, where $\mathcal{T}$ was introduced in Definition* 3.10 *and*

$$j_m \colon \lambda \in \overline{\mathbb{K}}_s \mapsto \mathrm{cl}(\widehat{\lambda}_*) \in \overline{\mathbb{K}} = \overline{\mathbb{K}}(\mathbb{R}^m),$$

being $\widehat{\lambda}$ an arbitrary representative of λ and $\widehat{\lambda}_*(\varphi) := \widehat{\lambda}(\min(1, i(\varphi)))$ for all $\varphi \in A_0 = A_0(\mathbb{R}^m; \mathbb{K})$.

Note that j_m is the natural injective ring-homomorphism of $\overline{\mathbb{K}}_s$ into $\overline{\mathbb{K}}$, hence we can identifies $\overline{\mathbb{K}}_s$ with $j_m(\overline{\mathbb{K}}_s)$ and write $\overline{\mathbb{K}}_s \subset \overline{\mathbb{K}}$. So we can rewrite the statement (b) of Theorem 3.13 by saying "$\mathcal{T}_s = \tau_s$ is the topology induced by $\mathcal{T}$".

PROOF. (a) For given $x_0 \in \overline{\mathbb{K}}_s$ and $\rho \in \mathbb{R}^*_+$ we set (see [4, Notation 1.5])

$$B_\rho(x_0) := \{x \in \overline{\mathbb{K}}_s \mid ||x - x_0|| < \rho\}.$$

Since the set of all balls $B_\rho(0)$ when ρ ranges over $\mathbb{R}^*_+$ is a fundamental system of τ_s-neighbourhoods of 0, it is enough to show the two following statements:

(3.7) for all $r \in \mathbb{R}$ there exists $\rho \in \mathbb{R}^*_+$ such that $B_\rho(0) \subset V_r(0)$

and

(3.8) for all $\rho \in \mathbb{R}^*_+$ there exists $r \in \mathbb{R}$ such that $V_r(0) \subset B_\rho(0)$.

In order to prove (3.7) fix an arbitrary $r \in \mathbb{R}$ and consider $\rho \in \mathbb{R}^*_+$ such that $\rho \leq e^{-r}$. We shall show that $B_\rho(0) \subset V_r(0)$. Indeed,

$$x \in B_\rho(0) \Leftrightarrow ||x|| = e^{-V(\widehat{x})} < \rho \Leftrightarrow V(\widehat{x}) > -\log \rho,$$

where $\widehat{x}$ is any representative of x. The preceding inequality and the definition of $V(\widehat{x})$ imply that there is $\eta \in \subset$ such that

$$\frac{|\widehat{x}(\varepsilon)|}{\varepsilon^{-\log \rho}} < 1 \quad \text{for all } \varepsilon \in \subset_\eta .$$

Hence, from the choice of ρ, it follows that

$$|\widehat{x}(\varepsilon)| \leq \widehat{\alpha}_{-\log \rho}(\varepsilon) \leq \widehat{\alpha}_r(\varepsilon) \quad \text{for all } \varepsilon \in \subset_\eta$$

which implies (see (3.6)) that $x \in V_r(0)$.

Let us now prove (3.8). To this end, fix an arbitrary $\rho \in \mathbb{R}^*_+$ and consider $r \in \mathbb{R}$ such that $e^{-r} < \rho$. We shall show that $V_r(0) \subset B_\rho(0)$. In the proof of the preceding inclusion we shall need the following trivial statement:

(3.9) if $z \in \overline{\mathbb{K}}_s$ and $|z| \leq 1$ then $V(\widehat{z}) \geq 0$ for every representative $\widehat{z}$ of z.

Now, for a given $x \in \overline{\mathbb{K}}_s$ we have

$$x \in V_r(0) \Leftrightarrow |x| \leq \alpha_r \Leftrightarrow |\alpha_{-r} x| \leq 1$$

and therefore, by (3.9), if $\widehat{x}$ is any representative of x we get (see [4, Proposition 1.3(c)]) $V(\widehat{\alpha}_{-r}.\widehat{x}) = -r + V(\widehat{x}) \geq 0$ thus $V(\widehat{x}) \geq r$. Therefore, from the choice of r, we then conclude that $||x|| = e^{-V(\widehat{x})} \leq e^{-r} < \rho$, i.e. $x \in B_\rho(0)$.

(b) It is enough to show that $j_m^{-1}(V_r[0]) = V_r(0)$ for all $r \in \mathbb{R}$, which follows, as usual, from (3.6) and (3.4). □

4. The full sharp topology

We use again the notation of Section 3. It will be convenient to give the following characterizations of the elements of $\mathcal{E}_M[\Omega]$ and $\mathcal{N}[\Omega]$ (of course, equivalent to the ones given in [2, Notation 2.1.1, (b) and (d)]): $u \in \mathcal{E}_M[\Omega]$ if and only if

$$\begin{cases} \text{for all } l, p \in \mathbb{N} \text{ there exist } N \in \mathbb{N} \text{ and } \sigma \in \mathbb{R} \\ \text{such that for all } \varphi \in A_N \text{ we have } ||\partial^\beta u(\varphi_\varepsilon, \cdot)||_l = o(\varepsilon^\sigma), \\ \text{if } \varepsilon \downarrow 0, \text{ whenever } |\beta| \leq p; \end{cases}$$

and $u \in \mathcal{N}[\Omega]$ if and only if

$$\begin{cases} \text{for all } l, p \in \mathbb{N} \text{ for all } \sigma \in \mathbb{R} \text{ there exists } N \in \mathbb{N} \\ \text{such that for all } \varphi \in A_N \text{we have } ||\partial^\beta u(\varphi_\varepsilon, \cdot)||_l = o(\varepsilon^\sigma), \\ \text{if } \varepsilon \downarrow 0, \text{ whenever } |\beta| \leq p. \end{cases}$$

We shall need the result below, whose trivial proof is omited.

LEMMA 4.1. *Let F, G and H be elements of $\mathcal{C}(\Omega; \mathbb{K})$ such that*

$$||F(x)| - |G(x)|| \leq |H(x)|, \quad \text{for all } x \in \Omega.$$

Then, for each $K \subset\subset \Omega$, we have $|||F||_K - ||G||_K| \leq ||H||_K$.

LEMMA 4.2. *For given $u \in \mathcal{E}_M[\Omega]$, $\beta \in \mathbb{N}^m$ and $l \in \mathbb{N}$, we have:*

(a) *the function $u_l^\beta : \varphi \in A_0 \mapsto ||\partial^\beta u(\varphi, \cdot)||_l \in \mathbb{R}_+$ is moderate (i.e. $u_l^\beta \in \mathcal{E}_M(\mathbb{R})$) and $\mathrm{cl}(u_l^\beta) \in \overline{\mathbb{R}}_+$;*

(b) *if $v \in \mathcal{E}_M[\Omega]$ and $u - v \in \mathcal{N}[\Omega]$, then $u_l^\beta - v_l^\beta \in \mathcal{N}(\mathbb{R})$ and hence $\mathrm{cl}(u_l^\beta) = \mathrm{cl}(v_l^\beta)$.*

PROOF. The assertion (a) is clear and the statement (b) follows at once from definitions and Lemma 4.1. □

The definition below makes sense from Lemma 4.2.

DEFINITION 4.3. Fix $\beta \in \mathbb{N}^m$ and $l \in \mathbb{N}$. For every $f \in \mathcal{G}(\Omega)$ we define

$$||f||_{\beta,l} := \mathrm{cl}(\widehat{f}_l^\beta) = \mathrm{cl}[\varphi \in A_0 \mapsto ||\partial^\beta \widehat{f}(\varphi, \cdot)||_l \in \mathbb{R}_+]$$

where $\widehat{f}$ is any representative of f. For every $f_0 \in \mathcal{G}(\Omega)$ and $r \in \mathbb{R}$ we define

$$W_{l,r}^\beta[f_0] := \{f \in \mathcal{G}(\Omega) \mid ||f - f_0||_{\sigma,l} \leq \alpha_r^\bullet, \text{ whenever } \sigma \leq \beta\}$$

and

$$\mathcal{B}_\Omega := \{W_{l,r}^\beta[0] \mid \beta \in \mathbb{N}^m, \ l \in \mathbb{N} \text{ and } r \in \mathbb{R}\}.$$

For the sake of simplicity, in the statement of the result below, we consider that $\overline{\mathbb{K}} \subset \mathcal{G}(\Omega)$ as the generalized constants.

LEMMA 4.4. *For all $\beta \in \mathbb{N}^m$, $l \in \mathbb{N}$ and $r \in \mathbb{R}$ we have $\overline{\mathbb{K}} \cap W^{\beta}_{l,r}[0] = V_r[0]$ (see Definition* 3.7*).*

PROOF. The inclusion $\overline{\mathbb{K}} \subset \mathcal{G}(\Omega)$ is given by the canonical homomorphism of $\overline{\mathbb{K}}$-algebras

$$\lambda = \mathrm{cl}(\widehat{\lambda}) \in \overline{\mathbb{K}} \mapsto \lambda^* := \mathrm{cl}[(\varphi, x) \in A_0 \times \Omega \mapsto \widehat{\lambda}(\varphi) \in \mathbb{R}_+] \in \mathcal{G}(\Omega).$$

Therefore

$$\lambda \in \overline{\mathbb{K}} \cap W^{\beta}_{l,r}[0] \Leftrightarrow \lambda \in \overline{\mathbb{K}} \quad \text{and} \quad ||\lambda^*||_{\sigma,l} \leq \alpha_r^{\bullet}, \quad \text{for all } \sigma \leq \beta$$

and since from Definition 4.3 we get (where δ_{ij} denotes the Kronecker δ):

$$||\lambda^*||_{\sigma,l} = |\lambda|.\delta_{0,|\sigma|} \quad (\text{for all } \sigma \leq \beta),$$

it follows that

$$\lambda \in \overline{\mathbb{K}} \cap W^{\beta}_{l,r}[0] \Leftrightarrow \lambda \in \overline{\mathbb{K}} \quad \text{and} \quad |\lambda| \leq \alpha_r^{\bullet} \Leftrightarrow \lambda \in V_r[0]. \qquad \square$$

REMARK 4.5. (a) From Lemma 3.1(a) it is easily seen that the statement $f \in W^{\beta}_{l,r}[0]$ is equivalent to the following sentence:

$$(4.1) \quad \begin{cases} \text{for every representative } \widehat{f} \text{ of } f, \text{ there exists } N \in \mathbb{N} \text{ such that} \\ \text{for all } b > 0 \text{ and all } \varphi \in A_N \text{ there exists } \eta = \eta(b, \varphi) \in \subset \\ \text{satisfying } ||\partial^{\sigma} \widehat{f}(\varphi_{\varepsilon}, \cdot)||_l \leq i(\varphi)^r \varepsilon^r + \varepsilon^b, \text{for all } \varepsilon \in \subset_{\eta} \text{ and all } \sigma \leq \beta. \end{cases}$$

(b) The following three statements hold:

(b1) If $\lambda \geq \lambda'$ then $W^{\lambda}_{p,r}[0] \subset W^{\lambda'}_{p,r}[0]$, for all $(p, r) \in \mathbb{N} \times \mathbb{R}$ (here, $\lambda \geq \lambda'$ means that $\lambda_j \geq \lambda'_j$, for all $j = 1, \ldots, m$).
(b2) If $r > r'$ then $W^{\lambda}_{p,r}[0] \subset W^{\lambda}_{p,r'}[0]$, for all $(\lambda, p) \in \mathbb{N}^m \times \mathbb{N}$.
(b3) If $p > q$ then $W^{\lambda}_{p,r}[0] \subset W^{\lambda}_{q,r}[0]$, for all $(\lambda, r) \in \mathbb{N}^m \times \mathbb{R}$.

THEOREM 4.6. (a) *The set $\mathcal{B}_{\Omega}$ (see Definition* 4.3*) is a filter basis on $\mathcal{G}(\Omega)$ which satisfies the four conditions* $(\mathrm{GA}'_{\mathrm{I}})$, $(\mathrm{GA}'_{\mathrm{II}})$, $(\mathrm{AV}'_{\mathrm{I}})$ *and* $(\mathrm{AV}'_{\mathrm{II}})$ *of Corollary* 2.3*.*

(b) *There exists a unique topology $\mathcal{T}_{\Omega}$ on $\mathcal{G}(\Omega)$ verifying the following conditions:*

(b1) *$\mathcal{T}_{\Omega}$ is compatible with the ring structure of $\mathcal{G}(\Omega)$. The topology induced by $\mathcal{T}_{\Omega}$ on $\overline{\mathbb{K}}$ coincides with the topology $\mathcal{T}$ (see Definition* 3.10*).*
(b2) *$\mathcal{G}(\Omega)$ is a $\overline{\mathbb{K}}$-topological algebra (when $\mathcal{G}(\Omega)$ and $\overline{\mathbb{K}}$ are endowed with $\mathcal{T}_{\Omega}$ and $\mathcal{T}$, respectively).*
(b3) *$\mathcal{B}_{\Omega}$ is a fundamental system of $\mathcal{T}_{\Omega}$-neighbourhoods of* 0*.*
(b4) *$\mathcal{T}_{\Omega}$ is metrizable.*

PROOF. We abbreviate notation by writing $W^{\beta}_{p,r}$ instead of $W^{\beta}_{p,r}[0]$.

(a) Clearly $\mathcal{B}_\Omega \neq \emptyset$ and $\emptyset \notin \mathcal{B}_\Omega$ since $0 \in W^{\beta}_{p,r}$ for all β, p, r. Given $W^{\beta}_{p,r}$ and $W^{\gamma}_{q,s}$ in $\mathcal{B}_\Omega$, if we define $\lambda = (\lambda_1, \dots, \lambda_m)$ where $\lambda_j := \max(\beta_j, \gamma_j)$ $(1 \leq j \leq m)$, $l := \max(p, q)$ and $t := \max(r, s)$, from Remark 4.5(b), we get $W^{\lambda}_{l,t} \subset W^{\beta}_{p,r} \cap W^{\gamma}_{q,s}$. Hence, $\mathcal{B}_\Omega$ is a filter basis on $\mathcal{G}(\Omega)$. The verification of that $\mathcal{B}_\Omega$ satisfies the four conditions of Corollary 2.3 is easy but rather tedious and here we only sketch these proofs.

Condition $(\mathrm{GA}'_{\mathrm{I}})$. Given any $W^{\beta}_{p,r}$ we have

$$W^{\beta}_{p,s} + W^{\beta}_{p,s} \subset W^{\beta}_{p,r}, \quad \text{for all } s > r.$$

Indeed, this follows easily from the implication

$$s > r \Rightarrow 2\alpha^{\bullet}_{s} \leq \alpha^{\bullet}_{r},$$

which is a consequence of Lemma 3.1(a).

Condition $(\mathrm{GA}'_{\mathrm{II}})$. Obvious, since $W^{\beta}_{p,r} = -W^{\beta}_{p,r}$ for all β, p, r.

Condition $(\mathrm{AV}'_{\mathrm{I}})$. Given $f_0 \in \mathcal{G}(\Omega)$ and $W^{\beta}_{p,r}$ there exists $s \in \mathbb{R}$ such that

$$f_0 W^{\beta}_{p,s} \subset W^{\beta}_{p,r}. \tag{4.2}$$

Fix a representative $\widehat{f_0}$ of f then, from the moderation we can find $N_1 \in \mathbb{N}$ and $r' < 0$ such that for all $\varphi \in A_{N_1}$ there exist $C > 0$ and $\eta_1 = \eta_1(\varphi) \in \subset$ satisfying

$$||\partial^{\mu} \widehat{f}_0(\varphi_\varepsilon, \cdot)||_p \leq C\varepsilon^{r'} \quad \text{for all } \varepsilon \in I_{\eta_1}, \text{ whenever } |\mu| \leq |\beta|. \tag{4.3}$$

Define $s := r - r' + 1 = r + |r'| + 1$. Now, the proof of (4.2) is easy: fix any representative $\widehat{g}$ of an arbitrary $g \in W^{\beta}_{p,s}$, we want to prove that $f_0 g \in W^{\beta}_{p,r}$. From $g \in W^{\beta}_{p,s}$ and (4.1) we have an inequality for $||\partial^{\sigma} \widehat{g}(\varphi_\varepsilon, \cdot)||_p$ $(\sigma \leq \beta)$ which jointly with (4.3) and the Leibnitz formula for derivation of a product, give an inequality for $||\partial^{\lambda}(\widehat{f_0}\widehat{g})(\varphi_\varepsilon, \cdot)||_p$ $(\lambda \leq \beta)$, which shows (by (4.1)) that $f_0 g \in W^{\beta}_{p,r}$.

Condition $(\mathrm{AV}'_{\mathrm{II}})$. Given $W^{\beta}_{p,r}$ there is $s \in \mathbb{R}$ such that $W^{\beta}_{p,s}.W^{\beta}_{p,s} \subset W^{\beta}_{p,r}$.

Define $s := (r + 1)/2$. In order to prove the above inclusion, for fixed arbitrary $f, g \in W^{\beta}_{p,s}$ we want to proof that $fg \in W^{\beta}_{p,r}$. From $f, g \in W^{\beta}_{p,s}$ and (4.1), for fixed arbitrary representatives $\widehat{f}, \widehat{g}$ of f, g, respectively, we get two inequalities for $||\partial^{\sigma} \widehat{f}(\varphi_\varepsilon, \cdot)||_p$ and $||\partial^{\sigma} \widehat{g}(\varphi_\varepsilon, \cdot)||_p$ $(\sigma \leq \beta)$. Next, we apply Leibnitz formula which, jointly with the above inequalities, leads to an inequality for $||\partial^{\lambda}(\widehat{f}\,\widehat{g})(\varphi_\varepsilon, \cdot)||_p$ $(\lambda \leq \beta)$, which shows (by (4.1)) that $f, g \in W^{\beta}_{p,r}$.

(b) In view of (a) and Corollary 2.3 we can conclude that there exists a unique topology $\mathcal{T}_\Omega$ on $\mathcal{G}(\Omega)$ satisfying the three following conditions:

(b1') $\mathcal{T}_\Omega$ is compatible with the ring structure of $\mathcal{G}(\Omega)$. The topology $\mathcal{T}_{\overline{\mathbb{K}}}$ induced by $\mathcal{T}_\Omega$ on $\overline{\mathbb{K}}$, is compatible with the ring structure of $\overline{\mathbb{K}}$;

(b2') $\mathcal{G}(\Omega)$ is a $\overline{\mathbb{K}}$-topological algebra ($\mathcal{G}(\Omega)$ and $\overline{\mathbb{K}}$ are endowed with $\mathcal{T}_\Omega$ and $\mathcal{T}_{\overline{\mathbb{K}}}$, respectively);

(b3') $\mathcal{B}_\Omega$ is a fundamental system of $\mathcal{T}_\Omega$-neighbourhoods of 0.

Now, by Lemma 4.4, it is clear that the topology induced by $\mathcal{T}_\Omega$ on $\overline{\mathbb{K}}$ is $\mathcal{T}$ (see Definition 3.10), hence $\mathcal{T}_{\overline{\mathbb{K}}} = \mathcal{T}$, and therefore (b1') implies (b1) and (b2') implies (b2).

Since $\lim_{r\to\infty} \alpha_r^\bullet = 0$, it is clear that the set

$$\{W_{p,r}^\beta \mid \beta \in \mathbb{N}^m,\ p \in \mathbb{N} \text{ and } r \in \mathbb{N}\}$$

is a countable fundamental system of $\mathcal{T}_\Omega$-neighbourhoods of 0 and so (b4) is true. □

Definition 4.7. The topology $\mathcal{T}_\Omega$ on $\mathcal{G}(\Omega)$ in Theorem 4.6 is called *full sharp topology*.

In the sequel, we shall show that the topology induced by $\mathcal{T}_\Omega$ on $\mathcal{G}_s(\Omega)$ (see notation in Section 3) is precisely the sharp topology τ_Ω (see [12], [13] and [4]). Here, we shall develop a procedure analogous to the one at the beginning of Section 4, proving the analog of Lemma 4.2 for $\mathcal{G}_s(\Omega)$, which allows us to define a filter basis $\mathcal{B}_{s,\Omega}$ on $\mathcal{G}_s(\Omega)$ which will determine a topology $\mathcal{T}_{s,\Omega}$ on $\mathcal{G}_s(\Omega)$. Next, we shall show that

$$W_{p,r}^\beta[0] \cap \mathcal{G}_s(\Omega) = W_{p,r}^\beta(0) \quad \text{for all } \beta,\ p,\ r$$

where $W_{p,r}^\beta(0)$ denote a general element of $\mathcal{B}_{s,\Omega}$. Hence $\mathcal{T}_{s,\Omega}$ will be the topology induced on $\mathcal{G}_s(\Omega)$ by $\mathcal{T}_\Omega$. Finally, we shall show that $\mathcal{T}_{s,\Omega} = \tau_\Omega$.

Lemma 4.8. *Given $u \in \mathcal{E}_{M,s}[\Omega]$, $\beta \in \mathbb{N}^m$ and $l \in \mathbb{N}$, we have:*

(a) *the function $u_l^\beta{:}\,\varepsilon \in\subset\mapsto ||\partial^\beta u(\varepsilon, \cdot)||_l \in \mathbb{R}_+$ is moderate (i.e. $u_l^\beta \in \mathcal{E}_{M,s}(\mathbb{R})$) and $\mathrm{cl}(u_l^\beta) \in \overline{\mathbb{R}}_{s,+}$;*

(b) *if $v \in \mathcal{E}_{M,s}[\Omega]$ and $u - v \in \mathcal{N}_s[\Omega]$, then $u_l^\beta - v_l^\beta \in \mathcal{N}_s(\mathbb{R})$ and therefore $\mathrm{cl}(u_l^\beta) = \mathrm{cl}(v_l^\beta)$.*

Proof. Follows at once from slight modifications in the proof of Lemma 4.2. □

Definition 4.9. Fix $\beta \in \mathbb{N}^m$ and $l \in \mathbb{N}$. For every $f \in \mathcal{G}_s(\Omega)$ we define

$$|f|_{\beta,l} = \mathrm{cl}(f_{*l}^{\ \beta}) = \mathrm{cl}[\varepsilon \in\subset\mapsto ||\partial^\beta f_*(\varepsilon, \cdot)||_l \in \mathbb{R}_+],$$

where f_* is any representative of f. For every $f_0 \in \mathcal{G}_s(\Omega)$ and $r \in \mathbb{R}$ we define

$$W_{l,r}^\beta(f_0) := \{f \in \mathcal{G}_s(\Omega) \mid |f - f_0|_{\sigma,l} \le \alpha_r \text{ whenever } \sigma \le \beta\}$$

and

$$\mathcal{B}_{s,\Omega} := \{W_{l,r}^\beta(0) \mid \beta \in \mathbb{N}^m,\ l \in \mathbb{N} \text{ and } r \in \mathbb{R}\}.$$

Here we have also remarks analogous to Remark 4.5.

REMARK 4.10. (a) From [5, Lemma 3.1(i)] it follows at once that the statement $f \in W^{\beta}_{l,r}(0)$ is equivalent to the following sentence

$$\begin{cases} \text{every representative } f_* \text{ of } f \text{ satisfies:} \\ \text{for all } b > 0 \text{ there exists } \eta_b \in \subset \text{ such that} \\ ||\partial^{\sigma} f_*(\varepsilon, \cdot)||_l \leq \varepsilon^r + \varepsilon^b, \text{ for all } \varepsilon \in \subset_{\eta_b} \text{ and all } \sigma \leq \beta. \end{cases} \tag{4.4}$$

(b) The three implications of Remark 4.5(b) hold for the sets $W^{\beta}_{l,r}(0)$, we omit them since it is suffices to change $W^{\beta}_{l,r}[0]$ by $W^{\beta}_{l,r}(0)$.

Since $\mathcal{G}_s(\Omega) \subset \mathcal{G}(\Omega)$, for given $f \in \mathcal{G}_s(\Omega)$, $\beta \in \mathbb{N}^m$ and $l \in \mathbb{N}$, the two generalized numbers $|f|_{\beta,l} \in \overline{\mathbb{K}}_s$ and $||f||_{\beta,l} \in \overline{\mathbb{K}}$ make sense. Moreover, since $\overline{\mathbb{K}}_s \subset \overline{\mathbb{K}}$, it is natural to ask the relation between these two generalized numbers. We shall show a nice answer to this question

$$|f|_{\beta,l} = ||f||_{\beta,l} \quad \text{for all } f, \beta, l. \tag{4.5}$$

More precisely, if $j_m : \overline{\mathbb{K}}_s \to \overline{\mathbb{K}}$ denote the canonical inclusion of $\overline{\mathbb{K}}_s$ into $\overline{\mathbb{K}}$ (see Theorem 3.13(b)) and

$$\Psi^*_{\Omega} : f \in \mathcal{G}_s(\Omega) \mapsto \mathrm{cl}[\Psi_{\Omega}(f_*)] \in \mathcal{G}(\Omega)$$

denotes the canonical inclusion of $\mathcal{G}_s(\Omega)$ into $\mathcal{G}(\Omega)$ (where $f_* \in \mathcal{E}_{M,s}[\Omega]$ is any representative of f and $\Psi_{\Omega} : u \in \mathcal{E}_{M,s}[\Omega] \mapsto u_1 \in \mathcal{E}_M[\Omega]$ is the ring homomorphism defined by $u_1(\varphi, x) := u(\min(1, i(\varphi)), x)$ for each $(\varphi, x) \in A_0 \times \Omega$) then we have:

LEMMA 4.11.

(a) $j_m(|f|_{\beta,l}) = ||\mathrm{cl}[\Psi_{\Omega}(f_*)]||_{\beta,l}$ *for every* $f \in \mathcal{G}_s(\Omega)$, $\beta \in \mathbb{N}^m$, $l \in \mathbb{N}$ *and any representative* f_* *of* f.

(b) $|f|_{\beta,l} \leq \alpha_r$ *if and only if* $||\mathrm{cl}[\Psi_{\Omega}(f_*)]||_{\beta,l} \leq \alpha^{\bullet}_r$, *for every* $f \in \mathcal{G}_s(\Omega)$, $\beta \in \mathbb{N}^m$, $l \in \mathbb{N}$ *and any representative* f_* *of* f.

(c) ${\Psi^*_{\Omega}}^{-1}(W^{\beta}_{p,r}[0]) = W^{\beta}_{p,r}(0)$ *for every* $\beta \in \mathbb{N}^m$, $p \in \mathbb{N}$ *and* $r \in \mathbb{R}$.

PROOF. (a) Fix f, β, l and any representative $f_* \in \mathcal{E}_{M,s}[\Omega]$ of f. The equality follows at once from the definitions of $|f|_{\beta,l}$, j_m, Ψ_{Ω} and $||\mathrm{cl}[\Psi_{\Omega}(f_*)]||_{\beta,l}$.

(b) If f_* is a representative of f, the inequality $|f|_{\beta,l} \leq \alpha_r$ is equivalent to the statement (4.4). Analogously, $||\mathrm{cl}[\Psi_{\Omega}(f_*)]||_{\beta,l} \leq \alpha^{\bullet}_r$ is equivalent to the statement (4.1) for $\widehat{f} := \Psi_{\Omega}(f_*)$. Now, the equivalence between the two inequalities follows as a tedious application of (4.4) and (4.1).

(c) Follows immediately from (b). □

Note that (4.5) is an abuse of notation whose correct meaning is given in Lemma 4.11(a). Analogously, the relation

$$W^{\beta}_{p,r}[0] \cap \mathcal{G}_s(\Omega) = W^{\beta}_{p,r}(0) \quad \text{for all } \beta, p, r$$

is an abuse of notation whose correct meaning is given in Lemma 4.11(c).

Theorem 4.12. (a) *The set* $\mathcal{B}_{s,\Omega}$ (*see Definition* 4.9) *is a filter basis on* $\mathcal{G}_s(\Omega)$ *which satisfies the four axioms* $(\mathrm{GA}'_{\mathrm{I}})$, $(\mathrm{GA}'_{\mathrm{II}})$, $(\mathrm{AV}'_{\mathrm{I}})$ *and* $(\mathrm{AV}'_{\mathrm{II}})$ *of Corollary* 2.3.

(b) *There exist a unique topology* $\mathcal{T}_{s,\Omega}$ *on* $\mathcal{G}_s(\Omega)$ *verifying the following conditions:*

(b1) $\mathcal{T}_{s,\Omega}$ *is compatible with the ring structure of* $\mathcal{G}_s(\Omega)$. *The topology induced by* $\mathcal{T}_{s,\Omega}$ *on* $\overline{\mathbb{K}}_s$ *coincides with the topology* $\mathcal{T}_s$ (*see Lemma* 3.12) *and hence, with the sharp topology* τ_s (*see Theorem* 3.13)*;*

(b2) $\mathcal{G}_s(\Omega)$ *is a* $\overline{\mathbb{K}}_s$*-topological algebra* (*here we assume that* $\mathcal{G}_s(\Omega)$ *and* $\overline{\mathbb{K}}_s$ *are endowed with the topologies* $\mathcal{T}_{s,\Omega}$ *and* $\mathcal{T}_s$*, respectively*)*;*

(b3) $\mathcal{B}_{s,\Omega}$ *is a fundamental system of* $\mathcal{T}_{s,\Omega}$*-neighbourhoods of* 0*;*

(b4) $\mathcal{T}_{s,\Omega}$ *coincides with the sharp topology* τ_Ω*;*

(b5) $\mathcal{T}_{s,\Omega}$ *coincides with the topology induced by* $\mathcal{T}_\Omega$ *on* $\mathcal{G}_s(\Omega)$.

Proof. By identifying $\mathcal{G}_s(\Omega)$ with his image $\mathcal{G}(\Omega)$ by the canonical map Ψ^*_Ω we can write $\mathcal{G}_s(\Omega) \subset \mathcal{G}(\Omega)$ and then, the Lemma 4.11(c) shows that

$$W^\beta_{p,r}[0] \cap \mathcal{G}_s(\Omega) = W^\beta_{p,r}(0) \quad \text{for all } \beta,\ p,\ r \tag{4.6}$$

which means that

$$\mathcal{B}_{s,\Omega} = \{W \cap \mathcal{G}_s(\Omega) \mid W \in \mathcal{B}_\Omega\}. \tag{4.6'}$$

The above relation (4.6'), together with Theorem 4.6(a), implies (a). Now, from (a) and Corollary 2.3 we get (b), the first statement of (b1), (b2) and (b3). The proof of the second statement of (b1) follows at once by noting that

$$W^\beta_{p,r}(0) \cap \overline{\mathbb{K}}_s = V_r(0) \quad \text{for all } \beta,\ p,\ r$$

which is the simplified version of Lemma 4.4 and easily proved. Note also that (b5) follows directly from (4.6').

(b4) Let recall that if $u \in \mathcal{E}_{M,s}[\Omega]$ and $(n,p) \in \mathbb{N}^2$, we set

$$S_{np}(u) := \{a \in \mathbb{R} \mid ||\partial^\beta u(\varepsilon, \cdot)||_n = o(\varepsilon^a) \text{ if } \varepsilon \downarrow 0 \text{ for all } |\beta| \le p\}$$

and $v_{np} := \sup S_{np}(u)$. Moreover, it is easy to see that v_{np} is constant on every equivalence class $u + \mathcal{N}_s[\Omega]$, hence for each $f \in \mathcal{G}_s(\Omega)$ the (extended) real number

$$V_{np}(f) := v_{np}(f_*) \in \,]-\infty, \infty]$$

is well defined, where f_* is any representative of f. The sharp topology τ_Ω on $\mathcal{G}_s(\Omega)$ is defined by the family of pseudo metrics

$$d_{np}(f,g) := \exp(-V_{np}(f-g)) \quad \text{for all } f, g \in \mathcal{G}_s(\Omega) \text{ and all } n, p \in \mathbb{N}.$$

For each $a > 0$ we define the d_{np}-ball of center 0 and radius a:

$$\overline{B}_a(d_{np}) := \{f \in \mathcal{G}_s(\Omega) \mid d_{np}(f,0) \le a\}.$$

Then, the collection of all finite intersections of these balls is a fundamental system of τ_Ω-neighbourhoods of 0 in $\mathcal{G}_s(\Omega)$. Therefore it suffices to prove the two following statements:

$$(4.7)\quad \begin{cases} \text{For every } a \in \mathbb{R}_+^* \text{ and } (n,p) \in \mathbb{R}^2 \text{ there is a finite sequence} \\ (W_{l,r}^\beta(0))_{\beta\in B} \text{ of elements in } \mathcal{B}_{s,\Omega} \text{ such that } \bigcap_{\beta\in B} W_{l,r}^\beta(0) \subset \overline{B}_a(d_{np}). \end{cases}$$

and

$$(4.8)\quad \begin{cases} \text{For each } \beta \in \mathbb{N}^m, l \in \mathbb{N} \text{ and } r \in \mathbb{R} \\ \text{there is } \overline{B}_a(d_{np}) \text{ such that } \overline{B}_a(d_{np}) \subset W_{l,r}^\beta(0). \end{cases}$$

To obtain (4.7) fix $a \in \mathbb{R}_+^*$ and $(n,p) \in \mathbb{N}^2$. Then, clearly we can supose without lost of generality that $0 < a < e^{-1}$. Now, defining $l := n$ and choosing $r \in \mathbb{R}$ such that $r \geq -\log a + 1$, if $B := \{\beta \in \mathbb{N}^m \mid |\beta| = p\}$ it follows (apply (4.4) as usual) that $\bigcap_{\beta\in B} W_{n,r}^\beta(0) \subset \overline{B}_a(d_{np})$. To prove (4.8) fix $\beta \in \mathbb{N}^m$, $l \in \mathbb{N}$ and $r \in \mathbb{R}$. If we define $n := l$, $p := |\beta|$ and take $a \in]0, e^{-r}[$, one proves easily the inclusion $\overline{B}_a(d_{lp}) \subset W_{l,r}^\beta(0)$. □

5. A few results of convergence

PROPOSITION 5.1. (a) *Let M be a Lebesgue-measurable set such that $\overline{M} \subset\subset \Omega$. Then the $\overline{\mathbb{K}}$-linear function*

$$J_M\colon f \in \mathcal{G}(\Omega) \mapsto \int_M f \in \overline{\mathbb{K}}$$

is $\mathcal{T}_\Omega$-$\mathcal{T}$-continuous;

(b) *For every $\alpha \in \mathbb{N}^m$ the $\overline{\mathbb{K}}$-linear function $\partial^\alpha\colon f \in \mathcal{G}(\Omega) \mapsto \partial^\alpha f \in \mathcal{G}(\Omega)$ is $\mathcal{T}_\Omega$-$\mathcal{T}_\Omega$-continuous.*

PROOF. (a) Fix $V_r[0]$ with $r \in \mathbb{N}$ arbitrary. Then, for $\beta := 0 = (0, \dots, 0)$ in $\mathbb{N}^m$, $s := r+1$ and $l \in \mathbb{N}$ such that $\overline{M} \subset \Omega_l$ one proves easily that

$$J_M(W_{l,r+1}^0[0]) \subset V_r[0].$$

(b) For $W_{l,r}^\beta[0]$ given arbitrarily it is clear that $\partial^\alpha(W_{l,r}^{\alpha+\beta}[0]) \subset W_{l,r}^\beta[0]$. □

COROLLARY 5.2. *If $P = \sum_{|\alpha|\leq m} a_\alpha(x)\partial^\alpha$ is a generalized* LPDO *(i.e. a_α in $\mathcal{G}(\Omega)$ for all $|\alpha| \leq m$), then P defines a linear application*

$$P\colon f \in \mathcal{G}(\Omega) \mapsto Pf \in \mathcal{G}(\Omega)$$

which is $\mathcal{T}_\Omega$-$\mathcal{T}_\Omega$-continuous.

PROOF. It is enough to show that if $(f_l)_{l\in\mathbb{N}}$ is a sequence in $\mathcal{G}(\Omega)$ such that $f_l \xrightarrow{\mathcal{T}_\Omega} f \in \mathcal{G}(\Omega)$ then $Pf_l \xrightarrow{\mathcal{T}_\Omega} Pf$.

The continuity of the multiplication in $\mathcal{G}(\Omega)$ proves that for every $a \in \mathcal{G}(\Omega)$ the function $M_a\colon f \in \mathcal{G}(\Omega) \longmapsto af \in \mathcal{G}(\Omega)$ is continuous. From Proposition 5.1(b) we get $\partial^\alpha f_l \xrightarrow{\mathcal{T}_\Omega} \partial^\alpha f$ and hence $M_a(\partial^\alpha f_l) \xrightarrow{\mathcal{T}_\Omega} M_a(\partial^\alpha f)$ which implies that $a_\alpha \partial^\alpha f_l \xrightarrow{\mathcal{T}_\Omega} a_\alpha \partial^\alpha f$ for all $|\alpha| \leq m$. Now, the continuity of the addition in $\mathcal{G}(\Omega)$ proves that $Pf_l \xrightarrow{\mathcal{T}_\Omega} Pf$. □

COROLLARY 5.3. *If Ω is an open subset of $\mathbb{C}^m$ then the subalgebra $\mathcal{HG}(\Omega)$ of all holomorphic generalized functions on Ω is a $\mathcal{T}_\Omega$-closed subalgebra of $\mathcal{G}(\Omega)$.*

PROOF. Let (f_l) be a sequence in $\mathcal{HG}(\Omega)$ and assume that $f_l \xrightarrow{\mathcal{T}_\Omega} f \in \mathcal{G}(\Omega)$. From Corollary 5.2 above we have, for each $j = 1, \dots, m$:

$$0 = \frac{\partial f_l}{\partial \overline{z}_j} \to \frac{\partial f}{\partial \overline{z}_j}, \quad \text{if } l \to \infty,$$

hence

$$\frac{\partial f}{\partial \overline{z}_j} = 0 \quad \text{for every } j = 1, \dots, m. \qquad \square$$

Appendix: On $\overline{\mathbb{K}}$-locally convex modules ($\overline{\mathbb{K}}$-LCM)

With the notation introduced in Definition 4.3 it is clear that for given $\beta \in \mathbb{N}^m$ and $l \in \mathbb{N}$ we have

$$\begin{aligned} ||f+g||_{\beta,l} &\leq ||f||_{\beta,l} + ||g||_{\beta,l} && \text{for all } f, g \in \mathcal{G}(\Omega); \\ ||af||_{\beta,l} &= |a|\,||f||_{\beta,l} && \text{for all } f \in \mathcal{G}(\Omega) \text{ and all } a \in \overline{\mathbb{K}}, \end{aligned}$$

which suggest that we can try to mimic some basic facts of the general theory of LCS by defining:

DEFINITION A.1. Let E be a $\overline{\mathbb{K}}$-module. A *generalized semi-norm* (or a *G-seminorm*, for short) on E is a function $p\colon E \to \overline{\mathbb{R}}_+$ verifying the two conditions:

$$\begin{aligned} &\text{(GSN1)} && p(x+y) \leq p(x) + p(y) && \text{for all } x, y \in E; \\ &\text{(GSN2)} && p(ax) = |a|\,p(x) && \text{for all } x \in E \text{ and all } a \in \overline{\mathbb{K}}. \end{aligned}$$

With the above notation clearly we have

$$|p(x) - p(y)| \leq p(x-y) \quad \text{for all } x, y \in E$$

since the classical proof works in this case.

The remark preceding Definition A.1 shows that for every $\beta \in \mathbb{N}^m$ and $l \in \mathbb{N}$, the function

$$||\cdot||_{\beta,l}\colon f \in \mathcal{G}(\Omega) \mapsto ||f||_{\beta,l} \in \overline{\mathbb{R}}_+$$

is a G-seminorm.

Next, fix a $\overline{\mathbb{K}}$-module E and a G-seminorm p on E. For each $r \in \mathbb{R}$ we can define the p-ball of center 0 and radius $\alpha_r^\bullet$ by

$$B_{p,r} = B_{p,r}(0) := \{x \in E \mid p(x) \leq \alpha_r^\bullet\}$$

and for $x_0 \in E$ we set $B_{p,r}(x_0) := x_0 + B_{p,r} = \{x \in E \mid p(x - x_0) \leq \alpha_r^\bullet\}$. We can also define the generalized segment of extremities 0 and 1 by

$$[0,1]_g := \{\lambda \in \overline{\mathbb{R}} \mid 0 \leq \lambda \leq 1\},$$

note that $\alpha_r^\bullet \in [0,1]_g$ for every $r \in \mathbb{R}_+$.

A subset A of (the $\overline{\mathbb{K}}$-module) E is G-convex (resp. G-absolutely convex) if $x, y \in A$, then $\lambda x + (1-\lambda)y \in A$ for all $\lambda \in [0,1]_g$ (resp. if $x, y \in A$, then $\alpha x + \beta y \in A$ for all $\alpha, \beta \in \overline{\mathbb{K}}$ such that $|\alpha| + |\beta| \leq 1$). So the definition below makes sense.

Definition A.2. Let E be a $\overline{\mathbb{K}}$-module. A module topology $\mathcal{T}$ on E (i.e. a topology compatible with $\overline{\mathbb{K}}$-module structure) is said to be *G-locally convex* if 0 has a fundamental system of G-absolutely convex neighbourhoods.

Let A be a $\mathbb{K}$-algebra and Γ a non void set of seminorms in A such that:

(a) $p(x) = 0$ for all $p \in \Gamma$ if and only if $x = 0$;
(b) For all $p_1, p_2 \in \Gamma$ there is $q \in \Gamma$ such that $p_i \leq q$ $(i = 1, 2)$;
(c) For each $p \in \Gamma$ and each $\alpha > 0$ we have $\alpha p \in \Gamma$.

By defining $B_p := \{x \in A \mid p(x) \leq 1\}$ for all $p \in \Gamma$, one proves trivially that the continuity of the multiplication

$$(x, y) \in A \times A \to xy \in A$$

is equivalent to each of the two following conditions:

(i) for all $p \in \Gamma$ there exists $q \in \Gamma$ such that $B_q B_q \subset B_p$;
(ii) for all $p \in \Gamma$ there exists $q \in \Gamma$ such that $p(xy) \leq q(x)q(y)$ for all $x, y \in A$.

In this case, the Hausdorff topological algebra (A, Γ) is said to be a *locally multiplicatively-convex algebra* (see [10], in this book the condition (ii) has a slight mistake). Note that the topology $\mathcal{T}_\Gamma$ determinated by Γ on A is given (see Corollary 2.3) by the filter basis $\mathcal{B}_\Gamma$ of all finite intersection of balls B_p.

In our case ($\overline{\mathbb{K}}$ instead of $\mathbb{K}$), if A is a $\overline{\mathbb{K}}$-algebra and Γ is a non-void set of G-seminorms on A it is easy to prove that, for the conditions (i), (ii) above, we have (ii) $\Rightarrow$ (i) but, seemingly (i) $\Rightarrow$ (ii) is false. Therefore, it is natural, in our case, to say that the topological algebra (A, Γ) is a *G-locally multiplicatively-convex algebra* if Γ satisfies the above condition (i). Note that the proof of Theorem 4.6(a) ($\mathrm{AV}'_{\mathrm{II}}$) shows that $(\mathcal{G}(\Omega), \Gamma) = (\mathcal{G}(\Omega), \mathcal{T}_\Omega)$, where $\Gamma := \{||\cdot||_{\beta,l} \mid \beta \in \mathbb{N}^m \text{ and } l \in \mathbb{N}\}$, is a G-locally multiplicatively-convex algebra.

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