Topological **M**ethods in **N**onlinear **A**nalysis
Journal of the Juliusz Schauder Center
Volume 34, 2009, 141–160

EXTENSION THEOREMS AND TOPOLOGICAL ESSENTIALITY IN α-WEAKLY CONVEX METRIC SPACES

F. S. de Blasi — L. Górniewicz — G. Pianigiani

Dedicated to Andrzej Granas on the occasion of his 80th birthday

1. Introduction

In the present paper we consider some extension properties for continuous maps in metric spaces and, using them, we develop in metric spaces some aspects of Granas theory of topological essentiality [10].

Let M be a metric space and A a closed subset of M. It is known that if $f: A \to X$ is a continuous map with values contained in a normed space X then, by Dugundji's theorem [6], f admits a continuous extension defined on all of M.

We shall prove some metric versions of this result under the assumption that X is an *α-weakly convex metric space* (see Definition 2.1). In this way we improve a result obtained by Himmelberg [12] under the stronger assumption that X is a *strictly equiconnected metric space* (see Definition 2.2). As consequence we obtain that any α-weakly convex metric space is an AR-space. A similar extension result is obtained when X is a *locally α-weakly convex metric space* (see Definition 4.1)) and any such space turns out to be an ANR-space.

By using the above results we will develop some aspects of Granas theory of topological essentiality in α-weakly convex metric spaces. Furthermore, a few applications to fixed point theory are given, including a metric version of the Leray–Schauder alternative theorem.

2000 *Mathematics Subject Classification.* Primary 54C55, 52A05, 54H25.
Key words and phrases. Extensor spaces, generalized convexity, topological essentiality.

In our method of approach to extension problems the notion of pseudo-barycenter plays an important role. This notion, introduced in [4] for α-convex metric spaces (see Definition 2.3), was used to prove in these spaces some corresponding metric versions of the theorems of Cellina [2] and Michael [13]. With minor modifications it can actually be used also in the more general case of the α-weakly convex metric spaces and in this setting it will prove useful to establish our metric versions of Dugundji's extension theorem.

The present paper consists of six sections, with the introduction. Section 2 contains notations and a review of the main properties of pseudo-barycenters in α-weakly convex metric spaces. Section 3 contains a version of Dugundji's extension theorem in these spaces. In Section 4 a similar result is proved in locally α-weakly convex metric spaces. In Section 5 essential maps and some some applications to fixed point theory are considered in α-weakly convex metric spaces.

2. Notations and preliminaries

In this section we introduce notations and terminology and review some properties of pseudo-barycenters in α-weakly convex metric spaces, which will be useful in the sequel.

Let Z be a metric space with distance d and let 2^Z be the set of all nonempty subsets of Z. For $z \in Z$ and $r > 0$, $B_Z(z,r)$ stands for an open ball in Z with center z and radius r, and $d(z,A) = \inf_{a\in A} d(z,a)$, $A \in 2^Z$. Moreover, $h(X,Y)$ denotes the Pompeiu–Hausdorff distance of two nonempty closed bounded subsets of Z, i.e.

$$h(X,Y) = \max\Big\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\Big\}.$$

For $A \subset Z$, $\overline{A}$ and ∂A (also written $\mathrm{cl}_Z A$ and $\partial_Z A$) denote the closure and the boundary of A. $\mathbb{N}$ denotes the set of integers $n \geq 1$. If A is a nonempty set, we put $A^n = A \times \ldots \times A$, $n \in \mathbb{N}$, and denote by $(a_1, \ldots, a_n)$ an element of A^n, i.e. an ordered n-tuple of points $a_i \in A$, $i = 1, \ldots, n$.

A hint for the following definition can be found in [14].

Definition 2.1. An *α-weakly convex metric space* (α-WCM space) is a metric space Y which is equipped with a continuous map $\alpha: Y \times Y \times [0,1] \to Y$, and with a family $\{\mathcal{B}_y\}_{y\in Y}$ where, for each $y \in Y$, $\mathcal{B}_y$ is a local base at y, such that the following properties are satisfied:

(i) $\alpha(y,y,t) = y$ for every $y \in Y$ and $t \in [0,1]$,
(ii) $\alpha(y_1,y_2,0) = y_1$, $\alpha(y_1,y_2,1) = y_2$ for every $(y_1,y_2) \in Y^2$,
(iii) for every $y \in Y$ and $B \in \mathcal{B}_y$ there exists $U \in \mathcal{B}_y$, with $U \subset B$, such that:

$$\text{(P)} \qquad y_1 \in B \text{ and } y_2 \in U \Rightarrow \alpha(y_1,y_2,t) \in B, \text{ for every } t \in [0,1].$$

For the following definition see [12] (comp. [7], [16]).

Definition 2.2. A *strictly equiconnected metric* space (SECM space) is a metric space Y which is equipped with a continuous map $\alpha\colon Y\times Y\times[0,1]\to Y$, and with family $\{\mathcal{B}_y\}_{y\in Y}$ where, for each $y\in Y$, $\mathcal{B}_y$ is a local base at y, such that the following properties are satisfied:

(i) $\alpha(y,y,t)=y$ for every $y\in Y$ and $t\in[0,1]$,

(ii) $\alpha(y_1,y_2,0)=y_1$, $\alpha(y_1,y_2,1)=y_2$ for every $(y_1,y_2)\in Y^2$,

(iii)' for every $y\in Y$ and $B\in\mathcal{B}_y$,

$$y_1,y_2\in B\Rightarrow \alpha(y_1,y_2,t)\in B,\ \text{for every } t\in[0,1]. \tag{P'}$$

The following definition was introduced in [4] (comp. [3], [14], [15], [18]).

Definition 2.3. An *α-convex metric space* (α-CM space) is a metric space Y equipped with a continuous map $\alpha\colon Y\times Y\times[0,1]\to Y$, such that the following properties are satisfied:

(i) $\alpha(y,y,t)=y$ for every $y\in Y$ and $t\in[0,1]$,

(ii) $\alpha(y_1,y_2,0)=y_1$, $\alpha(y_1,y_2,1)=y_2$ for every $(y_1,y_2)\in Y^2$,

(iii)" there is an $R>0$ such that for every $0<\varepsilon<R$ there exists $0<\delta\leq\varepsilon$ such that the following property is satisfied:

(P") for $(y_1,y_2),(\overline{y_1},\overline{y_2})\in Y^2$ with $d(y_1,\overline{y_1})<\varepsilon$ and $d(y_2,\overline{y_2})<\delta$, we have

$$h(\Lambda_\alpha(y_1,y_2),\Lambda_\alpha(\overline{y_1},\overline{y_2}))<\varepsilon,$$

where

$$\begin{aligned}\Lambda_\alpha(y_1,y_2)&=\{\alpha(y_1,y_2,t)\mid t\in[0,1]\},\\ \Lambda_\alpha(\overline{y_1},\overline{y_2})&=\{\alpha(\overline{y_1},\overline{y_2},t)\mid t\in[0,1]\}.\end{aligned}$$

The map α which occurs in each of the above spaces is called the *convexity map* of the space. In the sequel, when we say (for the sake of brevity) "Y is an α-WCM space" we actually mean that "Y is a metric space equipped with a map α and a family $\{\mathcal{B}_y\}_{y\in Y}$, satisfying the properties stated in Definition 2.1". The meaning of "Y is a SECM space" and "Y is an α-CM space", is to be understood in a similar manner.

In each of the above spaces, convex sets are naturally defined as follows.

Definition 2.4. A set $A\subset Y$ is called *convex* if $\alpha(y_1,y_2,t)\in A$ for every $(y_1,y_2)\in A^2$ and $t\in[0,1]$.

The empty set is assumed to be convex. Moreover, the intersection of convex sets is convex, and the closure of a convex set is convex.

Remark 2.5. Condition (P) in Definition 2.1 is certainly satisfied if for every $y \in Y$ the sets $B \in \mathcal{B}_y$ are convex.

Remark 2.6. In view of the following proposition and Examples 2.8 and 2.9 below, the notion of α-WCM space is strictly more general than either notion, that of SECM space and α-CM space.

Proposition 2.7.

(a) *If Y is a strictly equiconnected metric space, then Y is an α-weakly convex metric space.*

(b) *If Y is an α-convex metric space, then Y is an α-weakly convex metric space, where for each $y \in Y$ and $R > 0$ (independent of y), $\mathcal{B}_y = \{B_Y(y,r)\}_{r\in(0,R)}$.*

Proof. (a) is obvious, since condition (iii)' of Definition 2.2 implies condition (iii) of Definition 2.1.

(b) Let Y be an α-CM space. Let $y \in Y$ and $\varepsilon \in (0,R)$ be arbitrary (R as in (iii)"). By (iii)" there is $0 < \delta \leq \varepsilon$ such that, for $(y_1,y_2),(\overline{y_1},\overline{y_2}) \in Y^2$,

$$d(y_1,\overline{y}_1) < \varepsilon \text{ and } d(y_2,\overline{y}_2) < \delta \Rightarrow h(\Lambda_\alpha(y_1,y_2),\Lambda_\alpha(\overline{y}_1,\overline{y}_2)) < \varepsilon.$$

By taking $y_1 \in B_Y(y,\varepsilon)$, $y_2 \in B_Y(y,\delta)$, and $\overline{y}_1 = \overline{y}_2 = y$, we have

$$h(\Lambda_\alpha(y_1,y_2),\{y\}) < \varepsilon,$$

which implies $d(\alpha(y_1,y_2,t),y) < \varepsilon$ for all $t \in [0,1]$. Hence (iii) is verified, with $B = B_Y(y,\varepsilon)$ and $U = B_Y(y,\delta)$, and also (b) is proved. □

In the following Example 2.8 we construct an α-WCM space which is not a SECM space, while in Example 2.9 we present an α-WCM space which is not an α-CM space.

Example 2.8. Consider the space

$$Y = \{y = (u,v) \in \mathbb{R}^2 \mid v > \max\{1,|u|\}\},$$

and endow it with the metric $\|y_1 - y_2\| = \max\{|u_1 - u_2|,|v_1 - v_2|\}$, where $y_1 = (u_1,v_1) \in Y$ and $y_2 = (u_2,v_2) \in Y$. For $y_1,y_2 \in Y$, put

$$c = \frac{y_1 + y_2}{2}, \quad e = c + \frac{c}{\|c\|}\cdot\frac{\|y_1 - y_2\|}{4}, \tag{2.1}$$

and define

$$\alpha(y_1,y_2,t) = \begin{cases} (1-2t)y_1 + 2te & \text{for } t \in [0,1/2], \\ (2-2t)e + (2t-1)y_2 & \text{for } t \in [1/2,1]. \end{cases} \tag{2.2}$$

It is easy to see that (2.2) defines a map $\alpha\colon Y\times Y\times[0,1] \to Y$ which is continuous and satisfies conditions (i) and (ii) of Definition 2.1. For $x \in Y$ consider a local base at x given by $\mathcal{B}_x = \{B_Y(x,r)\}_{r\in(0,1)}$. We suppose that Y is equipped with the map α and the family $\{\mathcal{B}_x\}_{x\in Y}$.

Claim 1. Y is an α-WCM space.

It suffices to prove that condition (iii) of Definition 2.1 is satisfied. Let $x \in Y$ and $B_Y(x,r) \in \mathcal{B}_x$ be arbitrary and let $0 < \rho < r/8$. Then we have:

$$y_1 \in B_Y(x,r) \text{ and } y_2 \in B_Y(x,\rho) \Rightarrow \alpha(y_1,y_2,t) \in B_Y(x,r), \tag{2.3}$$

for every $t \in [0,1]$. In view of the definition of $\alpha(y_1,y_2,t)$, (2.3) is valid if we show that $e \in B_Y(x,r)$. Indeed, by virtue of (2.1), as $\|y_1 - y_2\| < r + \rho$, we have

$$\|e - x\| \le \|e - c\| + \|c - y_2\| + \|y_2 - x\| < \frac{3}{4}\|y_1 - y_2\| + \rho < r.$$

Hence $e \in B_Y(x,r)$ and Claim 1 is proved.

Claim 2. Y is not a SECM space.

To see this let $a = (0,2)$ and consider an arbitrary ball $B_Y(a,r) \in \mathcal{B}_x$. Take $y_1, y_2 \in B_Y(a,r)$ as follows

$$y_1 = \left(-\frac{3}{4}r, 2 + \frac{3}{4}r\right), \quad y_2 = \left(\frac{3}{4}r, 2 + \frac{3}{4}r\right).$$

With this choice of y_1 and y_2 we have $c = (0, 2 + 3r/4)$, $\|y_1 - y_2\| = 3r/2$ and thus, $\|e - c\| = 3r/8$ and $\|c - a\| = 3r/4$. Since

$$\|e - a\| = \|e - c\| + \|c - a\| = \frac{3}{8}r + \frac{3}{4}r = \frac{9}{8}r,$$

it follows that $e \notin B_Y(a, \frac{9}{8}r)$. Consequently for some $t \in (0,1)$ we have

$$\alpha(y_1,y_2,t) \notin B_Y(a,r),$$

which shows that condition (P') in Definition 2.2 is not satisfied. Hence Claim 2 is proved.

Example 2.9. Let S be the unit sphere of $\mathbb{R}^3$ with center $(0,0,0)$. Consider the space

$$Y = \{y = (y_1,y_2,y_3) \in S \mid y_1 \ge 0,\ y_2 \ge 0,\ y_3 \ne 1\},$$

and endow it with the metric induced by the Euclidean norm of $\mathbb{R}^3$. For $y, z \in Y$ set

$$\alpha(y,z,t) = \frac{(1-t)y + tz}{\|(1-t)y + tz\|}, \quad t \in [0,1]. \tag{2.4}$$

It is easy to see that (2.4) defines a continuous map $\alpha\colon Y \times Y \times [0,1] \to Y$ which satisfies conditions (i) and (ii) of Definition 2.1. For $x \in Y$ consider the local base at x given by $\mathcal{B}_x = \{B_Y(x,r)\}_{r\in(0,1)}$. We suppose that Y is equipped with the map α and the family $\{\mathcal{B}_x\}_{x\in Y}$.

Claim 1. Y is an α-WCM space.

In view of Remark 2.5 it suffices to show that, for any $x \in Y$ and $r \in (0,1)$, the set $B_Y(x,r) \in \mathcal{B}_x$ is convex. Indeed, let $y, z \in B_Y(x,r)$ be arbitrary. Clearly $\|y-x\|^2 < r^2$, $\|z-x\|^2 < r^2$, and thus

$$2\langle y, x\rangle > 2 - r^2, \quad 2\langle z, x\rangle > 2 - r^2,$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the inner product in $\mathbb{R}^3$. Since for every $t \in [0,1]$ we have

$$\left\|\frac{(1-t)y+tz}{\|(1-t)y+tz\|} - x\right\|^2 = 2 - \frac{2(1-t)\langle y,x\rangle + 2t\langle z,x\rangle}{\|(1-t)y+tz\|} < 2 - (2-r^2) = r^2,$$

it follows that $\alpha(y,z,t) \in B_Y(x,r)$ for all $t \in [0,1]$. Hence $B_Y(x,r)$ is convex and Claim 1 is proved.

Claim 2. Y is not an α-CM space.

Let us show that condition (iii)" of Definition 2.3 is not satisfied. Supposing the contrary, there is an $R > 0$ such that for every $0 < \varepsilon < R$ there exists $0 < \delta \leq \varepsilon$ such that (P") holds, i.e. for $(y,z), (\overline{y},\overline{z}) \in Y \times Y$, if $\|y-\overline{y}\| < \varepsilon$ and $\|z-\overline{z}\| < \delta$, then we have

$$h(\Lambda_\alpha(y,z), \Lambda_\alpha(\overline{y},\overline{z})) < \varepsilon.$$

Fix $0 < \varepsilon < \min\{1/\sqrt{2}, R\}$ and let $0 < \delta \leq \varepsilon$ correspond. Take now (y,z), $(\overline{y},\overline{z})$ in $Y \times Y$ as follows:

$$y = (\varepsilon^2, 0, \sqrt{1-\varepsilon^4}), \quad \overline{y} = (0, \varepsilon^2, \sqrt{1-\varepsilon^4}), \quad z = \overline{z} = a = (0,0,-1).$$

Clearly $\|y-\overline{y}\| = \sqrt{2}\varepsilon^2 < \varepsilon$ and $z = \overline{z}$, and thus

$$h(\Lambda_\alpha(y,z), \Lambda_\alpha(\overline{y},\overline{z})) < \varepsilon. \tag{2.5}$$

On the other hand, as $e = (1,0,0) \in \Lambda_\alpha(y,z)$ and

$$d(e, \Lambda_\alpha(\overline{y},\overline{z})) = \min_{t\in[0,1]} \left\| e - \frac{(1-t)\overline{y}+t\overline{z}}{\|(1-t)\overline{y}+t\overline{z}\|}\right\| = \sqrt{2},$$

it follows that

$$h(\Lambda_\alpha(y,z), \Lambda_\alpha(\overline{y},\overline{z})) \geq \sqrt{2},$$

which contradicts (2.5). Hence Claim 2 is proved.

For $n \in \mathbb{N}$ set

$$\Sigma^n = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \mid 0 \leq \lambda_i \leq 1,\ i = 1, \ldots, n,\ \lambda_1 + \ldots + \lambda_n = 1\}.$$

For the following definition of pseudo-barycenter and its properties (Propositions 2.7 and 2.11) see [4].

Definition 2.10. Let Y be an α-weakly convex metric space. For $(y_1,\dots y_n)$ in Y^n and $(\lambda_1,\dots,\lambda_n)$ in Σ^n the corresponding *pseudo-barycenter*

$$b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n)$$

is defined as follows: for $n=1$,

$$b_1(y_1,\lambda_1)=y_1 \tag{2.6}$$

and, for $n\ge 2$,

$$\begin{aligned} &b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n) \\ &\quad=\begin{cases} y_n & \text{if } \lambda_n=1,\\ \alpha\bigg(b_{n-1}\bigg(y_1,\dots,y_{n-1};\dfrac{\lambda_1}{1-\lambda_n},\dots,\dfrac{\lambda_{n-1}}{1-\lambda_n}\bigg),y_n,\lambda_n\bigg) & \text{if } \lambda_n<1.\end{cases}\end{aligned} \tag{2.7}$$

Clearly, when $n=2$, one has

$$b_2(y_1,y_2;\lambda_1,\lambda_2)=\alpha(y_1,y_2,\lambda_2),\quad \text{for } (y_1,y_2)\in Y^2 \text{ and } (\lambda_1,\lambda_2)\in\Sigma^2.$$

When we need emphasize the dependence of the pseudo-barycenter on the convexity mapping α, we shall write

$$b_n^\alpha(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n) \text{ instead of } b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n).$$

The following Propositions 2.11 and 2.12 below were proved in [4] for α-CM spaces, but by [4, Remark (4.8)] they remain valid also in α-WCM spaces.

Proposition 2.11. *Let Y be an α-weakly convex metric space. Then the function $b_n\colon Y^n\times\Sigma^n\to Y$, given by* (2.6) *if $n=1$ and by* (2.7) *if $n\ge 2$, is continuous on $Y^n\times\Sigma^n$.*

Proposition 2.12. *Let Y be an α-weakly convex metric space. Let $(y_1,\dots,y_n)\in Y^n$, $(\lambda_1,\dots,\lambda_n)\in\Sigma^n$, $n\ge 2$. Let $(i_1,\dots,i_k)$, $1\le k\le n-1$, be a subset of $(1,\dots,n)$, with $1\le i_1<\dots<i_k\le n$, such that*

$$\lambda_i>0 \quad \textit{if } \ i\in\{i_1,\dots,i_k\},\qquad \lambda_i=0 \quad \textit{if } \ i\in\{1,\dots,n\}\setminus\{i_1,\dots,i_k\}.$$

Then $b_n(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n)=b_k(y_{i_1},\dots,y_{i_k};\lambda_{i_1},\dots,\lambda_{i_k})$. Moreover, we have $b_n(y_1,\dots,y_n;1,\dots,0)=y_1,\dots,b_n(y_1,\dots,y_n;0,\dots,1)=y_n$.

Proposition 2.13. *Let Y and Y' be an α-weakly convex and α'-weakly convex metric space such that $Z=Y\cap Y'\ne\emptyset$. Suppose that*

$$\alpha(y_1,y_2,t)=\alpha'(y_1,y_2,t)\quad \textit{for every } (y_1,y_2)\in Z^2,\ t\in[0,1].$$

Then, for every $(y_1,\dots,y_n)\in Z^n$, $(\lambda_1,\dots,\lambda_n)\in\Sigma^n$, $n\in\mathbb{N}$ arbitrary, we have

$$b_n^\alpha(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n)=b_n^{\alpha'}(y_1,\dots,y_n;\lambda_1,\dots,\lambda_n).$$

Proof. The statement can be proved by induction since it holds for $n = 1, 2$. $\square$

Proposition 2.14. *Let Y be an α-weakly convex metric space. For $y \in Y$, let $B \in \mathcal{B}_y$ be arbitrary and let $U \in \mathcal{B}_y$, $U \subset B$, correspond as in condition* (iii) *of Definition* 2.1. *Then for $(y_1, \dots y_n) \in Y^n$, $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$, $n \geq 1$, we have:*

$$y_i \in U,\ i = 1, \dots, n \Rightarrow b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n) \in B.$$

Proof. The statement is true for $n = 1, 2$. Assuming that it holds for n, let us prove it for $n + 1$. Let $(y_1, \dots, y_{n+1}) \in Y^{n+1}$, $(\lambda_1, \dots, \lambda_{n+1}) \in \Sigma^{n+1}$ and suppose that $y_i \in U$, $i = 1, \dots, n + 1$. If $\lambda_{n+1} = 1$, we have $(\lambda_1, \dots, \lambda_{n+1}) = (0, \dots, 1)$ and thus $b_{n+1}(y_1, \dots, y_{n+1}; 0, \dots, 1) = y_{n+1} \in U \subset B$. Suppose $\lambda_{n+1} < 1$. By definition of pseudo-barycenter, we have

$$\begin{aligned} (2.8)\quad & b_{n+1}(y_1, \dots, y_{n+1}; \lambda_1, \dots, \lambda_{n+1}) \\ &= \alpha\bigg(b_n\bigg(y_1, \dots, y_n; \frac{\lambda_1}{1 - \lambda_{n+1}}, \dots, \frac{\lambda_n}{1 - \lambda_{n+1}}\bigg), y_{n+1}, \lambda_{n+1}\bigg). \end{aligned}$$

Now by the induction hypothesis, $b_n(y_1, \dots, y_n; \lambda_1/(1 - \lambda_{n+1}), \dots, \lambda_n/(1 - \lambda_{n+1})) \in B$, for $(\lambda_1/(1 - \lambda_{n+1}), \dots, \lambda_n/(1 - \lambda_{n+1})) \in \Sigma^n$. Moreover, $y_{n+1} \in U$ and thus, in view of Definition 2.1(iii), it follows that the right hand side of (2.8) is in B. This completes the proof. $\square$

Proposition 2.15 (Dugundji [6, p. 83]). *Let X, Z be topological spaces. Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a covering of X, where the sets $A_\lambda \subset X$ are open nonempty, and let $\{\psi_\lambda\}_{\lambda \in \Lambda}$ be a family of continuous functions $\psi_\lambda : A_\lambda \to Z$ such that, for every $\lambda, \lambda' \in \Lambda$, with $A_\lambda \cap A_{\lambda'} \neq \emptyset$,*

$$\psi_\lambda(x) = \psi_{\lambda'}(x) \quad \text{for every } x \in A_\lambda \cap A_{\lambda'}.$$

Then there is a unique continuous function $f: X \to Z$ which is an extension of each ψ_λ, that is, for each $\lambda \in \Lambda$

$$f(x) = \psi_\lambda(x) \quad \text{for every } x \in A_\lambda.$$

3. Extensions of maps with values in α-weakly convex metric spaces

In this section we prove a version of Dugundji's extension theorem for maps taking values in an α-weakly convex metric space.

Theorem 3.1. *Let X be a metric space and Y an α-weakly convex metric space. Let A be a nonempty closed subset of X and let C be a nonempty convex* *subset of Y. Then each continuous map $\varphi\colon A \to C$ has a continuous extension $f\colon X \to C$ defined on X.*

Proof. Following Dugundji's argument [6, p. 188], for every $x \in X \setminus A$ consider the open ball $B_x = B_X(x, d(x,A)/2)$. The family $\mathcal{B} = \{B_x\}_{x\in X\setminus A}$ is an open covering of $X \setminus A$, and thus it admits an open neighbourhood finite refinement $\mathcal{V}$. With each nonempty $V \in \mathcal{V}$ associate an $x_V \in V$ and $a_V \in A$ such that $d(x_V, a_V) < 2d(x_V, A)$. As in Dugundji [6, p. 188], one can prove that the family of the sets $V \in \mathcal{V}$, and the family of the corresponding points $a_V \in A$, have the following property:

(P) For every $a \in A$, $\rho > 0$ and $V \in \mathcal{V}$

$$V \cap B_X\left(a, \frac{\rho}{12}\right) \neq \emptyset \Rightarrow V \subset B_X(a,\rho) \text{ and } a_V \in B_X(a,\rho). \tag{3.1}$$

By Zermelo's well ordering theorem [6, p. 31], there exists a well ordering, say $\prec$, under which $\mathcal{V}$ is a well ordered set.

Let $\{p_V\}_{V\in\mathcal{V}}$ be a partition of unity subordinated to $\mathcal{V}$ (see [6, p. 170]), that is a family of continuous functions $p_V\colon X \setminus A \to [0,1]$ such that:

(j) $\operatorname{supp} p_V \subset V$ for every $V \in \mathcal{V}$,
(jj) $\{\operatorname{supp} p_V\}_{V\in\mathcal{V}}$ is a neighbourhood finite closed covering of $X \setminus A$,
(jjj) $\Sigma_{V\in\mathcal{V}} p_V(x) = 1$ for every $x \in X \setminus A$.

Let $u \in X \setminus A$. Since $\mathcal{V}$ is neighbourhood finite, there exists an open ball $W_u = B_X(u, \theta_u) \subset X \setminus A$, for some $\theta_u > 0$, such that the set $\mathcal{V}_{W_u} = \{V \in \mathcal{V} \mid V \cap W_u \neq \emptyset\}$ is finite and nonempty. Thus for some $1 \le n < \infty$ we have

$$\mathcal{V}_{W_u} = (V_1, \dots, V_n), \quad \text{where } V_1 \prec \dots \prec V_n. \tag{3.2}$$

Let $(a_{V_1}, \dots, a_{V_n})$ correspond, where $a_{V_i} \in A$, $i = 1, \dots, n$.

Define now $\psi_{W_u}\colon W_u \to C$ by

$$\psi_{W_u}(x) = b_n(\varphi(a_{V_1}), \dots, \varphi(a_{V_n}); p_{V_1}(x), \dots, p_{V_n}(x)). \tag{3.3}$$

By Proposition 2.11, ψ_{W_u} is well defined and continuous on W_u.

Claim 1. For every $u, u' \in X \setminus A$, with $W_u \cap W_{u'} \neq \emptyset$, we have

$$\psi_{W_u}(x) = \psi_{W_{u'}}(x), \quad \text{for every } x \in W_u \cap W_{u'}.$$

For $u, u' \in X \setminus A$ let $\mathcal{V}_{W_u}$ be given by (3.2) and, similarly, for some $n' \in \mathbb{N}$, let $\mathcal{V}_{W_{u'}} = (V_1', \dots, V_{n'}')$, where $V_1' \prec \dots \prec V_{n'}'$. Let $x \in W_u \cap W_{u'}$ be arbitrary. Set

$$\mathcal{V}_{W_u}^x = \{V \in \mathcal{V}_{W_u} \mid p_V(x) > 0\}, \quad \mathcal{V}_{W_{u'}}^x = \{V \in \mathcal{V}_{W_{u'}} \mid p_V(x) > 0\}.$$

We have $\mathcal{V}_{W_u}^x = \mathcal{V}_{W_{u'}}^x$. In fact let $V \in \mathcal{V}_{W_u}^x$. As $x \in \operatorname{supp} p_V \subset V$, by (j), and $x \in W_u \cap W_{u'}$, it follows that $V \cap W_{u'} \neq \emptyset$ and thus $V \in \mathcal{V}_{W_{u'}}$. As $p_V(x) > 0$

one has $V \in \mathcal{V}^x_{W_{u'}}$. By interchanging the roles of $\mathcal{V}^x_{W_u}$ and $\mathcal{V}^x_{W_{u'}}$ it follows that $\mathcal{V}^x_{W_u} = \mathcal{V}^x_{W_{u'}}$. This implies that, for some $1 \le k \le n$,

$$\mathcal{V}^x_{W_u} = (V_{i_1}, \dots, V_{i_k}) = \mathcal{V}^x_{W_{u'}}, \quad \text{where } 1 \le i_1 < \dots < i_k \le n. \tag{3.4}$$

By (3.3), (3.4) and Proposition 2.12, one has

$$\psi_{W_u}(x) = b_k(\varphi(a_{V_{i_1}}), \dots, \varphi(a_{V_{i_k}}); p_{V_{i_1}}(x), \dots, p_{V_{i_k}}(x)) = \psi_{W_{u'}}(x)$$

and hence, as $x \in W_u \cap W_{u'}$ is arbitrary, Claim 1 holds.

The family $\{W_u\}_{u \in X \setminus A}$ is an open covering of $X \setminus A$ and, moreover, the corresponding family $\{\psi_{W_u}\}_{u \in X \setminus A}$ of continuous maps ψ_{W_u} satisfies Claim 1. Therefore, by Proposition 2.15, there is a unique continuous function $\psi\colon X \setminus A \to C$ such that

$$\psi(x) = \psi_{W_u}(x), \quad \text{for every } x \in W_u \text{ and } u \in X \setminus A. \tag{3.5}$$

Claim 2. Define now $f\colon X \to C$ by

$$f(x) = \begin{cases} \varphi(x) & \text{if } x \in A, \\ \psi(x) & \text{if } x \in X \setminus A. \end{cases}$$

Then f is continuous on X.

Let $a \in A$ (if $a \in X \setminus A$ there is a nothing to prove). Let $B \in \mathcal{B}_{\varphi(a)}$ be arbitrary, where $\mathcal{B}_{\varphi(a)}$ is a local base at $\varphi(a)$, and let $U \in \mathcal{B}_{\varphi(a)}$, $U \subset B$, correspond according to Definition 2.1(iii). Then, by Proposition 2.14, for $(y_1, \dots, y_n) \in Y^n$, $(\lambda_1, \dots, \lambda_n) \in \Sigma^n$ and $n \in \mathbb{N}$,

$$y_i \in U, \ i = 1, \dots, n \Rightarrow b_n(y_1, \dots, y_n; \lambda_1, \dots, \lambda_n) \in B. \tag{3.6}$$

Since φ is continuous there is $\sigma > 0$ such that

$$x \in B_X(a, \sigma) \cap A \Rightarrow \varphi(x) \in U. \tag{3.7}$$

It will be shown that

$$x \in B_X\left(a, \frac{\sigma}{12}\right) \cap (X \setminus A) \Rightarrow \psi(x) \in B. \tag{3.8}$$

Let x be as in (3.8). Thus $x \in W_u$, for some $u \in X \setminus A$. Define $\mathcal{V}_{W_u}$ and $\mathcal{V}^x_{W_u}$ as before, and suppose that $\mathcal{V}_{W_u}$ and $\mathcal{V}^x_{W_u}$ are given by (3.2) and (3.4). As $x \in B_X(a, \sigma/12)$ and $x \in \operatorname{supp} p_{V_{i_r}} \subset V_{i_r}$, one has $V_{i_r} \cap B_X(a, \sigma/12) \neq \emptyset$, $r = 1, \dots, k$. Hence by (3.1) (with $\rho = \sigma$) it follows that $a_{V_{i_r}} \in B_X(a, \sigma)$, $r = 1, \dots, k$, and thus by (3.7),

$$\varphi(a_{V_{i_r}}) \in U, \quad r = 1, \dots, k.$$

Combining the latter with (3.6) gives

$$b_k(\varphi(a_{V_{i_1}}), \dots, \varphi(a_{V_{i_k}}); p_{V_{i_1}}(x), \dots, p_{V_{i_k}}(x)) \in B. \tag{3.9}$$

On the other hand, by (3.3) and Proposition 2.12, one has

$$\psi_{W_u}(x) = b_k(\varphi(a_{V_{i_1}}), \dots, \varphi(a_{V_{i_k}}); p_{V_{i_1}}(x), \dots, p_{V_{i_k}}(x)),$$

and thus by (3.5) and (3.9) it follows that $\psi(x) \in B$, proving (3.8). ¿From this and the continuity of φ at a, one has that ψ is continuous at a. Hence f is continuous on X, and Claim 2 holds. This completes the proof. □

By virtue of Borsuk [1, p. 87], we have:

Corollary 3.2. *Every α-weakly convex metric space is an* AR*-space.*

Since in an α-weakly convex metric space a Mazur type theorem is not available, the problem of extending a continuous compact map (considered in the next theorem) requires a different proof.

Theorem 3.3. *Let A be a nonempty closed subset of a metric space X and let Y be an α-weakly convex metric space. Then each continuous and compact map $\varphi: A \to Y$ has a continuous and compact extension $f: X \to Y$ defined on X.*

Proof. Set $B = \overline{\varphi(A)}$. Since B is a compact metric space it can be homeomorphically embedded into the Hilbert cube Q (see: [9, p. 8–9], [11, p. 597]). Denote by $h: B \to Q$ such an embedding map and by $h_1: h(B) \to B$ its inverse, and let $g: A \to Q$ be given by $g = h \circ \varphi$. We have the following commutative diagram

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "h\circ\varphi"'] & & Y \\ & h(B) \subset Q. \arrow[ur, "h_1"'] & \end{tikzcd}$$

Let $\widetilde{g}: X \to Q$ be a continuous extension of g to all of X. It is evident that $\widetilde{g}$ is compact. By Theorem 3.1, h_1 admits a continuous extension say $\widetilde{h}: Q \to Y$. Then the map $f: X \to Y$ given by $f = \widetilde{h} \circ \widetilde{g}$ is the desired continuous and compact extension of φ to X. This completes the proof. □

4. Extensions of maps with values in locally α-weakly convex metric spaces

In this section we prove a Dugundji type extension theorem for maps taking values in a locally α-weakly convex metric space.

Definition 4.1. Let Y be a connected metric space. Let $\mathcal{Y} = \{Y_\lambda\}_{\lambda \in \Lambda}$ be a family of nonempty open sets $Y_\lambda \subset Y$ such that, for each $\lambda \in \Lambda$, Y_λ is an α_λ-weakly convex metric space, and let $\alpha_\lambda: Y_\lambda \times Y_\lambda \times [0,1] \to Y_\lambda$ and $\{\mathcal{B}_x^\lambda\}_{x \in Y_\lambda}$ correspond to Y_λ, according to Definition 2.1. Suppose that:

(i) for every $y \in Y$ there exist $R_y > 0$ and $\lambda \in \Lambda$ such that $B_Y(y, R_y) \subset Y_\lambda$,

(ii) for every $Y_\lambda, Y_{\lambda'}$, with $Y_\lambda \cap Y_{\lambda'} \neq \emptyset$, and every $y_1, y_2 \in Y_\lambda \cap Y_{\lambda'}$ we have $\alpha_\lambda(y_1, y_2, t) = \alpha_{\lambda'}(y_1, y_2, t)$ for all $t \in [0,1]$.

Then Y, equipped with the family $\mathcal{Y}$, is called a *locally α-weakly convex metric space* (locally α-WCM space).

In the sequel, when we say (for the sake of brevity) "Y is a locally α-WCM space" we actually mean that "Y equipped with the family $\mathcal{Y}$ is a locally α-WCM space, according to Definition 4.1".

Remark 4.2. Condition (ii) of the above definition implies that for every $\lambda, \lambda' \in \Lambda$ the set $Y_\lambda \cap Y_{\lambda'}$ is a convex subset of Y_λ and $Y_{\lambda'}$.

Remark 4.3. Under appropriate assumptions (see Whitehead [17]) one can show that, locally, a Riemannian manifold Y with positive definite C^2 metric is an α-WCM space. If Y is compact, then Y turns out be a locally α-WCM space.

An example of a locally α-WCM space is given in the following

Example 4.4. Let $\mathbb{H}$ be a real Hilbert space with inner product $\langle \cdot\,, \cdot \rangle$ and norm $\| \cdot \|$. Let S be the unit sphere in $\mathbb{H}$ centered at 0 and suppose that S is endowed with the Euclidean metric of $\mathbb{H}$. For $u \in S$ and $0 < \sigma < 1$ consider the space

$$Y_u = \{x \in S \mid \langle x, u \rangle > 1 - \sigma\},$$

with the induced metric of S. Clearly Y_u is open in S.

For $y_1, y_2 \in Y_u$, set

$$\alpha_u(y_1, y_2, t) = \frac{(1-t)y_1 + ty_2}{\|(1-t)y_1 + ty_2\|}, \quad t \in [0,1]. \tag{4.1}$$

It is easy to see that (4.1) defines a continuous map $\alpha_u : Y_u \times Y_u \times [0,1] \to Y_u$ which satisfies the conditions (i) and (ii) of Definition 2.1. Equip now Y_u with the convexity map α_u and the family $\{\mathcal{B}_x^u\}_{x \in Y_u}$ where, for every $x \in Y_u$, $\mathcal{B}_x^u = \{B_{Y_u}(x, r)\}_{r \in (0,1)}$ is a local base at x. Clearly each ball $B_{Y_u}(x, r)$, with $x \in Y_u$ and $r \in (0,1)$, is convex and thus by Remark 2.5 also condition (iii) of Definition 2.1 is fulfilled. Hence Y_u is an α_u-WCM space. Set $\mathcal{Y} = \{Y_u\}_{u \in S}$. As conditions (i) and (ii) of Definition 4.1 are satisfied, it follows that the space S, equipped with the family $\mathcal{Y}$, is a locally α-WCM space.

Theorem 4.5. *Let X be a metric space and let Y be a locally α-weakly convex metric space. Let A be a nonempty closed subset of X. Then each continuous map $\varphi : A \to Y$ has a continuous extension $f : Z \to Y$ defined on some open set $Z \supset A$.*

Proof. By the second Hanner's theorem (see [11, p. 286]), any metric space which is locally an ANR-space is also an ANR-space. In our case, in view of

Corollary 3.2, the space Y is locally an ANR-space, and hence it is an ANR-space. Since any metric space is an ANR if and only if it has the local extension property (see [11]), it follows that the space Y has the local extension property. This completes the proof. $\square$

Corollary 4.6. *Every locally α-weakly convex metric space is an* ANR-*space.*

Remark 4.7. While Theorem 3.1 remains valid if X is paracompact and Y α-convex and complete (see [5]), it is not clear if, in Theorem 4.5, the space X can be taken not necessarily metric and Y locally α-convex and complete.

5. Topological essentiality of compact maps in α-weakly convex metric spaces

In this section we present some fixed point results which are consequences of the previous extension theorems, and we develop a Granas type theory for essential maps in α-weakly convex metric spaces. To this end we introduce some further notations.

For any two metric spaces X, Y we set

$$\begin{aligned} K(Y,X) &= \{f\colon Y \to X \mid f \text{ is continuous and compact}\},\\ C(Y,X) &= \{f\colon Y \to X \mid f \text{ is completely continuous}\}. \end{aligned}$$

Here f *completely continuous* means that f is continuous and, for each bounded set $A \subset Y$, the set $\overline{f(A)}$ is compact. Evidently,

$$K(Y,X) \subset C(Y,X).$$

Now, suppose that X is a locally α-weakly convex metric space and $U \subset X$ is a nonempty bounded open set. Then, define

$$K_{\partial U}(\overline{U},X) = \{f \in K(\overline{U},X) \mid f(x) \neq x \text{ for every } x \in \partial U\},$$

where ∂U denotes the boundary of U.

Definition 5.1. Let $f,g \in K_{\partial U}(\overline{U},X)$. We shall say that f and g are *homotopic* in $K_{\partial U}(\overline{U},X)$ (we write $f \sim_{\partial U} g$) if there exists a continuous and compact map $h\colon \overline{U} \times [0,1] \to X$, satisfying the following conditions:

(i) $h(x,t) \neq x$ for every $x \in \partial U$ and $t \in [0,1]$,
(ii) $h(x,0) = f(x)$ for every $x \in \overline{U}$,
(iii) $h(x,1) = g(x)$ for every $x \in \overline{U}$.

The map h is called a *homotopy* in $K_{\partial U}(\overline{U},X)$ joining f and g.

By Corollary 4.6 X is an ANR and thus the fixed point *index* function

$$\operatorname{ind}\colon K_{\partial U}(\overline{U},X) \to \mathbb{Z}$$

(where $\mathbb{Z}$ is the set of all integers) is well defined and has the usual properties of existence, unity, additivity, homotopy, commutativity, and normalization (see [8], [9] or [11]).

It is obvious that $K_{\partial U}(X,X) = K(X,X)$, if $U = X$. For our purposes it is enough to recall the following properties of the index.

- (P_1) (Homotopy) If f and g are homotopic in $K_{\partial U}(\overline{U},X)$, then $\mathrm{ind}(f) = \mathrm{ind}(g)$.
- (P_2) (Normalization) If $f \in K(X,X)$ then $\mathrm{ind}(f) = \lambda(f)$, where $\lambda(f)$ is the generalized Lefschetz number of f (see [8]).
- (P_3) (Existence) If $f \in K(X,X)$ has $\mathrm{ind}(f) \neq 0$, then there exists an $x \in X$ such that $f(x) = x$, i.e. the set $\mathrm{Fix}(f)$ of the fixed points of f is nonempty.

In particular, in view of (P_2) and (P_3), we have:

Corollary 5.2 (Lefschetz fixed point theorem). *Let $f \in K(X,X)$. Then $\lambda(f) \neq 0$ implies* $\mathrm{Fix}(f) \neq \emptyset$.

Let X be an α-weakly convex metric space and let $f \in K(X,X)$. By Corollary 3.2 X is an AR-space and thus $\lambda(f) = 1$. Therefore, we have:

Corollary 5.3 (Schauder fixed point theorem). *If X is an α-weakly convex metric space and $f \in K(X,X)$, then* $\mathrm{Fix}(f) \neq \emptyset$.

In 1962 A. Granas ([10]) introduced the notion of topological essentiality of a map. This notion was developed so far for maps $f\colon \overline{U} \to X$, with f continuous and compact X a convex set and U open in X, or f contractive X a complete metric space and U open in X. Later the multivalued case was studied by several authors (see [8] for details).

Now, we shall study the topological essentiality for continuous and compact maps $f\colon \overline{U} \to X$. In the sequel X is an α-weakly convex metric space (with distance d), and U an open subset of X.

Lemma 5.4. *Let $f, g \in K_{\partial U}(\overline{U},X)$. If, for every $x \in \partial U$ and $t \in [0,1]$*

$$x \neq \alpha(f(x), g(x), t),$$

then f and g are homotopic in $K_{\partial U}(\overline{U},X)$.

Proof. In fact the map $h\colon \overline{U} \times [0,1] \to X$ given by

$$h(x,t) = \alpha(f(x), g(x), t) \tag{5.1}$$

is continuous, compact and satisfies conditions (i)–(iii) of Definition 5.1. $\square$

Under the hypothesis of Lemma 5.4 the map h is called an *α-homotopy* in $K_{\partial U}(\overline{U},X)$ joining f and g.

Definition 5.5. Let $f \in K_{\partial U}(\overline{U}, X)$. We say that f is an *essential* map if every $g \in K_{\partial U}(\overline{U}, X)$, satisfying $g(x) = f(x)$ for every $x \in \partial U$, has a fixed point. A map f is called *inessential* if it is not essential.

Let us review some properties of essential maps.

Proposition 5.6 (Contraction property). *Let $f \in K_{\partial U}(\overline{U}, X)$ be an essential map such that $f(\overline{U}) \subset X_0$, where X_0 is a closed and convex subset of X, with $U \cap X_0 = U_0 \neq \emptyset$. Then the function $f_0 \colon \overline{U}_0 \to X$ given by $f_0(x) = f(x)$, for each $x \in \overline{U}_0$, is an essential map in $K_{\partial_{X_0} U_0}(\overline{U}_0, X_0)$.*

Proof. Let $\partial U_0 = \partial_{X_0} U_0$ be the boundary of U_0 in X_0. Assume that

$$g_0 \in K_{\partial U_0}(\overline{U}_0, X_0)$$

is an arbitrary map such that $g_0(x) = f_0(x)$ for every $x \in \partial_{X_0} U_0$. Since ∂U_0 is a closed subset of ∂U, the map $g : \overline{U}_0 \cup \partial U \to X_0$ defined by

$$g(x) = \begin{cases} g_0(x) & \text{if } x \in \overline{U}_0, \\ f(x) & \text{if } x \in \partial U, \end{cases}$$

is continuous and agrees with f on the boundary ∂U. By applying Theorem 3.3, we get a continuous and compact extension $\widetilde{g} \colon \overline{U} \to X_0$ of g.

Now, from the essentiality of f, we deduce that $\operatorname{Fix}(\widetilde{g}) \neq \emptyset$. Since $\operatorname{Fix}(\widetilde{g}) \subset X_0$ we obtain

$$\operatorname{Fix}(\widetilde{g}) = \operatorname{Fix}(g_0) \neq \emptyset.$$

This implies that f_0 is an essential map, which completes the proof. □

Proposition 5.7 (Localization property). *Set $U_r = B_X(x_0, r)$, $r > 0$. Suppose that $f \in K_{\partial U_r}(\overline{U}_r, X)$ is an essential map with the unique fixed point x_0. Then for every $0 < r_0 < r$ the map $f_0 \colon \overline{U}_{r_0} \to X$ given by $f_0(x) = f(x)$, for every $x \in \overline{U}_{r_0}$, is essential in $K_{\partial U_{r_0}}(\overline{U}_{r_0}, X)$.*

Proof. In the contrary case there exists a map $\widetilde{f}_0 \in K_{\partial U_{r_0}}(\overline{U}_{r_0}, X)$, satisfying $\widetilde{f}_0(x) = f_0(x)$ for every $x \in \partial U_{r_0}$, such that $\operatorname{Fix}(\widetilde{f}_0) = \emptyset$. Define $\widetilde{f} : \overline{U}_r \to X$ by

$$\widetilde{f}(x) = \begin{cases} \widetilde{f}_0(x), & \text{if } x \in \overline{U}_{r_0}, \\ f(x), & \text{if } x \in \overline{U}_r \setminus U_{r_0}. \end{cases}$$

The map $\widetilde{f}$ is well defined and continuous, for $\widetilde{f}_0$ and f agree on ∂U_{r_0}. Moreover, $\widetilde{f} \in K_{\partial U_r}(\overline{U}_r, X)$, since $\widetilde{f}(x) = f(x)$ for each $x \in \partial U_r$. As f is essential, for some $x_0 \in U_r$ we have $x_0 = f(x_0)$. This yields a contradiction, completing the proof. □

Lemma 5.8. *Let $f \in K_{\partial U}(\overline{U}, X)$. Then the following properties are equivalent:*

(i) *f is inessential,*
(ii) *there is a fixed point free map $g \in K_{\partial U}(\overline{U}, X)$ such that f and g are homotopic in $K_{\partial U}(\overline{U}, X)$,*
(iii) *there exists a fixed point free map $\widetilde{f} \in K_{\partial U}(\overline{U}, X)$ and a homotopy $\widetilde{h} : \overline{U} \times [0,1] \to X$ in $K_{\partial U}(\overline{U}, X)$, joining f and $\widetilde{f}$, satisfying the following property*

$$\widetilde{h}(x,t) = f(x), \quad \textit{for every } x \in \partial U,\ t \in [0,1]. \tag{5.2}$$

Proof. (i) $\Rightarrow$ (ii). Since f is inessential there is a fixed point free map $g \in K_{\partial U}(\overline{U}, X)$ such that $g(x) = f(x)$, for every $x \in \partial U$. Define $h: \overline{U} \times [0,1] \to X$ by

$$h(x,t) = \alpha(f(x), g(x), t), \quad \text{for every } x \in \overline{U},\ t \in [0,1].$$

Clearly h is continuous, compact and satisfies conditions (ii) and (iii) of Definition 5.1. Moreover h satisfies also (i), because $f(x) = g(x)$, if $x \in \partial U$, and thus

$$h(x,t) = \alpha(f(x), g(x), t) = f(x) \neq x, \quad \text{for every } x \in \partial U \text{ and } t \in [0,1].$$

Hence f and g are homotopic in $K_{\partial U}(\overline{U}, X)$.

(ii) $\Rightarrow$ (iii). Let $h: \overline{U} \times [0,1] \to X$ be a homotopy in $K_{\partial U}(\overline{U}, X)$ joining f and g, with $h(x,0) = g(x)$ and $h(x,1) = f(x)$, $x \in \overline{U}$. Set

$$A = \{x \in \overline{U} \mid h(x,t) = x,\ \text{for some } t \in [0,1]\}.$$

Suppose $A = \emptyset$. Hence $\mathrm{Fix}(f) = \emptyset$ and thus, setting $\widetilde{f} = f$, the map $\widetilde{h}: \overline{U} \times [0,1] \to X$ given by

$$\widetilde{h}(x,t) = \alpha(f(x), \widetilde{f}(x), t), \qquad \text{for every } x \in \overline{U} \text{ and } t \in [0,1], \tag{5.3}$$

satisfies (5.2) and furnishes the required homotopy in $K_{\partial U}(\overline{U}, X)$, joining f and $\widetilde{f}$. Suppose $A \neq \emptyset$. Clearly A is a closed (actually compact) set with $A \cap \partial U = \emptyset$. Let $s: \overline{U} \to [0,1]$ be a Uryshon function such that $s(x) = 0$, if $x \in A$, and $s(x) = 1$, if $x \in \partial U$. Define $\widetilde{f}: \overline{U} \to X$ by

$$\widetilde{f}(x) = h(x, s(x)), \quad x \in \overline{U}$$

and observe that $\widetilde{f} \in K_{\partial U}(\overline{U}, X)$, since $\widetilde{f}$ is continuous, compact and satisfies $\widetilde{f}(x) = h(x,1) \neq x$, for each $x \in \partial U$. Moreover, $\widetilde{f}$ is fixed point free. In the contrary case, for some $x \in U$ one has $x = \widetilde{f}(x) = h(x, s(x))$, which implies that $x \in A$. Hence $s(x) = 0$ and thus

$$x = h(x,0) = g(x),$$

a contradiction as g is fixed point free.

Define now $\widetilde{h}\colon \overline{U} \times [0,1] \to X$ by

$$\widetilde{h}(x,t) = h(x,(1-t)+ts(x)), \quad \text{for every } x \in \overline{U} \text{ and } t \in [0,1].$$

This map is continuous, compact and satisfies the following properties:

(a_1) $\widetilde{h}(x,0) = h(x,1) = f(x)$, $x \in \overline{U}$,
(a_2) $\widetilde{h}(x,1) = h(x,s(x)) = \widetilde{f}(x)$, $x \in \overline{U}$,
(a_3) if $x \in \partial U$ then $s(x) = 1$, and hence

$$\widetilde{h}(x,t) = h(x,(1-t)+ts(x)) = h(x,1) = f(x).$$

Thus $\widetilde{h}$ is the required homotopy in $K_{\partial U}(\overline{U},X)$ which joins f and $\widetilde{f}$ and satisfies (5.2).

(iii) $\Rightarrow$ (i). In fact, let $\widetilde{h}\colon \overline{U}\times[0,1] \to X$ be a homotopy in $K_{\partial U}(\overline{U},X)$ joining f and $\widetilde{f}$, with $\widetilde{h}(x,0) = f(x)$ and $\widetilde{h}(x,1) = \widetilde{f}(x)$, $x \in \overline{U}$, such that (5.2) holds. Since

$$\widetilde{f}(x) = \widetilde{h}(x,1) = f(x), \quad \text{for each } x \in \partial U$$

and $\widetilde{f}$ is fixed point free, it follows that f is inessential. This completes the proof. □

Proposition 5.9 (Homotopy property). *Let $f, g \in K_{\partial U}(\overline{U},X)$ be homotopic in $K_{\partial U}(\overline{U},X)$. Then f is essential if and only if g is essential.*

Proof. Suppose that f is inessential. Then, by Lemma 5.8(ii), there exists a fixed point free map $\varphi \in K_{\partial U}(\overline{U},X)$ homotopic to f in $K_{\partial U}(\overline{U},X)$. Since $f \sim_{\partial U} g$, it follows that $\varphi \sim_{\partial U} g$. Hence, by Lemma 5.8(iii), there exists a fixed point free map $\varphi^* \in K_{\partial U}(\overline{U},X)$ homotopic to g in $K_{\partial U}(\overline{U},X)$, such that $\varphi^*(x) = g(x)$ for every $x \in \partial U$. Consequently g is inessential and the proof is complete. □

Proposition 5.10 (Normalization property). *Let $f \in K_{\partial U}(\overline{U},X)$ be a map given by $f(x) = x_0$ for each $x \in \overline{U}$, where $x_0 \in X$. Then f is essential if and only if $x_0 \in U$.*

Proof. It suffices to show that, if $x_0 \in U$, then f is essential (the reverse implication is obvious). Let $g \in K_{\partial U}(\overline{U},X)$ satisfy $g(x) = f(x)$, $x \in \partial U$. Define $\widetilde{g}\colon X \to X$ by

$$\widetilde{g}(x) = \begin{cases} g(x) & \text{for } x \in \overline{U}, \\ x_0 & \text{for } x \in X \setminus \overline{U}. \end{cases}$$

It is evident that $\widetilde{g}$ is compact and continuous and thus, by Schauder's fixed point theorem (Corollary 5.3), $\operatorname{Fix}(\widetilde{g}) \neq \emptyset$. This implies that $\operatorname{Fix}(g) \neq \emptyset$, completing the proof. □

Proposition 5.11 (Perturbation property). *Let $f \in K_{\partial U}(\overline{U}, X)$ be an essential map. Then there exists an $\varepsilon > 0$ such that each $g \in K(\overline{U}, X)$ satisfying*

$$\rho_{\partial U}(g, f) = \sup_{x \in \partial U} d(g(x), f(x)) < \varepsilon \tag{5.4}$$

is an essential map.

Proof. Since $\overline{f(\partial U)}$ is compact, ∂U closed and $\overline{f(\partial U)} \cap \partial U = \emptyset$, there is an $\varepsilon_0 > 0$ such that each $g \in K(\overline{U}, X)$, with $\rho_{\partial U}(g, f) < \varepsilon_0$, is in $K_{\partial U}(\overline{U}, X)$.

We claim that there exists $0 < \varepsilon < \varepsilon_0$ such that, for each $g \in K(\overline{U}, X)$ satisfying (5.4) the map $h_g \colon \overline{U} \times [0,1] \to X$ given by $h_g(x,t) = \alpha(f(x), g(x), t)$ is an α-homotopy in $K_{\partial U}(\overline{U}, X)$ joining f and g. For this it suffices to show that for any such g one has $x \neq h_g(t, x)$, for every $x \in \partial U$ and $t \in [0,1]$. Arguing by contradiction, assume that there exist a sequence $\{g_n\} \subset K(\overline{U}, X)$, with $\rho_{\partial U}(g_n, f) \to 0$ as $n \to \infty$, and corresponding sequences $\{x_n\} \subset \partial U$ and $\{t_n\} \subset [0,1]$, such that

$$x_n = \alpha(f(x_n), g_n(x_n), t_n), \quad \text{for all } n \in \mathbb{N}. \tag{5.5}$$

Passing to subsequences, without change of notation, we can assume that for some $x \in \overline{U}$ and $t \in [0,1]$ we have $f(x_n) \to x$ and $t_n \to t$, as $n \to \infty$. Moreover, $g_n(x_n) \to x$ as $n \to \infty$, because

$$d(g_n(x_n), x) \leq d(g_n(x_n), f(x_n)) + d(f(x_n), x) \leq \rho_{\partial U}(g_n, f) + d(f(x_n), x),$$

for $n \in \mathbb{N}$. Letting $n \to \infty$, (5.5) implies that $x_n \to x$, for $\alpha(x, x, t) = x$. From $f(x_n) \to f(x)$ and $f(x_n) \to x$ one has $x = f(x)$. Moreover, $x \in \partial U$ and thus, from the contradiction, the claim follows.

Let $g \in K(\overline{U}, X)$ satisfy (5.3). As $\varepsilon < \varepsilon_0$ one has $g \in K_{\partial U}(\overline{U}, X)$ and, by the claim, $h_g \colon \overline{U} \times [0,1] \to X$ is an α-homotopy in $K_{\partial U}(\overline{U}, X)$ joining f and g. Then by Proposition 5.9 g is essential, completing the proof. □

The above results on the topological essentiality are now applied to study the equation $x = f(x)$ in an α-weakly convex metric space, where f is continuous and compact or, more generally, completely continuous.

In the sequel, X is an α-weakly convex metric space, with the convexity mapping $\alpha \colon X \times X \times [0,1] \to X$, and $U \subset X$ is a nonempty open and bounded set.

Theorem 5.12 (Nonlinear alternative). *Let $f \in K_{\partial U}(\overline{U}, X)$ and let $u_0 \in U$. Then at least one of the following properties holds:*

(i) *there exist $x_0 \in \partial U$ and $t \in (0,1)$ such that $x_0 = \alpha(u_0, f(x_0), t)$,*
(ii) $\mathrm{Fix}(f) \neq \emptyset$.

Proof. It suffices to show that, if (i) does not hold, one has $\mathrm{Fix}(f) \neq \emptyset$. Suppose that $x \neq \alpha(u_0, f(x), t)$ for every $x \in \partial U$ and $t \in [0,1]$. Here $t = 0$ and $t = 1$ have been included, since $x \in \partial U$ implies $x \neq u_0 = \alpha(u_0, f(x), 0)$ and $x \neq f(x) = \alpha(u_0, f(x), 1)$. Define $h\colon \overline{U} \times [0,1] \to X$ by

$$h(x,t) = \alpha(g(x), f(x), t), \quad x \in \overline{U},\ t \in [0,1],$$

where $g\colon \overline{U} \to X$ is given by $g(x) = u_0$, for each $x \in \overline{U}$. Clearly h is a homotopy in $K_{\partial U}(\overline{U}, X)$ joining g and f. By Proposition 5.10 g is essential and hence $\mathrm{Fix}(f) \neq \emptyset$, completing the proof. □

Remark 5.13. Let E be a normed space endowed with the natural convexity map

$$\alpha(x,y,t) = (1-t)x + ty, \quad x, y \in E,\ t \in [0,1],$$

and suppose that U is an open and bounded subset of E containing the origin 0 of E. Then, by taking $u_0 = 0$, one has the usual formulation of the above property (i), that is: $x_0 = tf(x_0)$ for some $x_0 \in \partial U$ and $t \in (0,1)$.

Theorem 5.14 (Leray–Schauder alternative). *Let $f \in C(X,X)$. For $u_0 \in X$, set*

$$\mathcal{E}(f) = \{x \in X \mid x = \alpha(u_0, f(x), t),\ \textit{for some } t \in (0,1)\}.$$

Then $\mathcal{E}(f)$ is unbounded or $\mathrm{Fix}(f) \neq \emptyset$.

Proof. It suffices to show that $\mathcal{E}(f)$ bounded implies $\mathrm{Fix}(f) \neq \emptyset$. In fact, if $\mathcal{E}(f)$ is bounded, for some $r > 0$ one has $\mathcal{E}(f) \subset U$, where $U = B_X(u_0, r)$. Define $f_0\colon \overline{U} \to X$ by $f_0(x) = f(x)$, $x \in \overline{U}$. As $f_0 \in K(\overline{U}, X)$ and, for this map, property (i) of Theorem 5.12 is not satisfied, it follows that $\mathrm{Fix}(f_0) \neq \emptyset$. Hence $\mathrm{Fix}(f) \neq \emptyset$, completing the proof. □

References

[1] K. Borsuk, *Theory of retracts*, PWN, Warszawa, 1967.

[2] A. Cellina, *A theorem on the approximation of compact multivalued mappings*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) **47** (1969), 429–433.

[3] D. W. Curtis, *Application of a selection theorem to hyperspace contractibility*, Canad. J. Math. **27** (1985), 747–759.

[4] F. S. De Blasi and G. Pianigiani, *Approximate selections in α-convex metric spaces and topological degree*, Topol. Methods Nonlinear Anal. **24** (2004), 347–375.

[5] ———, *Continuous selections in α-convex metric spaces*, Bull. Pol. Acad. Sci. Math. **52** (2004), 303–317.

[6] J. Dugundji, *Topology*, Allyn and Bacon, Boston, 1966.

[7] R. Fox, *On fibre spaces* II, Bull. Amer. Math. Soc. **49** (1943), 733–735.

[8] L. Górniewicz, *On the Lefschetz fixed point theorem*, Handbook of Topological Fixed Point Theory, Springer, Dordrecht, 2005, pp. 43–82.

[9] ———, *Topological Fixed Point Theory of Multivalued Mappings*, Springer–Verlag, Berlin, 2006.

[10] A. Granas, *The theory of compact vector fields and some of its applications to topology of function spaces*, Dissertationes Mathematicae, vol. 36, Warszawa, 1962.

[11] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer–Verlag, Berlin, 2003.

[12] C. J. Himmelberg, *Some theorems on equiconnected and locally equiconnected spaces*, Trans. Amer. Math. Soc. **115** (1965), 43–53.

[13] E. A. Michael, *Continuous selections* I, Ann. Math. **63** (1956), 361–382.

[14] ———, *Convex structures and continuous selections*, Canad. J. Math. **11** (1959), 556–575.

[15] L. Pasicki, *Retracts in metric spaces*, Proc. Amer. Math. Soc. **78** (1980), 595–600.

[16] J. P. Serre, *Homologie singuliére des espace fibrés*, Ann. of Math. **54** (1951), 425–505.

[17] J. H. C. Whitehead, *Convex regions in the geometry of paths*, Quart. J. Math. **3** (1932), 33–42.

[18] A. Wieczorek, *Compressibility: A property of topological spaces related to abstract convexity*, J. Math. Anal. Appl. **161** (1991), 9–19.

Manuscript received January 5, 2009

Francesco S. de Blasi
Dipartimento di Matematica
Università di Roma "Tor Vergata"
Via della Ricerca Scientifica 1
00133 Roma, ITALY

E-mail address: deblasi@mat.uniroma2.it

Lech Górniewicz
Schauder Center for Nonlinear Studies
Nicolaus Copernicus University
Chopina 12/18
87-100 Toruń, POLAND

E-mail address: gorn@mat.uni.torun.pl

Giulio Pianigiani
Dipartimento di Matematica per le Decisioni
Università di Firenze
Via Lombroso 6/17
50134 Firenze, ITALY

E-mail address: giulio.pianigiani@unifi.it